

Spectral theory and applications. An elementary introductory course. Version 2008-2009

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Abstract

We intend to present in this course the basic tools in spectral analysis and to illustrate the theory by presenting examples coming from the Schrödinger operator theory and from various branches of physics : statistical mechanics, superconductivity, fluid mechanics.

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1 Introduction

Our starting point could be the theory of Hermitian matrices, that is of the matrices satisfying $A^* = A$. If we look for eigenvalues and corresponding eigenvectors of A , that is for pairs (u, λ) with $u \in \mathbb{C}^k$, $u \neq 0$ and $\lambda \in \mathbb{C}$ such that $Au = \lambda u$, we know that the eigenvalues are real and that one can find an orthonormal basis of eigenvectors associated with real eigenvalues.

In order to extend this theory to the case of spaces with infinite dimension (that is replacing the space \mathbb{C}^m by a general Hilbert space \mathcal{H}), the first attempt consists in developing the theory of compact selfadjoint operators. But it is far to cover all the interesting cases that are present in Quantum Mechanics. So our aim is to present a general theory but it is perhaps good to start by looking at specific operators and to ask naive questions about the existence of pairs (u, λ) with u in some suitable domain, $u \neq 0$ and $\lambda \in \mathbb{C}$ such that $Au = \lambda u$. We shall discover in particular that the answer at these questions may depend strongly on the choice of the domain and on the precise definition of the operator.

1.1 The free Laplacian

The Laplacian $-\Delta$ has no eigenvectors in L^2 , but it has for any $\lambda \in \mathbb{R}^+$ an eigenvector in $\mathcal{S}'(\mathbb{R}^m)$ (actually in L^∞) and for any $\lambda \in \mathbb{C}$ an eigenvector in $\mathcal{D}'(\mathbb{R}^m)$. So what is the right way to extend the theory of Hermitian matrices on \mathbb{C}^k ?

On the other hand, it is easy to produce (take for simplicity $m = 1$) approximate eigenvectors in the form $u_n(x) = \frac{1}{\sqrt{n}} \exp ix \cdot \xi \chi(\frac{x-n^2}{n})$, where χ is a compactly supported function of L^2 -norm equal to 1.

1.2 The harmonic oscillator

As we shall see the harmonic oscillator

$$H = -d^2/dx^2 + x^2$$

plays a central role in the theory of quantum mechanics. When looking for eigenvectors in $\mathcal{S}(\mathbb{R})$, we obtain that there is a sequence of eigenvalues λ_n ($n \in \mathbb{N}$)

$$\lambda_n = (2n - 1) .$$

In particular the fundamental level (in other words the lowest eigenvalue) is

$$\lambda_1 = 1$$

and the splitting between the two first eigenvalues is 2.

The first eigenfunction is given by

$$\phi_1(x) = c_1 \exp -\frac{x^2}{2} \tag{1.2.1}$$

and the other eigenvectors are obtained by applying the creation operator

$$L^+ = -d/dx + x . \quad (1.2.2)$$

We observe indeed that

$$H = L^+ \cdot L^- + 1 , \quad (1.2.3)$$

where

$$L^- = d/dx + x , \quad (1.2.4)$$

and has the property

$$L^- \phi_1 = 0 . \quad (1.2.5)$$

Note that if $u \in L^2$ is a distributional solution of $L^+u = 0$, then $u = 0$. Note also that if $u \in L^2$ is a distributional solution of $L^-u = 0$, then $u = \mu\phi_1$ for some $\mu \in \mathbb{R}$.

The n^{th} -eigenfunction is then given by

$$\phi_n = c_n (L^+)^{n-1} \phi_1 .$$

This can be shown by recursion using the identity

$$L^+(H + 2) = HL^+ . \quad (1.2.6)$$

It is easy to see that $\phi_n(x) = P_n(x) \exp -\frac{x^2}{2}$ where $P_n(x)$ is a polynomial of order $n - 1$. One can also show that the ϕ_n are mutually orthogonal. The proof of this point is identical to the finite dimensional case, if we observe the following identity (expressing that H is symmetric) :

$$\langle Hu, v \rangle_{L^2} = \langle u, Hv \rangle_{L^2} , \quad \forall u \in \mathcal{S}(\mathbb{R}), \forall v \in \mathcal{S}(\mathbb{R}) , \quad (1.2.7)$$

which is obtained through an integration by parts.

Then it is a standard exercise to show that the family $(\phi_n)_{n \in \mathbb{N}}$ is total and that we have obtained an orthonormal hilbertian basis of L^2 , which in some sense permits to diagonalize the operator H .

Another way to understand the completeness is to show that starting of an eigenvector u in $\mathcal{S}'(\mathbb{R})$ associated with $\lambda \in \mathbb{R}$ solution (in the sense of distribution) of

$$Hu = \lambda u ,$$

then there exists $k \in \mathbb{N}$ and $c_k \neq 0$ such that $(L^-)^k u = c_k \phi_1$ and that the corresponding λ is equal to $(2k + 1)$.

For this proof, we admit that any eigenvector can be shown to be in $\mathcal{S}(\mathbb{R})$ and use the identity

$$L^-(H - 2) = HL^- , \quad (1.2.8)$$

and the inequality

$$\langle Hu, u \rangle \geq 0 , \quad \forall u \in \mathcal{S}(\mathbb{R}) . \quad (1.2.9)$$

This property is called "positivity" of the operator. Actually one can show by various ways that

$$\langle Hu, u \rangle \geq \|u\|^2, \forall u \in \mathcal{S}(\mathbb{R}). \quad (1.2.10)$$

One way is to first establish the Heisenberg Principle¹ :

$$\|u\|_{L^2(\mathbb{R})}^2 \leq 2\|xu\|_{L^2}\|u'\|_{L^2}, \forall u \in \mathcal{S}(\mathbb{R}). \quad (1.2.11)$$

The trick is to observe the identity

$$1 = \frac{d}{dx} \cdot x - x \cdot \frac{d}{dx}. \quad (1.2.12)$$

The inequality (1.2.10) is simply the consequence of the identity

$$\langle Hu, u \rangle = \|u'\|^2 + \|xu\|^2, \quad (1.2.13)$$

which is proved through an integration by parts, and of the application in (1.2.11) of Cauchy-Schwarz inequality.

Another way is to directly observe the identity

$$\langle Hu, u \rangle = \|L^-u\|^2 + \|u\|^2, \forall u \in \mathcal{S}. \quad (1.2.14)$$

1.3 The problem of the boundary

We mainly consider the operator $-\frac{d^2}{dx^2}$ and look at various problems that can be asked naively about the existence of eigenvectors for the problem in $L^2(]0, 1[)$.

1.3.1 Ill-posed problems

Look first at pairs $(u, \lambda) \in H^1(]0, 1[) \times \mathbb{C}$ ($u \neq 0$) such that

$$-du/dx = \lambda u, u(0) = 0.$$

It is immediate to see that no such pairs exist.

Look now at pairs $(u, \lambda) \in H^2(]0, 1[) \times \mathbb{C}$ ($u \neq 0$) such that

$$-d^2u/dx^2 = \lambda u.$$

We can find for any λ two linearly independent solutions.

¹Here is a more "physical" version. If u is normalized by $\|u\|_{L^2(\mathbb{R})} = 1$, $|u|^2 dx$ defines a probability measure. One can define $\langle x \rangle = \int x|u|^2 dx$, mean value of the position and the variance $\sigma_x = \langle (x - \langle x \rangle)^2 \rangle$. Similarly, we can consider: $\langle D_x \rangle := \int (D_x u) \cdot \bar{u}(x) dx$ and $\sigma_{D_x} := \|(D_x - \langle D_x \rangle)u\|^2$. Then (1.2.11) can be extended in the form :

$$\sigma_x \cdot \sigma_{D_x} \geq \frac{1}{4}.$$

1.3.2 The periodic problem

Here we consider pairs $(u, \lambda) \in H^{2,per}([0, 1]) \times \mathbb{C}$ ($u \neq 0$) such that

$$-d^2u/dx^2 = \lambda u .$$

Here

$$H^{2,per}([0, 1]) = \{u \in H^2([0, 1]) , u(0) = u(1) \text{ and } u'(0) = u'(1)\} .$$

Here we recall that $H^2([0, 1])$ is included in $C^1([0, 1])$ by the injection theorem of Sobolev. It is an easy exercise to show that the pairs are described by two families

- $\lambda = 4\pi^2 n^2 , u_n = \mu \cos 2\pi n x , \text{ for } n \in \mathbb{N} , \mu \in \mathbb{R} \setminus 0 ,$
- $\lambda = 4\pi^2 n^2 , v_n = \mu \sin 2\pi n x , \text{ for } n \in \mathbb{N}^* , \mu \in \mathbb{R} \setminus 0 .$

One observes that $\lambda = 0$ is the lowest eigenvalue and that its multiplicity is one. This means that the corresponding eigenspace is of dimension one (the other eigenspaces are of dimension 2). Moreover an eigenvector in this subspace neither vanishes in $]0, 1[$. This is quite evident because $u_0 = \mu \neq 0$.

One observes also that one can find an orthonormal basis in $L^2([0, 1])$ of eigenvectors by normalizing the family $(\cos 2\pi n x (n \in \mathbb{N}), \sin 2\pi n x (n \in \mathbb{N}^*))$ or the family $\exp 2\pi i n x (n \in \mathbb{Z})$.

We are just recovering the L^2 -theory of the Fourier series.

1.3.3 The Dirichlet problem

Here we consider pairs $(u, \lambda) \in H^{2,D}([0, 1]) \times \mathbb{C}$ ($u \neq 0$) such that $-d^2u/dx^2 = \lambda u$.

Here

$$H^{2,D}([0, 1]) = \{u \in H^2([0, 1]) , u(0) = u(1) = 0\} .$$

It is again an easy exercise to show that the pairs are described by

$$\lambda = \pi^2 n^2 , v_n = \mu \sin \pi n x , \text{ for } n \in \mathbb{N}^* , \mu \in \mathbb{R} \setminus 0 .$$

One observes that $\lambda = \pi^2$ is the lowest eigenvalue, that its multiplicity is one (Here all the eigenspaces are one-dimensional) and that an eigenvector in this subspace neither vanishes in $]0, 1[$.

1.3.4 The Neumann problem

Here we consider pairs

$(u, \lambda) \in H^{2,N}([0, 1]) \times \mathbb{C}$ ($u \neq 0$) such that

$$-d^2u/dx^2 = \lambda u .$$

Here

$$H^{2,N}(\]0, 1]) = \{u \in H^2(\]0, 1]) , u'(0) = u'(1) = 0\} .$$

It is again an easy exercise to show that the pairs are described by

$$\lambda = \pi^2 n^2 , v_n = \mu \cos \pi n x , \text{ for } n \in \mathbb{N} , \mu \in \mathbb{R} \setminus 0 .$$

One observes that $\lambda = 0$ is the lowest eigenvalue, that its multiplicity is one (Here all the eigenspaces are one-dimensional) and that the corresponding eigenspace is of dimension one and that an eigenvector in this subspace neither vanishes in $\]0, 1[$.

1.3.5 Conclusion

All these examples enter in the so called Sturm-Liouville theory. We emphasize on one property which was always verified in each case: the eigenspace corresponding to the lowest eigenvalue is one dimensional and one can find a strictly positive (in $\]0, 1[$ or in $\] - \infty , +\infty[$ in the case of the harmonic oscillator) corresponding eigenvector. **We suggest to the reader to come back at this introduction after have read the course. He will surely realize that the theory has permitted to clarify many badly posed problems.**

2 Unbounded operators, adjoints, Selfadjoint operators

2.1 Unbounded operators

We consider an Hilbert space \mathcal{H} . The scalar product will be denoted by $\langle u, v \rangle_{\mathcal{H}}$ or more simply by $\langle u, v \rangle$ when no confusion is possible. We take the convention that the scalar product is antilinear with respect to the second argument.

A linear operator (or more simply an operator) T in \mathcal{H} is a linear map $u \mapsto Tu$ defined on a subspace \mathcal{H}_0 of \mathcal{H} , denoted by $D(T)$ and which is called the domain of T . We shall also denote by $R(T)$ (or $\text{Im } T$ or $\text{Range}(T)$) the range of \mathcal{H}_0 by T . We shall say that T is bounded if it is continuous from $D(T)$ into \mathcal{H} . When $D(T) = \mathcal{H}$, we recover the notion of linear continuous operators on \mathcal{H} . We recall that with

$$\|T\|_{\mathcal{L}(\mathcal{H})} = \sup_{u \neq 0} \frac{\|Tu\|_{\mathcal{H}}}{\|u\|_{\mathcal{H}}} , \quad (2.1.1)$$

$\mathcal{L}(\mathcal{H})$ is a Banach space. When $D(T)$ is not equal to \mathcal{H} , we shall always assume that

$$D(T) \text{ is dense in } \mathcal{H}. \quad (2.1.2)$$

Note that, if T is bounded, then it admits a unique continuous extension to \mathcal{H} . In this case the generalized notion is not interesting.

We are mainly interested in extensions of this theory and would like to consider

unbounded operators.

– When using this word, we mean more precisely “non necessarily bounded operators”. –

The point is to find a natural notion replacing this notion of boundedness. This is the object of the next definition.

The operator is called **closed** if the graph $G(T)$ of T is closed in $\mathcal{H} \times \mathcal{H}$. We recall that

$$G(T) := \{(x, y) \in \mathcal{H} \times \mathcal{H}, x \in D(T), y = Tx\}. \quad (2.1.3)$$

Equivalently, we can say

Definition 2.1.1. (*Closed operators*).

Let T be an operator on \mathcal{H} with (dense) domain $D(T)$. We say that T is closed if the conditions

- $u_n \in D(T)$,
- $u_n \rightarrow u$ in \mathcal{H} ,
- $Tu_n \rightarrow v$ in \mathcal{H}

imply

- $u \in D(T)$,
- $v = Tu$.

Example 2.1.2.

1. $T_0 = -\Delta$ with $D(T_0) = C_0^\infty(\mathbb{R}^m)$ is not closed.

For this, it is enough to consider² some u in $H^2(\mathbb{R}^m)$ and not in $C_0^\infty(\mathbb{R}^m)$ and to consider a sequence $u_n \in C_0^\infty$ such that $u_n \rightarrow u$ in H^2 . The sequence $(u_n, -\Delta u_n)$ is contained in $G(T_0)$ and converges in $L^2 \times L^2$ to $(u, -\Delta u)$ which does not belong to $G(T_0)$.

2. $T_1 = -\Delta$ with $D(T_1) = H^2(\mathbb{R}^m)$ is closed.

We observe indeed that if $u_n \rightarrow u$ in L^2 and $(-\Delta u_n) \rightarrow v$ in L^2 then $-\Delta u = v \in L^2$. The last step is to observe that this implies that $u \in H^2(\mathbb{R}^m)$ (take the Fourier transform) and $(u, -\Delta u) \in G(T_1)$.

²We recall that the Sobolev space $H^s(\mathbb{R}^m)$ is defined as the space

$$H^s(\mathbb{R}^m) := \{u \in \mathcal{S}'(\mathbb{R}^m) \mid (1 + |\xi|^2)^{\frac{s}{2}} \hat{u} \in L^2(\mathbb{R}^m)\}.$$

Here \mathcal{S}' is the set of tempered distributions. $H^s(\mathbb{R}^m)$ is equipped with the natural Hilbertian norm :

$$\|u\|_{H^s}^2 := \int_{\mathbb{R}^m} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi.$$

By Hilbertian norm, we mean that the norm is associated to a scalar product.

When $s \in \mathbb{N}$, we can also describe H^s by

$$H^s(\mathbb{R}^m) := \{u \in L^2(\mathbb{R}^m) \mid D_x^\alpha u \in L^2, \forall \alpha \text{ s. t. } |\alpha| \leq s\}.$$

The natural norm associated with the second definition is equivalent to the first one.

This example suggests another definition.

Definition 2.1.3.

The operator T is called **closable** if the closure of the graph of T is a graph.

We can then define the closure \overline{T} of the operator by a limit procedure via its graph. We observe indeed that we can consider

$$D(\overline{T}) := \{x \in \mathcal{H} \mid \exists y \text{ s. t. } (x, y) \in \overline{G(T)}\}.$$

For any $x \in D(\overline{T})$, the assumption that $\overline{G(T)}$ is a graph says that y is unique. One can consequently define \overline{T} by

$$\overline{T}x = y .$$

In a more explicit way, the domain of \overline{T} is the set of the $x \in \mathcal{H}$ such that $x_n \rightarrow x \in \mathcal{H}$ and Tx_n is a Cauchy sequence and for such x 's we define \overline{T} by

$$\overline{T}x = \lim_{n \rightarrow +\infty} Tx_n .$$

Example 2.1.4.

$T_0 = -\Delta$ with $D(T_0) = C_0^\infty$ is closable and its closure is T_1 .

Let us prove it, as an exercise. Let $\overline{T_0}$ the closure of T_0 . Let $u \in L^2$ such that there exists $u_n \in C_0^\infty$ such that $u_n \rightarrow u$ in L^2 and $-\Delta u_n \rightarrow v$ in L^2 . We get by distribution theory that $u \in L^2$ satisfies $-\Delta u = v \in L^2$. By the ellipticity of the Laplacian (use the Fourier transform), we get that $u \in H^2$. We have consequently shown that $D(\overline{T_0}) \subset H^2$. But C_0^∞ is dense in H^2 and this gives the inverse inclusion : $H^2 \subset D(\overline{T_0})$. We have consequently,

$$H^2 = D(T_1) = D(\overline{T_0}) ,$$

and it is then easy to verify that $T_1 = \overline{T_0}$.

These examples lead to a more general question.

Realization of differential operators as unbounded operators. Let $\Omega \subset \mathbb{R}^n$ and let $P(x, D_x)$ be a partial differential operator with C^∞ coefficients in Ω . Then the operator P^Ω defined by

$$D(P^\Omega) = C_0^\infty(\Omega) , P^\Omega u = P(x, D_x)u, \forall u \in C_0^\infty(\Omega) ,$$

is closable. Here $\mathcal{H} = L^2(\Omega)$. We have indeed

$$\overline{G(P^\Omega)} \subset \tilde{G}_\Omega := \{(u, f) \in \mathcal{H} \times \mathcal{H} \mid P(x, D_x)u = f \text{ in } \mathcal{D}'(\Omega)\} .$$

The proof is then actually a simple exercise in distribution theory. This inclusion shows that $\overline{G(P^\Omega)}$ is a graph. Note that the corresponding operator is defined as P_{min}^Ω with domain

$$D(P_{min}^\Omega) = \left\{ u \in L^2(\Omega) \mid \exists \text{ a sequence } u_n \in C_0^\infty(\Omega) \text{ s.t. } \begin{cases} u_n \rightarrow u \text{ in } L^2(\Omega) \\ P(x, D_x)u_n \text{ converges in } L^2(\Omega) \end{cases} \right\}.$$

The operator P_{min}^Ω is then defined for such u by

$$P_{min}^\Omega u = \lim_{n \rightarrow +\infty} P(x, D_x)u_n.$$

Using the theory of distributions, this gives :

$$P_{min}^\Omega u = P(x, D_x)u.$$

Note that there exists also a natural closed operator whose graph is \tilde{G}_Ω and extending P^Ω : this is the operator \tilde{P}^Ω , with domain

$$\tilde{D}^\Omega := \{u \in L^2(\Omega), P(x, D_x)u \in L^2(\Omega)\},$$

and such that

$$\tilde{P}^\Omega u = P(x, D_x)u, \forall u \in \tilde{D}^\Omega,$$

where the last equality is in the distributional sense. Note that \tilde{P}^Ω is an extension of P_{min}^Ω in the sense that :

$$\tilde{P}^\Omega u = P_{min}^\Omega u, \forall u \in D(P_{min}^\Omega).$$

Conclusion.

We have associated with a differential operator $P(x, D_x)$ in an open set Ω three natural operators. It would be important to know better the connection between these three operators.

Remark 2.1.5. (*Link between continuity and closeness*).

If $\mathcal{H}_0 = \mathcal{H}$, the closed graph Theorem says that a closed operator T is continuous.

2.2 Adjoints.

When we have an operator T in $\mathcal{L}(\mathcal{H})$, it is easy to define the Hilbertian adjoint T^* by the identity :

$$\langle T^*u, v \rangle_{\mathcal{H}} = \langle u, Tv \rangle_{\mathcal{H}}, \forall u \in \mathcal{H}, \forall v \in \mathcal{H}. \quad (2.2.1)$$

The map $v \mapsto \langle u, Tv \rangle_{\mathcal{H}}$ defines a continuous antilinear map on \mathcal{H} and can be expressed, using Riesz's Theorem, by the scalar product by an element which is called T^*u . The linearity and the continuity of T^* is then easily proved using (2.2.1).

Let us now give the definition of the adjoint of an unbounded operator.

Definition 2.2.1. (*Adjoint*)

If T is an unbounded operator on \mathcal{H} whose domain $D(T)$ is dense in \mathcal{H} , we first define the domain of T^* by

$$D(T^*) = \{u \in \mathcal{H}, D(T) \ni v \mapsto \langle u, Tv \rangle, \\ \text{can be extended as an antilinear continuous form on } \mathcal{H}\} .$$

Using the Riesz's Theorem, there exists $f \in \mathcal{H}$ such that

$$(f, v) = (u, Tv) , \forall u \in D(T^*), \forall v \in D(T) .$$

The uniqueness of f is a consequence of the density of $D(T)$ in \mathcal{H} and we can then define T^*u by

$$T^*u = f .$$

Remark 2.2.2.

When $D(T) = \mathcal{H}$ and if T is bounded, then we recover as T^* the Hilbertian adjoint.

Example 2.2.3.

$$T_0^* = T_1 .$$

Let us treat in detail this example. We get

$$D(T_0^*) = \{u \in L^2 \mid \text{the map } C_0^\infty \ni v \mapsto \langle u, -\Delta v \rangle, \\ \text{can be extended as an antilinear continuous form on } L^2\} .$$

We observe that

$$\langle u, -\Delta v \rangle_{L^2} = \int_{\mathbb{R}^m} u(\overline{-\Delta v}) dx = (-\Delta u)(\bar{v}) .$$

The last equality just means that we are considering the distribution $(-\Delta u)$ on the test function \bar{v} . The condition appearing in the definition is just that this distribution is in $L^2(\mathbb{R}^m)$. Coming back to the definition of $D(T_0^*)$, we get

$$D(T_0^*) = \{u \in L^2 \mid -\Delta u \in L^2\} .$$

But as already seen, this gives

$$D(T_0^*) = H^2 , T_0^*u = -\Delta u , \forall u \in H^2 .$$

Proposition 2.2.4. .

T^* is a closed operator.

Proof.

Let (v_n) be a sequence in $D(T^*)$ such that $v_n \rightarrow v$ in \mathcal{H} and $T^*v_n \rightarrow w^*$ in \mathcal{H} for some pair (v, w^*) . We would like to show that (v, w^*) belongs to the graph of T^* .

For all $u \in D(T)$, we have :

$$\langle Tu, v \rangle = \lim_{n \rightarrow +\infty} \langle Tu, v_n \rangle = \lim_{n \rightarrow +\infty} \langle u, T^*v_n \rangle = \langle u, w^* \rangle . \quad (2.2.2)$$

Coming back to the definition of $D(T^*)$, we get from (2.2.2) that $v \in D(T^*)$ and $T^*v = w^*$. This means that (v, w^*) belongs to the graph of T^* .

Proposition 2.2.5. .

Let T be an operator in \mathcal{H} with domain $D(T)$. Then the graph $G(T^*)$ of T^* can be characterized by

$$G(T^*) = \{V(\overline{G(T)})\}^\perp, \quad (2.2.3)$$

where V is the unitary operator defined on $\mathcal{H} \times \mathcal{H}$ by

$$V\{u, v\} = \{v, -u\}. \quad (2.2.4)$$

Proof.

We just observe that for any $u \in D(T)$ and $(v, w^*) \in \mathcal{H} \times \mathcal{H}$ we have the identity

$$\langle V(u, Tu), (v, w^*) \rangle_{\mathcal{H} \times \mathcal{H}} = \langle Tu, v \rangle_{\mathcal{H}} - \langle u, w^* \rangle_{\mathcal{H}}.$$

The right hand side vanishes for all $u \in D(T)$ iff $v \in D(T^*)$ and $w^* = T^*v$, that is if (v, w^*) belongs to $G(T^*)$. The left hand side vanishes for all $u \in D(T)$ iff (v, w^*) belongs to $V(G(T))^\perp$.

Standard Hilbertian analysis, using the continuity of V and $V^{-1} = -V$, then shows that

$$\{V(G(T))\}^\perp = \{\overline{V(G(T))}\}^\perp = \{V(\overline{G(T)})\}^\perp.$$

End of proof We have not analyzed till now under which condition the domain of the adjoint is dense in \mathcal{H} . This is one of the objects of the next theorem.

Theorem 2.2.6.

Let T be a closable operator. Then we have

1. $D(T^*)$ is dense in \mathcal{H} ,
2. $T^{**} := (T^*)^* = \overline{T}$, where we have denoted by \overline{T} the operator whose graph is $\overline{G(T)}$.

Proof.

For the first point, let us assume that $D(T^*)$ is not dense in \mathcal{H} . Then there exists $w \neq 0$ such that w is orthogonal to $\overline{D(T^*)}$.

We consequently get that for any $v \in D(T^*)$, we have

$$\langle (0, w), (T^*v, -v) \rangle_{\mathcal{H} \times \mathcal{H}} = 0.$$

This shows that $(0, w)$ is orthogonal to $V(G(T^*))$.

But the previous proposition gives :

$$V(\overline{G(T)}) = G(T^*)^\perp.$$

We now apply V to this identity and get, using $V^2 = -I$,

$$V(G(T^*)^\perp) = \overline{G(T)}.$$

But, for any closed subspace $\mathcal{M} \subset \mathcal{H}$, we have

$$V(\mathcal{M}^\perp) = [V(\mathcal{M})]^\perp ,$$

as a consequence of the identity

$$\langle V(u, v) , (x, y) \rangle_{\mathcal{H} \times \mathcal{H}} = \langle (u, v) , V(x, y) \rangle_{\mathcal{H} \times \mathcal{H}} .$$

We finally obtain that $(0, w)$ belongs to the closure of the graph of T , that is the graph of \bar{T} because T is closable, and consequently that $w = 0$. This gives the contradiction.

For the second point, we first observe that, $D(T^*)$ being dense in \mathcal{H} , we can of course define $(T^*)^*$. Using again the proposition and the closeness of T^* , we obtain $G(T^{**}) = \overline{G(T)}$ and $T^{**} = \bar{T}$.

This means more explicitly that

$$D(T^{**}) = D(\bar{T}) , T^{**}u = \bar{T}u , \forall u \in D(\bar{T}) .$$

End of the proof.

2.3 Symmetric and selfadjoint operators.

Definition 2.3.1. (*symmetric operators*).

We shall say that $T : \mathcal{H}_0 \mapsto \mathcal{H}$ is **symmetric** if it satisfies

$$\langle Tu, v \rangle_{\mathcal{H}} = \langle u, Tv \rangle_{\mathcal{H}} , \forall u, v \in \mathcal{H}_0 .$$

Example 2.3.2. .

$T = -\Delta$ with $D(T) = C_0^\infty(\mathbb{R}^m)$.

If T is symmetric it is easy to see that

$$D(T) \subset D(T^*) \tag{2.3.1}$$

and that

$$Tu = T^*u , \forall u \in D(T) . \tag{2.3.2}$$

The two conditions (2.3.1) and (2.3.2) express the property that $(T^*, D(T^*))$ is an extension of $(T, D(T))$.

Exercise 2.3.3.

Show that a symmetric operator is closable.

Hint :

Show that, if u_n is a sequence in $D(T)$ such that, for some $\ell \in \mathcal{H}$, we have $u_n \rightarrow 0$ and $Tu_n \rightarrow \ell$, then $\ell = 0$.

For a symmetric operator, we have consequently two natural closed extensions:

- The **minimal** one denoted by T_{min} (or previously \overline{T}), which is obtained by taking the operator whose graph is the closure of the graph of T ,
- The **maximal** one denoted by T_{max} the adjoint of T .

If T^{sa} is a selfadjoint extension of T , then T^{sa} is automatically an extension of T_{min} and admits³ as an extension T_{max} .

Definition 2.3.4.

We shall say that T is **selfadjoint** if $T^* = T$, i. e.

$$D(T) = D(T^*) , \text{ and } Tu = T^*u , \quad \forall u \in D(T) .$$

Starting of a symmetric operator, it is a natural question to ask for the existence and the uniqueness of a selfadjoint extension. We shall see later that a natural way is to prove the equality between T_{min} and T_{max} .

Exercise 2.3.5. Analysis of differential operators.

Give simple criteria in the case of operators with constant coefficients for obtaining symmetric operators on $C_0^\infty(\mathbb{R}^m)$. In particular, verify that the operator $D_{x_j} = \frac{1}{i}\partial_{x_j}$ is symmetric.

Proposition 2.3.6.

A selfadjoint operator is closed.

This is immediate because T^* is closed.

Proposition 2.3.7.

Let T be a selfadjoint operator which is invertible. Then T^{-1} is also selfadjoint.

By invertible, we mean here that T admits an inverse T^{-1} from $R(T)$ into $D(T)$. Let us first show that $R(T)$ is dense in \mathcal{H} . Let $w \in \mathcal{H}$ such that $\langle Tu, w \rangle_{\mathcal{H}} = 0, \forall u \in D(T)$.

Coming back to the definition of T^* , this implies in particular that $w \in D(T^*)$ and $T^*w = 0$. But T is selfadjoint and injective and this implies that $w = 0$.

We consequently know that $D(T^{-1})$ is dense in \mathcal{H} .

Coming back to the analysis of the corresponding graphs it is now easy to show the second assertion by coming back to the corresponding graphs and by using Proposition 2.2.5.

Remark 2.3.8.

If T is selfadjoint $T + \lambda I$ is selfadjoint for any real λ .

³Use Proposition 2.2.5.

3 Representation theorems

We assume that the reader knows about this material but recall it for completeness⁴.

3.1 Riesz's Theorem.

Theorem 3.1.1. (*Riesz's Theorem*)

Let $u \mapsto F(u)$ a linear continuous form on \mathcal{H} . Then there exists a unique $w \in \mathcal{H}$ such that

$$F(u) = \langle u, w \rangle_{\mathcal{H}}, \quad \forall u \in \mathcal{H}. \quad (3.1.1)$$

There is a similar version with antilinear maps :

Theorem 3.1.2.

Let $u \mapsto F(u)$ a antilinear continuous form on \mathcal{H} . Then there exists a unique $w \in \mathcal{H}$ such that

$$F(u) = \langle w, u \rangle_{\mathcal{H}}, \quad \forall u \in \mathcal{H}. \quad (3.1.2)$$

3.2 Lax-Milgram's situation.

Let us now consider a continuous sesquilinear form a defined on $V \times V$:

$$(u, v) \mapsto a(u, v).$$

We recall that the continuity can be expressed by the existence of C such that

$$|a(u, v)| \leq C \|u\|_V \cdot \|v\|_V, \quad \forall u, v \in V. \quad (3.2.1)$$

It is immediate to associate, using the Riesz Theorem, a linear map $A \in \mathcal{L}(V)$ such that

$$a(u, v) = \langle Au, v \rangle_V. \quad (3.2.2)$$

Definition 3.2.1. (*V-ellipticity*)

We shall say that a is *V-elliptic*, if there exists $\alpha > 0$, such that

$$|a(u, u)| \geq \alpha \|u\|_V^2, \quad \forall u \in V. \quad (3.2.3)$$

Theorem 3.2.2. (*Lax-Milgram's Theorem*)

Let a be a continuous sesquilinear form on $V \times V$. If a is *V-elliptic*, then A is an isomorphism from V onto V .

The proof is in three steps.

Step 1.

A is injective. We get indeed from (3.2.3)

$$|\langle Au, u \rangle_V| \geq \alpha \|u\|_V^2, \quad \forall u \in V. \quad (3.2.4)$$

⁴Here we follow, almost verbatim, the book of D. Huet [Hu].

Using Cauchy-Schwarz in the left hand side, we first get

$$\|Au\|_V \cdot \|u\|_V \geq \alpha \|u\|_V^2, \quad \forall u \in V,$$

and consequently

$$\|Au\|_V \geq \alpha \|u\|_V, \quad \forall u \in V. \quad (3.2.5)$$

This gives clearly the injectivity but actually more.

Step 2.

$A(V)$ is dense in V .

Let us consider $u \in V$ such that $\langle Av, u \rangle_V = 0, \quad \forall v \in V$. In particular, we can take $v = u$. This gives $a(u, u) = 0$ and $u = 0$ using (3.2.3).

Step 3.

$R(A) := A(V)$ is closed in V .

Let v_n a Cauchy sequence in $A(V)$ and u_n the sequence such that $Au_n = v_n$. But using (3.2.5), we get that u_n is a Cauchy sequence which is consequently convergent to some $u \in V$. But the sequence Au_n tends to Au by continuity and this shows that $v_n \rightarrow v = Au$ and $v \in R(A)$.

Step 4.

The three previous steps show that A is bijective. The continuity of A^{-1} is a consequence of (3.2.5) or of the Banach Theorem.

Remark 3.2.3.

Let us suppose for simplicity that V is a real Hilbert space. Using the isomorphism \mathcal{I} between V and V' given by the Riesz Theorem, one gets also a natural operator \mathcal{A} from V onto V' such that

$$a(u, v) = (\mathcal{A}u)(v), \quad \forall v \in V. \quad (3.2.6)$$

We have

$$\mathcal{A} = \mathcal{I} \circ A.$$

3.3 An alternative point of view: V, \mathcal{H}, V' .

We now consider two Hilbert spaces V and \mathcal{H} such that

$$V \subset \mathcal{H}. \quad (3.3.1)$$

By this notation of inclusion, we mean also that the injection of V into \mathcal{H} is continuous or equivalently that there exists a constant $C > 0$ such that, for any $u \in V$, we have

$$\|u\|_{\mathcal{H}} \leq C \|u\|_V.$$

We also assume that

$$V \text{ is dense in } \mathcal{H}. \quad (3.3.2)$$

In this case, there exists a natural injection from \mathcal{H} into the space V' which is defined as the space of continuous linear forms on V . We observe indeed that if $h \in \mathcal{H}$ then $V \ni u \mapsto \langle u, h \rangle_{\mathcal{H}}$ is continuous on V . So there exists $\ell_h \in V'$ such that

$$\ell_h(u) = \langle u, h \rangle_{\mathcal{H}}, \forall u \in V .$$

The injectivity is a consequence of the density of V in \mathcal{H} .

We can also associate to the sesquilinear form a an unbounded operator S on \mathcal{H} .

We first define $D(S)$ by

$$D(S) = \{u \in V \mid v \mapsto a(u, v) \text{ is continuous on } V \text{ for the topology induced by } \mathcal{H}\}. \quad (3.3.3)$$

Using again the Riesz Theorem and assumption (3.3.2), this defines Su in \mathcal{H} by

$$a(u, v) = \langle Su, v \rangle_{\mathcal{H}}, \forall v \in V . \quad (3.3.4)$$

Theorem 3.2.2 is completed by

Theorem 3.3.1. .

Under the same assumptions, S is bijective from $D(S)$ onto \mathcal{H} and $S^{-1} \in \mathcal{L}(\mathcal{H})$. Moreover $D(S)$ is dense in \mathcal{H} .

Proof.

We first show that S is injective. This is a consequence of

$$\alpha \|u\|_{\mathcal{H}}^2 \leq C \alpha \|u\|_V^2 \leq C |a(u, u)| = C | \langle Su, u \rangle_{\mathcal{H}} | \leq C \|Su\|_{\mathcal{H}} \cdot \|u\|_{\mathcal{H}}, \forall u \in D(S),$$

which leads to

$$\alpha \|u\|_{\mathcal{H}} \leq C \|Su\|_{\mathcal{H}}, \forall u \in D(S) . \quad (3.3.5)$$

We get directly the surjectivity in the following way. If $h \in \mathcal{H}$ and if $w \in V$ is chosen such that

$$\langle h, v \rangle_{\mathcal{H}} = \langle w, v \rangle_V, \forall v \in V ,$$

(which follows from Riesz's Theorem), we can take $u = A^{-1}w$ in V , which is a solution of

$$a(u, v) = \langle w, v \rangle_V .$$

We then show that $u \in D(S)$, using the identity

$$a(u, v) = \langle h, v \rangle_{\mathcal{H}}, \forall v \in V ,$$

and get simultaneously

$$Su = h .$$

The continuity of S^{-1} is a consequence of (3.3.5).

Let us show the last statement of density.

Let $h \in \mathcal{H}$ s. t.

$$\langle u, h \rangle_{\mathcal{H}} = 0, \forall u \in D(S) .$$

By the surjectivity of S , there exists v s. t. :

$$Sv = h .$$

We get

$$\langle Sv, u \rangle_{\mathcal{H}} = 0 , \forall u \in D(S)$$

Taking $u = v$ and using the V – *ellipticity*, we get that $v = 0$ and consequently $h = 0$.

We now consider an hermitian sesquilinear form, that is satisfying

$$a(u, v) = \overline{a(v, u)} , \forall u, v \in V . \quad (3.3.6)$$

This property is transmitted to S in the following way

Theorem 3.3.2.

If a is hermitian and V -elliptic, we have

1. S is closed;
2. $S = S^*$;
3. $D(S)$ is dense in V .

Proof of 2.

We first observe that the assumption of Hermiticity gives

$$\langle Su, v \rangle_{\mathcal{H}} = \langle u, Sv \rangle_{\mathcal{H}} , \forall u \in D(S) , \forall v \in D(S) . \quad (3.3.7)$$

In other words S is symmetric. This means in particular that

$$D(S) \subset D(S^*) . \quad (3.3.8)$$

Let $v \in D(S^*)$. Using the surjectivity of S , there exists $v_0 \in D(S)$ such that

$$Sv_0 = S^*v .$$

For all $u \in D(S)$, we get that

$$\langle Su, v_0 \rangle_{\mathcal{H}} = \langle u, Sv_0 \rangle_{\mathcal{H}} = \langle u, S^*v \rangle_{\mathcal{H}} = \langle Su, v \rangle_{\mathcal{H}} .$$

Using again the surjectivity of S , we get $v = v_0 \in D(S)$. This shows that $D(S) = D(S^*)$ and $Sv = S^*v, \forall v \in D(S)$.

Proof of 1.

S is closed because S^* is closed and $S = S^*$.

Proof of 3.

Let $h \in V$ such that

$$\langle u, h \rangle_V = 0 , \forall u \in D(S) .$$

Let $f \in V$ such that $Af = h$ (A is an isomorphism from V onto V).

We then have

$$0 = \langle u, h \rangle_V = \langle u, Af \rangle_V = \overline{\langle Af, u \rangle_V} = \overline{a(f, u)} = a(u, f) = \langle Su, f \rangle_{\mathcal{H}} .$$

Using the surjectivity, we get $f = 0$ and consequently $h = 0$.

4 Semi-bounded operators and Friedrichs extension.

4.1 Definition

Definition 4.1.1.

Let T_0 be a symmetric unbounded operator of domain $D(T_0)$. We say that T_0 is semibounded (from below) if there exists a constant C such that

$$\langle T_0 u, u \rangle_{\mathcal{H}} \geq -C \|u\|_{\mathcal{H}}^2, \quad \forall u \in D(T_0). \quad (4.1.1)$$

Example 4.1.2. (The Schrödinger operator).

