On spectral minimal partitions, a magnetic characterization and other topics

B. Helffer (Université Paris-Sud 11)

Dedicated to Jim for his 70-th birthday.

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Abstract

Given a bounded open set Ω in $\mathbb{R}^n$ (or in a Riemannian manifold) and a partition $\mathcal{D}$ of by k open sets D_j, we can consider the quantity $\Lambda(\mathcal{D}) := max_j \lambda(D_j)$ where $\lambda(D_j)$ is the ground state energy of the Dirichlet realization of the Laplacian in D_j. If we denote by $\mathfrak{L}_k(\Omega)$ the infimum over all the k-partitions of $\Lambda(\mathcal{D})$ a minimal k-partition is then a partition which realizes the infimum. Although the analysis is rather standard when $k = 2$ (we find the nodal domains of a second eigenfunction), the analysis of higher k's becomes non trivial and quite interesting. We also extend the slides of the given talk by reporting on a recent contribution by Steinerberger.

In this talk, we consider the two-dimensional case and discuss the properties of minimal spectral partitions, illustrate the difficulties by considering simple cases like the rectangle and then give a "magnetic" characterization of these minimal partitions. This work has started in collaboration with T. Hoffmann-Ostenhof (with a preliminary work with M. and T. Hoffmann-Ostenhof and M. Owen) and has been continued with him and other coauthors : V. Bonnaillie-Noël, S. Terracini, G. Vial, P. Bérard, or PHD students: C. Lena

Section 1: Introduction to the mathematical problem

We consider mainly two-dimensional Laplacians operators in bounded domains. We would like to analyze the relations between the nodal domains of the eigenfunctions of the Dirichlet Laplacians and the partitions by k open sets D_i which are minimal in the sense that the maximum over the D_i's of the ground state energy of the Dirichlet realization of the Laplacian in D_i is minimal.

Let Ω be a regular bounded domain. Let $H(\Omega)$ be the Laplacian $-\Delta$ on $\Omega \subset \mathbb{R}^2$ with Dirichlet boundary condition ($u_{\partial\Omega} = 0$). In the case of a Riemannian manifold we will consider the Laplace Beltrami operator.

We could also consider other operators like the harmonic oscillator and $\Omega = \mathbb{R}^m$. This problem appears in the Bose-Einstein condensation theory. We will not continue in this direction in this talk.

We denote by $(\lambda_j(\Omega))_j$ the increasing sequence of its eigenvalues counted with multiplicity and by $(u_j)_j$ some associated orthonormal basis of eigenfunctions.

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For any $u \in C_0^0(\overline{\Omega})$, we introduce the nodal set of u by:

$$N(u) = \overline{\{x \in \Omega \mid u(x) = 0\}} \qquad (1)$$

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and call the components of $\Omega \setminus N(u)$ the nodal domains of u.
The $k = \mu(u)$ nodal domains define a partition of Ω.

We keep in mind the Courant nodal theorem and the Pleijel theorem. The main points in the proof of the Pleijel theorem are the Faber-Krahn inequality :

$$\lambda(\omega) \geq \frac{\pi j^2}{|\omega|}\,. \tag{2}$$

and the Weyl law for the counting function.

Partitions

We first introduce the notion of partition.

Definition 1

Let $1 \leq k \in \mathbb{N}$. We call **partition** (or k-partition for indicating the cardinal of the partition) of Ω a family $\mathcal{D} = \{D_i\}_{i=1}^{k}$ of mutually disjoint sets such that

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We call it **open** if the D_i are open sets of Ω, **connected** if the D_i are connected.

We denote by $\mathfrak{O}_k$ the set of open connected partitions.

Spectral minimal partitions

We now introduce the notion of spectral minimal partition sequence.

Definition 2

For any integer $k \geq 1$, and for $\mathcal{D}$ in $\mathfrak{O}_k$, we introduce the "energy" of $\mathcal{D}$:

$$\Lambda(\mathcal{D}) = \max_i \lambda(D_i). \tag{4}$$

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Then we define

$$\mathfrak{L}_k(\Omega) = \inf_{\mathcal{D} \in \mathfrak{O}_k} \Lambda(\mathcal{D}). \tag{5}$$

and call $\mathcal{D} \in \mathfrak{O}_k$ minimal if $\mathfrak{L}_k = \Lambda(\mathcal{D})$.