We consider on \mathbb{R}^m the operator

$$P_V(x, D_x) := -\Delta + V(x), \quad (4.1.2)$$

where $V(x)$ is a continuous function on \mathbb{R}^m (called the potential) such that there exists C s.t. :

$$V(x) \geq -C, \quad \forall x \in \mathbb{R}^m. \quad (4.1.3)$$

Then the operator T_0 defined by

$$D(T_0) = C_0^\infty(\mathbb{R}^m) \text{ and } T_0 u = P_V(x, D_x)u, \quad \forall u \in D(T_0),$$

is a symmetric, semibounded operator.

We have indeed, with $\mathcal{H} = L^2(\mathbb{R}^m)$,

$$\begin{aligned} \langle P(x, D_x)u, u \rangle_{\mathcal{H}} &= \int_{\mathbb{R}^m} (-\Delta u + V u) \cdot \bar{u} \, dx \\ &= \int_{\mathbb{R}^m} |\nabla u(x)|^2 \, dx + \int_{\mathbb{R}^m} V(x) |u(x)|^2 \, dx \\ &\geq -C \|u\|_{\mathcal{H}}^2. \end{aligned} \quad (4.1.4)$$

Exercise 4.1.3.

Let us consider on $L^2(\mathbb{R}^2; \mathbb{C}^2)$ the operator $\sum_{j=1}^2 \alpha_j D_{x_j}$ with domain $\mathcal{S}(\mathbb{R}^2; \mathbb{C}^2)$. Here the α_j are 2×2 Hermitian matrices such that :

$$\alpha_j \cdot \alpha_k + \alpha_k \cdot \alpha_j = 2\delta_{jk}.$$

Show that this operator symmetric but not semi-bounded. This operator is called the Dirac operator. Its domain is $H^1(\mathbb{R}^2, \mathbb{C}^2)$ and its square is the Laplacian :

$$\left(\sum_{j=1}^2 \alpha_j D_{x_j} \right)^2 = (-\Delta) \otimes I_{\mathbb{C}^2}.$$

4.2 Analysis of the Coulomb case.

There are two important inequalities which are useful when considering the Coulomb case which plays an important role in atomic Physics. By Coulomb case, we mean the analysis on \mathbb{R}^3 of the operator

$$S_Z := -\Delta - \frac{Z}{r},$$

or of the Klein-Gordon operator

$$K_Z := \sqrt{-\Delta + 1} - \frac{Z}{r} .$$

The operator $\sqrt{-\Delta + 1}$ can easily be defined on $\mathcal{S}(\mathbb{R}^3)$, using the Fourier transform \mathcal{F} , by

$$\mathcal{F}(\sqrt{-\Delta + 1} u)(p) = \sqrt{p^2 + 1}(\mathcal{F}u)(p) .$$

The first one is the Hardy Inequality (which can be found for example in the book of Kato ([Ka], p. 305-307)) :

$$\int_{\mathbb{R}^3} |x|^{-2} |u(x)|^2 dx \leq 4 \int_{\mathbb{R}^3} |p|^2 |\hat{u}(p)|^2 dp \quad (4.2.1)$$

and the second one is due to Kato and says that

$$\int_{\mathbb{R}^3} |x|^{-1} |u(x)|^2 dx \leq \frac{\pi}{2} \int_{\mathbb{R}^3} |p| |\hat{u}(p)|^2 dp . \quad (4.2.2)$$

One proof of the Hardy inequality consists in writing that, for any $\gamma \in \mathbb{R}$ and any $u \in C_0^\infty(\mathbb{R}^3; \mathbb{R})$, we have :

$$\int_{\mathbb{R}^3} |\nabla u + \gamma \frac{x}{|x|^2} u|^2 dx \geq 0 . \quad (4.2.3)$$

This leads to :

$$\int_{\mathbb{R}^3} |\nabla u|^2 + \gamma^2 \frac{1}{|x|^2} |u|^2 dx \geq -2\gamma \int_{\mathbb{R}^3} \nabla u \cdot \frac{x}{|x|^2} u dx .$$

But an integration by part gives :

$$\begin{aligned} -2 \int_{\mathbb{R}^3} \nabla u \cdot \frac{x}{|x|^2} u dx &= \int_{\mathbb{R}^3} |u(x)|^2 (\operatorname{div}(\frac{x}{|x|^2})) dx \\ &= \int_{\mathbb{R}^3} |u(x)|^2 (\frac{1}{|x|^2}) dx . \end{aligned}$$

Optimizing over γ leads to $\gamma = \frac{1}{2}$ and gives then the result.

Remark 4.2.1. .

The same idea works for $N \geq 3$ but fails for $N = 2$. So a good exercise⁵ is to look for substitutes in this case by starting from the inequality :

$$\int_{\mathbb{R}^3} |\nabla u - \gamma(x) \frac{x}{|x|^2} u|^2 dx \geq 0 . \quad (4.2.4)$$

The function $\gamma(x)$ can be assumed radial : $\gamma(x) = g(|x|)$ and the question is to find a differential inequality on g leading at a weaker Hardy's type inequality. One can for example try $g(r) = \ln(r)$.

⁵We thank M.J. Esteban for explaining to us the trick.

For the proof of Kato's inequality, there is another tricky nice estimate which, as far as we know, goes back to Hardy and Littlewood. In the case of the Coulomb potential, we can write, using the explicit computation for the Fourier-transform of $x \mapsto \frac{1}{|x|}$:

$$\begin{aligned} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \widehat{u}(p) \frac{1}{|p-p'|^2} \widetilde{u}(p') dp \cdot dp' &= \int_{\mathbb{R}^3 \times \mathbb{R}^3} \widehat{u}(p) \frac{h(p)}{h(p')} \cdot \frac{h(p')}{h(p)} \frac{1}{|p-p'|^2} \widetilde{u}(p') dp \cdot dp' \\ &= \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{1}{|p-p'|} \widehat{u}(p) \frac{h(p)}{h(p')} \cdot \frac{h(p')}{h(p)} \frac{1}{|p-p'|} \widetilde{u}(p') dp \cdot dp' , \end{aligned}$$

where h is a strictly positive measurable function to be determined later. We then use Cauchy-Schwarz in the last equality in order to get

$$\begin{aligned} &| \int \int \widehat{u}(p) \frac{1}{|p-p'|^2} \widetilde{u}(p') dp \cdot dp' | \\ &\leq \left(\int |\widehat{u}(p)|^2 \left| \frac{h(p)}{h(p')} \right|^2 \frac{1}{|p-p'|^2} dp' dp \right)^{\frac{1}{2}} \times \left(\int |\widetilde{u}(p')|^2 \left| \frac{h(p')}{h(p)} \right|^2 \frac{1}{|p-p'|^2} dp' dp \right)^{\frac{1}{2}} \\ &= \int |\widehat{u}(p)|^2 \left(\int \left| \frac{h(p)}{h(p')} \right|^2 \frac{1}{|p-p'|^2} dp' \right) dp \\ &= \int h(p)^2 |\widehat{u}(p)|^2 \left(\int \left| \frac{1}{h(p')} \right|^2 \frac{1}{|p-p'|^2} dp' \right) dp . \end{aligned}$$

We now write $|p'| = \omega'|p|$ in the integral $\int \left| \frac{1}{h(p')} \right|^2 \frac{1}{|p-p'|^2} dp'$. We then take

$$h(p) = |p| .$$

The integral becomes

$$\int \left| \frac{1}{h(p')} \right|^2 \frac{1}{|p-p'|^2} dp' = |p|^{-1} \int \frac{1}{|\omega'|^2 |\omega - \omega'|^2} d\omega' ,$$

with $p = \omega|p|$.

This is clearly a convergent integral. Moreover, observing the invariance by rotation, one can show that the integral is independent of $\omega \in \mathbb{S}^2$. Hence we can compute it with $\omega = (1, 0, 0)$.

We finally obtain the existence of an explicit constant C such that

$$\left| \int \int \widehat{u}(p) \frac{1}{|p-p'|^2} \widetilde{u}(p') dp \cdot dp' \right| \leq C \int_{\mathbb{R}^3} |p| |\widehat{u}(p)|^2 dp .$$

The optimization of the trick leads to $C = \frac{\pi}{2}$.

Let us now show how one can use these inequalities.

If we use (4.2.1), we get the semi-boundedness for any $Z > 0$ for the Schrödinger Coulomb operator (using Cauchy-Schwarz Inequality).

$$\int_{\mathbb{R}^3} \frac{1}{r} |u|^2 dx \leq \left(\int_{\mathbb{R}^3} \frac{1}{r^2} |u|^2 dx \right)^{\frac{1}{2}} \cdot \|u\| .$$

But we can rewrite the Hardy Inequality in the form

$$\int_{\mathbb{R}^3} \frac{1}{r^2} |u|^2 dx \leq 4 \langle -\Delta u, u \rangle_{L^2(\mathbb{R}^3)} .$$

So we get, for any $\epsilon > 0$,

$$\int_{\mathbb{R}^3} \frac{1}{r} |u|^2 dx \leq \epsilon \langle -\Delta u, u \rangle_{L^2} + \frac{1}{\epsilon} \|u\|^2. \quad (4.2.5)$$

This leads to :

$$\langle S_Z u, u \rangle_{L^2} \geq (1 - \epsilon Z) \langle -\Delta u, u \rangle - \frac{Z}{\epsilon} \|u\|^2.$$

Taking $\epsilon = \frac{1}{Z}$, we have finally shown that

$$\langle S_Z u, u \rangle_{L^2} \geq -Z^2 \|u\|^2. \quad (4.2.6)$$

Here we are probably not optimal⁶ but there is another way to see that the behavior with respect to Z is correct. We just observe some invariance of the model. Let us suppose that we have proved the inequality for $Z = 1$. In order to treat the general case, we make a change of variable $x = \rho y$. The operator S_Z becomes in the new coordinates :

$$\tilde{S}_Z = \rho^{-2} (-\Delta_y) - \frac{Z}{\rho y}.$$

Taking $\rho = Z^{-1}$, we obtain

$$\tilde{S}_Z = Z^2 \left(-\Delta_y - \frac{1}{y} \right).$$

The other inequality (4.2.2) is with this respect much better and quite important for the analysis of the relativistic case. Let us see what we obtain in the case of Klein-Gordon using Kato's Inequality.

We have

$$\langle K_Z u, u \rangle_{L^2} \geq \left(1 - Z \frac{\pi}{2}\right) \langle \sqrt{-\Delta + 1} u, u \rangle_{L^2}.$$

Here the nature of the result is different. The proof gives only that K_Z is semibounded if $Z \leq \frac{2}{\pi}$. This is actually more than a technical problem !

4.3 Friedrichs's extension

Theorem 4.3.1.

A symmetric semibounded operator T_0 on \mathcal{H} (with $D(T_0)$ dense in \mathcal{H}) admits a selfadjoint extension.

⁶It can be proven (see any standard book in quantum mechanics) that the negative spectrum of this operator is discrete and is described by a sequence of eigenvalues tending to 0 : $-\frac{Z^2}{4n^2}$ with $n \in \mathbb{N}^*$. An eigenfunction related to the lowest eigenvalue $-\frac{1}{4}$ ($Z = 1$) is given by $x \mapsto \exp -\frac{1}{2}|x|$.

The extension constructed in the proof is the so-called Friedrichs extension. The proof can be seen as a variant of Lax-Milgram's Lemma. We can assume indeed by possibly replacing T_0 by $T_0 + \lambda_0 Id$ that T_0 satisfies

$$\langle T_0 u, u \rangle_{\mathcal{H}} \geq \|u\|_{\mathcal{H}}^2, \quad \forall u \in D(T_0). \quad (4.3.1)$$

Let us consider the associated form a priori defined on $D(T_0) \times D(T_0)$:

$$(u, v) \mapsto a_0(u, v) := \langle T_0 u, v \rangle_{\mathcal{H}}. \quad (4.3.2)$$

The inequality (4.3.1) says that

$$a_0(u, u) \geq \|u\|_{\mathcal{H}}^2, \quad \forall u \in D(T_0). \quad (4.3.3)$$

We introduce V as the completion in \mathcal{H} of $D(T_0)$ for the norm

$$u \mapsto p_0(u) = \sqrt{a_0(u, u)}.$$

More concretely $u \in \mathcal{H}$ belongs to V , if there exists $u_n \in D(T_0)$ such that $u_n \rightarrow u$ in \mathcal{H} and u_n is a Cauchy sequence for the norm p_0 .

As a natural norm for V , we get as a candidate :

$$\|u\|_V = \lim_{n \rightarrow +\infty} p_0(u_n), \quad (4.3.4)$$

where u_n is a Cauchy sequence for p_0 tending to u in \mathcal{H} .

Let us show that the definition does not depend on the Cauchy sequence. This is the object of the

Lemma 4.3.2.

Let x_n a Cauchy sequence in $D(T_0)$ for p_0 such that $x_n \rightarrow 0$ in \mathcal{H} . Then $p_0(x_n) \rightarrow 0$.

Proof of the lemma.

First observe that $p_0(x_n)$ is a Cauchy sequence in \mathbb{R}^+ and consequently convergent in $\overline{\mathbb{R}^+}$.

Let us suppose by contradiction that

$$p_0(x_n) \rightarrow \alpha > 0. \quad (4.3.5)$$

We first observe that

$$a_0(x_n, x_m) = a_0(x_n, x_n) + a_0(x_n, x_m - x_n),$$

and that a Cauchy-Schwarz inequality is satisfied :

$$|a_0(x_n, x_m - x_n)| \leq \sqrt{a_0(x_n, x_n)} \cdot \sqrt{a_0(x_m - x_n, x_m - x_n)}.$$

Using also that x_n is a Cauchy sequence for p_0 , we obtain that

$$\forall \epsilon > 0, \exists N \text{ s. t. } \forall n \geq N, \forall m \geq N, |a_0(x_n, x_m) - \alpha^2| \leq \epsilon. \quad (4.3.6)$$

We take $\epsilon = \frac{\alpha^2}{2}$ and consider the corresponding N given by (4.3.6). Coming back to the definition of a_0 we obtain,

$$| \langle T_0 x_n, x_m \rangle | \geq \frac{1}{2} \alpha^2, \quad \forall n \geq N, \forall m \geq N. \quad (4.3.7)$$

But as $m \rightarrow +\infty$, the left hand side in (4.3.7) tends to 0 because $x_m \rightarrow 0$ and this gives a contradiction.

##.

We now observe that

$$\|u\|_V \geq \|u\|_{\mathcal{H}}, \quad (4.3.8)$$

as a consequence of (4.3.3) and (4.3.4).

This means that the injection of V in \mathcal{H} is continuous. Note also that V , which contains $D(T_0)$, is dense in \mathcal{H} , by density of $D(T_0)$ in \mathcal{H} . Moreover, we get a natural scalar product on V by extension of a_0 :

$$\langle u, v \rangle_V := \lim_{n \rightarrow +\infty} a_0(u_n, v_n), \quad (4.3.9)$$

where u_n and v_n are Cauchy sequences for p_0 tending respectively to u and v in \mathcal{H} .

By the second version of the Lax-Milgram Theorem (in the context V, \mathcal{H}, V') applied with

$$a(u, v) := \langle u, v \rangle_V,$$

we get an unbounded selfadjoint operator S on \mathcal{H} extending T_0 whose domain $D(S)$ satisfies $D(S) \subset V$.

Remark 4.3.3. (*Friedrichs extension starting from a sesquilinear form*)

One can also start directly from a semi-bounded sesquilinear form a_0 defined on a dense subspace of \mathcal{H} .

As we shall see below, this is actually the right way to proceed for the Neumann realization of the Laplacian, where we consider on $C^\infty(\overline{\Omega})$ the sesquilinear form

$$(u, v) \mapsto \langle \nabla u, \nabla v \rangle.$$

4.4 Applications

Application 1: The Dirichlet realization.

Let Ω be an open set in \mathbb{R}^m such that $\overline{\Omega}$ is compact and let T_0 be the unbounded operator defined by

$$D(T_0) = C_0^\infty(\Omega), \quad T_0 = -\Delta.$$

The involved Hilbert space is $\mathcal{H} = L^2(\Omega)$. It is clear that T_0 is symmetric and positive⁷ (hence semi-bounded). Following the previous general construction, we prefer to consider : $\tilde{T}_0 := T_0 + Id$.

⁷We shall in fact see later that it is strictly positive.

It is easy to see⁸ that V is the closure in $H^1(\Omega)$ of $C_0^\infty(\Omega)$. This means, at least if Ω is regular, the space $H_0^1(\Omega)$. The domain of S is then described as

$$D(S) := \{u \in H_0^1(\Omega) \mid -\Delta u \in L^2(\Omega)\}.$$

S is then the operator $(-\Delta + 1)$ acting in the sense of distributions. When Ω is regular, a standard regularity theorem (see [Lio2], [LiMa]) permits to show that

$$D(S) = H^2(\Omega) \cap H_0^1(\Omega) . \quad (4.4.1)$$

So we have shown the following theorem

Theorem 4.4.1.

The operator T_1 defined by

$$D(T_1) = H^2(\Omega) \cap H_0^1(\Omega) , \quad T_1 = -\Delta ,$$

is selfadjoint and called the Dirichlet realization of $-\Delta$ in Ω .

We have just to observe that $T_1 = S - 1$ and to use Remark 2.3.8. Note that T_1 is a selfadjoint extension of T_0 . Note that by the technique developed in Subsection ??, we have also constructed another selfadjoint extension of T_0 . So we have constructed, when Ω is relatively compact, two different selfadjoint realizations of T_0 . We say in this case that T_0 is not essentially selfadjoint.

Application 2: The harmonic oscillator.

We can start from

$$H_0 = -\Delta + |x|^2 + 1 ,$$

with domain

$$D(H_0) = C_0^\infty(\mathbb{R}^m) .$$

Following the scheme of the construction of the Friedrichs extension, we first get that

$$V = B^1(\mathbb{R}^m) := \{u \in H^1(\mathbb{R}^m) \mid x_j u \in L^2(\mathbb{R}^m) , \forall j \in [1, \dots, m]\} .$$

One can indeed first show that $V \subset B^1(\mathbb{R}^m)$ and then get the equality by proving the property that $C_0^\infty(\mathbb{R}^m)$ is dense in $B^1(\mathbb{R}^m)$. One can then determine the domain of S as

$$D(S) = \{u \in B^1(\mathbb{R}^m) \mid (-\Delta + |x|^2 + 1)u \in L^2(\mathbb{R}^m)\} .$$

⁸We recall that there are two ways for describing $H_0^1(\Omega)$. In the first definition we just take the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. In the second definition, we describe $\mathcal{H}_0^1(\Omega)$ as the subspace in $H^1(\Omega)$ of the distributions whose trace is zero at the boundary. This supposes that the boundary $\Gamma = \partial\Omega$ is regular. In this case, there exists a unique application γ_0 continuous from $H^1(\Omega)$ onto $H^{\frac{1}{2}}(\Gamma)$ extending the map $C^\infty(\bar{\Omega}) \ni u \mapsto u_\Gamma$. It is a standard result (cf Brézis [Br] or Lions-Magenes [LiMa]) that $\mathcal{H}_0^1(\Omega) = H_0^1(\Omega)$, when the boundary is regular.

By a regularity theorem (differential quotients method [LiMa]), one can show that

$$D(S) = B^2(\mathbb{R}^m) := \{u \in H^2(\mathbb{R}^m) \mid x^\alpha u \in L^2(\mathbb{R}^m), \forall \alpha \text{ s. t. } |\alpha| \leq 2\}.$$

Application 3 : Schrödinger operator with Coulomb potential

We consequently start from

$$D(T_0) = C_0^\infty(\mathbb{R}^3), T_0 = -\Delta - \frac{1}{r}. \quad (4.4.2)$$

We have seen that T_0 is semibounded and replacing T_0 by T_0+2 , the assumptions of the proof of Friedrich's extension theorem are satisfied. We now claim that

$$V = H^1(\mathbb{R}^3).$$

Having in mind that $C_0^\infty(\mathbb{R}^3)$ is dense in $H^1(\mathbb{R}^3)$, we have just to verify that the norm p_0 and the norm $\|\cdot\|_{H^1(\mathbb{R}^3)}$ are equivalent on $C_0^\infty(\mathbb{R}^3)$. This results immediately of (4.2.5).

With a little more effort, one gets that $D(S) = H^2(\mathbb{R}^3)$.

Application 4 : Neumann problem.

The reader should have some minimal knowledge of Sobolev Spaces and traces of distributions for reading this subsection (See for example Brézis [Br]). Let Ω be a bounded domain with regular boundary in \mathbb{R}^m . Take $\mathcal{H} = L^2(\Omega)$. Let us consider the sesquilinear form :

$$a_0(u, v) = \int_{\Omega} \langle \nabla u, \nabla v \rangle_{\mathbb{C}^m} dx + \int_{\Omega} u \bar{v} dx,$$

on $C^\infty(\bar{\Omega})$.

Using Remark 4.3.3 and the density of $C^\infty(\bar{\Omega})$ in $H^1(\Omega)$, we can extend the sesquilinear form to $V = H^1(\Omega)$ According to the definition of the domain of S , we observe that, for $u \in D(S)$, then it should exist some $f \in L^2(\Omega)$ such that, for all $v \in H^1(\Omega)$:

$$a(u, v) = \int_{\Omega} f(x) \overline{v(x)} dx. \quad (4.4.3)$$

Then, one can find first show (by taking in (4.4.3) $v \in C_0^\infty(\Omega)$) that, in the sense of distribution,

$$-\Delta u + u = f, \quad (4.4.4)$$

and consequently that :

$$D(S) \subset W(\Omega) := \{u \in H^1(\Omega) \mid -\Delta u \in L^2(\Omega)\}. \quad (4.4.5)$$

But this is not enough for characterizing the domain.

We refer to [Lio1], [Lio2], [LiMa] or better [DaLi], Vol. 4 (p. 1222-1225)) for a detailed explanation. We first recall the Green-Riemann Formula :

$$\int_{\Omega} \langle \nabla u \mid \nabla v \rangle = \int_{\Omega} (-\Delta u) \cdot \bar{v} dx + \int_{\partial\Omega} (\partial u / \partial \nu) \bar{v} d\mu_{\partial\Omega}, \quad (4.4.6)$$

where $d\mu_{\partial\Omega}$ is the induced measure on the boundary, which is clearly true for $u \in H^2(\Omega)$ (or for $u \in C^1(\overline{\Omega})$) and $v \in H^1(\Omega)$. We unfortunately do not know that $W(\Omega) \subset H^2(\Omega)$ and the inclusion in $H^1(\Omega)$ is not sufficient for defining the normal trace. But this formula can be extended in the following way.

We first observe that, for $v \in H_0^1(\Omega)$ and $u \in W$, we have :

$$\int_{\Omega} \langle \nabla u \mid \nabla v \rangle = \int_{\Omega} (-\Delta u) \cdot \bar{v} dx . \quad (4.4.7)$$

This shows that the expression

$$\Phi_u(v) := \int_{\Omega} \langle \nabla u \mid \nabla v \rangle - \int_{\Omega} (-\Delta u) \cdot \bar{v} dx ,$$

which is well defined for $u \in W$ and $v \in H^1(\Omega)$ depends only of the restriction of v to $\partial\Omega$.

If $v_0 \in C^\infty(\partial\Omega)$, we can then extend v_0 inside Ω as a function $v = Rv_0$ in $C^\infty(\overline{\Omega})$ and the distribution ⁹ is defined as the map $v_0 \mapsto \Phi_u(Rv_0)$.

One observes also that, when $u \in C^1(\overline{\Omega})$ or $u \in H^2(\Omega)$, the Green-Riemann formula shows that :

$$\Phi_u(v_0) = \int_{\partial\Omega} (\partial u / \partial \nu) \bar{v}_0 d\mu_{\partial\Omega} .$$

So we have found a natural way to extend the notion of trace of the normal derivative for $u \in W$ and we write :

$$\gamma_1 u = \Phi_u .$$

We then conclude (using (4.4.4) and again (4.4.3) this time in full generality) that :

$$D(S) = \{u \in W(\Omega) \mid \gamma_1 u = 0\} ,$$

and that

$$S = -\Delta + 1 .$$

The operator S is called the Neumann realization of the Laplacian in $L^2(\Omega)$.

Remark 4.4.2.

Another “standard” regularity theorem shows that

$$D(S) = \{u \in H^2(\Omega) \mid \gamma_1 u = 0\} ,$$

and the notion of normal trace $u \mapsto \gamma_1 u$ for $u \in H^2(\Omega)$ is more standard¹⁰.

⁹By choosing a more specific R continuous from $H^{\frac{1}{2}}(\partial\Omega)$ into $H^1(\Omega)$, we get that Φ_u can be extended as a linear form on $H^{\frac{1}{2}}(\partial\Omega)$.

¹⁰The trace is in $H^{\frac{1}{2}}(\partial\Omega)$.

5 Compact operators : general properties and examples.

5.1 Definition and properties.

We just recall here very briefly the basic properties of compact operators and their spectral theory. We will emphasize on examples. We refer to the book by H. Brézis [Br] (Chap. VI).

Let us recall that an operator T from a Banach E into a Banach F is compact if the range of the unit ball in E by T is relatively compact in F . We denote by $\mathcal{K}(E, F)$ the space of compact operators which is a closed subspace in $\mathcal{L}(E, F)$. There is an alternative equivalent definition in the case when E and F are Hilbert spaces by using sequences. The operator is compact if and only if, for any sequence x_n which converges weakly in E , Tx_n is a strongly convergent sequence in F . Here we recall that a sequence is said to be weakly convergent in \mathcal{H} , if, for any $y \in \mathcal{H}$, the sequence $\langle x_n, y \rangle_{\mathcal{H}}$ is convergent. Such a sequence is bounded (Banach-Steinhaus's Theorem) and we recall that, in this case, there exists a unique $x \in \mathcal{H}$, such that $\langle x_n, y \rangle_{\mathcal{H}} \rightarrow \langle x, y \rangle_{\mathcal{H}}$ for all $y \in \mathcal{H}$. In this case, we write : $x_n \rightharpoonup y$.

Let us recall that when one composes a continuous operator and a compact operator (in any order) one gets a compact operator. This could be one way to prove the compactness.

Another efficient way for proving compactness of an operator T is to show that it is the limit (for the norm convergence) of a sequence of continuous operators with finite rank (that is whose range is a finite dimensional space). We observe indeed that a continuous operator with finite rank is clearly a compact operator (in a finite dimensional space the closed bounded sets are compact).

5.2 Examples

Continuous kernels

The first example of this type is the operator T_K associated to the continuous kernel K on $[0, 1] \times [0, 1]$.

By this we mean that the operator T_K is defined by

$$E \ni u \mapsto (T_K u)(x) = \int_0^1 K(x, y)u(y)dy . \quad (5.2.1)$$

Here E could be the Banach space $C^0([0, 1])$ (with the Sup norm) or $L^2(]0, 1[)$.

Proposition 5.2.1.

If the kernel K is continuous, then the operator T_K is compact from E into E .

There are two standard proofs for this proposition. The first one is based on Ascoli's Theorem giving a criterion relating equicontinuity of a subset of continuous functions on a compact and relatively compact sets in $C^0([0, 1])$.

The other one is based on the Stone-Weierstrass Theorem permitting to recover the operator as the limit of a sequence of finite rank operators T_{K_n} associated to kernels K_n of the form $K_n(x, y) = \sum_{j=1}^{j_n} f_{j,n}(x)g_{j,n}(y)$.

Let us study three other examples.

The first example comes from statistical mechanics, the second one from the spectral theory for the Dirichlet realization of the Laplacian and the third one from Quantum Mechanics.

The transfer operator. The transfer operator is the operator associated with the kernel

$$K_t(x, y) = \exp -\frac{V(x)}{2} \exp -t|x - y|^2 \exp -\frac{V(y)}{2}, \quad (5.2.2)$$

where $t > 0$, and V is a $C^\infty(\mathbb{R})$ function such that

$$\int_{\mathbb{R}} \exp -V(x)dx < +\infty .$$

The L^2 - boundedness of operators with integral kernel is proven very often through the

Lemma 5.2.2. *Schur's Lemma*

Let \mathbf{K} an operator associated with an integral kernel K , that is a function $(x, y) \mapsto K(x, y)$ on $\mathbb{R}^m \times \mathbb{R}^m$, satisfying

$$\begin{aligned} M_1 &:= \sup_{x \in \mathbb{R}^m} \int_{\mathbb{R}^m} |K(x, y)| dy < +\infty, \\ M_2 &:= \sup_{y \in \mathbb{R}^m} \int_{\mathbb{R}^m} |K(x, y)| dx < +\infty. \end{aligned} \quad (5.2.3)$$

Then \mathbf{K} , initially defined for $u \in C_0^\infty(\mathbb{R}^m)$ by

$$(Ku)(x) = \int_{\mathbb{R}^m} K(x, y)u(y)dy ,$$

can be extended as a continuous linear operator in $\mathcal{L}(L^2(\mathbb{R}^m))$ (still denoted by \mathbf{K} or T_K), whose norm satisfies

$$\|\mathbf{K}\| \leq \sqrt{M_1 M_2} . \quad (5.2.4)$$

Proof:

By the Cauchy-Schwarz inequality, we have

$$|Ku(x)|^2 \leq \int |K(x, y)||u(y)|^2 dy \int |K(x, y)|dy \leq M_1 \int |K(x, y)||u(y)|^2 dy .$$

Integrating with respect to x and using Fubini's Theorem, we then obtain the result.

In our case, the operator T_K is actually an Hilbert-Schmidt operator, i.e. an operator whose integral kernel is in $L^2(\mathbb{R}^m \times \mathbb{R}^m)$ (with $m = 1$). One can indeed prove, using Cauchy-Schwarz's inequality, show that :

$$|Ku(x)|^2 \leq \int |u(y)|^2 dy \int |K(x, y)|^2 dy ,$$

and one obtains :

$$\|T_K\| \leq \|K\|_{L^2(\mathbb{R}^m \times \mathbb{R}^m)} . \quad (5.2.5)$$

It is then easy to show that T_K is a compact operator. Its kernel K is indeed the limit in L^2 of a sequence K_n such that T_{K_n} is of finite rank. If ϕ_j is an orthonormal basis in $L^2(\mathbb{R}^m)$, one first shows that the basis $\phi_k \otimes \overline{\phi_\ell}$ is an orthonormal basis in $L^2(\mathbb{R}^m \times \mathbb{R}^m)$. Here we have by definition :

$$(\phi_k \otimes \overline{\phi_\ell})(x, y) := \phi_k(x) \overline{\phi_\ell}(y) .$$

We then obtain

$$K(x, y) = \sum_{k, \ell} c_{k, \ell} \phi_k(x) \overline{\phi_\ell}(y) .$$

We now introduce

$$K_n(x, y) := \sum_{k+\ell \leq n} c_{k, \ell} \phi_k(x) \overline{\phi_\ell}(y) .$$

It is then easy to see that T_{K_n} is of finite rank because its range is included in the linear space generated by the ϕ_k 's ($k = 1, \dots, n$). Moreover, we have

$$\lim_{n \rightarrow +\infty} \|K - K_n\|_{L^2(\mathbb{R}^m \times \mathbb{R}^m)} = 0 .$$

Coming back to the corresponding operators, we get

$$\lim_{n \rightarrow +\infty} \|T_K - T_{K_n}\|_{\mathcal{L}(L^2(\mathbb{R}^m))} = 0 .$$

The inverse of the Dirichlet operator .

We come back to the operator S , which was introduced in the study of the Dirichlet realization. One can show the following

Proposition 5.2.3.

The operator S^{-1} is compact.

Proof.

The operator S^{-1} can indeed be considered as the composition of a continuous operator from L^2 into $V = H_0^1(\Omega)$ and of the continuous injection of V into $L^2(\Omega)$. If Ω is relatively compact, we know (cf [Br]) that we have compact injection from $H^1(\Omega)$ into $L^2(\Omega)$. For the continuity result, we observe that, for all $u \in D(S)$:

$$\|Su\|_{\mathcal{H}} \|u\|_{\mathcal{H}} \geq \langle Su | u \rangle = a(u, u) \geq \alpha \|u\|_V^2 \geq \alpha \|u\|_V \|u\|_{\mathcal{H}} .$$

This gives, for all $u \in D(S)$, the inequality :

$$\|Su\|_{\mathcal{H}} \geq \alpha \|u\|_{\mathcal{V}} . \quad (5.2.6)$$

Using the surjectivity of S , we get :

$$\|S^{-1}\|_{\mathcal{L}(\mathcal{H}, \mathcal{V})} \leq \frac{1}{\alpha} . \quad (5.2.7)$$

Note that in our example $\alpha = 1$ but that this part of the proof is completely general.

The inverse of the harmonic oscillator.

The analysis is analogous. We have seen that the Sobolev space $H_0^1(\mathbb{R})$ has to be replaced by the space

$$B^1(\mathbb{R}) := \{u \in L^2(\mathbb{R}), xu \in L^2 \text{ and } du/dx \in L^2\} .$$

We can then prove, using a standard precompactness criterion, that $B^1(\mathbb{R})$ has compact injection in $L^2(\mathbb{R})$. One has in particular to use the inequality :

$$\int_{|x| \geq R} |u(x)|^2 dx \leq \frac{1}{R^2} \|u\|_{B^1(\mathbb{R})}^2 . \quad (5.2.8)$$

It is very important to realize that the space $H^1(\mathbb{R})$ is not compactly injected in L^2 . To understand this point, it is enough to consider the sequence $u_n = \chi(x-n)$ where χ is a function in $C_0^\infty(\mathbb{R})$ with norm in L^2 equal to 1. It is a bounded sequence in H^1 , which converges weakly in H^1 to 0 and is not convergent in $L^2(\mathbb{R})$.

5.3 Adjoints and compact operators

We recall¹¹ that the adjoint of a bounded operator in the Hilbertian case is bounded. When E and F are different Hilbert spaces, the Hilbertian adjoint is defined through the identity :

$$\langle Tx, y \rangle_F = \langle x, T^*y \rangle_E, \quad \forall x \in E, \forall y \in F . \quad (5.3.1)$$

Example 5.3.1.

Let Ω be an open set in \mathbb{R}^m and Π_Ω the operator of restriction to $\Omega: L^2(\mathbb{R}^m) \ni u \mapsto u|_\Omega$. Then Π_Ω^* is the operator of extension by 0.

Exercise 5.3.2.

Let γ_0 be the trace operator on $x_m = 0$ defined from $H^1(\mathbb{R}_+^m)$ onto $H^{\frac{1}{2}}(\mathbb{R}^{m-1})$. Determine the adjoint.

¹¹We are mainly following Brézis's exposition [Br].

In an Hilbert space, we have

$$M^\perp = \overline{M}^\perp .$$

In particular, if M is a closed subspace, we have already used the property

$$(M^\perp)^\perp = M . \tag{5.3.2}$$

In the case of bounded operators ($T \in \mathcal{L}(E, F)$), one gets easily the properties

$$N(T^*) = R(T)^\perp , \tag{5.3.3}$$

and

$$\overline{R(T)} = (N(T^*))^\perp . \tag{5.3.4}$$

Let us also recall the proposition

Proposition 5.3.3.

The adjoint of a compact operator is compact.

5.4 Precompactness

We assume that the reader is aware of the basic results concerning compact sets in metric spaces. We in particular recall that in a complete metric space E , an efficient way to show that a subset M is relatively compact is to show, that for any $\epsilon > 0$, one can recover M by a finite family of balls of radius ϵ in E .

The second standard point is to remember Ascoli's Theorem, giving a criterion for a bounded subset in $C^0(K)$ (K compact in \mathbb{R}^m) to be relatively compact in term of uniform equicontinuity. Ascoli's Theorem gives in particular :

- the compact injection of $C^1(K)$ into $C^0(K)$
- the compact injection of $H^m(\Omega)$ into $C^0(K)$, for $m > \frac{n}{2}$ and with $K = \overline{\Omega}$.

Let us recall finally a general proposition permitting to show that a subset in L^2 is relatively compact.

Proposition 5.4.1. *Let $A \subset L^2(\mathbb{R}^m)$. Let us assume that :*

1. *A is bounded in $L^2(\mathbb{R}^m)$, that is there exists $M > 0$ such that :*

$$\|u\|_{L^2} \leq M , \forall u \in A .$$

2. *The expression $\epsilon(u, R) := \int_{|x| \geq R} |u(x)|^2 dx$ tends to zero as $R \rightarrow +\infty$ uniformly with respect to $u \in A$.*
3. *For $h \in \mathbb{R}^m$, let τ_h defined on L^2 by : $(\tau_h u)(x) = u(x - h)$. Then the expression $\delta(u, h) := \|\tau_h u - u\|_{L^2}$ tends to zero as $h \rightarrow 0$, uniformly with respect to A .*

Then A is relatively compact in L^2 .

This proposition can be applied for showing :

- the compact injection of $H_0^1(\Omega)$ in $L^2(\Omega)$ when Ω is regular and bounded,
- the compact injection of $H^1(\Omega)$ in $L^2(\Omega)$ when Ω is regular and bounded,
- the compact injection of $B^1(\mathbb{R}^m)$ in $L^2(\mathbb{R}^m)$.

6 Spectral theory for bounded operators.

6.1 Fredholm's alternative

Let us first recall Riesz's Theorem.

Theorem 6.1.1.

Let E be a normed linear space such that $\overline{B_E}$ is compact then E is finite dimensional.

Let us now describe Fredholm's alternative.

Theorem 6.1.2.

Let $T \in \mathcal{K}(E)$. Then

- (i) $N(I - T)$ is finite dimensional.
- (ii) $R(I - T)$ is closed (of finite codimension).
- (iii) $R(I - T) = E$ if and only if $N(I - T) = \{0\}$.

We shall only use this theorem in the Hilbertian framework, so $E = \mathcal{H}$, and we shall prove it for simplicity under the additional assumption that $T = T^*$.

Proof.

We divide the proof in successive steps.

Step 1.

(i) is a consequence of Riesz's Theorem.

Step 2.

Let us show that $R(I - T)$ is closed.

Let y_n a sequence in $R(I - T)$ such that $y_n \rightarrow y$ in \mathcal{H} . We would like to show that $y \in R(I - T)$.

Let x_n in $N(I - T)^\perp$ such that $y_n = x_n - Tx_n$.

Step 2a. Let us first show the weaker property that the sequence x_n is bounded.

Let us indeed suppose that there exists a subsequence x_{n_j} such that $\|x_{n_j}\| \rightarrow +\infty$. Considering $u_{n_j} = x_{n_j}/\|x_{n_j}\|$, we observe that

$$(*) \quad u_{n_j} - Tu_{n_j} \rightarrow 0.$$

The sequence being bounded, we observe that (after possibly extracting a subsequence) one can consider that the sequence u_{n_j} is weakly convergent. This implies that Tu_{n_j} is convergent (T is compact). Using now (*), we get the convergence of u_{n_j} to u :

$$u_{n_j} \rightarrow u, \quad Tu = u, \quad \|u\| = 1.$$

But $u \in N(I - T)^\perp$, hence we get $u = 0$ and a contradiction.

Step 2b. We have consequently obtained that the sequence x_n is bounded. One can consequently extract a subsequence x_{n_j} that weakly converges to x_∞ in \mathcal{H} . Using the compactness of T , we get Tx_{n_j} converges strongly to Tx_∞ .

Hence the sequence x_{n_j} tends strongly to $y + Tx_\infty$.
We have finally

$$y + Tx_\infty = x_\infty ,$$

and consequently proved that $y = x_\infty - Tx_\infty$.

Step 3.

If $N(I - T) = \{0\}$, then $N(I - T^*) = 0$ (here we use for simplification our additional assumption) and $R(I - T)$ being closed, we get

$$R(I - T) = N(I - T^*)^\perp = \mathcal{H} .$$

The converse is also immediate as $T = T^*$.

Step 4.

We have

$$R(I - T)^\perp = N(I - T^*) = N(I - T) .$$

This shows, according to (i) that $R(I - T)$ is of finite codimension (second statement of (ii)).

This ends the proof of Fredholm's alternative in the particular case that T is selfadjoint.

Remark 6.1.3.

Under the same assumptions, it is possible to show that

$$\dim N(I - T) = \dim N(I - T^*) .$$

6.2 Resolvent set for bounded operators

In this subsection, E could be a Banach on \mathbb{R} or \mathbb{C} , but we will essentially need the Hilbertian case in the applications treated here.

Definition 6.2.1. (*Resolvent set*)

For $T \in \mathcal{L}(E)$, the resolvent set is defined by

$$\varrho(T) = \{\lambda \in \mathbb{C} ; (T - \lambda I) \text{ is bijective from } E \text{ on } E\} . \quad (6.2.1)$$

Note that in this case $(T - \lambda I)^{-1}$ is continuous (Banach's Theorem). It is easy to see that $\varrho(T)$ is an open set in \mathbb{C} . If $\lambda_0 \in \varrho(T)$, we observe that

$$(T - \lambda) = (T - \lambda_0) (Id + (\lambda - \lambda_0)(T - \lambda_0)^{-1})$$

Hence $(T - \lambda)$ is invertible if $|\lambda - \lambda_0| < \|(T - \lambda_0)^{-1}\|^{-1}$. We also get the following identity for all $\lambda, \lambda_0 \in \varrho(T)$:

$$(T - \lambda)^{-1} - (T - \lambda_0)^{-1} = (\lambda - \lambda_0)(T - \lambda)^{-1}(T - \lambda_0)^{-1} . \quad (6.2.2)$$

Definition 6.2.2. (*Spectrum*)

The spectrum of T , $\sigma(T)$, is the complementary set of $\rho(T)$ in \mathbb{C} .

Note that $\sigma(T)$ is a closed set in \mathbb{C} . This is typically the case when T is a compact injective operator in a Banach space of infinite dimension.

We say that λ is an eigenvalue if $N(T - \lambda I) \neq 0$. $N(T - \lambda I)$ is called the eigenspace associated with λ .

Definition 6.2.3. (*Point spectrum*)

The point spectrum $\sigma_p(T)$ of T is defined as the set of the eigenvalues of T .

Example 6.2.4. (*Basic example*)

Let $\mathcal{H} = L^2(]0, 1[)$ and $f \in C^0([0, 1])$. Let T_f be the operator of multiplication by f . Then one has :

$$\sigma(T_f) = \text{Im } f =: \{ \lambda \in \mathbb{C} \mid \exists x \in [0, 1] \text{ with } f(x) = \lambda \} .$$

$$\sigma_p(T_f) = \text{Sta}(f) =: \{ \lambda \in \mathbb{C} \mid \text{meas}(f^{-1}(\lambda)) > 0 \} .$$

For the first assertion, it is first immediate to see that if $\lambda \notin \text{Im } f$, then $T_{(f-\lambda)^{-1}}$ is a continuous inverse of $T_f - \lambda$. On the other side, if $\lambda = f(x_0)$ for some $x_0 \in]0, 1[$ then we have $(T_f - \lambda)u_n \rightarrow 0$ and $\|u_n\| = 1$ for $u_n = \frac{1}{\sqrt{n}}\chi(\frac{x-x_0}{n})$, where χ is a C_0^∞ function such that $\|\chi\| = 1$. This shows that $f(]0, 1[) \subset \sigma(T_f)$ and we can conclude by considering the closure.

Note that the point spectrum is not necessarily closed. Note also that one can have a strict inclusion of the point spectrum in the spectrum as can be observed in the following example :

Example 6.2.5.

Let us consider $E = \ell^2(\mathbb{N})$ and let T be the shift operator defined by :

$$(Tu)_0 = 0, (Tu)_n = u_{n-1}, n > 0,$$

where $u = (u_0, \dots, u_n, \dots) \in \ell^2(\mathbb{N})$. Then it is easy to see that T est injective (so 0 is not an eigenvalue) and is not surjective (so 0 is in the spectrum of T).

As another interesting example, one can consider :

Example 6.2.6.

Let $E = \ell^2(\mathbb{Z}, \mathbb{C})$ and let T be the operator defined by :

$$(Tu)_n = \frac{1}{2}(u_{n-1} + u_{n+1}), n \in \mathbb{Z},$$

for $u \in \ell^2(\mathbb{Z})$. Then it is easy to see (by using expansion in Fourier series ¹²) that T has no eigenvalues and its spectrum is $[-1, +1]$.

¹²Using the isomorphism between $\ell^2(\mathbb{Z}; \mathbb{C})$ and $L^2(S^1; \mathbb{C})$, which associates to the sequence $(u_n)_{n \in \mathbb{Z}}$ the fonction $\sum_{n \in \mathbb{Z}} u_n \exp in\theta$, one has to analyze the operator of multiplication by $\cos \theta$:

$$f \mapsto Tf = \cos \theta f .$$

It is then analyzed as in example 6.2.4. One concludes that the spectrum of T is $[-1, +1]$.

Exercise 6.2.7.

Let $\alpha \in [0, 1]$. Let p and q integers which are mutually prime. Analyze the spectrum Σ_α of the operator H_α defined on $\ell^2(\mathbb{Z})$ by

$$(H_\alpha u)_n = \frac{1}{2}(u_{n-1} + u_{n+1}) + \cos 2\pi\left(\frac{p}{q}n + \alpha\right) u_n, \quad n \in \mathbb{Z}.$$

In order to make the analysis easier, one can admit (particular case of the so-called Floquet theory), that one has $\Sigma = \bigcap_{\theta \in [0, 1]} \Sigma_\theta$, where Σ_θ is the spectrum of H_α reduced to the space of the u 's in ℓ^∞ such that $u_{n+q} = \exp 2i\pi\theta u_n$, for $n \in \mathbb{Z}$.

This operator plays an important role in Solid State Physics and is called the Harper's operator.

We now replace the rational $\frac{p}{q}$ by an irrational number β . So we consider the operator $H_{\beta, \alpha} := \frac{1}{2}(\tau_{+1} + \tau_{-1}) + \cos 2\pi(\beta \cdot + \alpha)$ on $\ell^2(\mathbb{Z}, \mathbb{C})$, where τ_k ($k \in \mathbb{Z}$) is the operator defined on $\ell^2(\mathbb{Z}, \mathbb{C})$ by $(\tau_k u)_n = u_{n-k}$.

Show that, if $\beta \notin \mathbb{Q}$, then the spectrum of $H_{\beta, \alpha}$ is independent of α . For this, one can first prove that $H_{\beta, \alpha}$ is unitary equivalent with $H_{\beta, \alpha+k\beta}$ for any $k \in \mathbb{Z}$. Secondly, one can use the density of the set $\{\alpha + \beta\mathbb{Z} + \mathbb{Z}\}$ in \mathbb{R} . Finally, one can use the inequality

$$\|H_{\beta, \alpha} - H_{\beta, \alpha'}\| \leq 2\pi|\alpha - \alpha'|.$$

Proposition 6.2.8.

The spectrum $\sigma(T)$ is a compact set included in the ball $\overline{B}(0, \|T\|)$.

This proposition is immediate if we observe that $(I - \frac{T}{\lambda})$ is invertible if $|\lambda| > \|T\|$.

6.3 Spectral theory for compact operators

In the case of a compact operator, one has a more precise description of the spectrum.

Theorem 6.3.1.

Let $T \in \mathcal{K}(E)$ where E is an infinite dimensional Banach space. Then

1.

$$0 \in \sigma(T).$$

2.

$$\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}.$$

3. We are in one (and only one) of the following cases

- either $\sigma(T) = \{0\}$,
- either $\sigma(T) \setminus \{0\}$ is finite,
- or $\sigma(T) \setminus \{0\}$ can be described as a sequence of distinct points tending to 0.

Proof.

a) If $0 \notin \sigma(T)$, then T admits a continuous inverse T^{-1} and the identity, which can be considered as :

$$I = T \circ T^{-1} ,$$

is a compact operator, as the composition of the compact operator T and the continuous operator T^{-1} . Using Riesz's Theorem we get a contradiction in the case E is supposed of infinite dimension.

b) The fact that, if $\lambda \neq 0$ and $\lambda \in \sigma(T)$, then λ is an eigenvalue, comes directly from the Fredholm's alternative applied to $(I - \frac{T}{\lambda})$.

c) The last step comes essentially from the following lemma :

Lemma 6.3.2.

Let $(\lambda_n)_{n \geq 1}$ a sequence of distincts points $\lambda_n \rightarrow \lambda$ and $\lambda_n \in \sigma(T) \setminus \{0\}$, for all $n > 0$.

Then $\lambda = 0$.

Proof.

We just give the proof in the Hilbertian case. For all $n > 0$, let e_n be a normalized eigenvector such that $(T - \lambda_n)e_n = 0$ and let E_n be the vectorial space spanned by $\{e_1, e_2, \dots, e_n\}$. Let us show that we have a strict inclusion of E_n in E_{n+1} .

We prove this point by recursion. Let us assume the result up to order $n - 1$ and let us show it at order n . If $E_{n+1} = E_n$, then $e_{n+1} \in E_n$ and we can write

$$e_{n+1} = \sum_{j=1}^n \alpha_j e_j .$$

Let us apply T to this relation. Using the property that e_{n+1} is an eigenvector, we obtain

$$\lambda_{n+1} \left(\sum_{j=1}^n \alpha_j e_j \right) = \sum_{j=1}^n \alpha_j \lambda_j e_j .$$

Using the recursion assumption, $\{e_1, \dots, e_n\}$ is a basis of E_n and the λ_j being distincts, we obtain $\alpha_j = 0$ for all $j = 1, \dots, n$ and a contradiction with $\|e_{n+1}\| = 1$.

So we can find a sequence u_n such that $u_n \in E_n \cap E_{n-1}^\perp$ and $\|u_n\| = 1$. T being compact, one would extract a convergent subsequence (still denoted by Tu_n) from the sequence (Tu_n) and, if $\lambda_n \rightarrow \lambda \neq 0$, one would also have the convergence of this subsequence $(\frac{1}{\lambda_n} Tu_n)$ and consequently the Cauchy property.

Let us show that this leads to a contradiction. We remark that

$$(T - \lambda_n)E_n \subset E_{n-1} .$$

Let $n > m \geq 2$. We have

$$\begin{aligned} \left\| \frac{Tu_n}{\lambda_n} - \frac{Tu_m}{\lambda_m} \right\|^2 &= \left\| \frac{(T - \lambda_n)u_n}{\lambda_n} - \frac{(T - \lambda_m)u_m}{\lambda_m} + u_n - u_m \right\|^2 \\ &= \left\| \frac{(T - \lambda_n)u_n}{\lambda_n} - \frac{(T - \lambda_m)u_m}{\lambda_m} - u_m \right\|^2 + \|u_n\|^2 \\ &\geq \|u_n\|^2 = 1 . \end{aligned}$$

We can consequently not extract a Cauchy subsequence from the sequence $\frac{1}{\lambda_n}Tu_n$. This is in contradiction with the assumption $\lambda \neq 0$. This ends the proof of the lemma and of the theorem.

We shall now consider the Hilbertian case and see which new properties can be obtained by using the additional assumption that T is selfadjoint.

6.4 Spectrum of selfadjoint operators.

As $T = T^*$, the spectrum is real. If $\text{Im } \lambda \neq 0$, one immediately verifies that :

$$|\text{Im } \lambda| \|u\|^2 \leq |\text{Im } \langle (T - \lambda)u, u \rangle| \leq \|(T - \lambda)u\| \cdot \|u\|.$$

This shows immediately that the map $(T - \lambda)$ is injective and with close range. But the orthogonal of the range of $(T - \lambda)$ is the kernel of $(T - \bar{\lambda})$ which is reduced to 0. So $(T - \lambda)$ is bijective.

This was actually, a consequence of the Lax-Milgram theorem (in the simple case when $V = H$), once we have observed the inequality

$$|\langle (T - \lambda)u, u \rangle| \geq |\text{Im } \lambda| \|u\|^2.$$

Using again Lax-Milgram's theorem, we can show

Theorem 6.4.1.

Let $T \in \mathcal{L}(\mathcal{H})$ be a selfadjoint operator. Then the spectrum of T is contained in $[m, M]$ with $m = \inf \langle Tu, u \rangle / \|u\|^2$ and $M = \sup \langle Tu, u \rangle / \|u\|^2$. Moreover m and M belong to the spectrum of T .

Proof:

We have already mentioned that the spectrum is real. Now if $\lambda > M$, we can apply the Lax-Milgram to the sesquilinear-form $(u, v) \mapsto \lambda \langle u, v \rangle - \langle Tu, v \rangle$.

Let us now show that $M \in \sigma(T)$.

We observe that, by Cauchy-Schwarz applied to the scalar product $(u, v) \mapsto M \langle u, v \rangle - \langle Tu, v \rangle$, we have :

$$|\langle Mu - Tu, v \rangle| \leq \langle Mu - Tu, u \rangle^{\frac{1}{2}} \langle Mv - Tv, v \rangle^{\frac{1}{2}}.$$

In particular, we get :

$$\|Mu - Tu\|_{\mathcal{H}} \leq \|M - T\|_{\mathcal{L}(\mathcal{H})}^{\frac{1}{2}} \langle Mu - Tu, u \rangle^{\frac{1}{2}}. \quad (6.4.1)$$

Let $(u_n)_{n \in \mathbb{Z}}$ be a sequence such that $\|u_n\| = 1$ and $\langle Tu_n, u_n \rangle \rightarrow M$ as $n \rightarrow +\infty$. By (6.4.1), we get that $(T - M)u_n$ tends to 0 as $n \rightarrow +\infty$. This implies that $M \in \sigma(T)$. If not, we would get that $u_n = (M - T)^{-1}((M - T)u_n)$ tends to 0 in contradiction with $\|u_n\| = 1$.

This theorem admits the following important corollary

Corollary 6.4.2.

Let $T \in \mathcal{L}(\mathcal{H})$ be a selfadjoint operator such that $\sigma(T) = \{0\}$. Then $T = 0$.

We first indeed show that $m = M = 0$ and consequently that $\langle Tu, u \rangle_{\mathcal{H}} = 0$ for all $u \in \mathcal{H}$. But $\langle Tu, v \rangle$ can be written as a linear combination of terms of the type $\langle Tw, w \rangle_{\mathcal{H}}$ and this gives the result by taking $v = Tu$.

Another connected property which is useful is the

Proposition 6.4.3.

If T is positive and selfadjoint then $\|T\| = M$.

The proof is quite similar. We observe that :

$$|\langle Tu, v \rangle| \leq \langle Tu, u \rangle^{\frac{1}{2}} \langle Tv, v \rangle^{\frac{1}{2}} .$$

This implies, using Riesz's Theorem :

$$\|Tu\| \leq \|T\|^{\frac{1}{2}} (\langle Tu, u \rangle)^{\frac{1}{2}} .$$

Coming back to the definition of $\|T\|$, we obtain :

$$\|T\| \leq \|T\|^{\frac{1}{2}} M^{\frac{1}{2}} ,$$

and the inequality :

$$\|T\| \leq M .$$

But it is immediate that :

$$\langle Tu, u \rangle \leq \|T\| \|u\|^2 .$$

This gives the converse inequality and the proposition.

6.5 Spectral theory for compact selfadjoint operators

We have a very precise description of the selfadjoint compact operators.

Theorem 6.5.1.

Let \mathcal{H} be a separable Hilbert¹³ space and T a compact selfadjoint operator. Then \mathcal{H} admits an Hilbertian basis consisting of eigenvectors of T .

Proof.

Let $(\lambda_n)_{n \geq 1}$ be the sequence of disjoint eigenvalues of T , except 0. Their existence comes from Theorem 6.3.1 and we also observe that the eigenvalues are real.

Let us define $\lambda_0 = 0$.

¹³that is having a countable dense set.

We define $E_0 = N(T)$ and $E_n = N(T - \lambda_n I)$; We know (from Riesz's Theorem) that

$$0 < \dim E_n < +\infty .$$

Let us show that \mathcal{H} is the Hilbertian sum of the $(E_n)_{n \geq 0}$.

(i) The spaces (E_n) are mutually orthogonal. If $u \in E_m$ and $v \in E_n$ with $m \neq n$, we have

$$\langle Tu, v \rangle_{\mathcal{H}} = \lambda_m \langle u, v \rangle_{\mathcal{H}} = \langle u, Tv \rangle_{\mathcal{H}} = \lambda_n \langle u, v \rangle_{\mathcal{H}} ,$$

and consequently

$$\langle u, v \rangle_{\mathcal{H}} = 0 .$$

(ii) Let F be the linear space spanned by the $(E_n)_{n \geq 0}$. Let us verify that F is dense in \mathcal{H} . It is clear that $TF \subset F$ and, using the selfadjoint character of T , we have also $TF^\perp \subset F^\perp$. The operator \tilde{T} , obtained by restriction of T to F^\perp , is a compact selfadjoint operator. But one shows easily that $\sigma(\tilde{T}) = \{0\}$ and consequently $\tilde{T} = 0$. But $F^\perp \subset N(T) \subset F$ and hence $F^\perp = \{0\}$. F is consequently dense in \mathcal{H} .

(iii) To end the proof, one chooses in each E_n an Hilbertian basis. Taking the union of these bases, one obtains an Hilbertian basis of \mathcal{H} effectively formed with eigenvectors of T .

Remark 6.5.2.

If T is a compact selfadjoint operator, we can write any u in the form

$$u = \sum_{n=0}^{+\infty} u_n , \quad \text{with } u_n \in E_n .$$

This permits to write

$$Tu = \sum_{n=1}^{+\infty} \lambda_n u_n .$$

If, for $k \in \mathbb{N}^$, we define T_k by*

$$T_k u = \sum_{n=1}^k \lambda_n u_n ,$$

we easily see that T_k is of finite rank and that

$$\|T - T_k\| \leq \sup_{n \geq k+1} |\lambda_n| .$$

Hence the operator T appears as the limit in $\mathcal{L}(\mathcal{H})$ of the sequence T_k as $k \rightarrow +\infty$.

7 Examples.

We go back to our previous examples and analyze their properties.

7.1 The transfer operator.

7.1.1 Compactness

The transfer operator (which was introduced in (??) is compact and admits consequently a sequence of eigenvalues λ_n tending to 0 as $n \rightarrow +\infty$. Let us show the

Lemma 7.1.1.

The transfer operator is positive and injective.

Proof of the lemma.

Let $u \in L^2(\mathbb{R})$. We can write¹⁴, with $\phi(x) = \exp -\frac{V(x)}{2}u(x)$

$$\langle T_K u, u \rangle_{\mathcal{H}} = \int_{\mathbb{R}^2} \exp -t|x - y|^2 \phi(x) \overline{\phi(y)} dx dy .$$

Using the properties of the Fourier transform and of the convolution, we deduce

$$\langle T_K u, u \rangle_{\mathcal{H}} = c_t \int_{\mathbb{R}} \exp -\frac{|\xi|^2}{4t} |\widehat{\phi}(\xi)|^2 d\xi ,$$

where $c_t > 0$ is a normalization constant.

The spectrum is consequently the union of a sequence of positive eigenvalues and of its limit 0. T_K can be diagonalized in an orthonormal basis of eigenvectors associated with strictly positive eigenvalues. We emphasize that 0 is in the spectrum but not an eigenvalue.

Theorem 6.4.1 says also that $\|T_K\|$ is the largest eigenvalue and is isolated. A natural question is then to discuss the multiplicity of each eigenvalue, i. e. the dimension of each associated eigenspace. We shall see later (Krein-Rutman's Theorem) that the largest eigenvalue is of multiplicity 1.

7.1.2 About the physical origin of the problem.

Our initial problem was to find a rather general approach for the estimate of the decay of correlations attached to "gaussian like" measures of the type

$$\exp -\Phi(X) dX \tag{7.1.1}$$

on \mathbb{R}^n with Φ . One proof of this estimate (in the case when Φ has a particular structure) is based on the analysis of the transfer matrix method, originally

¹⁴If we make only the weak assumption that $\exp -V \in L^1(\mathbb{R})$, it is better to start for the proof of the positivity by considering u 's in $C_0^\infty(\mathbb{R})$ and then to treat the general case by using the density of C_0^∞ in L^2 .

due to Kramers-Wannier, that we have already seen for the study of the Ising model. We present here briefly the technique for our toy model. We shall only consider the case when $d = 1$ and treat the periodic case, that is the case when $\{1, \dots, n\}$ is a representation of $\Lambda^{per} = \mathbb{Z}/n\mathbb{Z}$.

We consider the particular potential Φ

$$\Phi^{(n)}(X) \equiv \Phi(X) \equiv \frac{1}{h} \left(\sum_{j=1}^n V(x_j) + \frac{|x_j - x_{j+1}|^2}{4} \right) \quad (7.1.2)$$

where we take the convention that $x_{n+1} = x_1$ and where h is possibly a semiclassical parameter which is sometimes chosen equal to one if we are not interested in the "semiclassical" aspects. More generally, we could more generally consider examples of the form:

$$\Phi_h(X) = \frac{1}{h} \left(\sum_{j=1}^n (V(x_j) + I(x_j, x_{j+1})) \right) \quad (7.1.3)$$

where I is a symmetric "interaction" potential on $\mathbb{R} \times \mathbb{R}$. Let us mention however that the example (7.1.2) appears naturally in quantum field theory when the so called "lattice approximation" is introduced. For this special class of potentials, we shall demonstrate that the informations given by the transfer operator method are complementary to the other approaches we have explained before. We shall present the "dictionary" between the properties of the measure $h^{-\frac{n}{2}} \cdot \exp -\Phi_h(X) dX$ on \mathbb{R}^n and the spectral properties of the transfer operator \mathbf{K}_V (which is also called Kac operator for some particular models introduced by M. Kac) whose integral kernel is real and given on $\mathbb{R} \times \mathbb{R}$ by

$$K_V(x, y) = h^{-\frac{1}{2}} \exp -\frac{V(x)}{2h} \cdot \exp -\frac{|x - y|^2}{4h} \cdot \exp -\frac{V(y)}{2h} . \quad (7.1.4)$$

By integral kernel (or distribution kernel), we mean¹⁵ a distribution in $\mathcal{D}'(\mathbb{R}^2)$ such that the operator is defined from $C_0^\infty(\mathbb{R})$ into $\mathcal{D}'(\mathbb{R})$ by the formula

$$\int_{\mathbb{R}} (\mathbf{K}_V u)(x) v(x) dx = \int_{\mathbb{R} \times \mathbb{R}} K_V(x, y) u(x) v(y) dx dy , \quad \forall u , v \in C_0^\infty(\mathbb{R}) . \quad (7.1.5)$$

This dictionary permits to obtain interesting connections between estimates for the quotient μ_2/μ_1 of the two first largest eigenvalues of the transfer operator and corresponding estimates controlling the speed of convergence of thermodynamic quantities. In particular this speed of convergence is exponentially rapid as $n \rightarrow +\infty$.

We know that when the operator \mathbf{K} is compact, symmetric and injective, then there exists a decreasing (in modulus) sequence μ_j of eigenvalues tending

¹⁵Here we shall always consider much more regular kernels which are in particular continuous. So the notation f can be interpreted in the usual sense. In general, this means that the distribution kernel K_V is applied to the test function $(x, y) \mapsto u(x) v(y)$.

to 0 and a corresponding sequence of eigenfunctions u_j which can be normalized in order to get an orthonormal basis of $L^2(\mathbb{R})$. Moreover, the operator becomes the limit in norm of the family of operators $\mathbf{K}^{(N)}$ whose corresponding kernel are defined by

$$K^{(N)}(x, y) = \sum_{j=1}^N \mu_j u_j(x) u_j(y). \quad (7.1.6)$$

We recall indeed that

$$\|\mathbf{K} - \mathbf{K}^{(N)}\| \leq \sup_{j>N} |\mu_j|. \quad (7.1.7)$$

The symmetric operators¹⁶ are called trace class if we have in addition the property that

$$\|\mathbf{K}\|_{tr} := \sum_j |\mu_j| < +\infty \quad (7.1.8)$$

In this case, we get that

$$\|\mathbf{K} - \mathbf{K}^{(N)}\|_{tr} \leq \sum_{j>N} |\mu_j|. \quad (7.1.9)$$

For a trace class symmetric operator, we can in particular define the trace as

$$\text{Tr } \mathbf{K} = \sum_j \mu_j, \quad (7.1.10)$$

and this operation is continuous on the space of the trace class operators

$$|\text{Tr } \mathbf{K}| \leq \|\mathbf{K}\|_{tr}. \quad (7.1.11)$$

This of course extends the usual notion of trace for matrices. We can actually compute directly the trace of a trace-class operator in the following way. We first observe that

$$\text{Tr } \mathbf{K} = \lim_{N \rightarrow +\infty} \text{Tr } \mathbf{K}^{(N)}. \quad (7.1.12)$$

Then we observe that

$$\text{Tr } \mathbf{K}^{(N)} = \sum_{j=1}^N \mu_j = \int_{\mathbb{R}} K^{(N)}(x, x) dx. \quad (7.1.13)$$

We consequently obtain that

$$\text{Tr } \mathbf{K} = \lim_{N \rightarrow +\infty} \int_{\mathbb{R}} K^{(N)}(x, x) dx. \quad (7.1.14)$$

¹⁶When \mathbf{K} is not symmetric, trace class operators can still be defined by considering $\sqrt{\mathbf{K}^* \mathbf{K}}$. Note that when K is trace class, one can compute the trace by considering any orthonormal basis e_i : $\text{Tr } \mathbf{K} = \sum_i \langle \mathbf{K} e_i, e_i \rangle$.

If we observe that $x \mapsto \sum_j |\mu_j| |u_j(x)|^2$ is in $L^1(\mathbb{R})$, then it is natural¹⁷ to hope (but this is not trivial!) that $x \mapsto K(x, x)$ is in L^1 and that

$$\mathrm{Tr} \mathbf{K} = \int_{\mathbb{R}} K(x, x) dx . \quad (7.1.15)$$

Note that it is only when \mathbf{K} is positive that the finiteness of the right hand side in (7.1.15) will imply the trace class property.

In the case the operator is Hilbert-Schmidt, that is with a kernel K in L^2 , then the operator is also compact and we have the identity

$$\sum_j \mu_j^2 = \int_{\mathbb{R} \times \mathbb{R}} |K(x, y)|^2 dx dy . \quad (7.1.16)$$

This gives an easy criterion for verifying the compactness of the operator. Let us first look at the thermodynamic limit. This means that we are interested in the limit $\lim_{n \rightarrow +\infty} \frac{1}{n} \ln \left(\int_{\mathbb{R}^n} \exp -\Phi(X) dX \right)$. We start from the decomposition:

$$\exp -\Phi(X) = K_V(x_1, x_2) \cdot K_V(x_2, x_3) \cdots K_V(x_{n-1}, x_n) \cdot K_V(x_n, x_1) \quad (7.1.17)$$

and we observe that

$$\int_{\mathbb{R}^n} \exp -\Phi(X) dX = \int_{\mathbb{R}} K_{V,n}(y, y) dy \quad (7.1.18)$$

where $K_{V,n}(x, y)$ is the distribution kernel of $(\mathbf{K}_V)^n$. Our assumption on V permits to see that $(K_V)^n$ is trace class¹⁸ and we rewrite (7.1.18) in the form:

$$\int_{\mathbb{R}^n} \exp -\Phi(X) dX = \mathrm{Tr} [(\mathbf{K}_V)^n] = \sum_j \mu_j^n , \quad (7.1.19)$$

where the μ_j are introduced in (7.1.24).

We note also for future use that

$$K_{V,n}(x, y) = \sum_j \mu_j^n u_j(x) u_j(y) . \quad (7.1.20)$$

¹⁷Let us sketch a proof of (7.1.15) in our particular case. \mathbf{K} being positive and with an explicit kernel : $\exp -\frac{V(x)}{2} \exp -\mathcal{J}|x-y|^2 \exp -\frac{V(y)}{2}$, one can find an Hilbert-Schmidt operator L satisfying $L^*L = K$. The kernel of L is given, for a suitable θ , by $L(x, y) = c_\theta \exp -\theta|x-y|^2 \exp -\frac{V(y)}{2}$. We note indeed that $c_\theta \exp -\theta|x-y|^2$ is the distribution kernel of $\exp t_\theta \Delta$ for a suitable $t_\theta > 0$.

Then we observe that L is Hilbert-Schmidt and that $\|L\|_{H.S}^2 = \sum_j \mu_j = \mathrm{Tr} K$. Using the previously mentioned formula for the Hilbert-Schmidt norm and the property that

$$K(x, x) = \int L^*(x, z) L(z, x) dz = \int L^*(z, x) L(z, x) dz ,$$

one obtains (7.1.15).

¹⁸See for example [Ro].

In particular we get for the thermodynamic limit:

$$\lim_{n \rightarrow \infty} \frac{\ln \int_{\mathbb{R}^n} \exp -\Phi(X) dX}{n} = \ln \mu_1 . \quad (7.1.21)$$

Moreover the speed of the convergence is easily estimated by:

$$-\ln \left| \frac{\ln \int_{\mathbb{R}^n} \exp -\Phi(X) dX}{n} - \ln \mu_1 \right| = -n \ln \left(\frac{\mu_2}{\mu_1} \right) - \ln k_2 + \ln n + \mathcal{O}(\exp -\delta_2 n) , \quad (7.1.22)$$

where k_2 is the multiplicity of μ_2 .

7.1.3 Krein-Rutman's Theorem.

We observe now that the kernel $(x, y) \mapsto K_V(x, y)$ satisfies the condition

$$K_V(x, y) > 0 , \forall x, y \in \mathbb{R} . \quad (7.1.23)$$

In particular it satisfies the assumptions of the extended Perron-Frobenius Theorem also called Krein-Rutman's Theorem and our positive operator \mathbf{K}_V admits consequently a largest eigenvalue μ_1 equal to $\|K_V\|$ which is simple and corresponds to a unique strictly positive normalized eigenfunction which we denote by u_1 . Let μ_j the sequence of eigenvalues that we order as a decreasing sequence tending to 0:

$$0 \leq \mu_{j+1} \leq \mu_j \leq \dots \leq \mu_2 < \mu_1 . \quad (7.1.24)$$

We shall denote by u_j a corresponding orthonormal basis of eigenvectors with

$$K_V u_j = \mu_j u_j , \|u_j\| = 1 . \quad (7.1.25)$$

Let us present the statements:

Definition 7.1.2.

Let A be a bounded positive operator on a Hilbert space $\mathcal{H} = L^2(X, d\nu)$ where (X, ν) is a measured space. Then we say that A has a strictly positive kernel if, for each choice of a non negative function $\theta \in \mathcal{H}$ ($\|\theta\| \neq 0$), we have

$$0 < A\theta$$

almost everywhere.

It is immediate to see that the transfer operator satisfies this condition. The theorem generalizing the Perron-Frobenius Theorem is then the following:

Theorem 7.1.3.

Let A be a bounded positive compact symmetric operator on \mathcal{H} having a strictly positive kernel and let¹⁹ $\|A\| = \lambda$ be the largest eigenvalue of A . Then λ has multiplicity 1 and the corresponding eigenvector u_λ can be chosen to be a strictly positive function.

¹⁹See Proposition 6.4.3

Proof:

Since A maps real functions into real functions, we may assume that u_λ is real. We now prove that

$$\langle Au_\lambda, u_\lambda \rangle \leq \langle A|u_\lambda|, |u_\lambda| \rangle.$$

This is an immediate consequence of the strict positivity of the kernel. We just write:

$$u_\lambda = u_\lambda^+ - u_\lambda^-$$

and

$$|u_\lambda| = u_\lambda^+ + u_\lambda^-$$

and the above inequality is then a consequence of

$$\langle Au_\lambda^+, u_\lambda^- \rangle \geq 0,$$

and

$$\langle u_\lambda^+, Au_\lambda^- \rangle \geq 0.$$

We then obtain

$$\lambda \|u_\lambda\|^2 = \langle Au_\lambda, u_\lambda \rangle \leq \langle A|u_\lambda|, |u_\lambda| \rangle \leq \|A\| \|u_\lambda\|^2 = \lambda \|u_\lambda\|^2.$$

This implies

$$\langle Au_\lambda, u_\lambda \rangle = \langle A|u_\lambda|, |u_\lambda| \rangle.$$

This equality means

$$\langle u_\lambda^+, Au_\lambda^- \rangle + \langle u_\lambda^-, Au_\lambda^+ \rangle = 0.$$

We then get a contradiction unless $u_\lambda^+ = 0$ or $u_\lambda^- = 0$. We can then assume $u_\lambda \geq 0$ and the assumption gives again

$$0 < \langle \theta, Au_\lambda \rangle = \lambda \langle \theta, u_\lambda \rangle,$$

for any positive θ . This gives

$$u_\lambda \geq 0 \quad a.e.$$

But

$$u_\lambda = \lambda^{-1} Au_\lambda$$

and this gives

$$u_\lambda > 0 \quad a.e.$$

Finally if there are two linearly independent eigenvectors v_λ and u_λ corresponding to λ , we would obtain the same property for v_λ by considering as new Hilbert space the orthogonal of u_λ in \mathcal{H} . But it is impossible to have two orthogonal vectors which are strictly positive.

q.e.d.

Remark 7.1.4.

In the case $t = 0$, we keep the positivity but lose the injectivity ! The spectrum is easy to determine. We are indeed dealing with the orthonormal projector associated to the function $x \mapsto \exp -\frac{V(x)}{2}$. The real number 1 is a simple eigenvalue and 0 is an eigenvalue whose corresponding eigenspace is infinite dimensional.

7.2 The Dirichlet realization, a model of operator with compact resolvent.

We can apply Theorem 6.5.1 to the operator $(-\Delta_D + Id)^{-1}$. We have seen that this operator is compact and it is clearly injective (by construction). It was also seen as a selfadjoint and positive. Moreover, the norm of this operator is less or equal to 1.

There exists consequently a sequence of distinct eigenvalues μ_n tending to 0 (with $0 < \mu_n \leq 1$) and a corresponding orthonormal basis of eigenvectors such that $(\Delta_D + I)^{-1}$ is diagonalized. If $\phi_{n,j}$ is a corresponding eigenvector associated with μ_n , that is, if

$$(-\Delta_D + I)^{-1} \phi_{n,j} = \mu_n \phi_{n,j} ,$$

we first observe that $\phi_{n,j} \in D(-\Delta_D + I)$; hence $\phi_{n,j} \in H_0^1(\Omega) \cap H^2(\Omega)$ (if Ω is relatively compact with regular boundary) and

$$-\Delta_D \phi_{n,j} = \left(\frac{1}{\mu_n} - 1 \right) \phi_{n,j} .$$

The function $\phi_{n,j}$ is consequently an eigenvector of $-\Delta_D$ associated with the eigenvalue $\lambda_n = \left(\frac{1}{\mu_n} - 1 \right)$.

Let us show, as easily imagined, that this basis $\phi_{n,j}$ permits effectively the diagonalization of $-\Delta_D$.

Let us indeed consider $u = \sum_{n,j} u_{n,j} \phi_{n,j}$ in the domain of $-\Delta_D$. Let us consider the scalar product $\langle -\Delta_D u, \phi_{m,\ell} \rangle_{\mathcal{H}}$. Using the selfadjoint character of $-\Delta_D$, we get

$$\langle -\Delta_D u, \phi_{m,\ell} \rangle_{\mathcal{H}} = \langle u, -\Delta_D \phi_{m,\ell} \rangle_{\mathcal{H}} = \lambda_m u_{m,\ell} .$$

Observing, that $D(\Delta_D) = R(S^{-1})$, one obtains that the domain of $-\Delta_D$ is characterized by

$$D(-\Delta_D) = \left\{ u \in L^2 \mid \sum_{n,j} \lambda_n^2 |u_{n,j}|^2 < +\infty \right\} .$$

Here we have used the property that for any N , we have the identity :

$$\sum_{n \leq N} u_{n,j} \phi_{n,j} = S \left(\sum_{n \leq N} \lambda_n u_{n,j} \phi_{n,j} \right) .$$

We have consequently given a meaning to the following diagonalization formula

$$-\Delta_D = \sum_n \lambda_n \Pi_{E_n} , \tag{7.2.1}$$

where Π_{E_n} is the orthogonal projector on the eigenspace E_n associated with the eigenvalue λ_n .

Let us remark that it results from the property that the sequence μ_n tends to 0 the property that λ_n tend to $+\infty$.

Let us also prove the

Lemma 7.2.1.

The lowest eigenvalue of the Dirichlet realization of the Laplacian in a relatively compact domain Ω is strictly positive :

$$\lambda_1 > 0 . \tag{7.2.2}$$

Proof.

We know that $\lambda_1 \geq 0$; the Dirichlet realization of the Laplacian is indeed positive. If $\lambda_1 = 0$, a corresponding normalized eigenvector ϕ_1 would satisfy

$$\langle -\Delta\phi_1, \phi_1 \rangle = 0 ,$$

and consequently

$$\nabla\phi_1 = 0 , \text{ dans } \Omega .$$

This leads first $\phi_1 = \text{Cste}$ in each connected component of Ω but because the trace of ϕ_1 on $\partial\Omega$ vanishes ($\phi_1 \in H_0^1(\Omega)$) implies that $\phi_1 = 0$. So we get a contradiction.

Finally it results from standard regularity theorems (See [Br], [LiMa]) that the eigenvectors belong (if Ω is regular) to $C^\infty(\bar{\Omega})$.

We now show the

Proposition 7.2.2.

The lowest eigenvalue is simple and one can choose the first eigenfunction to be strictly positive.

The natural idea is to apply Krein-Rutman's Theorem to $(-\Delta_D + 1)^{-1}$. One has to show that this operator is positivity improving. This is indeed the case if the domain is connected but the proof will not be given. We will only show that $(-\Delta_D + I)^{-1}$ is positivity preserving and observe that this implies (following the proof of Krein Rutman) that if Ω is an eigenfunction then $|\Omega|$ is an eigenfunction.

Lemma 7.2.3.

The operator $(-\Delta_D + I)^{-1}$ is positivity preserving.

The proof is a consequence of the Maximum principle. It is enough to show that

$$-\Delta u + u = f , \gamma_0 u = 0 \text{ and } f \geq 0 \text{ a.e. ,}$$

implies that $u \geq 0$ almost everywhere.

We introduce $u_+ = \max u, 0$ and $u_- = -\inf(u, 0)$. A standard proposition (see below) (see Lieb-Loss) shows that u_+ and u_- belong to $H_0^1(\Omega)$. Multiplying by u_- , we obtain

$$-\|\nabla u_-\|^2 - \|u_-\|^2 = \int_{\Omega} f u_- dx \geq 0 ,$$

which implies $u_- = 0$.

Proposition 7.2.4.