Remark A

If $k = 2$, it is rather well known (see [HH1] or [CTV3]) that $\mathfrak{L}_2 = \lambda_2$ and that the associated minimal 2-partition is a nodal partition.

We discuss briefly the notion of regular and strong partition.

Definition 3: strong partition

A partition $\mathcal{D} = \{D_i\}_{i=1}^k$ of Ω in $\mathfrak{O}_k$ is called **strong** if

$$\mathrm{Int}\,(\overline{\cup_i D_i}) \setminus \partial\Omega = \Omega \text{ and } \mathrm{Int}\,(\overline{D_i}) \setminus \partial\Omega = D_i \,. \tag{6}$$

Attached to a strong partition, we associate a closed set in $\overline{\Omega}$:

Definition 4: Boundary set

$$N(\mathcal{D}) = \overline{\cup_i \left(\partial D_i \cap \Omega\right)} \,. \tag{7}$$

$N(\mathcal{D})$ plays the role of the nodal set (in the case of a nodal partition).

Regular partitions

We now introduce the set $\mathcal{R}(\Omega)$ of regular partitions (or nodal like) through the properties of its associated boundary set N, which should satisfy :

Definition 5: regular boundary set

(i) Except finitely many distinct $x_i \in \Omega \cap N$ in the nbhd of which N is the union of $\nu_i = \nu(x_i)$ smooth curves ($\nu_i \geq 2$) with one end at x_i, N is locally diffeomorphic to a regular curve.
(ii) $\partial\Omega \cap N$ consists of a (possibly empty) finite set of points z_i. Moreover N is near z_i the union of ρ_i distinct smooth half-curves which hit z_i.
(iii) N has the equal angle meeting property

By equal angle meeting property, we mean that the half curves cross with equal angle at each critical point of N and also at the boundary together with the tangent to the boundary.

Partitions and bipartite property.

We say that D_i, D_j are neighbors or $D_i \sim D_j$, if $D_{i,j} := \mathrm{Int}\,(\overline{D_i \cup D_j}) \setminus \partial\Omega$ is connected.

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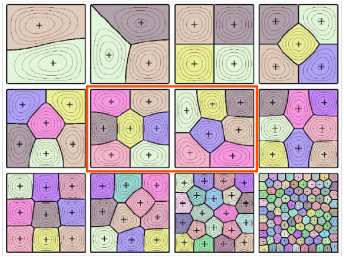
We recall that a collection of nodal domains of an eigenfunction is always bipartite.

Here are examples of regular partitions. These examples are supposed (unproved and not clearly stated) to correspond to minimal partitions of the square.

Multiple populations



“Minimization of the Renyi entropy production in the space-partitioning process”
Cybulski, Babin, and Holyst, Phys. Rev. E 71, 046130 (2005)

Section 2: Main results in the $2D$ case

It has been proved by Conti-Terracini-Verzini [CTV1, CTV2, CTV3] and Helffer–Hoffmann-Ostenhof–Terracini [HHOT1] that

Theorem 1

$\forall k \in \mathbb{N} \setminus \{0\}$, $\exists$ a minimal regular k-partition. Moreover any minimal k-partition has a regular representative.

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Theorem 1

$\forall k \in \mathbb{N} \setminus \{0\}$, $\exists$ a minimal regular k-partition. Moreover any minimal k-partition has a regular representative.

Other proofs of a somewhat weaker version of this statement have been given by Bucur-Buttazzo-Henrot [BBH], Caffarelli- F.H. Lin [CL].

Note that spectral minimal partitions are equi-partitions:

$$\lambda(D_i) = \mathfrak{L}_k(\Omega)\,.$$

Note also that for any pair of neighbours D_i, D_j

$$\lambda_2(D_{ij}) = \mathfrak{L}_k(\Omega)\,.$$

Hence minimal partitions satisfy the pair compatibility condition introduced in [HH1].

A natural question is whether a minimal partition of Ω is a nodal partition, i.e. the family of nodal domains of an eigenfunction of $H(\Omega)$.