Suppose that $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ with $\nabla f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then also $\nabla|f| \in L^1_{\text{loc}}(\mathbb{R}^n)$ and with the notation

$$\text{sign}z = \begin{cases} \frac{\bar{z}}{|z|}, & z \neq 0 \\ 0, & z = 0, \end{cases} \quad (7.2.3)$$

we have

$$\nabla|f|(x) = \text{Re}\{\text{sign}(f(x))\nabla f(x)\} \text{ almost everywhere.} \quad (7.2.4)$$

In particular,

$$|\nabla|f|| \leq |\nabla f|,$$

almost everywhere.

Proof of Proposition 7.2.4.

Suppose first that $u \in C^\infty(\mathbb{R}^n)$ and define $|z|_\epsilon = \sqrt{|z|^2 + \epsilon^2} - \epsilon$, for $z \in \mathbb{C}$ and $\epsilon > 0$. We observe that

$$0 \leq |z|_\epsilon \leq |z| \text{ and } \lim_{\epsilon \rightarrow 0} |z|_\epsilon = |z|.$$

Then $|u|_\epsilon \in C^\infty(\mathbb{R}^n)$ and

$$\nabla|u|_\epsilon = \frac{\text{Re}(\bar{u} \nabla u)}{\sqrt{|u|^2 + \epsilon^2}}. \quad (7.2.5)$$

Let now f be as in the proposition and define f_δ as the convolution

$$f_\delta = f * \rho_\delta$$

with ρ_δ being a standard approximation of the unity for convolution. Explicitly, we take a $\rho \in C^\infty_0(\mathbb{R}^n)$ with

$$\rho \geq 0, \quad \int_{\mathbb{R}^n} \rho(x) dx = 1,$$

and define $\rho_\delta(x) := \delta^{-n} \rho(x/\delta)$, for $x \in \mathbb{R}^n$ and $\delta > 0$.

Then $f_\delta \rightarrow f$, $|f_\delta| \rightarrow |f|$ and $\nabla f_\delta \rightarrow \nabla f$ in $L^1_{\text{loc}}(\mathbb{R}^n)$ as $\delta \rightarrow 0$.

Take a test function $\phi \in C^\infty_0(\mathbb{R}^n)$. We may extract a subsequence $\{\delta_k\}_{k \in \mathbb{N}}$ (with $\delta_k \rightarrow 0$ for $k \rightarrow \infty$) such that $f_{\delta_k}(x) \rightarrow f(x)$ for almost every $x \in \text{supp } \phi$. We restrict our attention to this subsequence. For simplicity of notation we omit the k from the notation and write $\lim_{\delta \rightarrow 0}$ instead of $\lim_{k \rightarrow \infty}$.

We now calculate, using dominated convergence and (7.2.5),

$$\begin{aligned} \int (\nabla \phi) |f| dx &= \lim_{\epsilon \rightarrow 0} \int (\nabla \phi) |f|_\epsilon dx \\ &= \lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \int (\nabla \phi) |f_\delta|_\epsilon dx \\ &= - \lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \int \phi \frac{\text{Re}(\overline{f_\delta} \nabla f_\delta)}{\sqrt{|f_\delta|^2 + \epsilon^2}} dx \end{aligned}$$

Using the pointwise convergence of $f_\delta(x)$ and $\|\nabla f_\delta - \nabla f\|_{L^1(\text{supp } \phi)} \rightarrow 0$, we can take the limit $\delta \rightarrow 0$ and get

$$\int (\nabla \phi) |f| dx = - \lim_{\epsilon \rightarrow 0} \int \phi \frac{\text{Re}(\bar{f} \nabla f)}{\sqrt{|f|^2 + \epsilon^2}} dx. \quad (7.2.6)$$

Now, $\phi \nabla f \in L^1(\mathbb{R}^n)$ and $\frac{\overline{f(x)}}{\sqrt{|f|^2 + \epsilon^2}} \rightarrow \text{sign} f(x)$ as $\epsilon \rightarrow 0$, so we get (7.2.4) from (7.2.6) by dominated convergence. ■

We can now look at the property of a positive eigenfunction. The eigenfunction is positive, in $C^\infty(\bar{\Omega})$ by a regularity theorem and satisfies

$$-\Delta u = \lambda u \geq 0.$$

Hence u is superharmonic and (see Lieb-Loss) satisfies the mean value property : For all $y \in \Omega$, for all $R > 0$ such that $B(y, R) \in \Omega$, then

$$u(y) \geq \frac{1}{B(y, R)} \int_{B(y, R)} u(z) dz.$$

Moreover, we know that $\inf u = 0$. Applying this mean value property with y (if any) such that $u(y) = 0$, we obtain that $u = 0$ in $B(y, R)$. Using in addition a connectedness argument, we obtain that in a connected open set u is either identically 0 or strictly positive.

Let us come back to what appears in the proof of Krein-Rutamn. u_λ^+ and u_λ^- are either 0 or strictly positive eigenfunctions and we have also $\langle u_\lambda^+, u_\lambda^- \rangle = 0$. Hence u_λ^+ or u_λ^- should vanish.

7.3 Extension to operators with compact resolvent

What we have done for the analysis of the Dirichlet realization is indeed quite general. It can be applied to selfadjoint operators, which are bounded from below and with compact resolvent.

We show that in this case, there exists an infinite sequence (if the Hilbert space is infinite dimensional) of real eigenvalues λ_n tending to $+\infty$ such that the corresponding eigenspaces are mutually orthogonal, of finite dimension and such that their corresponding Hilbertian sum is equal to \mathcal{H} .

Typically, one can apply the method to the Neumann realization of the Laplacian in a relatively compact domain Ω or to the harmonic oscillator in \mathbb{R}^m .

7.4 Operators with compact resolvent : the Schrödinger operator in an unbounded domain.

We just recall some criteria of compactness for the resolvent of the Schrödinger operator $P = -\Delta + V$ in \mathbb{R}^m in connection with the precompactness criterion.

In the case of the Schrödinger equation on \mathbb{R}^m and if V is C^∞ and bounded from below, the domain of the selfadjoint extension is always contained in

$$Q(P) := H_V^1(\mathbb{R}^m) = \{u \in H^1(\mathbb{R}^m) \mid (V + C)^{\frac{1}{2}}u \in L^2(\mathbb{R}^m)\} .$$

$Q(P)$ is usually called the form domain of the form

$$u \mapsto \int_{\mathbb{R}^m} |\nabla u|^2 dx + \int_{\mathbb{R}^m} V(x)|u(x)|^2 dx .$$

It is then easy to see that, if V tends to ∞ , then the injection of H_V^1 in L^2 is compact (using a criterion of precompactness). We then obtain, observing that $(P + 1)^{-1}$ is continuous from L^2 into H_V^1 , that the resolvent $(P + \lambda)^{-1}$ is compact for $\lambda \notin \sigma(P)$.

In the case of a compact manifold M and if we consider the Laplace-Beltrami operator on M , then the compactness of the resolvent is obtain without additional assumption on V . The domain of the operator is $H^2(M)$ and we have compact injection from $H^2(M)$ into $L^2(M)$.

The condition that $V \rightarrow \infty$ as $|x| \rightarrow \infty$ is not a necessary condition. We can indeed replace it by the weaker sufficient condition

Proposition 7.4.1.

Let us assume that the injection of $H_V^1(\mathbb{R}^m)$ into $L^2(\mathbb{R}^m)$ is compact then P is with compact resolvent.

More concretely, the way to verify this criterion is to show the existence of a continuous function $x \mapsto \rho(x)$ tending to ∞ as $|x| \rightarrow +\infty$ such that

$$H_V^1(\mathbb{R}^m) \subset L_\rho^2(\mathbb{R}^m) . \tag{7.4.1}$$

Of course, the preceding case corresponds to $\rho = V$, but, as typical example of this strategy, we shall show in exercise 7.4.3 that the Schrödinger operator on \mathbb{R}^2 , $-\Delta + x^2 \cdot y^2 + 1$, is with compact inverse.

On the other hand, the criterion that $V \rightarrow +\infty$ as $|x| \rightarrow +\infty$ is not too far from optimality.

We can indeed prove

Lemma 7.4.2.

Suppose that $V \geq 0$ and that there exists $r > 0$ and a sequence σ_n such that $|\sigma_n| \rightarrow +\infty$ and such that

$$\sup_{n \in \mathbb{N}} \sup_{x \in B(\sigma_n, r)} V(x) < +\infty . \tag{7.4.2}$$

Then $-\Delta + V + 1$ is not with compact inverse.

Proof.

Let us consider the sequence

$$\phi_n(x) = \psi(x - \sigma_n) . \tag{7.4.3}$$

Here ψ is a compactly supported function of L^2 norm 1 and with support in $B(0, r)$.

We observe that the ϕ_n are an orthogonal sequence (after possibly extracting a subsequence for obtaining that the supports of ϕ_n and $\phi_{n'}$ are disjoint for $n \neq n'$) which satisfies for some constant C

$$\|\phi_n\|_{H^2}^2 + \|V\phi_n\|_{L^2}^2 \leq C. \quad (7.4.4)$$

In particular, there exists C such that :

$$\|(-\Delta + V + 1)\phi_n\|_{L^2} \leq C, \quad \forall n \in \mathbb{N}.$$

But we can not extract from this sequence a strongly convergent sequence in L^2 , because ϕ_n is weakly convergent to 0 and $\|\phi_n\| = 1$. So the operator $(-\Delta + V + I)^{-1}$ can not be a compact operator.

Exercise 7.4.3.

Show that the unbounded operator on $L^2(\mathbb{R}^2)$

$$P := -\frac{d^2}{dx^2} - \frac{d^2}{dy^2} + x^2y^2,$$

is with compact resolvent.

Hint.

One can introduce

$$X_1 = \frac{1}{i}\partial_x, \quad X_2 = \frac{1}{i}\partial_y, \quad X_3 = xy,$$

and show, for $j = 1, 2$ and for a suitable constant C , the following inequality :

$$\|(x^2 + y^2 + 1)^{-\frac{1}{4}}[X_j, X_3]u\|^2 \leq C (\langle Pu, u \rangle_{L^2(\mathbb{R}^2)} + \|u\|^2),$$

for all $u \in C_0^\infty(\mathbb{R}^2)$.

One can also observe that

$$i[X_1, X_3] = y, \quad i[X_2, X_3] = x,$$

and then

$$\begin{aligned} & \| (x^2 + y^2 + 1)^{-\frac{1}{4}}[X_1, X_3]u \|^2 + \| (x^2 + y^2 + 1)^{-\frac{1}{4}}[X_2, X_3]u \|^2 + \|u\|^2 \\ & \geq \| (x^2 + y^2 + 1)^{\frac{1}{4}}u \|^2. \end{aligned}$$

For the control of $\|(x^2 + y^2 + 1)^{-\frac{1}{4}}[X_1, X_3]u\|^2$, one can remark that

$$\|(x^2 + y^2 + 1)^{-\frac{1}{4}}[X_1, X_3]u\|^2 = \left\langle \frac{y}{(x^2 + y^2 + 1)^{\frac{1}{4}}}u \mid (X_1 X_3 - X_3 X_1)u \right\rangle,$$

perform an integration by parts, and control a commutator.

7.5 The Schrödinger operator with magnetic field

We can consider on \mathbb{R}^m the so-called Schrödinger operator with magnetic field :

$$P_{A,V} := -\Delta_A + V , \quad (7.5.1)$$

where

$$-\Delta_A := \sum_{j=1}^m \left(\frac{1}{i} \partial_{x_j} - A_j(x) \right)^2 . \quad (7.5.2)$$

Here $x \mapsto \vec{A} = (A_1(x), \dots, A_n(x))$ is a vector field on \mathbb{R}^m called the “magnetic potential” and V is called the electric potential. It is easy to see that, when V is semi-bounded, the operator is symmetric and semi-bounded on $C_0^\infty(\mathbb{R}^m)$. We can therefore consider the Friedrichs extension and analyze the property of this selfadjoint extension.

A general question arises if one can get operator of the type $P_{A,V}$ which are with compact resolvent if $V = 0$. This is the problem which is called the problem of the magnetic bottle.

The “heuristic” idea is that the module of the magnetic field can play in some sense the role of the electric potential if it does not oscillate too rapidly ($m \geq 2$). For defining the magnetic field it is probably easier to consider the magnetic potential as a one-form

$$\sigma_A = \sum_{j=1}^n A_j(x) dx_j .$$

The magnetic field is then defined as the two form

$$\omega_B = d\sigma_A = \sum_{j < k} (\partial_{x_j} A_k - \partial_{x_k} A_j) dx_j \wedge dx_k .$$

The case when $m = 2$ is particularly simple. In this case,

$$\omega_B = B dx_1 \wedge dx_2 ,$$

and we can identify ω_B with the function²⁰ $x \mapsto B(x) = \text{curl}(\vec{A})(x)$.

The proof is particularly simple in the case when $B(x)$ has a constant sign (say $B(x) \geq 0$). In this case, we immediately have the inequality

$$\int_{\mathbb{R}^2} B(x) |u(x)|^2 dx \leq \langle -\Delta_A u, u \rangle_{L^2} . \quad (7.5.3)$$

We observe indeed the following identities between operators

$$B(x) = \frac{1}{i} [X_1, X_2] , \quad -\Delta_A = X_1^2 + X_2^2 . \quad (7.5.4)$$

Here

$$X_1 = \frac{1}{i} \partial_{x_1} - A_1(x) , \quad X_2 = \frac{1}{i} \partial_{x_2} - A_2(x) .$$

²⁰Here, for french readers, curl denotes the rotational (in french “rotationnel”) rot .

Note also that :

$$\langle -\Delta_A u, u \rangle = \|X_1 u\|^2 + \|X_2 u\|^2,$$

for all $u \in C_0^\infty(\mathbb{R}^2)$.

To obtain (7.5.3) is then easy through an integration by parts. One can also use, introducing $Z = X_1 + iX_2$, the positivity of Z^*Z or ZZ^* .

We then easily obtain as in the previous example that the operator is with compact resolvent if $B(x) \rightarrow +\infty$.

As a simple example, one can think of

$$\vec{A} = (-x_1^2 x_2, +x_1 x_2^2),$$

which gives

$$B(x) = x_1^2 + x_2^2.$$

Note that the case $m = 2$ is rather particular and it is more difficult to treat $m > 2$. We have indeed to introduce partition of unity.

7.6 Laplace Beltrami operators on a Riemannian compact manifold

If M is a compact riemannian manifold, it is well known that in this case one can canonically define a measure $d\mu_M$ on M and consequently the Hilbertian space $L^2(M)$. We have also a canonical definition of the gradient. At each point x of M , we have indeed a scalar product on $T_x M$ giving an isomorphism between $T_x M$ and $T_x^* M$. Using this family of isomorphisms we have a natural identification between the C^∞ -vector fields on M and the 1-forms on M . In this identification, the vector field $\text{grad } u$ associated to a C^∞ function on M corresponds to the 1-form du .

Considering on $C^\infty(M) \times C^\infty(M)$ the sesquilinear form

$$(u, v) \mapsto a_0(u, v) := \int_M \langle \text{grad } u(x), \text{grad } v(x) \rangle_{T_x M} d\mu_M.$$

There is a natural differential operator $-\Delta_M$ called the Laplace-Beltrami operator on M such that

$$a_0(u, v) = \langle -\Delta_M u, v \rangle_{L^2(M)}.$$

In this context, it is not difficult to define the Friedrichs extension and to get a selfadjoint extension of $-\Delta_M$ as a selfadjoint operator on $L^2(M)$. The domain is easily characterized as being $H^2(M)$, the Sobolev space naturally associated to $L^2(M)$ and one can show that the injection of $H^2(M)$ into $L^2(M)$ is compact because M is compact. The selfadjoint extension of $-\Delta_M$ is with compact resolvent and the general theory can be applied to this example.

The case on the circle \mathbb{S}^1

The simplest model is the operator $-d^2/d\theta^2$ on the circle of radius one whose spectrum is $\{n^2, n \in \mathbb{N}\}$. For $n > 0$ the multiplicity is 2. An orthonormal basis is given by the functions $\theta \mapsto (2\pi)^{-\frac{1}{2}} \exp in\theta$ for $n \in \mathbb{Z}$. Here the form domain of the operator is $H^{1,per}(S^1)$ and the domain of the operator is $H^{2,per}(S^1)$. These spaces have two descriptions. One is to describe these operators as $H^{1,per} := \{u \in H^1(]0, 2\pi[\mid u(0) = u(2\pi)\}$ and $H^{2,per} := \{u \in H^2(]0, 2\pi[\mid u(0) = u(2\pi), u'(0) = u'(2\pi)\}$.

The other way is to consider the Fourier coefficients of u . The Fourier coefficients of $u \in H^{k,per}$ are in \mathbf{h}^k . Here

$$\mathbf{h}^k := \{u_n \in \ell^2(\mathbb{Z}) \mid n^k u_n \in \ell^2(\mathbb{Z})\}.$$

It is then easy to prove the compact injection from $H^{1,per}$ in $L^2(S^1)$ or equivalently from \mathbf{h}^1 into ℓ^2 .

More generally, elliptic symmetric positive operators of order $m > 0$ admit a selfadjoint extension with compact resolvent. We refer to the book by Berger-Gauduchon-Mazet [BGM] for this central subject in Riemannian geometry.

The Laplacian on \mathbb{S}^2

One can also consider the Laplacian on \mathbb{S}^2 . We describe as usual \mathbb{S}^2 by the spherical coordinates, with

$$x = \cos \phi \sin \theta, y = \sin \phi \sin \theta, z = \cos \theta, \text{ with } \phi \in [-\pi, \pi[, \theta \in]0, \pi[, \quad (7.6.1)$$

and we add the two poles ‘‘North’’ and ‘‘South’’, corresponding to the two points $(0, 0, 1)$ and $(0, 0, -1)$.

We are looking for eigenfunctions of the Friedrichs extension of

$$\mathbf{L}^2 = -\frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} \quad (7.6.2)$$

in $L^2(\sin \theta d\theta d\phi)$, satisfying

$$\mathbf{L}^2 Y_{\ell m} = \ell(\ell + 1) Y_{\ell m}. \quad (7.6.3)$$

The standard spherical harmonics, corresponding to $\ell \geq 0$ and for an integer $m \in \{-\ell, \dots, \ell\}$, are defined by

$$Y_{\ell m}(\theta, \phi) = c_{\ell, m} \exp im\phi \frac{1}{\sin^m \theta} \left(-\frac{1}{\sin \theta} \frac{d}{d\theta}\right)^{\ell-m} \sin^{2\ell} \theta, \quad (7.6.4)$$

where $c_{\ell, m}$ is an explicit normalization constant.

For future extensions, we prefer to take this as a definition for $m \geq 0$ and then to observe that

$$Y_{\ell, -m} = \hat{c}_{\ell, m} \overline{Y_{\ell, m}}. \quad (7.6.5)$$

For $\ell = 0$, we get $m = 0$ and the constant. For $\ell = 1$, we obtain, for $m = 1$, the function $(\theta, \phi) \mapsto \sin \theta \exp i\phi$ and for $m = -1$, the function $\sin \theta \exp -i\phi$ and for $m = 0$ the function $\cos \theta$, which shows that the multiplicity is 3 for the eigenvalue 2.

To show the completeness it is enough to show that, for given $m \geq 0$, the orthogonal family (indexed by $\ell \in \{m + \mathbb{N}\}$) of functions $\theta \mapsto \psi_{\ell, m}(\theta) := \frac{1}{\sin^m \theta} \left(-\frac{1}{\sin \theta} \frac{d}{d\theta}\right)^{\ell-m} \sin^{2\ell} \theta$ span all $L^2(]0, \pi[, \sin \theta d\theta)$. For this, we consider $\chi \in C_0^\infty(]0, \pi[)$ and assume that

$$\int_0^\pi \chi(\theta) \psi_{\ell, m}(\theta) \sin \theta d\theta = 0, \forall \ell \in \{m + \mathbb{N}\}.$$

We would like to deduce that this implies $\chi = 0$. After a change of variable $t = \cos \theta$ and an integration by parts, we obtain that this problem is equivalent to the problem to show that, if

$$\int_{-1}^1 \psi(t) ((1-t^2)^\ell)^{(\ell-m)}(t) dt = 0, \forall \ell \in \{m + \mathbb{N}\},$$

then $\psi = 0$.

Observing that the space spanned by the functions $(1-t^2)^{-m} ((1-t^2)^\ell)^{(\ell-m)}$ (which are actually polynomials of exact order ℓ) is the space of all polynomials we can conclude the completeness.

8 Selfadjoint unbounded operators and spectral theory.

8.1 Introduction

We assume that \mathcal{H} is an Hilbert space. Once we have a selfadjoint operator we can apply the basic spectral decomposition, which we shall now describe without to give complete proofs. Before to explain the general case, let us come back to the spectral theorem for compact operators T or operators with compact resolvent. This will permit us to introduce a new vocabulary.

We have seen that one can obtain a decomposition of \mathcal{H} in the form

$$\mathcal{H} = \bigoplus_{k \in \mathbb{N}} V_k, \quad (8.1.1)$$

such that

$$Tu_k = \lambda_k u_k, \quad \text{if } u_k \in V_k. \quad (8.1.2)$$

Hence we have decomposed \mathcal{H} into a direct sum of orthogonal subspaces V_k in which the selfadjoint operator T is reduced to multiplication by λ_k .

If P_k denotes the orthogonal projection operator onto V_k , we can write

$$I = \sum_k P_k, \quad (8.1.3)$$

(the limit is in the strong-convergence sense) and

$$Tu = \sum_k \lambda_k P_k u, \quad \forall u \in D(T). \quad (8.1.4)$$

Here we recall the definition :

Definition 8.1.1.

An operator $P \in \mathcal{L}(\mathcal{H})$ is called an orthogonal projection if $P = P^$ and $P^2 = P$.*

If we assume that T is semibounded (with compact resolvent²¹), we can introduce for any $\lambda \in \mathbb{R}$

$$G_\lambda = \bigoplus_{\lambda_k \leq \lambda} V_k, \quad (8.1.5)$$

and E_λ is the orthogonal projection onto G_λ :

$$E_\lambda = \sum_{\lambda_k \leq \lambda} P_k. \quad (8.1.6)$$

It is easy to see that the function $\lambda \mapsto E_\lambda$ has values in $\mathcal{L}(\mathcal{H})$ and satisfy the following properties :

- $E_\lambda = E_\lambda^*$;
- $E_\lambda \cdot E_\mu = E_{\inf(\lambda, \mu)}$;

²¹Note that if $(T - \lambda_0)^{-1}$ is compact for some $\lambda_0 \in \rho(T)$, then it is true for any $\lambda \in \rho(T)$.

- for all λ , $E_{\lambda+0} = E_\lambda$;
- $\lim_{\lambda \rightarrow -\infty} E_\lambda = 0$, $\lim_{\lambda \rightarrow +\infty} E_\lambda = Id$.
- $E_\lambda \geq 0$

All the limits above are in the sense of the strong convergence.

We also observe that

$$E_{\lambda_k} - E_{\lambda_k-0} = P_k .$$

Then in the sense of vectorvalued distributions, we have

$$dE_\lambda = \sum_k \delta_{\lambda_k} \otimes P_k , \quad (8.1.7)$$

where δ_{λ_k} is the Dirac measure at the point λ_k . Hence, in the sense of Stieltjes integrals (this will be explained in more detail below), one can write

$$x = \int_{-\infty}^{+\infty} dE_\lambda(x) ,$$

and

$$T = \int_{-\infty}^{+\infty} \lambda dE_\lambda .$$

This is in this form that we shall generalize the previous formulas to the case of any selfadjoint operator T .

Functional calculus for operators with compact resolvent :

If f is a continuous (or piecewise continuous function) one can also define $f(T)$ as

$$f(T) = \sum_k f(\lambda_k) \cdot P_k ,$$

as an unbounded operator whose domain is

$$D(f(T)) = \{x \in \mathcal{H} \mid \sum_k |f(\lambda_k)|^2 \|x_k\|^2 < +\infty\} ,$$

where $x_k = P_k x$.

We can also write $f(T)$ in the form :

$$\langle f(T)x, y \rangle_{\mathcal{H}} = \int_{\mathbb{R}} f(\lambda) d \langle E_\lambda x, y \rangle ,$$

where the domain of $f(T)$ is described as

$$D(f(T)) = \{x \in \mathcal{H} \mid \int_{\mathbb{R}} |f(\lambda)|^2 d \langle E_\lambda x, x \rangle_{\mathcal{H}} < +\infty\} .$$

Remark 8.1.2.

There are, for semibounded operators with compact resolvent, two possible conventions for the notation of the eigenvalues. The first one is to classify them into an increasing sequence

$$\mu_j \leq \mu_{j+1}$$

counting each eigenvalue according to its multiplicity. The second one is to describe them as a strictly increasing sequence λ_k with λ_k eigenvalue of multiplicity m_k .

We now present a list of properties which are easy to verify in this particular case and which will be still true in the general case.

1. If f and g coincide on $\sigma(T)$, then $f(T) = g(T)$. For any $(x, y) \in \mathcal{H} \times \mathcal{H}$, the support of the measure $d\langle E_\lambda x, y \rangle$ is contained²² in $\sigma(T)$.
2. If f and g are functions on \mathbb{R} ,

$$f(T)g(T) = (f \cdot g)(T) .$$

In particular, if $(T - z)$ is invertible, the inverse is given by $f(T)$ where f is a continuous function such that $f(\lambda) = (\lambda - z)^{-1}$, $\forall \lambda \in \sigma(T)$.

3. If f is bounded, then $f(T)$ is bounded and we have

$$\|f(T)\| \leq \sup_{\lambda \in \sigma(T)} |f(\lambda)| . \quad (8.1.8)$$

4. The function f may be complex. Note that, in this case, we get

$$f(T)^* = \overline{f}(T) . \quad (8.1.9)$$

An interesting case is, for $z \in \mathbb{C} \setminus \mathbb{R}$, the function $\lambda \mapsto (\lambda - z)^{-1}$. Then we get from (8.1.8)

$$\|(T - z)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq |\operatorname{Im} z|^{-1} . \quad (8.1.10)$$

5. More generally, this works also for $z \in \mathbb{R} \setminus \sigma(T)$. We then obtain in this case the spectral theorem

$$\|(T - z)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq d(z, \sigma(T))^{-1} . \quad (8.1.11)$$

6. If $f \in C_0^\infty(\mathbb{R})$, we have :

$$f(T) = \frac{1}{\pi} \lim_{\epsilon \rightarrow 0^+} \int_{|\operatorname{Im} z| > \epsilon} \left(\frac{\partial \tilde{f}}{\partial \bar{z}} \right) (T - z)^{-1} dx . dy . \quad (8.1.12)$$

²²At the moment, we have not defined $\sigma(T)$ when T is unbounded. Think here simply of the set of the eigenvalues !

Here \tilde{f} is defined by :

$$\tilde{f}(x, y) = (f(x) + iyf'(x))\chi(y) ,$$

where $\chi(y)$ is equal to 1 in a neighborhood of 0 and with compact support. This formula can be proven using the Green-Formula (first prove it with T replaced by the scalar λ) or using that :

$$\partial_{\bar{z}} \frac{1}{z - \lambda} = \pi \delta_{(\lambda, 0)} ,$$

where $\delta_{(\lambda, 0)}$ is the Dirac measure at $(\lambda, 0) \in \mathbb{R}^2$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$. One should observe that \tilde{f} is not holomorphic but “almost” holomorphic in the sense that :

$$\partial \tilde{f} / \partial \bar{z} = \mathcal{O}(y) ,$$

as $y \rightarrow 0$.

Exercise 8.1.3.

Show that, one can also find, for any $N \geq 1$, \tilde{f}^N such that in addition

$$\partial \tilde{f} / \partial \bar{z} = \mathcal{O}(y^N) ,$$

as $y \rightarrow 0$.

8.2 Spectrum.

We now come back to the notion of spectrum that we have only met for bounded operators.

Definition 8.2.1.

The resolvent set of a closed operator T is the set of the λ in \mathbb{C} such that the range of $(T - \lambda)$ is equal to \mathcal{H} and such that $(T - \lambda)$ admits a continuous operator denoted by $R(\lambda)$ whose range is included in $D(T)$ such that :

$$R(\lambda)(T - \lambda) = I_{D(T)} ,$$

and

$$(T - \lambda)R(\lambda) = I_{\mathcal{H}} .$$

As in the bounded case, we observe that the resolvent set is open. Note also that the continuity of $R(\lambda)$ is actually a consequence of the property that the graph of $R(\lambda)$ is closed (using that T is closed) and that $R(\lambda)$ is defined on \mathcal{H} .

Definition 8.2.2.

The spectrum of a closed operator T is defined as the complementary set in \mathbb{C} of the resolvent set.

It is then rather easy to show that the spectrum $\sigma(T)$ is closed in \mathbb{C} . The proof of the fact that the spectrum is contained in \mathbb{R} if T is selfadjoint is very close to the bounded case.

The spectrum $\sigma(T)$ is not empty if T is selfadjoint.

The proof is by contradiction. If T has empty spectrum T^{-1} is a bounded selfadjoint operator with spectrum equal to $\{0\}$. We observe indeed that, for $\lambda \neq 0$, the inverse of $T^{-1} - \lambda$ is given, if $\lambda^{-1} \in \rho(T)$, by $\lambda^{-1}T(T - \lambda^{-1})^{-1}$. Hence T^{-1} should be the 0 operator, which contradicts $T \circ T^{-1} = I$. This is no longer true in the non selfadjoint case. At the end of the chapter, we give some example appearing naturally in various questions in Fluid Mechanics.

8.3 Spectral family and resolution of the identity.

Definition 8.3.1.

A family of orthogonal projectors $E(\lambda)$ (or E_λ), $-\infty < \lambda < \infty$ in an Hilbert space \mathcal{H} is called a resolution of the identity (or spectral family) if it satisfies the following conditions :

- $$E(\lambda)E(\mu) = E(\min(\lambda, \mu)) , \quad (8.3.1)$$

- $$E(-\infty) = 0 , E(+\infty) = I , \quad (8.3.2)$$

where $E(\pm\infty)$ is defined ²³ by

$$E(\pm\infty)x = \lim_{\lambda \rightarrow \pm\infty} E(\lambda)x \quad (8.3.3)$$

for all x in \mathcal{H} ,

- $$E(\lambda + 0) = E(\lambda) , \quad (8.3.4)$$

where $E(\lambda + 0)$ is defined by

$$E(\lambda + 0)x = \lim_{\mu \rightarrow \lambda , \mu > \lambda} E(\mu)x . \quad (8.3.5)$$

Remark 8.3.2.

We have shown an example of such a family in the previous subsection.

Proposition 8.3.3.

Let $E(\lambda)$ be a resolution of identity (=spectral family); then for all $x, y \in \mathcal{H}$, the function

$$\lambda \mapsto \langle E(\lambda)x, y \rangle \quad (8.3.6)$$

is a function of bounded variation whose total variation²⁴ satisfies

$$V(x, y) \leq \|x\| \cdot \|y\| , \quad \forall x, y \in \mathcal{H} . \quad (8.3.7)$$

²³(8.3.1) gives the existence of the limit (cf also Lemma 8.3.4). The limit in (8.3.3) is taken in \mathcal{H} . We observe indeed that $\lambda \mapsto \langle E(\lambda)x, x \rangle = \|E(\lambda)x\|^2$ is monotonically increasing.

²⁴See the definition in (8.3.9)

Proof.

Let $\lambda_1 < \lambda_2 < \dots < \lambda_n$. We first get from the assumption (8.3.1) that

$$E_{] \alpha, \beta]} = E_\beta - E_\alpha$$

is an orthogonal projection. From the Cauchy-Schwarz inequality we have

$$\begin{aligned} \sum_{j=2}^n | \langle E_{] \lambda_{j-1}, \lambda_j]} x, y \rangle | &= \sum_{j=2}^n | \langle E_{] \lambda_{j-1}, \lambda_j]} x, E_{] \lambda_{j-1}, \lambda_j]} y \rangle | \\ &\leq \sum_{j=2}^n \| E_{] \lambda_{j-1}, \lambda_j]} x \| \| E_{] \lambda_{j-1}, \lambda_j]} y \| \\ &\leq \left(\sum_{j=2}^n \| E_{] \lambda_{j-1}, \lambda_j]} x \|^2 \right)^{\frac{1}{2}} \left(\sum_{j=2}^n \| E_{] \lambda_{j-1}, \lambda_j]} y \|^2 \right)^{\frac{1}{2}} \\ &= \left(\| E_{] \lambda_1, \lambda_n]} x \|^2 \right)^{\frac{1}{2}} \left(\| E_{] \lambda_1, \lambda_n]} y \|^2 \right)^{\frac{1}{2}} . \end{aligned}$$

But for $m > n$, we get

$$\|x\|^2 \geq \|E_{] \lambda_n, \lambda_m]} x\|^2 = \sum_{i=n}^{m-1} \|E_{] \lambda_i, \lambda_{i+1}]} x\|^2 . \quad (8.3.8)$$

We finally obtain that, for any finite sequence $\lambda_1 < \lambda_2 < \dots < \lambda_n$, we have

$$\sum_{j=2}^n | \langle E_{] \lambda_{j-1}, \lambda_j]} x, y \rangle | \leq \|x\| \cdot \|y\| .$$

This shows the bounded variation and the estimate of the total variation defined as

$$V(x, y) := \sup_{\lambda_1, \dots, \lambda_n} \sum_{j=2}^n | \langle E_{] \lambda_{j-1}, \lambda_j]} x, y \rangle | . \quad (8.3.9)$$

Hence we have shown that, for all x and y in \mathcal{H} , the function $\lambda \mapsto \langle E(\lambda)x, y \rangle$ is with bounded variation and we can then show the existence of $E(\lambda + 0)$ and $E(\lambda - 0)$. This is the object of

Lemma 8.3.4.

If $E(\lambda)$ is a family of projectors satisfying (8.3.1) and (8.3.2), then, for all $\lambda \in \mathbb{R}$, the operators

$$E_{\lambda+0} = \lim_{\mu \rightarrow \lambda} \lim_{\mu > \lambda} E(\mu) , \quad E_{\lambda-0} = \lim_{\mu \rightarrow \lambda} \lim_{\mu < \lambda} E(\mu) , \quad (8.3.10)$$

are well defined when considering the limit for the strong convergence topology.

Proof.

Let us show the existence of the left limit. From (8.3.8), we get that, for any $\epsilon > 0$, there exists $\lambda_0 < \lambda$ such that, $\forall \lambda', \forall \lambda'' \in [\lambda_0, \lambda[$, such that $\lambda' < \lambda''$

$$\|E_{] \lambda', \lambda'']} x\|^2 \leq \epsilon .$$

It is then easy to show that $E_{\lambda-\frac{1}{n}} x$ is a Cauchy sequence converging to a limit and that this limit does not depend on the choice of the sequence tending to λ .

The proof of the existence of the limit from the right is the same. This ends the proof of the lemma.

It is then classical (Stieltjes integrals) that one can define for any continuous complex valued function $\lambda \mapsto f(\lambda)$ the integrals $\int_a^b f(\lambda) d\langle E(\lambda)x, y \rangle$ as a limit²⁵ of Riemann sums.

Proposition 8.3.5.

Let f be a continuous function on \mathbb{R} with complex values and let $x \in \mathcal{H}$. Then it is possible to define for $\alpha < \beta$, the integral

$$\int_{\alpha}^{\beta} f(\lambda) dE_{\lambda}x$$

as the strong limit in \mathcal{H} of the Riemann sum :

$$\sum_j f(\lambda'_j)(E_{\lambda_{j+1}} - E_{\lambda_j})x, \tag{8.3.11}$$

where

$$\alpha = \lambda_1 < \lambda_2 < \dots < \lambda_n = \beta,$$

and

$$\lambda'_j \in]\lambda_j, \lambda_{j+1}],$$

when $\max_j |\lambda_{j+1} - \lambda_j| \rightarrow 0$.

The proof is easy using the uniform continuity of f . Note also that the notation could be misleading. May be $\int_{] \alpha, \beta]} f(\lambda) dE_{\lambda}x$ is less ambiguous.

We now arrive like in the standard theory to the generalized integral.

Definition 8.3.6.

For any given $x \in \mathcal{H}$ and any continuous function f on \mathbb{R} , the integral :

$$\int_{-\infty}^{+\infty} f(\lambda) dE_{\lambda}x$$

is defined as the strong limit in \mathcal{H} , if it exists of $\int_{\alpha}^{\beta} f(\lambda) dE_{\lambda}x$ when $\alpha \rightarrow -\infty$ and $\beta \rightarrow +\infty$.

Remark 8.3.7.

The theory works more generally for any borelian function (cf Reed-Simon, Vol. 1 [RS-I]). This can be important, because we are in particular interested in the case when $f(t) = 1_{]-\infty, \lambda]}(t)$.

One possibility for the reader who wants to understand how this can be made is to look at Rudin's book [Ru1], which gives the following theorem (Theorem 8.14, p. 173)

²⁵The best is to first consider the case when $x = y$ and then use a depolarisation formula, in the same way that, when we have an Hilbertian norm, we can recover the scalar product from the norm.

Theorem 8.3.8. .

1. If μ is a complex Borel measure on \mathbb{R} and if

$$(\star) \quad f(x) = \mu([-\infty, x]), \quad \forall x \in \mathbb{R},$$

then f is a normalized function with bounded variation (NBV). By NBV we mean, with bounded variation, but also continuous from the right and such that $\lim_{x \rightarrow -\infty} f(x) = 0$.

2. Conversely, to every $f \in \text{NBV}$, there corresponds a unique complex Borel measure μ such that (\star) is satisfied.

Theorem 8.3.9.

For x given in \mathcal{H} and if f is a complex valued continuous function on \mathbb{R} , the following conditions are equivalent

- $$\int_{-\infty}^{+\infty} f(\lambda) dE_\lambda x \text{ exists}; \quad (8.3.12)$$

- $$\int_{-\infty}^{+\infty} |f(\lambda)|^2 d\|E_\lambda x\|^2 < +\infty; \quad (8.3.13)$$

- $$y \mapsto \int_{-\infty}^{+\infty} f(\lambda) d\langle E_\lambda y, x \rangle_{\mathcal{H}} \quad (8.3.14)$$

is a continuous linear form.

Hint for the proof.

a) (8.3.12) implies (8.3.14) essentially by using repeatedly the Banach-Steinhaus Theorem (also called Uniform Boundedness Theorem) and the definition of the integral.

b) Let us prove that (8.3.14) implies (8.3.13).

Let F be the linear form appearing in (8.3.14). If we introduce

$$y = \int_{\alpha}^{\beta} \overline{f(\lambda)} dE_\lambda x,$$

then we first observe (coming back to the Riemann integrals) that

$$y = E_{] \alpha, \beta]} y.$$

It is then not too difficult to show that

$$\begin{aligned}
\overline{F(y)} &= \int_{-\infty}^{+\infty} \overline{f(\lambda)} d \langle E_\lambda x, y \rangle \\
&= \int_{-\infty}^{+\infty} \overline{f(\lambda)} d \langle E_\lambda x, E_{] \alpha, \beta] } y \rangle \\
&= \int_{-\infty}^{+\infty} \overline{f(\lambda)} d \langle E_{] \alpha, \beta] } E_\lambda x, y \rangle \\
&= \int_\alpha^\beta \overline{f(\lambda)} d \langle E_\lambda x, y \rangle \\
&= \|y\|^2 .
\end{aligned}$$

Using (8.3.14), we get $\|y\|^2 \leq \|F\| \cdot \|y\|$ and consequently

$$\|y\| \leq \|F\| . \quad (8.3.15)$$

Here we observe that the r.h.s. is independent of α and β .

On the other hand, coming back to Riemann sums, we get

$$\|y\|^2 = \int_\alpha^\beta |f(\lambda)|^2 d \|E_\lambda x\|^2 .$$

We finally obtain

$$\int_\alpha^\beta |f(\lambda)|^2 d \|E_\lambda x\|^2 \leq \|F\|^2 . \quad (8.3.16)$$

Hence, taking the limits $\alpha \rightarrow -\infty$ and $\beta \rightarrow +\infty$, we obtain (8.3.13).

c)

For the last implication, it is enough to observe that, for $\alpha' < \alpha < \beta < \beta'$, we have

$$\| \int_{\alpha'}^{\beta'} f(\lambda) d E_\lambda x - \int_\alpha^\beta f(\lambda) d E_\lambda x \|^2 = \int_{\alpha'}^\alpha |f(\lambda)|^2 d \|E_\lambda x\|^2 + \int_\beta^{\beta'} |f(\lambda)|^2 d \|E_\lambda x\|^2 .$$

Theorem 8.3.10.

Let $\lambda \mapsto f(\lambda)$ be a real-valued continuous function. Let

$$D_f := \{ x \in \mathcal{H}, \int_{-\infty}^{+\infty} |f(\lambda)|^2 d \langle E(\lambda)x, x \rangle < \infty, \} .$$

Then D_f is dense in \mathcal{H} and we can define T_f whose domain is defined by

$$D(T_f) = D_f ,$$

and

$$\langle T_f x, y \rangle = \int_{-\infty}^{+\infty} f(\lambda) d \langle E(\lambda)x, y \rangle$$

for all x in $D(T_f)$ and y in \mathcal{H} .

The operator T_f is selfadjoint.

Finally, we have

$$T_f E_\lambda \text{ is an extension of } E_\lambda T_f . \quad (8.3.17)$$

Proof of the theorem.

From property (8.3.2), we obtain that, for any y in \mathcal{H} , there exists a sequence (α_n, β_n) such that $E_{] \alpha_n, \beta_n]} y \rightarrow y$ as $n \rightarrow +\infty$.

But $E_{] \alpha, \beta]} y$ belongs to D_f , for any α, β , and this shows the density of D_f in \mathcal{H} . We now observe that f being real and E_λ being symmetric the symmetry is clear.

We observe that, for $f_0 = 1$, we have $T_{f_0} = I$ and for $f_1(\lambda) = \lambda$, we obtain a selfadjoint operator $T_{f_1} := T$.

In this case we say that

$$T = \int_{-\infty}^{+\infty} \lambda dE(\lambda)$$

is a spectral decomposition of T and we shall note that

$$\|Tx\|^2 = \int_{-\infty}^{+\infty} \lambda^2 d\langle E(\lambda)x, x \rangle = \int_{-\infty}^{+\infty} \lambda^2 d\|E(\lambda)x\|^2,$$

for $x \in D(T)$.

More generally

$$\|T_f x\|^2 = \int_{-\infty}^{+\infty} |f(\lambda)|^2 d(\langle E(\lambda)x, x \rangle) = \int_{-\infty}^{+\infty} |f(\lambda)|^2 d(\|E(\lambda)x\|^2)$$

for $x \in D(T_f)$.

Conclusion. We have consequently seen in this subsection how one can associate to a spectral family of projectors a selfadjoint operator. We have seen in the introduction that the converse was true for a compact operator or an operator with compact resolvent. It remains to prove that this is true in the general case.

8.4 The spectral decomposition Theorem.

The spectral decomposition Theorem makes explicit that the preceding situation is actually the general one.

Theorem 8.4.1.

Any selfadjoint operator T in an Hilbert space \mathcal{H} admits a spectral decomposition such that

$$\begin{aligned} \langle Tx, y \rangle &= \int_{\mathbb{R}} \lambda d \langle E_\lambda x, y \rangle_{\mathcal{H}}, \\ Tx &= \int_{\mathbb{R}} \lambda d(E_\lambda x). \end{aligned} \tag{8.4.1}$$

”Proof”.

We shall only give the main points of the proof. We refer to [Hu], [Le-Br] or [DaLi] for detailed proofs or to [RS-I] for another proof which we describe now shortly. Another interesting proof is based on Formula 8.1.12 and presented in

the book of Davies [Da].

Step 1.

It is rather natural to imagine that it is essentially enough to treat the case when T is a bounded selfadjoint operator (or at least a normal bounded operator, that is satisfying $T^*T = TT^*$). If A is indeed a general semibounded selfadjoint operator, one can come back to the bounded case by considering $(A + \lambda_0)^{-1}$, with λ_0 real, which is bounded and selfadjoint. In the general case²⁶, one can consider $(A + i)^{-1}$.

Step 2.

We analyze first the spectrum of $P(T)$ where P is a polynomial.

Lemma 8.4.2.

If P is a polynomial, then

$$\sigma(P(T)) = \{P(\lambda) \mid \lambda \in \sigma(T)\} \tag{8.4.2}$$

Proof.

We start from the identity $P(x) - P(\lambda) = (x - \lambda)Q_\lambda(x)$ and from the corresponding identity between bounded operators $P(T) - P(\lambda) = (T - \lambda)Q_\lambda(T)$. This permits to construct the inverse of $(T - \lambda)$ if one knows the inverse of $P(T) - P(\lambda)$.

Conversely, we observe that, if $z \in \mathbb{C}$ and if $\lambda_j(z)$ are the roots of $\lambda \mapsto (P(\lambda) - z)$, then we can write :

$$(P(T) - z) = c \prod_j (T - \lambda_j(z)) .$$

This permits to construct the inverse of $(P(T) - z)$ if one has the inverses of $(T - \lambda_j(z))$ (for all j).

Lemma 8.4.3.

Let T be a bounded self-adjoint operator. Then

$$\|P(T)\| = \sup_{\lambda \in \sigma(T)} |P(\lambda)| . \tag{8.4.3}$$

We first observe that

$$\|P(T)\|^2 = \|P(T)^*P(T)\|$$

This is the consequence of the general property for bounded linear operators that :

$$\|A^*A\| = \|A\|^2 .$$

²⁶Here we recall that an example of operator which is not semibounded is given in Exercise 4.1.3

We recall that the proof is obtained by observing first that :

$$\begin{aligned} \|A^*A\| &= \sup_{\|x\|\leq 1, \|y\|\leq 1} |\langle A^*Ax, y \rangle| \\ &= \sup_{x,y, \|x\|\leq 1, \|y\|\leq 1} |\langle Ax, Ay \rangle| \\ &\leq \|A\|^2, \end{aligned}$$

and secondly that :

$$\|A\|^2 = \sup_{\|x\|\leq 1} \langle Ax, Ax \rangle = \sup_{\|x\|\leq 1} \langle A^*Ax, x \rangle \leq \|A^*A\|.$$

We then observe that :

$$\begin{aligned} \|P(T)\|^2 &= \|(\bar{P}P)(T)\| \\ &= \sup_{\mu \in \sigma(\bar{P}P)(T)} |\mu| \quad (\text{using Theorem 6.4.1}) \\ &= \sup_{\lambda \in \sigma(T)} |(\bar{P}P)(\lambda)| \quad (\text{using Lemma 8.4.2}) \\ &= \left(\sup_{\lambda \in \sigma(T)} |P(\lambda)|^2 \right). \end{aligned}$$

Step 3.

We have defined a map Φ from the set of polynomials into $\mathcal{L}(\mathcal{H})$ defined by

$$P \mapsto \Phi(P) = P(T), \quad (8.4.4)$$

which is continuous

$$\|\Phi(P)\|_{\mathcal{L}(\mathcal{H})} = \sup_{\lambda \in \sigma(T)} |P(\lambda)|. \quad (8.4.5)$$

The set $\sigma(T)$ is a compact in \mathbb{R} and using the Stone-Weierstrass theorem (which states the density of the polynomials in $C^0(\sigma(T))$), the map Φ can be uniquely extended to $C^0(\sigma(T))$. We still denote by Φ this extension. The properties of Φ are described in the following theorem

Theorem 8.4.4.

Let T be as selfadjoint continuous operator on \mathcal{H} . Then there exists a unique map Φ from $C^0(\sigma(T))$ into $\mathcal{L}(\mathcal{H})$ with the following properties :

1.
$$\begin{aligned} \Phi(f+g) &= \Phi(f) + \Phi(g), & \Phi(\lambda f) &= \lambda \Phi(f); \\ \Phi(1) &= Id, & \Phi(\bar{f}) &= \Phi(f)^*; \\ \Phi(fg) &= \Phi(f) \circ \Phi(g). \end{aligned}$$

2.
$$\|\Phi(f)\| = \sup_{\lambda \in \sigma(T)} |f(\lambda)|.$$

3. *If f is defined by $f(\lambda) = \lambda$, then $\Phi(f) = T$.*

4.
$$\sigma(\Phi(f)) = \{f(\lambda) \mid \lambda \in \sigma(T)\}.$$

5. *If ψ satisfies $T\psi = \lambda\psi$, then $\Phi(f)\psi = f(\lambda)\psi$.*

6. If $f \geq 0$, then $\Phi(f) \geq 0$.

All these properties are obtained by showing first the properties for polynomials P and then extending the properties by continuity to continuous functions. For the last item, note that :

$$\Phi(f) = \Phi(\sqrt{f}) \cdot \Phi(\sqrt{f}) = \Phi(\sqrt{f})^* \cdot \Phi(\sqrt{f}) .$$

Step 4.

We are now ready to introduce the measures. Let $\psi \in \mathcal{H}$. Then

$$f \mapsto \langle \psi, f(T)\psi \rangle_{\mathcal{H}} = \langle \psi, \Phi(f)\psi \rangle_{\mathcal{H}}$$

is a positive linear functional on $C^0(\sigma(T))$. By measure theory (Riesz Theorem) (cf Rudin [Ru1]), there exists a unique measure μ_{ψ} on $\sigma(T)$, such that

$$\langle \psi, f(T)\psi \rangle_{\mathcal{H}} = \int_{\sigma(T)} f(\lambda) d\mu_{\psi}(\lambda) . \tag{8.4.6}$$

This measure is called the spectral measure associated with the vector $\psi \in \mathcal{H}$. This measure is a Borel measure. This means that we can extend the map Φ and (8.4.6) to Borelian functions.

Using the standard Hilbert calculus (that is the link between sesquilinear form and quadratic forms) we can also construct for any x and y in \mathcal{H} a complex measure $d\mu_{x,y}$ such that

$$\langle x, \Phi(f)y \rangle_{\mathcal{H}} = \int_{\sigma(T)} f(\lambda) d\mu_{x,y}(\lambda) . \tag{8.4.7}$$

Using the Riesz representation Theorem (Theorem 3.1.1), this gives as, when f is bounded, an operator $f(T)$. If $f = 1_{]-\infty, \mu]}$, we recover the operator $E_{\mu} = f(T)$ which permits to construct indeed the spectral family announced in Theorem 8.4.1.

Remark 8.4.5.

Modulo some care concerning the domains of the operator, the properties mentioned at the first subsection of this section for operators with compact resolvent are preserved in the case of an unbounded selfadjoint operator.

8.5 Applications of the spectral theorem:

One of the first applications of the spectral theorem (Property 2.) is the following property :

Proposition 8.5.1.

$$d(\lambda, \sigma(T)) \|x\| \leq \|(T - \lambda)x\| , \tag{8.5.1}$$

for all x in $D(T)$.

This proposition is frequently used in the following context. Except very special cases like the harmonic oscillator, it is usually difficult to get explicitly the values of the eigenvalues of an operator. One consequently tries to localize these eigenvalues by using approximations. Let us suppose for example that one has found λ_0 and y in $D(L)$ such that

$$\|(T - \lambda_0)y\| \leq \epsilon \tag{8.5.2}$$

and $\|y\| = 1$ then we deduce the existence of λ in the spectrum of L such that $|\lambda - \lambda_0| \leq \epsilon$.

Standard examples are the case of hermitian matrices or the case of the anharmonic oscillator $T := -h^2 \frac{d^2}{dx^2} + x^2 + x^4$. In the second case the first eigenfunction of the harmonic oscillator $-h^2 \frac{d^2}{dx^2} + x^2$ can be used as approximate eigenfunction y in (8.5.2) with $\lambda_0 = h$. We then find (8.5.2) with $\epsilon = \mathcal{O}(h^2)$.

Another application is, using the property that the spectrum is real, the following inequality

$$|\operatorname{Im} \lambda| \|x\| \leq \|(T - \lambda)x\| \tag{8.5.3}$$

and this gives an upper bound on the norm of $(T - \lambda)^{-1}$ in $\mathcal{L}(\mathcal{H})$ by $1/|\operatorname{Im} \lambda|$. One can also consider the operator $T_\epsilon = -d^2/dx^2 + x^2 + \epsilon x^4$. One can show that near each eigenvalue of the harmonic oscillator $(2n + 1)$, then there exists, when $\epsilon > 0$ is small enough, an eigenvalue $\lambda_n(\epsilon)$ of T_ϵ .

Another good example to analyze is the construction of a sequence of approximate eigenvectors considered in Subsection 1.1. From the construction of u_n such that, with $T = -\Delta$,

$$\|(T - \xi^2)u_n\|_{L^2(\mathbb{R}^m)} = \mathcal{O}\left(\frac{1}{n}\right),$$

one obtains that

$$d(\sigma(T), \xi^2) \leq \frac{C}{n}, \quad \forall n \in \mathbb{N}.$$

As $n \rightarrow +\infty$, we get $\xi^2 \in \sigma(T)$.

It is then easy to show that

$$\sigma(P) = [0, +\infty[.$$

It is enough to prove indeed, using the Fourier transform, that, for any $b > 0$, $(-\Delta + b)$ has an inverse $(-\Delta + b)^{-1}$ sending L^2 onto H^2 .

Here we have followed in a particular case the proof of the following general theorem

Theorem 8.5.2.

Let T be a selfadjoint operator. Then $\lambda \in \sigma(P)$ if and only if there exists a sequence $(u_n)_{n \in \mathbb{N}}$, $u_n \in D(T)$ such that $\|u_n\| = 1$ and $\|(T - \lambda)u_n\| \rightarrow 0$ as $n \rightarrow +\infty$.

Exercise 8.5.3.

Show the “only if”, after reading of the proof of Proposition 8.5.4.

Note also the following characterization.

Proposition 8.5.4.

$$\sigma(T) = \{\lambda \in \mathbb{R}, \text{ s.t. } \forall \epsilon > 0, E(] \lambda - \epsilon, \lambda + \epsilon[) \neq 0\} \quad (8.5.4)$$

The proof uses in one direction the explicit construction of $(T - \lambda)$ through Proposition 8.5.1. If λ and $\epsilon_0 > 0$ are such $E_{] \lambda - \epsilon_0, \lambda + \epsilon_0[} = 0$. Then there exists a continuous function f on \mathbb{R} , such that $f(t) = (t - \lambda)^{-1}$ on the support of the measure dE_λ . This permits to construct the inverse and to show that λ is in the resolvent set of T .

Conversely, let λ in the set defined by the r.h.s of (8.5.4) later denoted by $\tilde{\sigma}(T)$. For any $n \in \mathbb{N}^*$, let us take x_n such that $\|x_n\| = 1$ and $E(] \lambda - \frac{1}{n}, \lambda + \frac{1}{n}[)x_n = x_n$. Using property 2., with the function $f_n(t) = (t - \lambda)1_{] \lambda - \frac{1}{n}, \lambda + \frac{1}{n}[}(t)$, we get :

$$\|(T - \lambda)x_n\| = \|(T - \lambda)E(] \lambda - \frac{1}{n}, \lambda + \frac{1}{n}[)x_n\| \leq \frac{1}{n}\|x_n\| = \frac{1}{n}.$$

Applying Proposition 8.5.1, we obtain :

$$\tilde{\sigma}(T) \subset \sigma(T).$$

8.6 Examples of functions of a selfadjoint operator

We shall for example meet usually in spectral theory the functions

1. f is the characteristic function of $] - \infty, \lambda]$, $Y_{] - \infty, \lambda]}$; $\Phi(f) = f(T)$ is then $\Phi(f) = E(\lambda)$.
2. f is the characteristic function of $] - \infty, \lambda[$, $Y_{] - \infty, \lambda[}$; $f(T)$ is then $\Phi(f) = E(\lambda - 0)$.
3. f is a compactly supported continuous function. $f(T)$ will be an operator whose spectrum is localized in the support of f .
4. $f_t(\lambda) = \exp(it\lambda)$ with t real.
 $f_t(T)$ is then a solution of the functional equation

$$\begin{aligned} (\partial_t - iT)(f(t, T)) &= 0 \\ f(0, T) &= Id \end{aligned}$$

We note here that, for all real t , $f_t(T) = \exp(itT)$ is a bounded unitary operator .

5. $g_t(\lambda) = \exp(-t\lambda)$ with t real positive. $g_t(T)$ is then a solution of the functional equation

$$\begin{aligned} (\partial_t + T)(g(t, T)) &= 0, \text{ for } t \geq 0 \\ g(0, T) &= Id. \end{aligned}$$

We have discussed in the introduction the case of an operator with compact resolvent. The other case to understand for the beginner is of course the case of the free Laplacian $-\Delta$ on \mathbb{R}^n . Using the Fourier transform \mathcal{F} , we get as unbounded operator the operator of multiplication by ξ^2 . It is not difficult to define directly the functional calculus which simply becomes for a borelian function ϕ :

$$\phi(-\Delta) = \mathcal{F}^{-1}\phi(\xi^2)\mathcal{F} . \quad (8.6.1)$$

One possibility is to start from $(-\Delta + 1)^{-1}$, for which this formula is true and to then follow what was our construction of the functional calculus. Another possibility is to use the Formula (8.1.12) and to use that (8.6.1) is satisfied for $(-\Delta + z)^{-1}$, with $z \in \mathbb{C} \setminus \mathbb{R}$.

The spectral family is then defined by

$$\langle E(\lambda)f, g \rangle_{L^2(\mathbb{R}^m)} = \int_{\xi^2 \leq \lambda} \hat{f}(\xi) \cdot \bar{\hat{g}}(\xi) d\xi .$$

8.7 Spectrum and spectral measures

Another interest of the spectral theorem is to permit the study of the different properties of the spectrum according to the nature of the spectral measure and this leads to the definition of the continuous spectrum and of the pure point spectrum. Let us briefly discuss (without proof) these notions.

Starting of a selfadjoint operator T , one defines \mathcal{H}_{pp} (pure point subspace) as the set defined as

$$\mathcal{H}_{pp} = \{ \psi \in \mathcal{H} \mid f \mapsto \langle f(T)\psi, \psi \rangle_{\mathcal{H}} \text{ is a pure point measure} \} \quad (8.7.1)$$

We recall that a measure on X is pure point if

$$\mu(X) = \sum_{x \in X} \mu(\{x\}) . \quad (8.7.2)$$

One can verify that H_{pp} is a closed subspace of \mathcal{H} and that the corresponding orthogonal projection $\Pi_{\mathcal{H}_{pp}}$ satisfies

$$\Pi_{\mathcal{H}_{pp}} D(T) \subset D(T) .$$

In this case $T|_{\mathcal{H}_{pp}}$ is naturally defined as unbounded operator on \mathcal{H}_{pp} and one defines the pure point spectrum of T by

$$\sigma_{pp}(T) = \sigma(T|_{\mathcal{H}_{pp}}) . \quad (8.7.3)$$

We can similarly define \mathcal{H}_c (continuous subspace) as the set defined as

$$\mathcal{H}_c = \{ \psi \in \mathcal{H} \mid f \mapsto \langle f(T)\psi, \psi \rangle_{\mathcal{H}} \text{ is a continuous measure} \} \quad (8.7.4)$$

We recall that a measure on X is continuous if

$$\mu(\{x\}) = 0 , \quad \forall x \in X . \quad (8.7.5)$$

One can verify that H_c is a closed subspace of \mathcal{H} and that the corresponding orthogonal projection $\Pi_{\mathcal{H}_c}$ satisfies

$$\Pi_{\mathcal{H}_c} D(T) \subset D(T) .$$

Moreover, it can be shown (See [RS-I]), that

$$\mathcal{H} = \mathcal{H}_{pp} \oplus \mathcal{H}_c . \tag{8.7.6}$$

In this case $T|_{\mathcal{H}_c}$ is naturally defined as unbounded operator on \mathcal{H}_c and one defines the continuous spectrum of T by

$$\sigma_c(T) = \sigma(T|_{\mathcal{H}_c}) . \tag{8.7.7}$$

Example 8.7.1.

The spectrum of $-\Delta$ is continuous.

We observe indeed that

$$\lim_{\epsilon \rightarrow 0, \epsilon > 0} \int_{||\xi|^2 - \lambda| \leq \epsilon} |\hat{f}(\xi)|^2 d\xi = 0 , \forall f \in L^2(\mathbb{R}^m) .$$

One can still refine this discussion by using the natural decomposition of the measure given by the Radon-Nikodym Theorem. This leads to the notion of absolutely continuous spectrum and of singularly continuous spectrum.

9 Non-self adjoint operators and ϵ -pseudospectrum

When the operators are not selfadjoint, one should think that the spectrum is not the right object because it becomes very unstable by perturbation. It has been realized in the recent years that a family of sets (parametrized by $\epsilon > 0$) in \mathbb{C} called the ϵ -pseudospectrum is the right object for getting this stability.

9.1 Main definitions and properties

Here we follow Chapter 4 in the book by L.N. Trefethen and M. Embree [TrEm].

Definition 9.1.1.

If A is a closed operator with dense domain $D(A)$ in an Hilbert space \mathcal{H} , the ϵ -pseudospectrum $\sigma_\epsilon(A)$ of A is defined by

$$\sigma_\epsilon(A) := \{z \in \mathbb{C} \mid \|(zI - A)^{-1}\| > \frac{1}{\epsilon}\}.$$

Remark 9.1.2.

In one part of the literature, $>$ is replaced by \geq in the above definition. We have chosen the definition which leads to the simplest equivalent definitions.

We take the convention that $\|(zI - A)^{-1}\| = +\infty$ if $z \in \sigma(A)$, where $\sigma(A)$ denotes the spectrum of A , so it is clear that we always have :

$$\sigma(A) \subset \sigma_\epsilon(A) .$$

When A is selfadjoint (or more generally normal), $\sigma_\epsilon(A)$ is, by the Spectral Theorem, given by,

$$\sigma_\epsilon(A) = \{z \in \mathbb{C} \mid d(z, \sigma(A)) < \epsilon\} .$$

So this is only in the case of non self-adjoint operators that this new concept (first appearing in numerical analysis, see Trefethen [Tr1, Tr2]) becomes interesting. Although formulated in a rather abstract way, the following (easy version of) a result by Roch-Silbermann [RoSi] explains rather well to what corresponds the ϵ -pseudospectrum :

Proposition 9.1.3.

$$\sigma_\epsilon(A) = \bigcup_{\{\delta A \in \mathcal{L}(\mathcal{H}) \text{ s. t. } \|\delta A\|_{\mathcal{L}(\mathcal{H})} < \epsilon\}} \sigma(A + \delta A) .$$

In other words, z is in the ϵ -pseudospectrum of A if z is in the spectrum of some perturbation $A + \delta A$ of A with $\|\delta A\| < \epsilon$. This is indeed a natural notion thinking of the fact that the models we are analyzing are only approximations of the real problem and of the fact that the numerical analysis of the model goes through the analysis of explicitly computable approximated problems. Numerical examples are treated in [Tr2].

Proof

Let us show the easy part of this characterization of the ϵ -pseudospectrum. If $\|(z - A)^{-1}\| \leq \frac{1}{\epsilon}$, it is clear that for any δA such that $\|\delta A\| < \epsilon$, $(A + \delta A - z)$ is invertible. Its inverse is obtained by observing that

$$(z - A)^{-1}(z - A - \delta A) = I - (z - A)^{-1}\delta A.$$

But the left hand side is invertible because

$$\|(z - A)^{-1}\delta A\| \leq \|(z - A)^{-1}\| \|\delta A\| < 1.$$

The inverse is consequently given by

$$(z - A - \delta A)^{-1} = \left(\sum_j ((z - A)^{-1}\delta A)^j \right) (z - A)^{-1}.$$

The converse is not very difficult. If $\|(z - A)^{-1}\| > \frac{1}{\epsilon}$, by definition of the norm, there exists $u \in \mathcal{H}$ such that $\|u\| = 1$ and

$$\|(z - A)^{-1}u\| = \mu > \frac{1}{\epsilon}.$$

Let $v = (z - A)^{-1}u$. Let (δA) the linear bounded operator such that

$$(\delta A)x = -\mu^{-2}u \langle v | x \rangle, \forall x \in \mathcal{H}.$$

It is clear that v is an eigenvector of $A + \delta A$ associated to z and that $\|\delta A\| = \frac{1}{\mu} < \epsilon$.

Hence we have found a perturbation $A + \delta A$ of A such that $z \in \sigma(A + \delta A)$ and $\|\delta A\| < \epsilon$.

Another presentation for defining the ϵ -pseudospectrum is to say that $z \in \sigma_\epsilon(A)$ iff $z \in \sigma(A)$ or if there exists an ϵ -pseudoeigenvector that is $u \in D(A)$ such that $\|u\| = 1$ and $\|(z - A)u\| < \epsilon$.

Theorem 9.1.4 (ϵ -Pseudospectrum of the adjoint).

For any closed densely defined operator A and any $\epsilon > 0$, we have

$$\sigma_\epsilon(A^*) = \overline{\sigma_\epsilon(A)},$$

where for a subset Σ in \mathbb{C} we denote by $\overline{\Sigma}$ the set

$$\overline{\Sigma} = \{z \in \mathbb{C} \mid \bar{z} \in \Sigma\}.$$

Proof

This is immediate using that, if $z \notin \sigma(A)$,

$$\|(z - A)^{-1}\| = \|(\bar{z} - A^*)^{-1}\|.$$

9.2 ϵ -Pseudospectrum : complete analysis of the differentiation operator.

We consider the operator A defined on $L^2(]0, 1[)$ by

$$D(A) = \{u \in H^1(]0, 1[), u(1) = 0\},$$

and

$$Au = u', \forall u \in D(A).$$

This is clearly a closed operator with dense domain.

The adjoint of A is defined on $L^2(]0, 1[)$ by

$$D(A^*) = \{u \in H^1(]0, 1[), u(0) = 0\},$$

and

$$Au = -u', \forall u \in D(A^*).$$

Lemma 9.2.1.

$\sigma(A) = \emptyset$ and A has compact resolvent.

First we can observe that $(A - z)$ is injective on $D(A)$ for any $z \in \mathbb{C}$. Then one easily verifies that for any $z \in \mathbb{C}$, the inverse is given by

$$(z - A)^{-1}f = \int_x^1 \exp z(x - s) f(s) ds.$$

It is also clear that this operator is compact.

To analyze the ϵ -pseudospectrum is more interesting. For this we need to estimate

$$\psi(z) := \|(z - A)^{-1}\|.$$

The first remark is that ψ depends only on $\operatorname{Re} z$. For this we can observe that the map $u \mapsto \exp i\alpha x u$ is a unitary transform on $L^2(]0, 1[)$, which maps $D(A)$ onto $D(A)$.

The main result is the following

Theorem 9.2.2.

The function ψ is a subharmonic function which depends only of $\operatorname{Re} z$ and satisfies

$$\psi(z) \leq \frac{1}{\operatorname{Re} z}, \text{ for } \operatorname{Re} z > 0, \quad (9.2.1)$$

and

$$\psi(z) = -\frac{\exp - \operatorname{Re} z}{2 \operatorname{Re} z} + \mathcal{O}\left(\frac{1}{|\operatorname{Re} z|}\right), \text{ for } \operatorname{Re} z < 0. \quad (9.2.2)$$

This implies for the ϵ -pseudospectrum of A :

Corollary 9.2.3.

For $\epsilon > 0$, the ϵ -pseudospectrum of A is an hyperplane of the form

$$\sigma_\epsilon(A) = \{z \in \mathbb{C} \mid \operatorname{Re} z < c_\epsilon\} \quad (9.2.3)$$

with

$$c_\epsilon \sim \begin{cases} (\ln \epsilon) & \text{as } \epsilon \rightarrow 0 \\ \epsilon & \text{as } \epsilon \rightarrow +\infty. \end{cases} \quad (9.2.4)$$

Rough estimate from below for $\operatorname{Re} z < 0$

A first (non optimal) step is to think²⁷ “semi-classical”. Let us take z real and let us consider

$$x \mapsto \phi_z(x) := |2z|^{\frac{1}{2}} \exp zx.$$

This function is not in $\operatorname{Ker}(z - A)$ because it does not satisfies the boundary condition. But when $z \rightarrow -\infty$, the boundary condition at $x = 1$ is “almost satisfied”. Actually ϕ_z lives very close to 0. Moreover, for $z < 0$,

$$\|\phi_z\|^2 = 1 - \exp 2z$$

Hence the norm tends to 1 as $z \rightarrow -\infty$.

Let us indeed consider for $\eta > 0$ a cut-off function χ_η such that $\chi_\eta = 1$ on $[0, 1 - \eta]$ and $\chi_\eta = 0$ on $[1 - \frac{\eta}{2}, 1]$ and let us introduce

$$\phi_{z,\eta}(x) = \chi_\eta \phi_z.$$

We now observe that $\phi_{z,\eta} \in D(A)$ and that

$$(z - A)\phi_{z,\eta}(x) = -\chi'_\eta \phi_z.$$

The L^2 norm of the right hand side is exponentially small like $\exp(1 - \eta)z$. This shows that, for any $\eta > 0$ there exists $z_\eta < 0$ such that

$$\frac{1}{C_\eta} \exp -(1 - \eta)z \leq \psi(z), \quad \text{for } z < z_\eta. \quad (9.2.5)$$

This is not as good as in the statement of the theorem but this suggests a rather general point of view.

Rough estimate from above for $\operatorname{Re} z > 0$

Here we will try to estimate $\psi(z)$ from above by using an a priori estimate for $(A - z)$ (with z real).

For $u \in D(A)$, we have

$$\langle (A - z)u \mid u \rangle = -z\|u\|^2 + \int_0^1 u'(t)\bar{u}(t)dt.$$

²⁷The semi-classical parameter is $h = \frac{1}{\operatorname{Re} z}$

But by an integration by parts, we observe that

$$\int_0^1 u'(t)\bar{u}(t) dt = - \int_0^1 u(t)\bar{u}'(t)dt - |u(0)|^2.$$

Coming back to the previous equality and taking the real part, we obtain

$$- \operatorname{Re} \langle (A - z)u | u \rangle = z\|u\|^2 + \frac{1}{2}|u(0)|^2 \geq z\|u\|^2.$$

Then we obtain

$$\|(A - z)u\| \geq z\|u\|, \forall u \in D(A), \quad (9.2.6)$$

which implies (9.2.1).

Schur's Lemma

The operator $(A - z)^{-1}$ being an operator associated to an integral kernel, one can analyze what is given by Schur's lemma or by the Hilbert-Schmidt criterion. The kernel is defined by

$$K(x, s) = \begin{cases} 0 & \text{for } s < x \\ \exp z(x - s) & \text{for } x < s \end{cases}$$

According to Schur's Lemma, we have to consider $\sup_x \int K(x, s)ds$ and $\sup_s \int K(x, s)dx$. If we are interested with the Hilbert-Schmidt norm, we have to compute $\int \int K(x, s)^2 dx ds$. All these computations can be done rather explicitly !

For $z \neq 0$, we have

$$\sup_x \int K(x, s)ds = \frac{1}{z}(1 - \exp -z),$$

and

$$\sup_s \int K(x, s)dx = \frac{1}{z}(1 - \exp -z)$$

This gives

$$\psi(z) \leq \frac{1}{z}(1 - \exp -z). \quad (9.2.7)$$

This is actually an improved version of (9.2.1) for $z > 0$ and for $z < 0$, it is better to write it in the form

$$\psi(z) \leq \frac{-1}{z}(\exp -z - 1), \quad (9.2.8)$$

and to compare it to the lower bound obtained in (9.2.5).

A more accurate estimate for $z < 0$

We can rewrite $(z - A)^{-1}$ in the form

$$(z - A)^{-1} = R_1 - R_2,$$

with

$$R_1 v(x) := \int_0^1 \exp z(x - s) v(s), ds$$

and

$$R_2 v(x) := \int_0^x \exp z(x - s) v(s), ds$$

Observing that R_2^* can be treated as for the proof of (9.2.7), we first obtain

$$\|R_2\| \leq -\frac{1}{\operatorname{Re} z}.$$

It remains to control $\|R_1\|$. This norm can be computed explicitly. We have indeed

$$\|R_1 v\| = \|\exp zx\| \left| \int_0^1 \exp -zs v(s) ds \right|$$

Hence we have just to compute the norm of the linear form

$$v \mapsto \int_0^1 \exp -zs v(s) ds$$

which is the L^2 -norm of $s \mapsto \exp -zs$.

This gives

$$\|R_1\| = \|\exp zx\| \cdot \|\exp -zx\| = -\frac{1}{2z} \sqrt{(1 - e^{2z})(e^{-2z} - 1)} = -\frac{1}{2z} e^{-z} (1 - e^{2z}).$$

Combining the estimates of $\|R_1\|$ and $\|R_2\|$ leads to the upper bound for (9.2.2).

Remark 9.2.4.

One can discretize the preceding problem by considering, for $n \in \mathbb{N}^$, the matrix $A_n = nA_1$ with $A_1 = I + J$ where J is the $n \times n$ matrix such that $J_{i,j} = \delta_{i+1,j}$. One can observe that the spectrum of A_n is $-n$. It is also interesting to analyze the pseudospectrum.*

Remark 9.2.5.

There is a semi-classical version of the pseudospectrum for families A_h . One can then relate the ϵ appearing in the definition of the ϵ -pseudospectrum with the parameter h (which could typically be in $]0, h_0]$). For example, we can consider $\epsilon(h) = h^N$.

Exercise 9.2.6.

Analyze the pseudospectrum of $\frac{d}{d\theta} + \lambda g(\theta)$ on the circle.

9.3 Another example of non selfadjoint operator without spectrum

We consider the spectrum of the operator

$$A = \frac{d}{dx} + x^2,$$

on the line.

We can take as domain $D(A)$ the space of the $u \in L^2(\mathbb{R})$ such that $Au \in L^2(\mathbb{R})$. We note that $C_0^\infty(\mathbb{R})$ is dense for the graph norm in $D(A)$. Hence A is the closed extension of the differential operator $\frac{d}{dx} + x^2$ with domain $C_0^\infty(\mathbb{R})$. We note that the operator is not selfadjoint. The adjoint is $-\frac{d}{dx} + x^2$.

The two following inequalities can be useful.

$$\operatorname{Re} \langle Au | u \rangle \geq \|xu\|^2 \geq 0. \quad (9.3.1)$$

This inequality is first proved for $u \in C_0^\infty(\mathbb{R})$ and then extended to $u \in D(A)$ using the density of $C_0^\infty(\mathbb{R})$. This implies that $(D(A))$ has continuous injection in the weighted space L_ρ^2 with $\rho(x) = |x|$. Together with the fact that $D(A) \subset H_{loc}^1(\mathbb{R})$, this implies that $(A + I)$ is invertible and that the inverse is compact.

The second inequality is

$$\|x^2u\|^2 + \|u'\|^2 \leq C (\|Au\|^2 + \|u\|^2). \quad (9.3.2)$$

To prove it, we observe that

$$\|Au\|^2 = \|u'\|^2 + \|x^2u\|^2 - 2 \operatorname{Re} \langle u', x^2u \rangle.$$

Using an integration by parts, we get

$$2 \operatorname{Re} \langle u', x^2u \rangle = 2 \langle xu, u \rangle.$$

Hence we get (using Cauchy-Schwarz)

$$\|Au\|^2 \geq \|u'\|^2 + \|x^2u\|^2 - 2\|xu\|\|u\|.$$

One can then use the consequence of our first inequality (9.3.1)

$$\|xu\|^2 \leq \frac{1}{2} (\|Au\|^2 + \|u\|^2)$$

to get the conclusion.

(9.3.2) permits to obtain that

$$D(A) = \{u \in H^1(\mathbb{R}), x^2u \in L^2(\mathbb{R})\}, \quad (9.3.3)$$

which is not obvious at all.

One will show that A has an inverse, that the inverse is compact but that its spectrum is empty.

We consider on \mathbb{R} , the differential equation

$$u' + x^2u = f . \quad (9.3.4)$$

For all $f \in L^2(\mathbb{R})$, let us show that there exists a unique solution in $L^2(\mathbb{R})$ de (9.3.4). An elementary calculus gives

$$u(x) = \exp -\frac{1}{3}x^3 \int_{-\infty}^x \exp \frac{1}{3}y^3 f(y) dy .$$

One has to work a little for showing that $u \in L^2$ (this is easier if f is compactly supported). If we denote by \mathbf{K} the operator which associates to f the solution u , the distribution kernel of \mathbf{K} is given by

$$K(x, y) = \begin{cases} 0 & \text{if } y \geq x \\ \exp \frac{1}{3}(y^3 - x^3) & \text{if } y < x \end{cases} .$$

We note that if there exists an eigenvalue of \mathbf{K} $\lambda \neq 0$ and if u_λ is a corresponding eigenfunction, then u_λ satisfies

$$u'_\lambda + x^2u_\lambda = \frac{1}{\lambda}u_\lambda . \quad (9.3.5)$$

From this we deduce that \mathbf{K} has no non zero eigenvalue. One can indeed solve explicitly (9.3.5) :

$$u_\lambda(x) = C \exp -\frac{x^3}{3} \exp \frac{1}{\lambda}x .$$

It is then easy to see that noone can be in $L^2(\mathbb{R})$ when $C \neq 0$.

To show that \mathbf{K} is compact, we can actually show that \mathbf{K} is Hilbert-Schmidt. That is, we will show that $K(x, y)$ is in $L^2(\mathbb{R}^2)$. We have

$$\int_{y < x} \exp \frac{2}{3}(y^3 - x^3) dx dy = \int_{-\infty}^{+\infty} \left(\int_{-\infty}^x \exp \frac{2}{3}(y^3 - x^3) dy \right) dx .$$

Dividing the domain of integration in two parts, we first consider :

$$\int_{-1}^{+1} \left(\int_{-\infty}^x \exp \frac{2}{3}(y^3 - x^3) dy \right) dx ,$$

which is bounded from above by

$$2e \int_{-\infty}^1 \exp \frac{2}{3}y^3 dy .$$

Then we look at

$$\int_{|x| > 1} \left(\int_{-\infty}^x \exp \frac{2}{3}(y^3 - x^3) dy \right) dx .$$

Here we observe that

$$\exp \frac{2}{3}(y^3 - x^3) = \exp \frac{2}{3}(y - x)(y^2 + yx + x^2),$$

and that

$$(y^2 + yx + x^2) \geq \frac{1}{2}(y^2 + x^2) \geq \frac{1}{2}x^2.$$

This leads, as $y \leq x$, to the upper bound la majoration

$$\exp \frac{2}{3}(y^3 - x^3) \leq \exp \frac{1}{3}(y - x)x^2.$$

and to

$$\int_{|x|>1} \left(\int_{-\infty}^x \exp \frac{2}{3}(y^3 - x^3) dy \right) dx \leq 3 \int_{|x|>1} \frac{1}{x^2} dx < +\infty.$$

This implies that the spectrum of \mathbf{K} is 0.

Non self-adjoint effects.

We can also try to estimate the “solution” operator \mathbf{K}_λ corresponding to the equation

$$u' + x^2 u = \lambda f. \tag{9.3.6}$$

It is easy, to see that we can reduce the computation to the case when λ réel. We have indeed

$$\mathbf{K}_\lambda = \exp -i \operatorname{Im} \lambda x \mathbf{K}_{\operatorname{Re} \lambda} \exp i \operatorname{Im} \lambda x.$$

For $\lambda < 0$, we easily find, observing that $u \in \mathbb{S}(\mathbb{R})$ and

$$\operatorname{Re} \langle u' + x^2 u - \lambda | u \rangle_{L^2} = \langle x^2 u - \lambda | u \rangle \geq -\lambda \|u\|^2,$$

the estimate

$$\|\mathbf{K}_\lambda\|_{\mathcal{L}(L^2)} \leq -\frac{1}{\lambda}.$$

The case $\lambda = 0$ has been treated before. Without to much effort, we get the same result for $|\operatorname{Re} \lambda| \leq 1$.

So we have to consider the case when $\lambda \geq 1$ and control the estimate as $\lambda \rightarrow +\infty$. Proceeding as in the case $\lambda = 0$, we first obtain

$$K_\lambda(x, y) = \begin{cases} 0 & \text{si } y \geq x \\ \exp \left(\frac{1}{3}(y^3 - x^3) - \lambda(y - x) \right) & \text{si } y < x \end{cases}.$$

Again, we see that $K_\lambda(x, y)$ is in $L^2(\mathbb{R}^2)$:

$$\|K_\lambda\|_{L^2(\mathbb{R}^2)}^2 = \int_{-\infty}^{+\infty} \left(\int_{-\infty}^x \exp \left(\frac{2}{3}(y^3 - x^3) - 2\lambda(y - x) \right) dy \right) dx.$$

The proof is similar to the case $\lambda = 0$. We cut the domain of integration.

$$\int_{-1-4\sqrt{\lambda}}^{-1} \left(\int_{-\infty}^x \exp \left(\frac{2}{3}(y^3 - x^3) - 2\lambda(y - x) \right) dy \right) dx ,$$

is bounded after a new partition of the domain of integration by

$$4\sqrt{\lambda} e^{\frac{2}{3}(1+4\lambda^{\frac{1}{2}})^3} \int_{-\infty}^{-1-4\sqrt{\lambda}} \exp \left(\frac{2}{3}y^3 - 2\lambda y \right) dy \leq 3\sqrt{\lambda} e^{\frac{2}{3}(1+2\lambda^{\frac{1}{2}})^3} ,$$

and by

$$\begin{aligned} & \int_{-1-4\sqrt{\lambda}}^{-1} \left(\int_{-1-4\sqrt{\lambda}}^x \exp \left(\frac{2}{3}(y^3 - x^3) - 2\lambda(y - x) \right) dy \right) dx \\ & \leq 4\sqrt{\lambda} \sup_{x \in]-1-4\sqrt{\lambda}, -1[} \left(\int_{-1-4\sqrt{\lambda}}^x \exp -2\lambda(y - x) dy \right) \\ & \leq 2\lambda^{-\frac{1}{2}} e^{8\lambda^{\frac{3}{2}}} . \end{aligned}$$

Then we consider

$$\int_{-1}^{+1} \left(\int_{-\infty}^x \exp \left(\frac{2}{3}(y^3 - x^3) - 2\lambda(y - x) \right) dy \right) dx ,$$

which is bounded by

$$\begin{aligned} & 2e^{\frac{2}{3}+2\lambda} \int_{-\infty}^{-1-4\sqrt{\lambda}} \exp \left(\frac{2}{3}y^3 - 2\lambda y \right) dy \\ & \leq 2e^{\frac{2}{3}+2\lambda} \\ & \leq Ce^{\frac{2}{3}+2\lambda} . \end{aligned}$$

and by

$$\begin{aligned} & \int_{-1}^{+1} \left(\int_{-1-4\sqrt{\lambda}}^x \exp \left(\frac{2}{3}(y^3 - x^3) - 2\lambda(y - x) \right) dy \right) dx \\ & \int_{-1}^{+1} \left(\int_{-1-4\sqrt{\lambda}}^x \exp -2\lambda(y - x) dy \right) dx \\ & \leq \frac{1}{\lambda} \exp(4\lambda(1 + 2\sqrt{\lambda})) \end{aligned}$$

We look now at

$$\int_1^{1+4\sqrt{\lambda}} \left(\int_{-\infty}^x \exp \left(\frac{2}{3}(y^3 - x^3) - 2\lambda(y - x) \right) dy \right) dx ,$$

which is similarly controlled by

$$4\sqrt{\lambda} \exp 2\lambda(1 + 4\sqrt{\lambda}) \int_{-\infty}^{-1-4\sqrt{\lambda}} \exp \left(\frac{2}{3}y^3 - 2\lambda y \right) dy$$

and by

$$\int_1^{1+4\sqrt{\lambda}} \left(\int_{-1-4\sqrt{\lambda}}^x \exp -2\lambda(y - x) dy \right) dx .$$

Finally, we have to estimate

$$\int_{|x| \geq 1+4\sqrt{\lambda}} \left(\int_{-\infty}^x \exp \left(\frac{2}{3}(y^3 - x^3) - 2\lambda(y - x) \right) dy \right) dx ,$$

Here we observe that

$$\exp \frac{2}{3}(y^3 - x^3) = \exp \frac{2}{3}(y - x)(y^2 + yx + x^2) ,$$

and that

$$(y^2 + yx + x^2) \geq \frac{1}{2}(y^2 + x^2) \geq \frac{1}{2}x^2 .$$

This leads to the upper bound

$$\exp \frac{2}{3}(y^3 - x^3) \exp -2\lambda(y - x) \leq \exp \frac{1}{3}(y - x)(x^2 - 6\lambda) .$$

By simple integration, we get :

$$\begin{aligned} & \int_{|x| > 1+4\sqrt{\lambda}} \left(\int_{-\infty}^x \exp \left(\frac{2}{3}(y^3 - x^3) - 2\lambda(y - x) \right) dy \right) dx \\ & \leq 3 \int_{|x| > 1+4\sqrt{\lambda}} \frac{1}{x^2 - 6\lambda} dx < +\infty . \end{aligned}$$

The last bound can be controlled independently of $\operatorname{Re} \lambda$. Hence, we have finally proved the existence of $C > 0$ such that, for $\operatorname{Re} \lambda \geq 1$, we have

$$\|\mathbf{K}_\lambda\|_{HS} \leq C|\lambda|^C \exp C \operatorname{Re} \lambda^{\frac{3}{2}} .$$

Remark 9.3.1.

Using the Laplace integral method, one can get the asymptotics of $\|\mathbf{K}_\lambda\|_{HS}$.

Note that $\|\mathbf{K}_\lambda\|_{HS} \geq \|\mathbf{K}_\lambda\|_{\mathcal{L}(L^2)}$.

One can also find a lower bound of $\|\mathbf{K}_\lambda\|_{\mathcal{L}(L^2)}$ using quasimodes.

*Note that \mathbf{K}_λ is the resolvent of the unbounded operator $-\frac{d}{dx} + x^2$. This operator is **not** selfadjoint. He is with compact resolvent and has empty spectrum. Moreover, the norm of the resolvent depends only of $\operatorname{Re} \lambda$.*

This example shows that when the operators are not selfadjoint many things can occur. In particular the fact that if $z \in \rho(T)$, then $\|(T - z)^{-1}\|$ is controlled by $\frac{1}{d(\lambda, \sigma(T))}$ and that in particular that if $z \notin \mathbb{R}$ then $\|(T - z)^{-1}\| \leq \frac{1}{|\operatorname{Im} z|}$ can become completely wrong.

In this context, the notion of pseudo-spectrum can be helpful. We leave this as an exercise to the reader.

Remark 9.3.2.

Other interesting examples to analyze in the same spirit are for example the complex harmonic oscillator $(-\frac{d^2}{dx^2} + ix^2)$ with $c \notin \mathbb{R}$ (See Davies (1999), [Tr2], [Zw] and references therein). The spectrum can be seen as the spectrum of $-\frac{d^2}{dx^2}$ rotated by $\frac{\pi}{4}$ in the complex plane.

10 Essentially selfadjoint operators

10.1 Introduction

In most of the examples which were presented, the abstract operators are associated with differential operators. These differential operators are naturally defined on $C_0^\infty(\Omega)$ or $\mathcal{D}'(\Omega)$. Most of the time (for suitable potentials increasing slowly at ∞) they are also defined (when $\Omega = \mathbb{R}^m$) on $\mathcal{S}(\mathbb{R}^m)$ or $\mathcal{S}'(\mathbb{R}^m)$. It is important to understand how the abstract point of view can be related to the PDE point of view. The theory of the essential selfadjointness gives a clear understanding of the problem. The question is to decide if, starting from a symmetric operator T , whose domain $D(T) = \mathcal{H}_0$ is dense in \mathcal{H} , there exists a unique selfadjoint extension T^{ext} of T . We recall that it means that $D(T) \subset D(T^{ext})$ and $T^{ext}u = Tu$, $\forall u \in D(T)$. This leads to

Definition 10.1.1.

A symmetric operator T with domain \mathcal{H}_0 is called essentially selfadjoint if its closure is selfadjoint.

Proposition 10.1.2.

If T is essentially selfadjoint, then its selfadjoint extension²⁸ is unique.

Indeed suppose that S is a selfadjoint extension of T . Then S is closed and being an extension of T , is also an extension of its smallest extension \overline{T} . We recall from Theorem 2.2.6 that $\overline{T} = T^{**}$. Thus, $S = S^* \subset (T^{**})^* = T^{**}$, and so $S = T^{**}$.

Example 10.1.3.

Here we give a list of examples and counter-examples which will be analyzed later.

1. The differential operator $-\Delta$ with domain $\mathcal{H}_0 = C_0^\infty(\mathbb{R}^m)$ is essentially selfadjoint (see later).
2. The differential operators $-\Delta + |x|^2$ with domain $\mathcal{H}_0 = C_0^\infty(\mathbb{R}^m)$ or $\mathcal{H}_1 = \mathcal{S}(\mathbb{R}^m)$ are essentially selfadjoint and admit consequently a unique selfadjoint extension. This extension is the same for the two operators. The domain can be explicitly described as

$$B^2(\mathbb{R}^m) = \{u \in L^2(\mathbb{R}^m) \mid x^\alpha D_x^\beta u \in L^2(\mathbb{R}^m), \forall \alpha, \beta \in \mathbb{N}^m \text{ with } |\alpha| + |\beta| \leq 2\}.$$

3. The differential operator $-\Delta$ with domain $C_0^\infty(\Omega)$ (where Ω is an open bounded set with smooth boundary) is not essentially selfadjoint. There exists a lot of selfadjoint extensions related to the choice of a boundary problem. As we have seen before, we have already met two such extensions :

²⁸Although, it will not help in this course, note that the converse is true. See for example the book [Ro], in which the essential selfadjointness is defined differently.

- the Dirichlet realization whose domain is the set $H_0^1(\Omega) \cap H^2(\Omega)$,
 - the Neumann realization whose domain is the set : $\{u \in H^2(\Omega) \mid (\partial u / \partial \nu)_{/\partial \Omega} = 0\}$
4. The Laplace Beltrami operator on a compact manifold M with domain $C^\infty(M)$ is essentially selfadjoint on $L^2(M)$. The domain of the selfadjoint extension can be described as $H^2(M)$ (which can be described using local charts).

10.2 Basic criteria.

We now give some criteria in order to verify that an operator is essentially selfadjoint. As already mentioned, one can prove essential selfadjointness by proving that the minimal closed extension $T_{min} := \overline{T}$ coincides with T^* . One easily verifies (see Proposition 2.2.5 in conjunction with the definition of \overline{T}) that $T^* = \overline{T}^*$ and we recall that $T^{**} = \overline{T}$.

We now observe the

Proposition 10.2.1.

Any closed symmetric extension of T is a restriction of T^ .*

Proof.

Let S is a closed symmetric extension of T .

We have indeed $T \subset S \subset S^*$ and, observing that $S^* \subset T^*$, we consequently get $S \subset T^*$.

In particular, if \overline{T} is selfadjoint then \overline{T} is the unique selfadjoint extension of T .

We can characterize the selfadjointness through the following general criterion

Theorem 10.2.2.

Let T be a closed symmetric operator. Then the following statements are equivalent :

1. T is selfadjoint.
2. $\text{Ker}(T^* \pm i) = \{0\}$;
3. $\text{Range}(T \pm i) = \mathcal{H}$.

Proof.

1. implies 2.

This property was already observed (because $T = T^*$ and $\pm i \notin \sigma(T)$).

2. implies 3. .

We first observe that the property that $\text{ker}(T^* + i) = \{0\}$ implies that $R(T - i)$ is dense in \mathcal{H} . Note that the converse is also true. For getting 3., it remains

to show that $R(T - i)$ is closed. But, for all ϕ in $D(T)$, we have (using the symmetry of T)

$$\|(T - i)\phi\|^2 = \|T\phi\|^2 + \|\phi\|^2. \quad (10.2.1)$$

If ϕ_n is a sequence in $D(T)$ such that $(T + i)\phi_n$ converges to some ψ_∞ . The previous identity shows that ϕ_n is a Cauchy sequence, so there exists ϕ_∞ such that $\phi_n \rightarrow \phi_\infty$ in \mathcal{H} . But $T\phi_n = (T + i)\phi_n - i\phi_n$ is convergent and using that the graph is closed, we obtain that $\phi_\infty \in D(T)$ and $T\phi_\infty = \psi_\infty$.

3. implies 1. .

Let $\phi \in D(T^*)$. Let $\eta \in D(T)$ such that $(T - i)\eta = (T^* - i)\phi$. T being symmetric, we have also $(T^* - i)(\eta - \phi) = 0$. But, if $(T + i)$ is surjective, then $(T^* - i)$ is injective and we get $\phi = \eta$. This proves that $\phi \in D(T)$.

Remark 10.2.3.

Here we have used and proved during the proof of the assertion “2. implies 3.” the following lemma

Lemma 10.2.4.

If T is closed and symmetric, then $R(T \pm i)$ is closed.

This theorem gives as a corollary a criterion for essential selfadjointness in the form

Corollary 10.2.5.

Let A with domain $D(A)$ be a symmetric operator. Then the following are equivalent

1. A is essentially selfadjoint.
2. $\text{Ker}(A^* \pm i) = \{0\}$.
3. The two spaces $R(A \pm i)$ are dense in \mathcal{H} .

We have indeed essentially to apply the previous theorem to \overline{A} , observing in addition that \overline{A} is symmetric and that Lemma 10.2.4 is true.

Let us here emphasize that in this case, to specify the operator A , it is not necessary to give the exact domain of A but a core for A that is a subspace D such that the closure of $A|_D$ is \overline{A} .

In the same spirit, we have in the semibounded case the following

Theorem 10.2.6.

Let T be a positive, symmetric operator. Then the following statements are equivalent :

1. T is essentially selfadjoint.
2.
$$\text{Ker}(T^* + b) = \{0\} \text{ for some } b > 0.$$
3. $\text{Range}(T + b)$ is dense for some $b > 0$.

The proof essentially the same as for the previous corollary, if one observes that if T is positive then the following trivial estimate is a good substitute for (10.2.1) :

$$\langle (T + b)u, u \rangle \geq b\|u\|^2 .$$

Example 10.2.7. (*The free Laplacian*)

The operator $-\Delta$ with domain C_0^∞ is essentially selfadjoint. Its selfadjoint extension is $-\Delta$ with domain H^2 .

10.3 The Kato -Rellich theorem

We would like to consider the case when $P = -\Delta + V$ when V is regular and tends to 0 as $|x| \rightarrow +\infty$. Here V can be considered as a perturbation of the Laplacian. One can then apply a general theorem due to Kato-Rellich.

Theorem 10.3.1.

Let A be a selfadjoint operator, B be a symmetric operator whose domain contains $D(A)$. Let us assume the existence of a and b such that $0 \leq a < 1$ and $b \geq 0$ such that

$$\|Bu\| \leq a \|Au\| + b \|u\| \tag{10.3.1}$$

for all $u \in D(A)$. Then $A + B$ is selfadjoint on $D(A)$.

If A is essentially selfadjoint on²⁹ $D \subset D(A)$, then $A+B$ has the same property.

Proof.

Step 1.

We start from the following identity which only uses that $(A + B)$ with domain $D(A)$ is symmetric.

$$\|(A + B \pm i\lambda)u\|^2 = \|(A + B)u\|^2 + \lambda^2 \|u\|^2, \quad \forall u \in D(A) . \tag{10.3.2}$$

By the triangle inequality and the symmetry of $A + B$, we get for a real $\lambda > 0$, and for any $u \in D(A)$:

$$\begin{aligned} \sqrt{2}\|(A + B - i\lambda)u\| &\geq \|(A + B)u\| + \lambda\|u\| \\ &\geq \|Au\| - \|Bu\| + \lambda\|u\| \\ &\geq (1 - a)\|Au\| + (\lambda - b)\|u\|. \end{aligned} \tag{10.3.3}$$

We now choose $\lambda > b$.

Step 2.

Let us show that $(A + B)$ with domain $D(A)$ is closed. If we start indeed from a pair (u_n, f_n) with $u_n \in D(A)$ and $f_n = (A + B)u_n$ such that $(u_n, f_n) \rightarrow (u, f)$ in \mathcal{H} . From (10.3.3), we get that Au_n is a Cauchy sequence in \mathcal{H} . A being closed, we get $u \in D(A)$ and the existence of g such that $Au_n \rightarrow g = Au$.

Now from (10.3.2) and (10.3.1), we get also that Bu_n is a Cauchy sequence and

²⁹By this we mean that the closure of $A|_D$ is A .

there exists v such that $Bu_n \rightarrow v$ in \mathcal{H} .

We claim that $Bu = v$. We have indeed, for any $h \in D(A)$,

$$\langle v, h \rangle_{\mathcal{H}} = \lim_{n \rightarrow +\infty} \langle Bu_n, h \rangle_{\mathcal{H}} = \lim_{n \rightarrow +\infty} \langle u_n, Bh \rangle_{\mathcal{H}} = \langle u, Bh \rangle_{\mathcal{H}} = \langle Bu, h \rangle_{\mathcal{H}} .$$

Using the density of $D(A)$, we get $v = Bu$. (We could have also used that B is closable).

We conclude by observing that $(A + B)u = f$ (with $f = g + v$) as expected.

Step 3.

In order to apply Theorem 10.2.2, we have to show that $(A + B \pm i\lambda)$ is surjective. The main element in the proof is the following

Lemma 10.3.2.

For $\lambda > 0$ large enough, we have

$$\|B(A \pm i\lambda)^{-1}\| < 1 . \tag{10.3.4}$$

Proof.

We observe that, for $u \in D(A)$,

$$\|(A \pm i\lambda)u\|^2 = \|Au\|^2 + \lambda^2\|u\|^2 . \tag{10.3.5}$$

For $u \in D(A)$, we have, using two times (10.3.5) and then (10.3.1)

$$\begin{aligned} \|Bu\| &\leq a\|Au\| + b\|u\| \\ &\leq a\|(A + i\lambda)u\| + \frac{b}{\lambda}\|(A + i\lambda)u\| \\ &\leq \left(a + \frac{b}{\lambda}\right)\|(A + i\lambda)u\| . \end{aligned} \tag{10.3.6}$$

It is then enough to choose $\lambda > 0$ large enough such that

$$\left(a + \frac{b}{\lambda}\right) < 1 .$$

Writing

$$A + B - i\lambda = [I + B(A - i\lambda)^{-1}](A - i\lambda) , \tag{10.3.7}$$

it is easy to deduce the surjectivity using the lemma and the surjectivity of $(A - i\lambda)$.

Application.

As an application, let us treat the case of the Schrödinger operator with Coulomb potential.

Proposition 10.3.3.

The operator $-\Delta - \frac{1}{|x|}$ with domain $C_0^\infty(\mathbb{R}^3)$ is essentially selfadjoint.

We recall that the operator is well defined because $\frac{1}{r}$ belongs to $L_{loc}^2(\mathbb{R}^3)$. We first observe a Sobolev type inequality.

Lemma 10.3.4. :

There exists a constant C such that for all $u \in H^2(\mathbb{R}^3)$, all $a > 0$ and all $x \in \mathbb{R}^3$, we have

$$|u(x)| \leq C(a\|\Delta u\|_0 + a^{-3}\|u\|_0).$$

(Prove first the inequality³⁰ for all u with $x = 0$ and $a = 1$, then use translation and dilation.)

In the second step, we show that the potential $V = -\frac{1}{r}$ is a perturbation of the Laplacian.

There exists indeed a constant C such that for all $u \in H^2(\mathbb{R}^3)$ and all $b > 0$, we have

$$\|Vu\|_0 \leq C(b\|\Delta u\|_0 + b^{-3}\|u\|_0).$$

For this proof we observe that, for any $R > 0$,

$$\int V(x)^2|u(x)|^2dx = \int_{|x| \leq R} V(x)^2|u(x)|^2dx + \int_{|x| \geq R} V(x)^2|u(x)|^2dx,$$

and treat the first term of the right hand side using the Sobolev's type inequality and the second term by the trivial estimate $|V(x)| \leq \frac{1}{R}$.

Remark 10.3.5.

We note that the same proof shows that $-\Delta + V$ is essentially selfadjoint starting from $C_0^\infty(\mathbb{R}^m)$ if $V \in L^2 + L^\infty$, that is if $V = V_1 + V_2$ with $V_1 \in L^2$ and $V_2 \in L^\infty$.

10.4 Other criteria of selfadjointness for Schrödinger operators

We present in this subsection two criteria which are specific of the Schrödinger case. The first one seems due to Rellich (See [Sima]) and we present it in the easy case when the potential is regular. The second one permits to treat singular potentials and is due to Kato (cf [HiSi] or [Ro]).

The first theorem is adapted to operators which are already know to be positive on $C_0^\infty(\mathbb{R}^m)$.

Theorem 10.4.1.

A Schrödinger operator $T = -\Delta + V$ on \mathbb{R}^n associated with a C^0 potential V , which is semibounded on $C_0^\infty(\mathbb{R}^n)$, is essentially selfadjoint. In other words, the Friedrichs extension is the unique selfadjoint extension starting from $C_0^\infty(\mathbb{R}^m)$.

This theorem is complementary to the second theorem (Theorem 10.4.4) which will be stated at the end of this subsection because we do not have to assume the positivity of the potential but only the semi-boundedness of the operator T .

³⁰In this case, this is just the Sobolev's injection theorem of $H^2(\mathbb{R}^3)$ into $C_b^0(\mathbb{R}^3)$, where $C_b^0(\mathbb{R}^3)$ is the space of the continuous bounded functions.

Proof.

Let T be our operator. Possibly by adding a constant, we can assume that

$$\langle Tu, u \rangle_{\mathcal{H}} \geq \|u\|^2, \quad \forall u \in C_0^\infty(\mathbb{R}^m). \quad (10.4.1)$$

Of course this inequality can be rewritten in the form :

$$\|\nabla u\|^2 + \int_{\mathbb{R}^m} V(x)|u(x)|^2 dx \geq \|u\|^2, \quad \forall u \in C_0^\infty(\mathbb{R}^m).$$

In this form, the inequality can be extended to the elements of $H_{comp}^1(\mathbb{R}^m)$, corresponding to the distributions of $H^1(\mathbb{R}^m)$ with compact support :

$$\|\nabla u\|^2 + \int_{\mathbb{R}^m} V(x)|u(x)|^2 dx \geq \|u\|^2, \quad \forall u \in H_{comp}^1(\mathbb{R}^m). \quad (10.4.2)$$

According to the general criterion of essential selfadjointness (cf Theorem 10.2.6), it is enough to verify that $R(T)$ is dense. Let us show this property.

Let $f \in L^2(\mathbb{R}^m)$, such that

$$\langle f, Tu \rangle_{\mathcal{H}} = 0, \quad \forall u \in C_0^\infty(\mathbb{R}^m). \quad (10.4.3)$$

We have to show that $f = 0$.

Because T is real, one can assume that f is real.

We first observe that (10.4.3) implies that : $(-\Delta + V)f = 0$ in $\mathcal{D}'(\mathbb{R}^m)$. A standard regularity theorem for the Laplacian³¹ implies that $f \in H_{loc}^2(\mathbb{R}^m)$.

We now introduce a family of cutoff functions ζ_k by

$$\zeta_k := \zeta(x/k), \quad \forall k \in \mathbb{N}, \quad (10.4.4)$$

where ζ is a C^∞ function satisfying $0 \leq \zeta \leq 1$, $\zeta = 1$ on $B(0, 1)$ and $\text{supp } \zeta \subset B(0, 2)$.

For any $u \in C^\infty$ and any $f \in H_{loc}^2$, we have the identity

$$\begin{aligned} & \int \nabla(\zeta_k f) \cdot \nabla(\zeta_k u) dx + \int \zeta_k(x)^2 V(x) u(x) f(x) dx \\ &= \int |(\nabla \zeta_k)(x)|^2 u(x) f(x) dx + \sum_{i=1}^m \int (f(\partial_i u) - u(\partial_i f))(x) \zeta_k(x) (\partial_i \zeta_k)(x) dx \\ &+ \langle f(x), T \zeta_k^2 u \rangle. \end{aligned} \quad (10.4.5)$$

When f satisfies (10.4.3), we get :

$$\begin{aligned} & \int_{\mathbb{R}^m} \nabla(\zeta_k f) \cdot \nabla(\zeta_k u) dx + \int \zeta_k(x)^2 V(x) u(x) f(x) dx \\ &= \int |(\nabla \zeta_k)(x)|^2 u(x) f(x) dx + \sum_{i=1}^m \int (f(\partial_i u) - u(\partial_i f))(x) \zeta_k(x) (\partial_i \zeta_k)(x) dx, \end{aligned} \quad (10.4.6)$$

for all $u \in C^\infty(\mathbb{R}^m)$.

This formula can be extended to functions $u \in H_{loc}^1$. In particular, we can take $u = f$.

We obtain

$$\langle \nabla(\zeta_k f), \nabla(\zeta_k f) \rangle + \int \zeta_k^2 V(x) |f(x)|^2 dx = \int |\nabla \zeta_k|^2 |f(x)|^2 dx. \quad (10.4.7)$$

³¹This is the property that $f \in L_{loc}^2(\mathbb{R}^m)$, $\Delta f \in L_{loc}^2(\mathbb{R}^m)$ implies that $f \in H_{loc}^2(\mathbb{R}^m)$.

Using (10.4.1), (10.4.7) and taking the limit $k \rightarrow +\infty$, we get

$$\begin{aligned} \|f\|^2 &= \lim_{k \rightarrow +\infty} \|\zeta_k f\|^2 \\ &\leq \limsup_{k \rightarrow +\infty} (\langle \nabla(\zeta_k f), \nabla(\zeta_k f) \rangle + \int \zeta_k^2 V(x) |f(x)|^2 dx) \\ &= \limsup_{k \rightarrow +\infty} \int f(x)^2 |(\nabla \zeta_k)(x)|^2 dx = 0. \end{aligned} \tag{10.4.8}$$

This proves the theorem.

Remark 10.4.2.

When V is C^∞ , we get, in the previous proof, that $f \in C^\infty$ and we immediately can prove (10.4.7) without going through the previous discussion.

Example 10.4.3.

- If $V \geq 0$ and C^∞ , then the Schrödinger operator $-\Delta + V$ with domain $C_0^\infty(\mathbb{R}^m)$ is essentially selfadjoint. The operator $-\Delta + V$ is indeed positive.
- If ϕ is C^∞ on \mathbb{R}^m , then the operators $-\Delta + |\nabla\phi|^2 \pm \Delta\phi$ are essentially selfadjoint. They are indeed positive on $C_0^\infty(\mathbb{R}^m)$. They can actually be written in the form $\sum_j Z_j^* Z_j$ with $Z_j = \partial_{x_j} \mp \partial_{x_j} \phi$. These operators appear naturally in statistical mechanics.

Let us now mention, without proof, a quite general theorem due to Kato (See for example [Ro]).

Theorem 10.4.4.

Let V in $L_{loc}^2(\mathbb{R}^m)$ such that $V \geq 0$ almost everywhere on \mathbb{R}^m . Then $-\Delta + V$ with domain $C_0^\infty(\mathbb{R}^m)$ is essentially selfadjoint.

This theorem is based on the so called Kato's inequality.

Remark 10.4.5.

This last theorem may be extended to the case of the Schrödinger operator with magnetic regular potential A (See Subsection 7.5).

10.5 Non-selfadjoint case : Maximal accretivity and application to the Fokker-Planck operator

10.5.1 Accretive operators

We collect here some material on accretive operators. The references could be the books by Dautray-Lions (Vol. 5, Chapter XVII), Reed-Simon or the book of B. Davies. Let \mathcal{H} be a complex (or real) Hilbert space.

Definition 10.5.1.

Let A be an unbounded operator in \mathcal{H} with domain $D(A)$. We say that A is accretive if

$$Re \langle Ax | x \rangle_{\mathcal{H}} \geq 0, \quad \forall x \in D(A). \tag{10.5.1}$$

Definition 10.5.2.

An accretive operator A is maximally accretive if it does not exist an accretive extension \tilde{A} with strict inclusion of $D(A)$ in $D(\tilde{A})$.

Proposition 10.5.3.

Let A be an accretive operator with domain $D(A)$ dense in \mathcal{H} . Then A is closable and its closed extension \bar{A} is accretive.

For the analysis of the Fokker-Planck operator, the following criterion, which extends the standard criterion of essential self-adjointness, will be the most suitable

Theorem 10.5.4.

For an accretive operator A , the following conditions are equivalent

1. \bar{A} is maximally accretive.
2. There exists $\lambda_0 > 0$ such that $A^* + \lambda_0 I$ is injective.
3. There exists $\lambda_1 > 0$ such that the range of $A + \lambda_1 I$ is dense in \mathcal{H} .

Note that in this case $-\bar{A}$ is the infinitesimal generator of a contraction semi-group.

10.5.2 Application to the Fokker-Planck operator

We would like to show

Proposition 10.5.5.

Let V be a C^∞ potential on \mathbb{R}^n , then the closure \bar{K} of the Fokker-Planck operator defined on $C_0^\infty(\mathbb{R}^{2n})$ by

$$K := -\Delta_v + \frac{1}{4}|v|^2 - \frac{n}{2} + X_0, \quad (10.5.2)$$

where

$$X_0 := -\nabla V(x) \cdot \partial_v + v \cdot \partial_x \quad (10.5.3)$$

is maximally accretive.

Moreover K^* is also maximally accretive.

The idea is to adapt the proof that a semi-bounded Schrödinger operator with regular potential is essentially self-adjoint on $L^2(\mathbb{R}^n)$.

Proof:

We apply the abstract criterion taking $\mathcal{H} = L^2(\mathbb{R}^n \times \mathbb{R}^n)$ and $A = K$. The operators being real, we can consider everywhere real functions. The accretivity on $C_0^\infty(\mathbb{R}^{2n})$ is clear. We can then consider the closure \bar{K} .

Changing K in $T := K + (\frac{n}{2} + 1)I$, we would like to show that its range is dense.

Let $f \in L^2(\mathbb{R}^m)$, with $m = 2n$, such that

$$\langle f | Tu \rangle_{\mathcal{H}} = 0, \quad \forall u \in C_0^\infty(\mathbb{R}^m). \quad (10.5.4)$$

We have to show that $f = 0$.

Because K is real, one can assume that f is real.

We first observe that (10.5.4) implies that :

$$(-\Delta_v + v^2/4 + 1 - X_0)f = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^m).$$

The standard hypoellipticity theorem for the Hörmander operators³² of type 2 implies that $f \in C^\infty(\mathbb{R}^m)$.

We now introduce a family of cut-off functions $\zeta_k := \zeta_{k_1, k_2}$ by

$$\zeta_{k_1, k_2}(x, v) := \zeta(x/k_1)\zeta(v/k_2), \quad \forall k \in \mathbb{N}^2, \quad (10.5.5)$$

where ζ is a C^∞ function satisfying $0 \leq \zeta \leq 1$, $\zeta = 1$ on $B(0, 1)$ and $\text{supp } \zeta \subset B(0, 2)$.

For any $u \in C_0^\infty$, we have the identity

$$\begin{aligned} & \int \nabla_v(\zeta_k f) \cdot \nabla_v(\zeta_k u) \, dx dv + \int \zeta_k(x, v)^2 (v^2/4 + 1) u(x, v) f(x, v) \, dx dv \\ & + \int f(x, v) (X_0(\zeta_k^2 u))(x, v) \, dx dv \\ & = \int |\nabla_v \zeta_k(x, v)|^2 u(x, v) f(x, v) \, dx dv \\ & \quad + \sum_{i=1}^m \int (f(\partial_{v_i} u) - u(\partial_{v_i} f))(x, v) \zeta_k(x, v) (\partial_{v_i} \zeta_k)(x, v) \, dx dv \\ & \quad + \langle f(x, v) | T \zeta_k^2 u \rangle. \end{aligned} \quad (10.5.6)$$

When f satisfies (10.5.4), we get :

$$\begin{aligned} & \int_{\mathbb{R}^m} \nabla_v(\zeta_k f) \cdot \nabla_v(\zeta_k u) \, dx dv + \int \zeta_k^2 (v^2/4 + 1) u(x, v) f(x, v) \, dx dv \\ & + \int f(x, v) (X_0(\zeta_k^2 u))(x, v) \, dx dv \\ & = \int |\nabla_y \zeta_k(x)|^2 u(x, v) f(x, v) \, dx dv \\ & \quad + \sum_{i=1}^m \int (f(\partial_{v_i} u) - u(\partial_{v_i} f))(x, v) \zeta_k(x, v) (\partial_{v_i} \zeta_k)(x, v) \, dx dv, \end{aligned} \quad (10.5.7)$$

for all $u \in C^\infty(\mathbb{R}^m)$. In particular, we can take $u = f$.

We obtain

$$\begin{aligned} & \langle \nabla_v(\zeta_k f) | \nabla_v(\zeta_k f) \rangle + \int \zeta_k^2 (v^2/4 + 1) |f(x, v)|^2 \, dx dv \\ & + \int f(x, v) (X_0(\zeta_k^2 f))(x, v) \, dx dv \\ & = \int |\nabla_v \zeta_k|^2 |f(x, v)|^2 \, dx dv. \end{aligned} \quad (10.5.8)$$

With an additional integration by part, we get

$$\begin{aligned} & \langle \nabla_v(\zeta_k f) | \nabla_v(\zeta_k f) \rangle + \int \zeta_k^2 (v^2/4 + 1) |f(x, v)|^2 \, dx dv \\ & + \int \zeta_k f(x, v)^2 (X_0 \zeta_k)(x, v) \, dx dv \\ & = \int |\nabla_v \zeta_k|^2 |f(x, v)|^2 \, dx dv. \end{aligned} \quad (10.5.9)$$

³²These operators are in the form $P = -\sum_{j=1}^k X_j^2 + X_0 + a(x)$, where the X_j are real vectorfield. If the X_j together with the brackets $[X_\ell, X_m]$ span at each point x the whole tangent space, then one can show that the corresponding operator is hypoelliptic. P is said hypoelliptic if for any $u \in \mathcal{D}'$ and any open set ω , $Pu \in C^\infty(\omega)$ implies that $u \in C^\infty(\omega)$.

This leads to the existence of a constant C such that, for all k ,

$$\begin{aligned} & \|\zeta_k f\|^2 + \frac{1}{4}\|\zeta_k v f\|^2 \\ & \leq C \frac{1}{k_2^2} \|f\|^2 + C \frac{1}{k_1} \|v \zeta_k f\| \|f\| + C \frac{1}{k_2} \|\nabla V(x) \zeta_k f\| \|f\|. \end{aligned} \quad (10.5.10)$$

(The constant C will possibly be changed from line to line). This leads to

$$\|\zeta_k f\|^2 + \frac{1}{8}\|\zeta_k v f\|^2 \leq C \left(\frac{1}{k_2^2} + \frac{1}{k_1^2} \right) \|f\|^2 + C(k_1) \frac{1}{k_2} \|\zeta_k f\| \|f\|, \quad (10.5.11)$$

where

$$C(k_1) = \sup_{|x| \leq 2k_1} |\nabla_x V(x)|$$

This implies

$$\|\zeta_k f\|^2 \leq C \left(\frac{\tilde{C}(k_1)}{k_2^2} + \frac{1}{k_1^2} \right) \|f\|^2. \quad (10.5.12)$$

This finally leads to $f = 0$. For example, one can take first the limit $k_2 \rightarrow +\infty$, which leads to

$$\|\zeta\left(\frac{x}{k_1}\right) f\|^2 \leq \frac{C}{k_1^2} \|f\|^2,$$

and then the limit $k_1 \rightarrow +\infty$.

11 Discrete spectrum, essential spectrum

11.1 Discrete spectrum

We have already recalled in Proposition 8.5.4 a characterization of the spectrum. Let us now complete this characterization by introducing different spectra.

Definition 11.1.1.

If T is a selfadjoint operator, we shall call discrete spectrum of T the set

$$\sigma_{disc}(T) = \{\lambda \in \sigma(T) \text{ s. t. } \exists \epsilon > 0, \dim \text{range}(E(] \lambda - \epsilon, \lambda + \epsilon[)) < +\infty\}.$$

With this new definition, we can say that, for a selfadjoint operator with compact resolvent, the spectrum is reduced to the discrete spectrum.

For a compact selfadjoint operator, the spectrum is discrete outside 0. We see in this case that the discrete spectrum is not closed.

Equivalently, let us observe now give another characterization :

Proposition 11.1.2.

Let T be a selfadjoint operator. A real λ is in the discrete spectrum if and only if :

λ is an isolated point in $\sigma(T)$ and if λ is an eigenvalue of finite multiplicity.

Proof.

If $\lambda \in \sigma_{disc}(T)$, we immediately see that there exists ϵ_0 such that, $\forall \epsilon$ such that $0 < \epsilon < \epsilon_0$, $E_{] \lambda - \epsilon, \lambda + \epsilon[}$ becomes a projector independent of ϵ with finite range. This is actually the projector $\Pi_\lambda = 1_{\{\lambda\}}(T)$ and we observe moreover $E_{] \lambda, \lambda + \epsilon[} = 0$ and $E_{] \lambda - \epsilon, \lambda[} = 0$. This shows that λ is an isolated point in $\sigma(T)$. Using the spectral representation of T , one immediately get that, if $x = \Pi_\lambda x$ ($x \neq 0$), then x is an eigenvector of T . Moreover, one easily obtains that $(T - \lambda)$ is invertible on $R(I - \Pi_\lambda)$. One can indeed find a continuous bounded f such that $f(T)(T - \lambda)(I - \Pi_\lambda) = (I - \Pi_\lambda)$.

Conversely, let λ be isolated. The previous proof as already shown that in this case the range of Π_λ is an eigenspace. The assumption of finite multiplicity permits then to conclude.

Remark 11.1.3.

The discrete spectrum is not a closed set ! If we consider in \mathbb{R}^3 , the Schrödinger operator with coulomb potential, the discrete spectrum is a sequence of eigenvalues tending to 0 but 0 does not belong to the discrete spectrum.

11.2 Essential spectrum

Definition 11.2.1.

The essential spectrum is the complementary in the spectrum of the discrete spectrum.

Intuitively, a point of the essential spectrum corresponds

- either to a point in the continuous spectrum,
- or to a limit point of a sequence of eigenvalues with finite multiplicity,
- or to an eigenvalue of infinite multiplicity.

The discrete spectrum being composed of isolated points, we get

Proposition 11.2.2.

The essential spectrum of a selfadjoint operator T is closed in \mathbb{R} .

11.3 Basic examples:

1. The essential spectrum of a compact selfadjoint operator is reduced to 0.
2. The essential spectrum of an operator with compact resolvent is empty.
3. The Laplacian on $\mathbb{R}^n - \Delta$ is a selfadjoint operator on $L^2(\mathbb{R}^n)$ whose domain is the Sobolev space $H^2(\mathbb{R}^n)$. The spectrum is continuous and equal to $\overline{\mathbb{R}^+}$. The essential spectrum is also $\overline{\mathbb{R}^+}$ and the operator has no discrete spectrum.
4. The Schrödinger operator with constant magnetic field ($B \neq 0$) in \mathbb{R}^2 :

$$S_B := \left(D_{x_1} - \frac{Bx_2}{2}\right)^2 + \left(D_{x_2} + \frac{Bx_1}{2}\right)^2, \quad (11.3.1)$$

with

$$D_{x_j} := \frac{1}{i} \partial_{x_j} = \frac{1}{i} \frac{\partial}{\partial x_j}.$$

The spectrum is formed with eigenvalues $(2k + 1)|B|$ but the spectrum is not discrete because each eigenvalue is with infinite multiplicity. This example will be analyzed later.

11.4 Weyl's criterion:

We have already mentioned that the essential spectrum is a closed set. In order to determine the essential spectrum it is useful to have theorems proving the invariance by perturbation. The following characterization is in this spirit quite useful.

Theorem 11.4.1.

Let T be a selfadjoint operator. Then λ belongs to the essential spectrum iff there exists a sequence u_n in $D(T)$ with $\|u_n\| = 1$ such that u_n tends weakly³³ to 0 and such that $\|(T - \lambda)u_n\| \rightarrow 0$ as $n \rightarrow \infty$.

³³We recall that we say that a sequence u_n in a separable Hilbert space \mathcal{H} is weakly convergent if, for any g in \mathcal{H} , $\langle u_n | g \rangle_{\mathcal{H}}$ is convergent. In this case, there exists a unique f such that $\langle u_n | g \rangle_{\mathcal{H}} \rightarrow \langle f | g \rangle_{\mathcal{H}}$ and $\|u_n\|$ is a bounded sequence.

Let us give a partial proof of the theorem and we refer to [Ro], [HiSi] or [RS-IV] for a complete proof.

The sequence appearing in the theorem is called a Weyl sequence. A point λ such that there exists an associated Weyl sequence is said to belong to the Weyl spectrum $W(T)$. Let us show the inclusion

$$W(T) \subset \sigma_{ess}(T) . \quad (11.4.1)$$

We have already seen that

$$W(T) \subset \sigma(T) . \quad (11.4.2)$$

Let us suppose by contradiction that $\lambda \in \sigma_{disc}(T)$. Let $\Pi_\lambda := E_{\{\lambda\}}$ be the associated spectral projector. We first observe that, Π_λ being finite range, hence compact, we have :

$$\Pi_\lambda u_n \rightarrow 0 \in \mathcal{H} . \quad (11.4.3)$$

Let us define

$$w_n = (I - \Pi_\lambda)u_n .$$

We get $\|w_n\| \rightarrow +1$ and $(T - \lambda)w_n = (I - \Pi_\lambda)(T - \lambda)u_n \rightarrow 0$.

But $(T - \lambda)$ is invertible on $R(I - \Pi_\lambda)$, so we get $w_n \rightarrow 0$ and the contradiction. This shows the announced inclusion (11.4.1).

Corollary 11.4.2.

The operator $-\hbar^2\Delta + V$ with V a continuous function tending to 0 as $|x| \rightarrow \infty$ ($x \in \mathbb{R}^D$) has $\overline{\mathbb{R}^+}$ as essential spectrum.

For proving the inclusion of $\overline{\mathbb{R}^+}$ in the essential spectrum, we can indeed consider the sequence

$$u_n(x) = \exp(ix \cdot \xi) n^{-(D/2)} \cdot \chi((x - R_n)/n)$$

with $\chi \geq 0$ and supported in the ball $B(0, 1)$ and equal to one on say $B(0, \frac{1}{2})$. The sequence R_n is chosen such that $|R_n|$ (for example $|R_n| = n^2$) tends to ∞ and such that the support of the u_n are disjoint.

This is a particular case of a Weyl sequence (called in [HiSi] a Zhislin sequence). The converse can be obtained by abstract analysis and the fact that we know that the essential spectrum of $-\Delta$ is $[0, +\infty[$. This idea is formalized through the notion of relative compactness.

Definition 11.4.3.

If T is a closed operator with a dense domain D_T , we shall say that the operator V is relatively compact with respect to T or T -compact if $D_T \subset D_V$ and if the image by V of a closed ball in D_T (for the graph-norm $u \mapsto \sqrt{\|u\|^2 + \|Tu\|^2}$) is relatively compact in \mathcal{H} .

In other words, we shall say that V is T -compact, if, from each sequence u_n in D_T bounded in \mathcal{H} and such that Tu_n is bounded in \mathcal{H} , one can extract a subsequence u_{n_i} such that Vu_{n_i} is convergent in \mathcal{H} . Here we recall (exercise) that when T is closed, then $D(T)$ equipped with the graph norm is an Hilbert space.

Example 11.4.4.

If V is the multiplication operator by a continuous function V tending to 0 then V is $(-\Delta)$ -compact. This is a consequence of Proposition 5.4.1 and of the uniform continuity of V on each compact.

Weyl's Theorem says

Theorem 11.4.5.

Let T be a selfadjoint operator, and V be symmetric and T -compact, then $T + V$ is selfadjoint and the essential spectrum of $T + V$ is equal to the essential spectrum of T .

The first part can be deduced from what was discussed in the previous subsection (See Theorem 10.3.1). We observe indeed the following variant of Lions's Lemma :

Lemma 11.4.6.

If V is T -compact and closable, then, for any $a > 0$, there exists $b > 0$ such that

$$\|Vu\| \leq a\|Tu\| + b\|u\|, \quad \forall u \in D(T). \quad (11.4.4)$$

Proof of the lemma.

The proof is by contradiction. If (11.4.4) is not true, then there exists $a > 0$ such that, $\forall n \in \mathbb{N}^*$, there exists $u_n \in D(T)$ such that

$$a\|Tu_n\| + n\|u_n\| < \|Vu_n\|. \quad (11.4.5)$$

Observing that $\|Vu_n\| \neq 0$ and that the inequation is homogeneous, we can in addition assume that u_n satisfies the condition :

$$\|Vu_n\| = 1. \quad (11.4.6)$$

From these two properties we get that the sequence Tu_n is bounded and that $u_n \rightarrow 0$.

On the other hand, by T -compactness, we can extract a subsequence u_{n_k} such that Vu_{n_k} is convergent to v with $\|v\| = 1$. But $(0, v)$ is in the closure of the graph of V , hence in the graph of the closure of V and consequently $v = 0$ (contradiction).

For the second part we can use the Theorem 11.4.1. If we take a Weyl's sequence u_n such that $u_n \rightarrow 0$ (weakly) and $(T - \lambda)u_n \rightarrow 0$ strongly, let us consider $(T + V - \lambda)u_n$. We have simply to show that one can extract a subsequence u_{n_k} such that $(T + V - \lambda)u_{n_k} \rightarrow 0$.

But Tu_n is a bounded sequence. By the T -compactness, we can extract a subsequence such that Vu_{n_k} converges strongly to some v in \mathcal{H} . It remains to show that $v = 0$. But here we can observe that for any $f \in D(T)$, we have

$$\langle v, f \rangle = \lim_{k \rightarrow +\infty} \langle Vu_{n_k}, f \rangle = \lim_{k \rightarrow +\infty} \langle u_{n_k}, Vf \rangle = 0.$$

Here we have used the symmetry of V and the weak convergence of u_n to 0. Using the density of $D(T)$ in \mathcal{H} , we obtain $v = 0$.

This shows that a Weyl sequence for T is a Weyl sequence for $T + V$. For the converse, one can intertwine the roles of T and $T + V$, once we have shown that V is $(T + V)$ -compact. For this, we can use Lemma 11.4.6, and observe that the following inequality is true :

$$\|Tu\| \leq \frac{1}{1-a} (\|(T+V)u\| + b\|u\|) . \quad (11.4.7)$$

This shows that if u_n is a sequence such that $(\|u_n\| + \|(T+V)u_n\|)$ is bounded, then this sequence has also the property that $(\|u_n\| + \|Tu_n\|)$ is bounded.

12 The max-min principle

12.1 Introduction

The max-min principle is an alternative way for describing the lowest part of the spectrum when it is discrete. It gives also an efficient way to localize these eigenvalues or to follow their dependence on various parameters.

12.2 On positivity

We first recall the following definition

Definition 12.2.1.

Let A be a symmetric operator. We say that A is positive (and we write $A \geq 0$), if

$$\langle Au, u \rangle \geq 0, \quad \forall u \in D(A). \quad (12.2.1)$$

The following proposition relates the positivity with the spectrum

Proposition 12.2.2.

Let A be a selfadjoint operator. Then $A \geq 0$ if and only if $\sigma(A) \subset [0, +\infty[$.

Proof.

It is clear that if the spectrum is in \mathbb{R}^+ , then the operator is positive. This can be seen for example through the spectral representation :

$$\langle Au, u \rangle = \int_{\lambda \in \sigma(A)} \lambda d\|E_\lambda u\|^2.$$

Now, if $A \geq 0$, then, for any $a > 0$, $A + a$ is invertible. We have indeed

$$a\|u\|^2 \leq \langle (A + a)u, u \rangle \leq \|(A + a)u\| \|u\|,$$

which leads to

$$a\|u\| \leq \|(A + a)u\|, \quad \forall u \in D(A). \quad (12.2.2)$$

This inequality gives the closed range and the injectivity. A being selfadjoint, we get also from the injectivity, the density of the image of $(A + a)$. This shows that $-a$ is not in the spectrum of A .

Example 12.2.3.

Let us consider the Schrödinger operator $P := -\Delta + V$, with $V \in C^\infty$ and semi-bounded, then

$$\sigma(P) \subset [\inf V, +\infty[. \quad (12.2.3)$$

12.3 Variational characterization of the discrete spectrum

Theorem 12.3.1.

Let A be a selfadjoint semibounded operator. Let $\Sigma := \inf \sigma_{\text{ess}}(A)$ and let us consider $\sigma(A) \cap]-\infty, \Sigma[$, described as a sequence (finite or infinite) of eigenvalues that we write in the form

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n \cdots .$$

Then we have

$$\lambda_1 = \inf_{\phi \in D(A), \phi \neq 0} \|\phi\|^{-2} \langle A\phi, \phi \rangle , \quad (12.3.1)$$

$$\lambda_2 = \inf_{\phi \in D(A) \cap K_1^\perp, \phi \neq 0} \|\phi\|^{-2} \langle A\phi, \phi \rangle , \quad (12.3.2)$$

and, for $n \geq 2$,

$$\lambda_n = \inf_{\phi \in D(A) \cap K_{n-1}^\perp, \phi \neq 0} \|\phi\|^{-2} \langle A\phi, \phi \rangle , \quad (12.3.3)$$

where

$$K_j = \bigoplus_{i \leq j} \text{Ker} (A - \lambda_i) .$$

Proof.

Step 1. Let us start, with the lowest eigenvalue. Let us define $\mu_1(A)$ by

$$\mu_1(A) := \inf_{\phi \in D(A), \phi \neq 0} \|\phi\|^{-2} \langle A\phi, \phi \rangle . \quad (12.3.4)$$

If ϕ_1 is an eigenvector associated to λ_1 , we get immediately the inequality

$$\mu_1(A) \leq \lambda_1(A) . \quad (12.3.5)$$

Let us prove the converse inequality. Using the spectral theorem, one get immediately that $A \geq \inf \sigma(A)$.

So we get

$$\inf \sigma(A) \leq \mu_1(A) . \quad (12.3.6)$$

Now, if the spectrum below Σ is not empty, we get

$$\lambda_1(A) \leq \mu_1(A) .$$

We have consequently the equality. We have actually a little more.

We have indeed proved that, if $\mu_1(A) < \Sigma$, then, the spectrum below Σ is not empty, and the lowest eigenvalue is $\mu_1(A)$.

Step 2. The proof is by recursion, applying Step 1 to $A_{/D(A) \cap K_{n-1}^\perp}$.

This ends the proof of Theorem 12.3.1.

Example 12.3.2. (*Payne-Polya-Weinberger Inequality.*)

Let $P = -\Delta + V$ with $V \in C^\infty$ positive and $V \rightarrow +\infty$ as $|x| \rightarrow +\infty$.
Let us assume that V is even

$$V(x) = V(-x) . \quad (12.3.7)$$

Then λ_2 satisfies

$$\lambda_2 \leq \inf_{\phi \in Q(P), \phi \text{ odd}} \langle P\phi, \phi \rangle \quad (12.3.8)$$

Let u_1 be the first normalized eigenvector. We admit that the lowest eigenvalue of the Schrödinger operator is simple (variant of the Krein-Rutman's Theorem) and that the first eigenvector can be chosen strictly positive, with exponential decay at ∞ together with ∇u_1 (this is a consequence of Agmon's inequality [Ag]). Then it is not difficult to verify that u_1 is even. Let us consider $v_j := x_j u_1$. v_j is in the form domain of P . We observe that

$$P(x_j u_1) = \lambda_1 x_j u_1 - 2\partial_j u_1 .$$

Taking the scalar product with $x_j u_1$, we then obtain

$$\begin{aligned} (\lambda_2 - \lambda_1) \|x_j u_1\|^2 &\leq -2 \langle \partial_j u_1, x_j u_1 \rangle \\ &\leq \|u_1\|^2 \\ &\leq 1 . \end{aligned} \quad (12.3.9)$$

We now use the uncertainty principle (1.2.11) and get :

$$(\lambda_2 - \lambda_1) \leq 4 \|\partial_j u_1\|^2 . \quad (12.3.10)$$

On the other hand,

$$\|\nabla u_1\|^2 + \int_{\mathbb{R}^m} V(x) |u_1(x)|^2 dx = \lambda_1 , \quad (12.3.11)$$

and this gives

$$\|\nabla u_1\|^2 \leq \lambda_1 . \quad (12.3.12)$$

Putting the inequalities (12.3.9) and (12.3.12), we get, summing over j ,

$$\lambda_2 - \lambda_1 \leq \frac{4}{m} \lambda_1 . \quad (12.3.13)$$

This inequality is not optimal, in the sense that for $m = 1$ and $V(x) = x^2$, we have $\lambda_2 - \lambda_1 = 2\lambda_1$.

Example 12.3.3.

Let us consider $S_h := -h^2\Delta + V$ on \mathbb{R}^m where V is a C^∞ potential tending to 0 at ∞ and such that $\inf_{x \in \mathbb{R}^m} V(x) < 0$.

Then if $h > 0$ is small enough, there exists at least one eigenvalue for S_h . We note that the essential spectrum is $[0, +\infty[$. The proof of the existence of this eigenvalue is elementary. If x_{min} is one point such that $V(x_{min}) = \inf_x V(x)$, it is enough to show that, with $\phi_h(x) = \exp -\frac{\lambda}{h}|x - x_{min}|^2$, the quotient $\frac{\langle S_h \phi_h, \phi_h \rangle}{\|\phi_h\|^2}$ tends as $h \rightarrow 0$ to $V(x_{min}) < 0$.

12.4 Max-min principle

We now give a more flexible criterion for the determination of the bottom of the spectrum and for the bottom of the essential spectrum. This flexibility comes from the fact that we do not need an explicit knowledge of the various eigenspaces.

Theorem 12.4.1.

Let \mathcal{H} an Hilbert space of infinite dimension³⁴ and A be a selfadjoint semi-bounded operator of domain $D(A) \subset \mathcal{H}$. Let us introduce

$$\mu_n(A) = \sup_{\psi_1, \psi_2, \dots, \psi_{n-1}} \left\{ \begin{array}{l} \inf \\ \phi \in [span(\psi_1, \dots, \psi_{n-1})]^\perp; \\ \phi \in D(A) \text{ and } \|\phi\| = 1 \end{array} \right\} \langle A\phi | \phi \rangle_{\mathcal{H}}. \quad (12.4.1)$$

Then either

(a) $\mu_n(A)$ is the n -th eigenvalue when ordering the eigenvalues in increasing order (and counting the multiplicity) and A has a discrete spectrum in $] -\infty, \mu_n(A)]$

or

(b) $\mu_n(A)$ corresponds to the bottom of the essential spectrum. In this case, we have $\mu_j(A) = \mu_n(A)$ for all $j \geq n$.

Remark 12.4.2.

In the case when the operator is with compact resolvent, case (b) does not occur and the supremum in (12.4.1) is a maximum. Similarly the infimum is a minimum. This explains the traditional terminology “Max-Min principle” for this theorem.

Proof.

If Ω is a borelian, let E_Ω be the projection-valued measure for A (see Remark 8.3.7).

We first prove that

$$\dim (\text{Range } E_{]-\infty, a[}) < n \text{ if } a < \mu_n(A). \quad (12.4.2)$$

$$\dim (\text{Range } E_{]a, \infty[}) \geq n \text{ if } a > \mu_n(A). \quad (12.4.3)$$

³⁴In the case of a finite dimensional Hilbert space of dimension d , the minimax principle holds for $n \leq d$.

Notons que la conjonction de (12.4.2) et (12.4.3) montre que $\mu_n(A)$ est dans le spectre de A .

Step 1: Proof of (12.4.2).

Let a and n be given such that $a < \mu_n(A)$. Let us prove (12.4.2) by contradiction. If it was false, then we would have $\dim (\text{Range}(E_{]-\infty, a[})) \geq n$ and we could find an n -dimensional space $V \subset \text{Range}(E_{]-\infty, a[})$. Note now, that A being bounded from below, $\text{Range}(E_{]-\infty, a[})$ is included in $D(A)$. So we can find an n -dimensional space $V \subset D(A)$, such that

$$\forall \phi \in V, \langle A\phi, \phi \rangle \leq a \|\phi\|^2. \quad (12.4.4)$$

But then given any $\psi_1, \dots, \psi_{n-1}$ in \mathcal{H} , we can find $\phi \in V \cap \{\psi_1, \dots, \psi_{n-1}\}^\perp$ such that $\|\phi\| = 1$ and $\langle A\phi, \phi \rangle \leq a$. Coming back to the definition, this shows that $\mu_n(A) \leq a$ and a contradiction.

Note that we have proved in this step the following proposition

Proposition 12.4.3.

Suppose that there exists a and an n -dimensional subspace $V \subset D(A)$ such that (12.4.4) is satisfied. Then we have the inequality :

$$\mu_n(A) \leq a. \quad (12.4.5)$$

Modulo the complete proof of the theorem, we obtain

Corollary 12.4.4.

Under the same assumption as in Proposition 12.4.3, if a is below the bottom of the essential spectrum of A , then A has at least n eigenvalues (counted with multiplicity).

Exercise 12.4.5.

In continuation of Example 12.3.3, show that for any $\epsilon > 0$ and any N , there exists $h_0 > 0$ such that for $h \in]0, h_0]$, S_h has at least N eigenvalues in $[\inf V, \inf V + \epsilon]$. One can treat first the case when V has a unique non degenerate minimum at 0.

Step 2 : Proof of (12.4.3).

Suppose that (12.4.3) is false. Then $\dim (\text{Range}(E_{]-\infty, a[})) \leq n - 1$, so we can find $(n - 1)$ generators $\psi_1, \dots, \psi_{n-1}$ of this space. Then any $\phi \in D(A) \cap \text{span}\{\psi_1, \dots, \psi_{n-1}\}^\perp$ is in $\text{Range}(E_{[a, +\infty[})$, so

$$\langle A\phi, \phi \rangle \geq a \|\phi\|^2.$$

Therefore, coming back to the definition of $\mu_n(A)$, we get $\mu_n(A) \geq a$ in contradiction with our initial assumption.

Before to continue the proof, let us emphasize on one point.

Remark 12.4.6.

In the definition of $\mu_n(A)$, $\psi_1, \dots, \psi_{n-1}$ are only assumed to belong to the Hilbert space \mathcal{H} .

Step 3 : $\mu_n(A) < +\infty$.

First the semi-boundedness from below of A gives a uniform lower bound.

Secondly, if $\mu_n(A) = +\infty$, this would mean by (12.4.2) that :

$\dim(\text{Range}(E_{]-\infty, a[})) < n$ for all a ,

and consequently that \mathcal{H} is finite dimensional. This is a contradiction, if \mathcal{H} is infinite dimensional. But the finite case is trivial, we have indeed $\mu_n(A) \leq \|A\|$, in this case.

As the statement of the theorem suggests, there are two cases to consider and this will be the object of the two next steps.

Step 4.

Let us first assume (with $\mu_n = \mu_n(A)$) that

$$\dim(\text{Range}(E_{]-\infty, \mu_n + \epsilon[})) = +\infty, \forall \epsilon > 0. \quad (12.4.6)$$

We claim that, in this case, we are in the second alternative in the theorem. Using (12.4.2) and (12.4.6), we get indeed

$$\dim(\text{Range}(E_{] \mu_n - \epsilon, \mu_n + \epsilon[})) = +\infty, \forall \epsilon > 0. \quad (12.4.7)$$

This shows that $\mu_n(A) \in \sigma_{ess}(A)$.

On the other hand, using again (12.4.2), we immediately get that $]-\infty, \mu_n(A)[$ does not contain any point in the essential spectrum. Thus $\mu_n(A) = \inf\{\lambda \mid \lambda \in \sigma_{ess}(A)\}$.

Let us show now that $\mu_{n+1} = \mu_n$ in this case. From the definition of the $\mu_k(A)$, it is clear that $\mu_{n+1} \geq \mu_n$, since one can take $\psi_n = \psi_{n-1}$.

But if $\mu_{n+1} > \mu_n$, (12.4.2) would also be satisfied for μ_{n+1} , and this is in contradiction with (12.4.6).

Step 5.

Let us now assume that

$$\dim(\text{Range}(E_{]-\infty, \mu_n + \epsilon_0[})) < \infty, \text{ for some } \epsilon_0 > 0. \quad (12.4.8)$$

Then it is clear, that the spectrum is discrete in $]-\infty, \mu_n + \epsilon_0[$. Therefore μ_n is an eigenvalue. Then, for $\epsilon_1 > 0$ small enough,

$$\text{Range}(E_{]-\infty, \mu_n[})) = \text{Range}(E_{]-\infty, \mu_n + \epsilon_1[})) ,$$

and by (12.4.3)

$$\dim(\text{Range} E_{]-\infty, \mu_n[})) \geq n. \quad (12.4.9)$$

So there are at least n eigenvalues $E_1 \leq E_2 \leq \dots \leq E_n \leq \mu_n$ for A . If E_n were strictly less than μ_n , $\dim(\text{Range} E_{]-\infty, E_n[}))$ would equal n in contradiction with

(12.4.2).

This ends the proof of Theorem 12.4.1.

A first natural extension of Theorem 12.4.1 is obtained by

Theorem 12.4.7.

Let A be a selfadjoint semibounded operator and $Q(A)$ its form domain ³⁵. Then

$$\mu_n(A) = \sup_{\psi_1, \psi_2, \dots, \psi_{n-1}} \inf_{\left\{ \begin{array}{l} \phi \in [\text{span}(\psi_1, \dots, \psi_{n-1})]^\perp; \\ \phi \in Q(A) \text{ and } \|\phi\| = 1 \end{array} \right\}} \langle A\phi | \phi \rangle_{\mathcal{H}}. \quad (12.4.10)$$

Proof.

Let $\tilde{\mu}_n$ be the right hand side of (12.4.10). By imitating the proof of the previous theorem, we get that each $\tilde{\mu}_n$ obeys one of the two conditions. These conditions determine μ_n and consequently $\mu_n = \tilde{\mu}_n$.

One can also note (see Subsection 3.3) that, when constructing the Friedrichs extension, one has shown that the domain of the Friedrichs extension is dense in the form domain.

Applications

- It is very often useful to apply the max-min principle by taking the minimum over a dense set in $Q(A)$.
- The max-min principle permits to control the continuity of the eigenvalues with respect to parameters. For example the lowest eigenvalue $\lambda_1(\epsilon)$ of $-\frac{d^2}{dx^2} + x^2 + \epsilon x^4$ increases with respect to ϵ . Show that $\epsilon \mapsto \lambda_1(\epsilon)$ is right continuous on $[0, +\infty[$. (The reader can admit that the corresponding eigenfunction is in $\mathcal{S}(\mathbb{R})$ for $\epsilon \geq 0$).
- The max-min principle permits to give an upperbound on the bottom of the spectrum and the comparison between the spectrum of two operators. If $A \leq B$ in the sense that, $Q(B) \subset Q(A)$ and³⁶

$$\langle Au, u \rangle \leq \langle Bu, u \rangle, \quad \forall u \in Q(B),$$

then

$$\lambda_n(A) \leq \lambda_n(B).$$

Similar conclusions occur if we have $D(B) \subset D(A)$.

³⁵associated by completion with the form $u \mapsto \langle u|Au \rangle_{\mathcal{H}}$ initially defined on $D(A)$.

³⁶It is enough to verify the inequality on a dense set in $Q(B)$.

Example 12.4.8. (*Comparison between Dirichlet and Neumann*)

Let Ω be a bounded regular connected open set in \mathbb{R}^m . Then the N -th eigenvalue of the Neumann realization of $-\Delta + V$ is less or equal to the N -th eigenvalue of the Dirichlet realization. It is indeed enough to observe the inclusion of the form domains.

Example 12.4.9. (*monotonicity with respect to the domain*)

Let $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^m$ two bounded regular open sets. Then the n -th eigenvalue of the Dirichlet realization of the Schrödinger operator in Ω_2 is less or equal to the n -th eigenvalue of the Dirichlet realization of the Schrödinger operator in Ω_1 . We observe that we can indeed identify $H_0^1(\Omega_1)$ with a subspace of $H_0^1(\Omega_2)$ by just an extension by 0 in $\Omega_2 \setminus \Omega_1$.

We then have

$$\begin{aligned}
\lambda_n(\Omega_2) &= \sup_{\{\psi_1, \dots, \psi_{n-1} \in L^2(\Omega_2)\}} \inf \left\{ \begin{array}{l} \phi \in H_0^1(\Omega_2) \\ \langle \phi, \psi_j \rangle_{L^2(\Omega_2)} \text{ and } \|\phi\|=1 \end{array} \right\} \|\nabla \phi\|_{L^2(\Omega_2)}^2 \\
&\leq \sup_{\{\psi_1, \dots, \psi_{n-1} \in L^2(\Omega_2)\}} \inf \left\{ \begin{array}{l} \phi \in H_0^1(\Omega_1) \\ \langle \phi, \psi_j \rangle_{L^2(\Omega_2)} \text{ and } \|\phi\|=1 \end{array} \right\} \|\nabla \phi\|_{L^2(\Omega_2)}^2 \\
&= \sup_{\{\psi_1, \dots, \psi_{n-1} \in L^2(\Omega_2)\}} \inf \left\{ \begin{array}{l} \phi \in H_0^1(\Omega_1) \\ \langle \phi, \psi_j \rangle_{L^2(\Omega_1)} \text{ and } \|\phi\|=1 \end{array} \right\} \|\nabla \phi\|_{L^2(\Omega_1)}^2 \\
&= \sup_{\{\psi_1, \dots, \psi_{n-1} \in L^2(\Omega_1)\}} \inf \left\{ \begin{array}{l} \phi \in H_0^1(\Omega_1) \\ \langle \phi, \psi_j \rangle_{L^2(\Omega_1)} \text{ and } \|\phi\|=1 \end{array} \right\} \|\nabla \phi\|_{L^2(\Omega_1)}^2 \\
&= \lambda_n(\Omega_1).
\end{aligned}$$

Note that this argument is not valid for the Neumann realization.

12.5 CLR inequality

In order to complete the picture, let us mention (confer [RS-IV], p. 101) that, if $m \geq 3$, then the following theorem due to Cwickel-Lieb-Rozenbljum is true :

Theorem 12.5.1.

There exists a constant L_m , such that, for any V such that $V_- \in L^{\frac{m}{2}}$, and if $m \geq 3$, the number of strictly negative eigenvalues of $S_1 N_-$ is finite and bounded by

$$N_- \leq L_m \int_{V(x) \leq 0} (-V)^{\frac{m}{2}} dx. \quad (12.5.1)$$

This shows that when $m \geq 3$, we could have examples of negative potentials V (which are not identically zero) and such that the corresponding Schrödinger operator S_1 has no eigenvalues. A sufficient condition is indeed

$$L_m \int_{V \leq 0} (-V)^{\frac{m}{2}} dx < 1.$$

In the other direction, we have³⁷ the following results.

³⁷These counterexamples come back (when $m = 1$ to Avron-Herbst-Simon [AHS] and when $m = 2$ to Blanchard-Stubbe [BS]).

Proposition 12.5.2.

Let V be in $L^1(\mathbb{R}^m)$ ($m = 1, 2$). Then $S_1 = -\Delta + V$ has a negative eigenvalue if the following condition holds :

$$\int_{\mathbb{R}^m} V(x) dx < 0 . \quad (12.5.2)$$

Proof.

We just treat the case when $V \in C_0^\infty(\mathbb{R}^m)$.

We first observe that the the essential spectrum is $[0, +\infty[$. For the proof of the proposition, it is then enough to find $\psi \in D(S_1)$ such that

$$\langle S_1 \psi, \psi \rangle_{L^2(\mathbb{R}^m)} < 0 .$$

When $m = 1$, taking $\psi_a = \exp -a|x|$, $a > 0$, we find that

$$\int_{\mathbb{R}} |\psi'_a(x)|^2 dx = a ,$$

and

$$\lim_{a \rightarrow 0} \int_{\mathbb{R}} V(x) |\psi_a(x)|^2 dx = \int_{\mathbb{R}} V(x) dx < 0 ,$$

by the dominated convergence Theorem.

When $m = 2$, we can take $\psi_a(x) = \exp -\frac{1}{2}|x|^a$, $a > 0$, then

$$\int_{\mathbb{R}^2} \|\nabla \psi_a(x)\|^2 dx = \frac{\pi}{2} a ,$$

and

$$\lim_{a \rightarrow 0} \int_{\mathbb{R}^2} V(x) |\psi_a(x)|^2 dx = \int_{\mathbb{R}^2} V(x) dx < 0 .$$

12.6 Essential spectrum and Persson's Theorem

We refer to Agmon's book [Ag] for details.

Theorem 12.6.1.

Let V be a real-valued potential in the Kato-Rellich class³⁸, and let $H = -\Delta + V$ be the corresponding self-adjoint, semibounded Schrödinger operator with domain $H^2(\mathbb{R}^m)$. Then, the bottom of the essential spectrum is given by

$$\inf \sigma_{ess}(H) = \Sigma(H) , \quad (12.6.1)$$

where

$$\Sigma(H) := \sup_{\mathcal{K} \subset \mathbb{R}^m} \left[\inf_{\|\phi\|=1} \{ \langle \phi, H\phi \rangle \mid \phi \in C_0^\infty(\mathbb{R}^m \setminus \mathcal{K}) \} \right] , \quad (12.6.2)$$

where the supremum is over all compact subset $\mathcal{K} \subset \mathbb{R}^m$.

³⁸Cf Theorem 10.3.1.

Essentially this is a corollary of Weyl's Theorem 11.4.5. We will indeed play with the fact that

Lemma 12.6.2.

$$\sigma_{ess}(H) = \sigma_{ess}(H + W) ,$$

for any regular potential W with compact support.

13 On the Schrödinger Operator with constant magnetic field

The Schrödinger operator with magnetic field has been briefly introduced in Subsection 7.5. We have in particular given examples where this operator was with compact resolvent. In this section, we analyze more in detail the properties of the Schrödinger operator with constant magnetic field in dimension 2 and 3. This appears to play an important role in Superconductivity theory.

13.1 Dimension 2

We would like to analyze the spectrum of :

$$S_B := (D_{x_1} - \frac{B}{2}x_2)^2 + (D_{x_2} + \frac{B}{2}x_1)^2 . \quad (13.1.1)$$

13.1.1 The case of \mathbb{R}^2

We first look at the selfadjoint realization in \mathbb{R}^2 . Let us show briefly, how one can analyze its spectrum. We leave as an exercise to show that the spectrum (or the discrete spectrum) of two selfadjoints operators S and T are the same if there exists a unitary operator U such that $U(S \pm i)^{-1}U^{-1} = (T \pm i)^{-1}$. We note that this implies that U sends the domain of S onto the domain of T .

In order to determine the spectrum of the operator S_B , we perform a succession of unitary conjugations. The first one is called a gauge transformation. We introduce U_1 on $L^2(\mathbb{R}^2)$ defined, for $f \in L^2$ by

$$U_1 f = \exp iB \frac{x_1 x_2}{2} f . \quad (13.1.2)$$

It satisfies

$$S_B U_1 f = U_1 S_B^1 f , \quad \forall f \in \mathcal{S}(\mathbb{R}^2) , \quad (13.1.3)$$

with

$$S_B^1 := (D_{x_1})^2 + (D_{x_2} + Bx_1)^2 . \quad (13.1.4)$$

Remark 13.1.1.

U_1 is a very special case of what is called a gauge transform. More generally, we can consider $U = \exp i\phi$ where ϕ is C^∞ . If $\Delta_A := \sum_j (D_{x_j} - A_j)^2$ is a general Schrödinger operator associated with the magnetic potential A , then $U^{-1}\Delta_A U = \Delta_{\tilde{A}}$ where $\tilde{A} = A + \text{grad } \phi$. Here we observe that $B := \text{rot } A = \text{rot } \tilde{A}$. The associated magnetic field is unchanged in a gauge transformation. We are discussing in our example the very special case (but important!) when the magnetic potential is constant.

We have now to analyze the spectrum of S_B^1 . Observing that the operator is with constant coefficients with respect to the x_2 -variable, we perform a partial Fourier transform with respect to the x_2 variable

$$U_2 = \mathcal{F}_{x_2 \rightarrow \xi_2} , \quad (13.1.5)$$

and get by conjugation, on $L^2(\mathbb{R}_{x_1, \xi_2}^2)$,

$$S_B^2 := (D_{x_1})^2 + (\xi_2 + Bx_1)^2 . \quad (13.1.6)$$

We now introduce a third unitary transform U_3

$$(U_3 f)(y_1, \xi_2) = f(x_1, \xi_2) , \quad \text{with } y_1 = x_1 + \frac{\xi_2}{B} , \quad (13.1.7)$$

and we obtain the operator

$$S_B^3 := D_y^2 + B^2 y^2 , \quad (13.1.8)$$

operating on $L^2(\mathbb{R}_{y, \xi_2}^2)$.

The operator depends only on the y variable. It is easy to find for this operator an orthonormal basis of eigenvectors. We observe indeed that if $f \in L^2(\mathbb{R}_{\xi_2})$, and if ϕ_n is the $(n+1)$ -th eigenfunction of the harmonic oscillator, then

$$(x, \xi_2) \mapsto |B|^{\frac{1}{4}} f(\xi_2) \cdot \phi_n(|B|^{\frac{1}{2}} y)$$

is an eigenvector corresponding to the eigenvalue $(2n+1)|B|$. So each eigenspace has an infinite dimension. An orthonormal basis of this eigenvalue can be given by vectors $e_j(\xi_2)|B|^{\frac{1}{4}} f(\xi_2) \phi_n(|B|^{\frac{1}{2}} y)$ where e_j ($j \in \mathbb{N}$) is a basis of $L^2(\mathbb{R})$. We have consequently an empty discrete spectrum. The eigenvalues are usually called Landau levels.

13.2 Magnetic Schrödinger operators in dimension 3

We only consider the Schrödinger operator with constant magnetic field in \mathbb{R}^3 . After some rotation in \mathbb{R}^3 , we arrive to the model :

$$P(h, \vec{b}) = D_{x_1}^2 + (D_{x_2} - bx_1)^2 + D_{x_3}^2 , \quad (13.2.1)$$

with

$$b = ||B|| .$$

This time, we can take the Partial Fourier transform, with respect to x_2 and x_3 in order to get the operator

$$D_{x_1}^2 + (\xi_2 - bx_1)^2 + \xi_3^2 .$$

When $b \neq 0$, we can translate in the x_1 variable and get the operator on $L^2(\mathbb{R}^3)$

$$D_{y_1}^2 + (|b|y_1)^2 + \xi_3^2 .$$

It is then easy to see that the spectrum is $[|b|, +\infty[$.

13.3 The case of \mathbb{R}_2^+ .

13.3.1 Preliminaries

For the analysis of the spectrum of the Neumann realization of the Schrödinger operator with constant magnetic field S_B in $\mathbb{R}^{2,+}$, we start like in the case of \mathbb{R}^2 till (13.1.6). Then we can no more perform the translation and hence we have to look at the family (parametrized by ξ) $D_x^2 + (x + \xi)^2$ on the half line, with Neumann condition and more precisely to the groundstate energy $\mu(\xi)$. The bottom of the spectrum is effectively given by :

$$\inf \sigma(S_B^{N, \mathbb{R}^{2,+}}) = |B| \inf \mu(\xi) = \Theta_0 |B|. \quad (13.3.1)$$

Similarly, for the Dirichlet realization, we find :

$$\inf \sigma(S_B^{D, \mathbb{R}^{2,+}}) = |B| \inf \lambda(\xi) = |B|. \quad (13.3.2)$$

13.3.2 Main results for a 1D-model

Let us begin with the analysis of a family of ordinary differential operators, whose study will play an important role in the analysis of various examples. We consider the Neumann realization $H^{N, \xi}$ in $L^2(\mathbb{R}^+)$ associated to the operator $D_x^2 + (x - \xi)^2$. It is easy to see that the operator is with compact resolvent and that the lowest eigenvalue $\mu(\xi)$ of $H^{N, \xi}$ is simple. For the second point, the following simple argument can be used. Suppose by contradiction that the eigenspace is of dimension 2. Then, we can find in this eigenspace an eigenvector such that u such that $u(0) = u'(0) = 0$. But then it should be identically 0 by Cauchy uniqueness.

We denote by φ_ξ the corresponding strictly positive normal eigenvector. The minimax shows that $\mu(\xi)$ is continuous as a function of ξ . It is a little more work (admitted (see Kato [Ka]) to show that the function is C^∞ and actually analytic (see in the next Subsubsection). It is immediate to show that $\mu(\xi) \rightarrow +\infty$ as $\xi \rightarrow -\infty$. We can indeed compare by monotonicity with $D_x^2 + x^2 + \xi^2$.

The second remark is that $\mu(0) = 1$. For this, we use the fact that the lowest eigenvalue of the Neumann realization of $D_t^2 + t^2$ in \mathbb{R}^+ is the same as the lowest eigenvalue of $D_t^2 + t^2$ in \mathbb{R} , but restricted to the even functions, which is also the same as the lowest eigenvalue of $D_t^2 + t^2$ in \mathbb{R} .

Moreover the derivative of μ at 0 is strictly negative. It is a little more difficult to show that

$$\lim_{\xi \rightarrow +\infty} \mu(\xi) = 1.$$

The proof can be done in the following way. For the upper bound, we observe that $\mu(\xi) \leq \lambda(\xi)$ where $\lambda(\xi)$ is the eigenvalue of the Dirichlet realization. By monotonicity of $\lambda(\xi)$, it is easy to see that $\lambda(\xi)$ is larger than one and tend to 1 as $\xi \rightarrow +\infty$. Another way is to use the function $\exp -\frac{1}{2}(x - \xi)^2$ as a test

function.

For the converse, we start from the eigenfunction $\phi_\xi(x)$, show some uniform decay of $\phi_\xi(x)$ near 0 as $\xi \rightarrow +\infty$ and use $x \mapsto \chi(x + \xi)\phi_\xi(x + \xi)$ as a test function for the harmonic oscillator in \mathbb{R} .

All these remarks lead to the observation that the infimum $\inf_{\xi \in \mathbb{R}} \inf \text{Sp}(H^{N,\xi})$ is actually a minimum [DaHe] and strictly less than 1. Moreover one can see that $\mu(\xi) > 0$, for any ξ , so the minimum is strictly positive. To be more precise on the variation of μ , let us first establish

$$\mu'(\xi) = -[\mu(\xi) - \xi^2]\varphi_\xi(0)^2. \quad (13.3.3)$$

To get (13.3.3), we observe that, if $\tau > 0$, then

$$\begin{aligned} 0 &= \int_{\mathbb{R}^+} [D_t^2 \varphi_\xi(t) + (t - \xi)^2 \varphi_\xi(t)] \varphi_{\xi+\tau}(t + \tau) dt \\ &= -\varphi_\xi(0) \varphi'_{\xi+\tau}(\tau) + (\mu(\xi + \tau) - \mu(\xi)) \int_{\mathbb{R}^+} \varphi_\xi(t) \varphi_{\xi+\tau}(t + \tau) dt. \end{aligned}$$

We then take the limit $\tau \rightarrow 0$ to get the formula.

From (13.3.3), it comes that, for any critical point ξ_c of μ in \mathbb{R}^+

$$\mu''(\xi_c) = 2\xi_c \varphi_{\xi_c}^2(0) > 0. \quad (13.3.4)$$

So the critical points are necessarily non degenerate local minima. It is then easy to deduce that there exists a unique $\xi_0 > 0$ such that $\mu(\xi)$ continues to decay monotonically till some value $\Theta_0 < 1$ and is then increasing monotonically and tending to 1 at $+\infty$. Moreover

$$\Theta_0 = \xi_0^2. \quad (13.3.5)$$

Finally, it is easy to see that $\varphi_\xi(x)$ decays exponentially at ∞ .

Let us show additional remarks on the properties of μ and $\varphi_\xi(x)$ which are related to the Feynman-Heilmann formula. We admit again (See Kato [Ka] and the next subsection) that we can “freely” differentiate with respect to ξ .

Let us start³⁹ from :

$$H^N(\xi)\varphi(\cdot; \xi) = \mu(\xi)\varphi(\cdot; \xi). \quad (13.3.6)$$

Differentiating with respect to ξ , we obtain :

$$(\partial_\xi H^N(\xi) - \mu'(\xi))\varphi(\cdot; \xi) + (H^N(\xi)\varphi(\cdot; \xi) - \mu(\xi))(\partial_\xi \varphi)(\cdot; \xi) = 0. \quad (13.3.7)$$

Taking the scalar product with φ_ξ in $L^2(\mathbb{R}^+)$, we obtain the so-called Feynman-Heilmann Formula

$$\mu'(\xi) = \langle (\partial_\xi H^N(\xi)\varphi_\xi \mid \varphi_\xi) = -2 \int_0^{+\infty} (t - \xi) |\varphi_\xi(t)|^2 dt. \quad (13.3.8)$$

³⁹We change a little the notations for $H^{N,\xi}$ (this becomes $H^N(\xi)$) and φ_ξ in order to have an easier way for the differentiation.

Taking the scalar product with $\partial_\xi \varphi(\cdot; \xi)$, we obtain the identity :

$$\langle (\partial_\xi H^N(\xi) - \mu'(\xi))\varphi(\cdot; \xi) \mid \partial_\xi \varphi(\cdot; \xi) \rangle + \langle (H^N(\xi)\varphi(\cdot; \xi) - \mu(\xi))(\partial_\xi \varphi)(\cdot; \xi) \mid (\partial_\xi \varphi)(\cdot; \xi) \rangle = 0. \quad (13.3.9)$$

In particular, we obtain for $\xi = \xi_0$ that :

$$\langle (\partial_\xi H^N(\xi_0)\varphi(\cdot; \xi_0) \mid \partial_\xi \varphi(\cdot; \xi_0)) \rangle + \langle (H^N(\xi_0)\varphi(\cdot; \xi_0) - \mu(\xi_0))(\partial_\xi \varphi)(\cdot; \xi_0) \mid (\partial_\xi \varphi)(\cdot; \xi_0) \rangle = 0. \quad (13.3.10)$$

We observe that the second term is positive (and with some extra work coming back to (13.3.7) strictly positive) :

$$\langle (\partial_\xi H^N(\xi_0)\varphi(\cdot; \xi_0) \mid \partial_\xi \varphi(\cdot; \xi_0)) \rangle < 0. \quad (13.3.11)$$

Let us differentiate one more (13.3.7) with respect to ξ .

$$\begin{aligned} & 2(\partial_\xi H^N(\xi) - \mu'(\xi))\partial_\xi \varphi(\cdot; \xi) \\ & + (H^N(\xi) - \mu(\xi))(\partial_\xi^2 \varphi)(\cdot; \xi) \\ & + (\partial_\xi^2 H^N(\xi) - \mu''(\xi))\varphi(\cdot; \xi) = 0. \end{aligned} \quad (13.3.12)$$

Taking the scalar product with φ_ξ and $\xi = \xi_0$, we obtain

$$\mu''(\xi_0) = 2 + \langle (\partial_\xi H^N(\xi_0)\varphi(\cdot; \xi_0) \mid \partial_\xi \varphi(\cdot; \xi_0)) \rangle < 2. \quad (13.3.13)$$

13.3.3 Regularity

Proposition 13.3.1. *The eigenvalue $\mu(\xi)$ and the corresponding eigenvector ϕ_ξ are of class C^∞ with respect to ξ .*

Proof :

Step 1 :

Let $\mu(\xi_0)$ the lowest eigenvalue of $H^N(\xi_0)$. We recall that it is simple. Let φ_{ξ_0} the normalized eigenvector attached to $\mu(\xi_0)$. Let us denote by $\varphi_{\xi_0}^*$ the orthogonal projection on varphi_{ξ_0} . The domain of the operator can be seen as

$$D(H^N(\xi)) = \{u \in B^2(\mathbb{R}^+) \mid u'(0) = 0\},$$

and we observe that it is independent of ξ . We will use a variant of the so called Grushin's method. Let us introduce the unbounded operator on $L^2(\mathbb{R}^+) \times \mathbb{C}$

$$M_0 := \begin{pmatrix} H^N(\xi_0) - \mu(\xi_0) & \varphi_{\xi_0} \\ \varphi_{\xi_0}^* & 0 \end{pmatrix}$$

with domain $D(H^N(\xi_0))$. Let us show its invertibility. By elementary algebra, we get that the inverse is

$$R_0 = \begin{pmatrix} E_0 & E_0^+ \\ E_0^- & E_0^{+-} \end{pmatrix},$$

with :

$$E_0 = ((H^N(\xi_0) - \mu(\xi_0))|_{\varphi_{\xi_0}^\pm})^{-1} \quad (13.3.14)$$

$$E_0^+ = \varphi_{\xi_0} \quad (13.3.15)$$

$$E_0^- = \varphi_{\xi_0}^* \quad (13.3.16)$$

$$E_0^{+-} = 0. \quad (13.3.17)$$

Step 2 :

Let us now introduce

$$M(\xi, \mu) = \begin{pmatrix} H^N(\xi) - \mu & \varphi_{\xi_0} \\ \varphi_{\xi_0}^* & 0 \end{pmatrix}.$$

Let us show that the invertibility is stable when ξ remains near ξ_0 , and μ remains near $\mu(\xi_0)$. We observe that :

$$\begin{aligned} M(\xi, \mu) &= M_0 + \begin{pmatrix} (H(\xi) - H(\xi_0)) - (\mu - \mu(\xi_0)) & 0 \\ 0 & 0 \end{pmatrix} \\ &= M_0 \left(\text{Id} + R_0 \begin{pmatrix} (H^N(\xi) - H^N(\xi_0)) - (\mu - \mu(\xi_0)) & 0 \\ 0 & 0 \end{pmatrix} \right). \end{aligned}$$

But the map $\xi \mapsto H^N(\xi)$ is continuous from \mathbb{C} into $\mathcal{L}(D(H^N(\xi_0)), L^2)$. So the result is clear and the inverse can be given by the convergent Neumann series :

$$M(\xi, \mu)^{-1} = \sum_{j \in \mathbb{N}} (-1)^j \left(R_0 \begin{pmatrix} (H(\xi) - H(\xi_0)) - (\mu - \mu(\xi_0)) & 0 \\ 0 & 0 \end{pmatrix} \right)^j R_0. \quad (13.3.18)$$

Let us denote by

$$R(\xi, \mu) = \begin{pmatrix} E(\xi, \mu) & E^+(\xi, \mu) \\ E^-(\xi, \mu) & E^{+-}(\xi, \mu) \end{pmatrix}$$

the inverse of $M(\xi, \mu)$. The following result is standard :

Lemma 13.3.2.

The inverse of $M(\xi, \mu)$ is a C^∞ map in a neighborhood of $(\xi_0, \mu(\xi_0))$ with value in $D(H^N(\xi_0)) \times \mathbb{C}$.

Proof of Lemma 13.3.2 :

It is clear that

$$T(\xi, \mu) = \left(R_0 \begin{pmatrix} (H(\xi) - H(\xi_0)) - (\mu - \mu(\xi_0)) & 0 \\ 0 & 0 \end{pmatrix} \right) \quad (13.3.19)$$

is C^∞ . Let us also observe that :

$$M(\xi, \mu)^{-1} = \sum_{j \in \mathbb{N}} (-1)^j T^j(\xi, \mu) R_0. \quad (13.3.20)$$

Consider

$$r(\xi, \mu) = (H(\xi) - H(\xi_0)) - (\mu - \mu(\xi_0)).$$

The derivatives of $r(\xi, \mu)$ are given by :

$$\begin{aligned} \partial_\xi r(\xi, \mu) &= \partial_\xi H(\xi) = -2(t - \xi) \\ \partial_\mu r(\xi, \mu) &= -1. \end{aligned} \quad (13.3.21)$$

In view of (13.3.20), we will estimate $T(\xi, \mu)^j$:

$$\begin{aligned} \left(R_0 \begin{pmatrix} r(\xi, \mu) & 0 \\ 0 & 0 \end{pmatrix} \right)^j &= \left(\begin{pmatrix} E_0 & \varphi_{\xi_0} \\ \varphi_{\xi_0}^* & 0 \end{pmatrix} \begin{pmatrix} r(\xi, \mu) & 0 \\ 0 & 0 \end{pmatrix} \right)^j \\ &= \begin{pmatrix} (E_0 r(\xi, \mu))^j & 0 \\ (\varphi_{\xi_0}^* r(\xi, \mu))(E_0 r(\xi, \mu))^{j-1} & 0 \end{pmatrix}. \end{aligned} \quad (13.3.22)$$

So

$$M(\xi, \mu)^{-1} = \sum_{j \in \mathbb{N}} (-1)^j \begin{pmatrix} (E_0 r(\xi, \mu))^j & 0 \\ (\varphi_{\xi_0}^* r(\xi, \mu))(E_0 r(\xi, \mu))^{j-1} & 0 \end{pmatrix} R_0. \quad (13.3.23)$$

It is then easy to show the C^∞ property.

Lemma 13.3.3.

μ is an eigenvalue of $H^N(\xi)$ if and only if $E^{+-}(\xi, \mu) = 0$.

Moreover, if μ is an eigenvalue, $E^+(\xi, \mu)$ is the associated eigenvector in a neighborhood of $(\xi_0, \mu(\xi_0))$.

Proof of 13.3.3 :

Again this is simple linear algebra. Expressing that $R(\xi, \mu)$ is the inverse of $M(\xi, \mu)$ gives :

$$E(\xi, \mu)(H(\xi) - \mu) + E^+(\xi, \mu)\varphi_{\xi_0}^* = \text{Id} \quad (13.3.24)$$

$$E^-(\xi, \mu)\varphi_{\xi_0} = 1 \quad (13.3.25)$$

$$E(\xi, \mu)\varphi_{\xi_0} = 0 \quad (13.3.26)$$

$$E^-(\xi, \mu)(H(\xi) - \mu) + E^{+-}(\xi, \mu)\varphi_{\xi_0}^* = 0 \quad (13.3.27)$$

$$(H(\xi) - \mu)E(\xi, \mu) + \varphi_{\xi_0} E^-(\xi, \mu) = \text{Id} \quad (13.3.28)$$

$$\varphi_{\xi_0}^* E_{\xi, \mu}^+ = 1 \quad (13.3.29)$$

$$(H(\xi) - \mu)E^+(\xi, \mu) + \varphi_{\xi_0} E^{+-}(\xi, \mu) = 0 \quad (13.3.30)$$

$$\varphi_{\xi_0}^* E(\xi, \mu) = 0. \quad (13.3.31)$$

Taking the composition of $E^+(\xi, \mu)$ with (13.3.27) on the left, we get

$$E^+(\xi, \mu)E^-(\xi, \mu)(H(\xi) - \mu) + E^{+-}(\xi, \mu)E^+(\xi, \mu)\varphi_{\xi_0}^* = 0,$$

so

$$E^+(\xi, \mu)\varphi_{\xi_0}^* = -\frac{E^+(\xi, \mu)E^-(\xi, \mu)(H(\xi) - \mu)}{E^{+-}(\xi, \mu)}. \quad (13.3.32)$$

This quantity (13.3.32) is well defined if $E_{\xi,\mu}^{+-} \neq 0$. So using (13.3.24) et(13.3.32) we get,

$$(H(\xi) - \mu)^{-1} = E(\xi, \mu) - \frac{E^+(\xi, \mu)E^-(\xi, \mu)}{E^{+-}(\xi, \mu)}. \quad (13.3.33)$$

So we have shown that if $E_{\xi,\mu}^{+-} \neq 0$, then $H(\xi) - \mu$ is invertible.

Conversely, let us assume that $E_{\xi,\mu}^{+-} = 0$. Then using (13.3.30), $E^+(\xi, \mu)$ is an eigenvector as soon that $E^+(\xi, \mu)$ is different from 0. But $E_{\xi_0, \mu(\xi_0)}^+ = E_0^+ = \varphi_{\xi_0}$ is non zero, so by continuity it is also true for $E^+(\xi, \mu)$.

Step 3 : Analysis of the equation $E^{+-}(\xi, \mu) = 0$.

We have just to compute the implicit function theorem in the neighborhood of $(\xi_0, \mu(\xi_0))$. But by elementary computations, we have

$$E^{+-}(\xi, \mu) = \sum_{j \geq 1} (-1)^j (\varphi_{\xi_0}^* r_{\xi, \mu}) (E_0 r(\xi, \mu))^{j-1} \varphi_{\xi_0}. \quad (13.3.34)$$

The derivatives at $(\xi_0, \mu(\xi_0))$ are easily computed as :

$$\partial_{\xi} E(\xi, \mu)^{+-}(\xi_0, \mu(\xi_0)) = -\varphi_{\xi_0}^* \partial_{\xi} r(\xi_0, \mu(\xi_0)) \varphi_{\xi_0} \quad (13.3.35)$$

$$= 2 \int_{\mathbb{R}^+} (t - \xi_0) \varphi_{\xi_0}^2(t) dt \quad (13.3.36)$$

$$\partial_{\mu} E_{\xi, \mu}^{+-}(\xi, \mu(\xi_0)) = -1. \quad (13.3.37)$$

In particular $\partial_{\mu} E_{\xi, \mu}^{+-}(\xi_0, \mu(\xi_0)) \neq 0$ and the implicit function theorem leads to

Lemma 13.3.4.

There exists $\eta > 0$ and a C^{∞} map $\tilde{\mu}$ on $]\xi_0 - \eta, \xi_0 + \eta[$ such that

$$\forall \xi \in]\xi_0 - \eta, \xi_0 + \eta[, \forall \mu \in]\mu(\xi_0) - \eta, \mu(\xi_0) + \eta[, E_{\xi, \mu}^{+-} = 0 \iff \mu = \tilde{\mu}(\xi).$$

We then obtain a C^{∞} function $\xi \mapsto \tilde{\mu}(\xi)$ such that $\tilde{\mu}(\xi)$ is an eigenvalue of $H^N(\xi)$ and which is equal to $\mu(\xi)$ at ξ_0 . By uniqueness, we get that for $|\xi - \xi_0|$ small enough $\tilde{\mu}(\xi) = \mu(\xi)$.

13.4 Notes

Old results are due to Kato [Ka] and Avron-Herbst-Simon [AHS], but we have also added more recent results of Dauge-Helffer [DaHe].

14 Exercises and Problems

14.1 Introduction

We present some exercises or problems proposed in the last years. They sometimes strongly intersect with the course.

14.2 Exercises

Exercise 14.2.1. (After Effros, Avron-Seiler-Simon).

Let P and Q two selfadjoint projectors in a Hilbert space \mathcal{H} . i) Let us assume that $A = P - Q$ is compact. Show that if $\lambda \neq \pm 1$ is in the spectrum, then $-\lambda$ is in the spectrum with the same multiplicity. For this, one can first show that with $B = I - P - Q$,

$$A^2 + B^2 = I, \quad AB + BA = 0.$$

ii) Assume now that A is in addition trace class, that is that the series $|\mu_j|$, where μ_j are the non zero eigenvalues of A , counted with multiplicity. Compute $\text{Tr } A := \sum_j \mu_j$ and show that it is an integer.

Exercise 14.2.2. (Temple's inequality).

Let A be a selfadjoint operator on an Hilbert space and $\psi \in D(A)$ such that $\|\psi\| = 1$.

Suppose that λ is the unique eigenvalue of A in some interval $]\alpha, \beta[$ and that $\eta = \langle \psi | A\psi \rangle$ belongs to the interval $]\alpha, \beta[$. Then show that :

$$\eta - \frac{\epsilon^2}{\beta - \eta} \leq \lambda \leq \eta + \frac{\epsilon^2}{\eta - \alpha}.$$

with :

$$\epsilon^2 = \|(A - \eta)\psi\|^2.$$

As a preliminary result, one can show that $(A - \alpha)(A - \lambda)$ and $(A - \beta)(A - \lambda)$ are positive operators. Then apply the inequalities with ψ .

Show that this inequality is an improvement if $\epsilon^2 \leq (\beta - \eta)(\eta - \alpha)$.

Exercise 14.2.3. .

Let $A(x_1, x_2) = (A_1(x_1, x_2), A_2(x_1, x_2))$ be a C^∞ vector field on \mathbb{R}^2 . Let V be a C^∞ positive function on \mathbb{R}^2 .

Let $P := (D_{x_1} - A_1(x_1, x_2))^2 + (D_{x_2} - A_2(x_1, x_2))^2 + V(x)$ the differential operator defined on $C_0^\infty(\mathbb{R}^2)$.

a) Show that P admits a selfadjoint extension in $L^2(\mathbb{R}^2)$.

b) Show that P is essentially selfadjoint.

Exercise 14.2.4. .

We admit the results of Exercise 14.2.3. Show that the selfadjoint extension in $L^2(\mathbb{R}^2)$ of

$$T := -\left(\frac{d}{dx_1} - ix_2x_1^2\right)^2 - \frac{d^2}{dx_2^2} + x_2^2,$$

is with compact resolvent.

Exercise 14.2.5. .

Let V be a C^∞ positive potential in \mathbb{R}^2 . Let us consider, with $B \in \mathbb{R} \setminus \{0\}$, the operator

$$P = D_{x_1}^2 + (D_{x_2} + Bx_1)^2 + V(x).$$

- a) Recall briefly the spectrum of its selfadjoint extension in the case $V = 0$.
- b) We assume that V tends to 0 as $|x| \rightarrow +\infty$. Determine the essential spectrum of \bar{P} .

Exercise 14.2.6. .

Let K be a kernel in $\mathcal{S}(\mathbb{R}^2)$ which is strictly positive and symmetric.

- a) Show that the associated operator \mathcal{K} which is defined on $\mathcal{S}(\mathbb{R})$ by

$$(\mathcal{K}u)(x) = \int_{\mathbb{R}} K(x, y)u(y)dy,$$

can be extended as a compact operator on $L^2(\mathbb{R})$. b) Let I be an open interval in \mathbb{R} and let us denote by \mathcal{K}_I the operator on $L^2(I)$ defined by

$$(\mathcal{K}_I u)(x) = \int_I K(x, y)u(y)dy.$$

Let λ_I^1 be the largest eigenvalue of \mathcal{K}_I . Show that

$$\lambda_I^1 \leq \lambda_{\mathbb{R}}^1.$$

Show that we have strict inequality when I is not \mathbb{R} .

- c) Let u^1 be a normalized eigenfunction of \mathcal{K} associated with $\lambda_{\mathbb{R}}^1$. Using its restriction to I , show the inequality :

$$\lambda_{\mathbb{R}}^1 \leq \lambda_I^1 (1 - \|u^1\|_{L^2(I^c)})^{-1}.$$

- d) Let $I_n = [-n, n]$. Show that $\lambda_{I_n}^1$ converges rapidly to $\lambda_{\mathbb{R}}^1$ as $n \rightarrow +\infty$. More precisely, show that, for all $j \in \mathbb{N}$, there exists a constant C_j such that :

$$|\lambda_{\mathbb{R}}^1 - \lambda_{I_n}^1| \leq C_j n^{-j}, \forall n \in \mathbb{N}^*.$$

Exercise 14.2.7. .

Let us consider in $\Omega =]0, 1[\times \mathbb{R}$, a positive C^∞ function V and let S_0 be the Schrödinger operator $S_0 = -\Delta + V$ defined on $C_0^\infty(\Omega)$.

- (a) Show that S_0 admits a selfadjoint extension on $L^2(\Omega)$. Let S this extension.
- (b) Determine if S is with compact resolvent in the following cases :

1. $V(x) = 0$,
2. $V(x) = x_1^2 + x_2^2$,
3. $V(x) = x_1^2$,
4. $V(x) = x_2^2$

5. $V(x) = (x_1 - x_2)^2$.

Determine the spectrum in the cases (1) and (4). One can first determine the spectrum of the Dirichlet realization (or of Neumann) of $-d^2/dx^2$ on $]0, 1[$.

Exercise 14.2.8. .

We consider in \mathbb{R}^2 the operator defined on $\mathcal{S}(\mathbb{R}^2; \mathbb{C}^2)$ by

$$\mathcal{D}_0 = \alpha_1 D_{x_1} + \alpha_2 D_{x_2} + \alpha_3 .$$

Here the matrices α_j are hermitian 2×2 matrices such that :

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij} ,$$

and we recall that $D_{x_j} = \frac{1}{i} \frac{\partial}{\partial x_j}$ for $j = 1, 2$.

a) Is \mathcal{D}_0 symmetric? semi-bounded ? It is suggested to use the Fourier transform.

b) Compute \mathcal{D}_0^2 .

c) Show that \mathcal{D}_0 admits a selfadjoint extension \mathcal{D}_1 in $L^2(\mathbb{R}^2; \mathbb{C}^2)$, and determine its domain.

d) Determine the spectrum of \mathcal{D}_1 .

e) We suppose that, for all $x \in \mathbb{R}^2$, $V(x)$ is a 2×2 hermitian matrix, with bounded C^∞ coefficients. Show that $\mathcal{D}_V = \mathcal{D}_0 + V$ admits a selfadjoint extension and determine its domain.

Exercise 14.2.9. .

Let H_a be the Dirichlet realization of $-d^2/dx^2 + x^2$ in $] - a, +a[$. Show that the lowest eigenvalue $\lambda_1(a)$ of H_a is strictly positive, monotonically decreasing as $a \rightarrow +\infty$ and tend exponentially fast to 1 as $a \rightarrow +\infty$. Give an estimate as fine as possible of $|\lambda_1(a) - 1|$.

Exercise 14.2.10. .

We consider on $C_0^\infty(\mathbb{R}^2)$, the operator

$$P_0 := (D_{x_1} - x_2)^2 + (D_{x_2} - x_1)^2 .$$

We recall that $D_{x_j} = \frac{1}{i} \partial_{x_j}$.

Show that its natural selfadjoint extension P is unitary equivalent to the operator $-\Delta$ (of domain H^2). Determine its spectrum and its essential spectrum.

Exercise 14.2.11. .

Show that one can associate to the differential operator on $C_0^\infty(\mathbb{R} \times]0, 1[)$:

$$T_0 := (D_{x_1} - x_2 x_1^2)^2 + (D_{x_2})^2 ,$$

an unbounded selfadjoint operator T on $L^2(\mathbb{R} \times]0, 1[)$ whose spectrum is with compact resolvent.

Exercise 14.2.12.

Let ϕ be a C^2 -function on \mathbb{R}^m such that $|\nabla\phi(x)| \rightarrow +\infty$ as $|x| \rightarrow +\infty$ and with uniformly bounded second derivatives. Let us consider the differential operator on $C_0^\infty(\mathbb{R}^m) - \Delta + 2\nabla\phi \cdot \nabla$. We consider this operator as an unbounded operator on $\mathcal{H} = L^2(\mathbb{R}^m, \exp -2\phi dx)$. Show that it admits a selfadjoint extension and that its spectrum is discrete.

We assume in addition that $\int_{\mathbb{R}^m} \exp -2\phi dx < +\infty$. Show that its lowest eigenvalue is simple and determine a corresponding eigenvector.

Exercise 14.2.13.

We consider in \mathbb{R}^3 the differential operator $S_0 := -\Delta - \frac{1}{r}$, a priori defined on $C_0^\infty(\mathbb{R}^3)$.

a) Show that the operator admits a selfadjoint extension S .

b) Show the continuous injection of $H^2(\mathbb{R}^3)$ into the space of the Hölder functions $C^s(\mathbb{R}^3)$, with $s \in]0, \frac{1}{2}[$, and the compact injection for all compact K of $C^s(K)$ into $C^0(K)$.

c) Determine the essential spectrum of S . One possibility is to start with the analysis of $S_\chi = -\Delta - \frac{\chi}{r}$ where χ is C^∞ with compact support.

d) Show using the minimax-principle that S has at least one eigenvalue. One can try to minimize over a $u \mapsto \langle S_0 u, u \rangle / \|u\|^2$ with $u(x) = \exp -ar$.

e) Determine this lowest eigenvalue (using the property that the groundstate should be radial).

Exercise 14.2.14.

a) Let g be a continuous function on \mathbb{R} such that $g(0) = 0$. Analyze the convergence of the sequence $(g(t)u_n(t))_{n \geq 1}$ in $L^2(\mathbb{R})$ where $u_n(t) = \sqrt{n}\chi(nt)$ and χ is a C^∞ function with compact support.

b) let $f \in C^0([0, 1]; \mathbb{R})$. Let T_f be the multiplication operator by f defined on $L^2(]0, 1[)$: $u \mapsto T_f u = fu$.

Determine the spectrum of T_f . Discuss in function of f the possible existence of eigenvalues. Determine the essential spectrum of T_f .

Exercise 14.2.15.

Discuss in function of $\alpha \geq 0$ the possibility of associating to the differential operator define on $C_0^\infty(\mathbb{R}^3)$

$$-\Delta - r^{-\alpha}$$

a selfadjoint operator on $L^2(\mathbb{R}^3)$.

Exercise 14.2.16.

Let Ω be a non empty open subset in \mathbb{R}^d and let us consider the multiplication operator on $L^2(\mathbb{R}^d)$ defined by the multiplication by χ_Ω where χ_Ω is equal to 1 in Ω and 0 outside. Determine the spectrum, the essential spectrum, the discrete spectrum.

Exercise 14.2.17.

Show that the spectrum in \mathbb{R}^2 of $P = D_x^2 + x^2 + D_y^2$ is $[1, +\infty[$.

Exercise 14.2.18.

Let $V \in \mathcal{S}(\mathbb{R}^d)$ be positive and let us consider the operator :

$$T := (-\Delta + 1)^{-\frac{1}{2}} V (-\Delta + 1)^{-\frac{1}{2}} .$$

- Explain how to define $(-\Delta + 1)^{-\frac{1}{2}}$, as an operator on $L^2(\mathbb{R}^d)$.
- Show that T is a bounded, selfadjoint, positive, compact operator on $L^2(\mathbb{R}^d)$.
- Discuss the injectivity in function of V ?
- Establish a link with the research of pairs (u, μ) in $H^2(\mathbb{R}^d) \times \mathbb{R}^+$ such that :

$$(-\Delta + 1 - \mu V)u = 0 .$$

Exercise 14.2.19.

Let $\delta \in \mathbb{R}$.

- Show that the operator P_δ defined on $C_0^\infty(\mathbb{R}^2)$ by

$$P_\delta := D_x^2 + D_y^2 + x^2 y^4 + x^4 y^2 + \delta(x + y)$$

is semibounded.

One can first show the inequality :

$$\langle P_0 u, u \rangle \geq \|xu\|^2 + \|yu\|^2 .$$

- Show that there exists a natural selfadjoint extension of P_δ .
- What is the corresponding form domain ?
- Show that the selfadjoint extension is with compact resolvent.

Exercise 14.2.20.

Let us consider in \mathbb{R}^+ , the Neumann realization in \mathbb{R}^+ of $P_0(\xi) := D_t^2 + (t - \xi)^2$, where ξ is a parameter in \mathbb{R} . We would like to find an upper bound for $\Theta_0 = \inf_\xi \mu(\xi)$ where $\mu(\xi)$ is the smallest eigenvalue of $P_0(\xi)$. Following the physicist Kittel, one can proceed by minimizing $\langle P_0(\xi)\phi(\cdot; \rho) | \phi(\cdot; \rho) \rangle$ over the normalized functions $\phi(t; \rho) := c_\rho \exp -\rho t^2$ ($\rho > 0$). For which value of ξ is this quantity minimal ?? Deduce the inequality :

$$\Theta_0 < \sqrt{1 - \frac{2}{\pi}} .$$

14.3 Problems**Problem 14.3.1.**

Let us consider in the disk of \mathbb{R}^2 $\Omega := D(0, R)$ the Dirichlet realization of the Schrödinger operator

$$S(h) := -\Delta + \frac{1}{h^2} V(x) , \tag{14.3.1}$$

where V is a C^∞ potential on $\bar{\Omega}$ satisfying :

$$V(x) \geq 0 . \tag{14.3.2}$$

Here $h > 0$ is a parameter.

- a) Show that this operator is with compact resolvent.
 b) Let $\lambda_1(h)$ be the lowest eigenvalue of $S(h)$. We would like to analyze the behavior of $\lambda_1(h)$ as $h \rightarrow 0$. Show that $h \rightarrow \lambda_1(h)$ is monotonically increasing.
 c) Let us assume that $V > 0$ on $\bar{\Omega}$; show that there exists $\epsilon > 0$ such that

$$h^2 \lambda_1(h) \geq \epsilon . \quad (14.3.3)$$

- d) We assume now that $V = 0$ in an open set ω in Ω . Show that there exists a constant $C > 0$ such that, for any $h > 0$,

$$\lambda_1(h) \leq C . \quad (14.3.4)$$

One can use the study of the Dirichlet realization of $-\Delta$ in ω .

- e) Let us assume that :

$$V > 0 \text{ almost everywhere in } \Omega . \quad (14.3.5)$$

Show that, under this assumption :

$$\lim_{h \rightarrow 0} \lambda_1(h) = +\infty . \quad (14.3.6)$$

One could proceed by contradiction supposing that there exists C such that

$$\lambda_1(h) \leq C , \forall h \text{ such that } 1 \geq h > 0 . \quad (14.3.7)$$

and establishing the following properties.

- For $h > 0$, let us denote by $x \mapsto u_1(h)(x)$ an L^2 -normalized eigenfunction associated with $\lambda_1(h)$. Show that the family $u_1(h)$ ($0 < h \leq 1$) is bounded in $H^1(\Omega)$.
- Show the existence of a sequence h_n ($n \in \mathbb{N}$) tending to 0 as $n \rightarrow +\infty$ and $u_\infty \in L^2(\Omega)$ such that

$$\lim_{n \rightarrow +\infty} u_1(h_n) = u_\infty$$

in $L^2(\Omega)$.

- Deduce that :

$$\int_{\Omega} V(x) u_\infty(x)^2 dx = 0 .$$

- Deduce that $u_\infty = 0$ and make explicite the contradiction.

- f) Let us assume that $V(0) = 0$; show that there exists a constant C , such that :

$$\lambda_1(h) \leq \frac{C}{h} .$$

- g) Let us assume that $V(x) = \mathcal{O}(|x|^4)$ près de 0. Show that in this case :

$$\lambda_1(h) \leq \frac{C}{h^{\frac{2}{3}}} .$$

h) We assume that $V(x) \sim |x|^2$ near 0; discuss if one can hope a lower bound in the form

$$\lambda_1(h) \geq \frac{1}{Ch}.$$

Justify the answer by illustrating the arguments by examples and counterexamples.

Problem 14.3.2.

We consider on \mathbb{R} and for $\epsilon \in I := [-\frac{1}{4}, +\infty[$ the operator $H_\epsilon = -d^2/dx^2 + x^2 + \epsilon|x|$.

- Determine the form domain of H_ϵ and show that it is independent of ϵ .
- What is the nature of the spectrum of the associated selfadjoint operator?
- Let $\lambda_1(\epsilon)$ the smallest eigenvalue. Give rough estimates permitting to estimate from above or below $\lambda_1(\epsilon)$ independently of ϵ on every compact interval of I .
- Show that, for any compact sub-interval J of I , there exists a constant C_J such that, for all $\epsilon \in J$, any L^2 -normalized eigenfunction u_ϵ of H_ϵ associated with $\lambda_1(\epsilon)$ satisfies :

$$\|u_\epsilon\|_{B^1(\mathbb{R})} \leq C_J.$$

For this, one can play with : $\langle H_\epsilon u_\epsilon, u_\epsilon \rangle_{L^2(\mathbb{R})}$.

- Show that the lowest eigenvalue is a monotonically increasing sequence of $\epsilon \in I$.
- Show that the lowest eigenvalue is a locally Lipschitzian function of $\epsilon \in I$. On utilisera de nouveau le principe du max-min.
- Show that $\lambda(\epsilon) \rightarrow +\infty$, as $\epsilon \rightarrow +\infty$ and estimate the asymptotic behavior.
- Discuss the same questions for the case $H_\epsilon = -d^2/dx^2 + x^2 + \epsilon x^4$ (with $\epsilon \geq 0$).

Problem 14.3.3.

The aim of this problem is to analyze the spectrum $\Sigma^D(P)$ of the Dirichlet realization of the operator $P := (D_{x_1} - \frac{1}{2}x_2)^2 + (D_{x_2} + \frac{1}{2}x_1)^2$ in $\mathbb{R}^+ \times \mathbb{R}$.

- Show that one can a priori compare the infimum of the spectrum of P in \mathbb{R}^2 and the infimum of $\Sigma^D(P)$.
- Compare $\Sigma^D(P)$ with the spectrum $\Sigma^D(Q)$ of the Dirichlet realization of $Q := D_{y_1}^2 + (y_1 - y_2)^2$ in $\mathbb{R}^+ \times \mathbb{R}$.
- We first consider the following family of Dirichlet problems associated with the family of differential operators : $\alpha \mapsto H(\alpha)$ defined on $]0, +\infty[$ by :

$$H(\alpha) = D_t^2 + (t - \alpha)^2.$$

Compare with the Dirichlet realization of the harmonic oscillator in $] -\alpha, +\infty[$.

- Show that the lowest eigenvalue $\lambda(\alpha)$ of $H(\alpha)$ is a monotonic function of $\alpha \in \mathbb{R}$.

5. Show that $\alpha \mapsto \lambda(\alpha)$ is a continuous function on \mathbb{R} .
6. Analyze the limit of $\lambda(\alpha)$ as $\alpha \rightarrow -\infty$.
7. Analyze the limit of $\lambda(\alpha)$ as $\alpha \rightarrow +\infty$.
8. Compute $\lambda(0)$. For this, one can compare the spectrum of $H(0)$ with the spectrum of the harmonic oscillator restricted to the odd functions.
9. Let $t \mapsto u(t; \alpha)$ the positive L^2 -normalized eigenfunction associated with $\lambda(\alpha)$. Let us admit that this is the restriction to \mathbb{R}^+ of a function in $\mathcal{S}(\mathbb{R})$. Let, for $\alpha \in \mathbb{R}$, T_α be the distribution in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R})$ defined by

$$\phi \mapsto T_\alpha(\phi) = \int_0^{+\infty} \phi(y_1, \alpha) u_\alpha(y_1) dy_1 .$$

Compute QT_α .

10. By constructing starting from T_α a suitable sequence of L^2 -functions tending to T_α , show that $\lambda(\alpha) \in \Sigma^D(Q)$.
11. Determine $\Sigma^D(P)$.

Problem 14.3.4.

Let H_a be the Dirichlet realization of $-d^2/dx^2 + x^2$ in $] -a, +a[$.

- (a) Briefly recall the results concerning the case $a = +\infty$.
- (b) Show that the lowest eigenvalue $\lambda_1(a)$ of H_a is decreasing for $a \in]0, +\infty[$ and larger than 1.
- (c) Show that $\lambda_1(a)$ tends exponentially fast to 1 as $a \rightarrow +\infty$. One can use a suitable construction of approximate eigenvectors.
- (d) What is the behavior of $\lambda_1(a)$ as $a \rightarrow 0$. One can use the change of variable $x = ay$ and analyze the limit $\lim_{a \rightarrow 0} a^2 \lambda_1(a)$.
- (e) Let $\mu_1(a)$ be the smallest eigenvalue of the Neumann realization in $] -a, +a[$. Show that $\mu_1(a) \leq \lambda_1(a)$.
- (f) Show that, if u_a is a normalized eigenfunction associated with $\mu_1(a)$, then there exists a constant C such that, for all $a \geq 1$, we have :

$$\|xu_a\|_{L^2(]-a, +a])} \leq C .$$

- (g) Show that, for u in $C^2(]-a, +a])$ and χ in $C_0^2(]-a, +a])$, we have :

$$-\int_{-a}^{+a} \chi^2 u''(t) u(t) dt = \int_{-a}^{+a} |(\chi u)'(t)|^2 dt - \int_{-a}^{+a} \chi'(t)^2 u(t)^2 dt .$$

- (h) Using this identity with $u = u_a$, a suitable χ which should be equal to 1 on $[-a+1, a-1]$, the estimate obtained in (f) and the minimax principle, show that there exists C such that, for $a \geq 1$, we have :

$$\lambda_1(a) \leq \mu_1(a) + Ca^{-2} .$$

Deduce the limit of $\mu_1(a)$ as $a \rightarrow +\infty$.

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