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We have first the following converse theorem ([HH1], [HHOT1]):

Theorem 2

If the minimal partition is bipartite this is a nodal partition.

A natural question is now to determine how general this previous situation is.

Surprisingly this only occurs in the so called Courant-sharp situation. We say that:

Definition 6: Courant-sharp

A pair (u, λ_k) is Courant-sharp if
$u \in E(\lambda_k) \setminus \{0\}$ and $\mu(u) = k$.

An eigenvalue is
called Courant-sharp if there exists an associated Courant-sharp pair.

For any integer $k \geq 1$, we denote by $L_k(\Omega)$ the smallest eigenvalue whose eigenspace contains an eigenfunction of $H(\Omega)$ with k nodal domains. We set $L_k = \infty$, if there are no eigenfunctions with k nodal domains.

In general, one can show, that

$$\lambda_k(\Omega) \leq \mathfrak{L}_k(\Omega) \leq L_k(\Omega)\ . \tag{8}$$

The last result gives the full picture of the equality cases :

Theorem 3

Suppose $\Omega \subset \mathbb{R}^2$ is regular.
If $\mathfrak{L}_k = L_k$ or $\mathfrak{L}_k = \lambda_k$ then

$$\lambda_k = \mathfrak{L}_k = L_k\ .$$

In addition, one can find a Courant-sharp pair (u, λ_k).

This answers a question by K. Burdzy, R. Holyst, D. Ingerman, and P. March in [BHIM] (Section 7).

The defect $k - \mu(u_k)$ has a nice interpretation in term of stability : see Berkolaiko, Colin de Verdière and coauthors [Berk] (and references therein including U. Smilansky and coauthors).

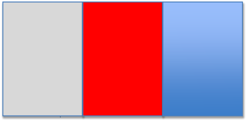
Section 3: Examples of k-minimal partitions for special domains

If in addition the domain has some symmetries and we assume that a minimal partition keeps some of these symmetries, then we find natural candidates for minimal partitions.

The case of a rectangle

Using Theorem 3, it is now easier to analyze the situation for rectangles (at least in the irrational case), since we have just to look for Courant-sharp pairs.

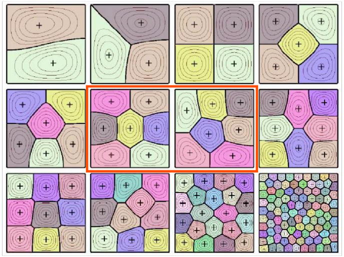
In the long rectangle $]0, a[\times]0, 1[$ the eigenfunction $\sin(k\pi x/a)\sin \pi y$ is Courant-sharp for $a \geq \sqrt{(k^2 - 1)/3}$. See the nodal domain for $k = 3$.



The case of the square

We verify that $\mathfrak{L}_2 = \lambda_2$.
It is not to difficult to see that $\mathfrak{L}_3$ is strictly less than L_3. We observe indeed that there is no eigenfunction corresponding to $\lambda_2 = \lambda_3$ with three nodal domains (by Courant's Theorem).
Finally λ_4 is Courant-sharp, so $\mathfrak{L}_4 = \lambda_4$.

Multiple populations



"Minimization of the Renyi entropy production in the space-partitioning process"

Assuming that there is a minimal partition which is symmetric with one of the symmetry axes of the square perpendicular to two opposite sides, one is reduced to analyze a family of Dirichlet-Neumann problems.

Figure 3

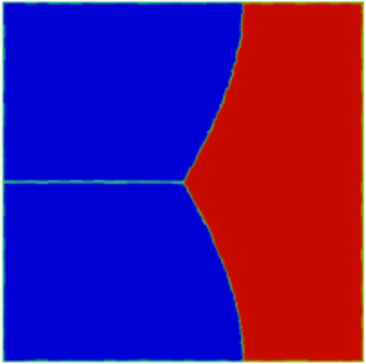


Figure: Trace on the half-square of the candidate for the 3-partition of the square. The complete structure is obtained from the half square by symmetry with respect to the horizontal axis.

See http://www.bretagne.ens-cachan.fr/math/Simulations/MinimalPartitions/

The case of the square: $k = 3$ continued

In the case of the square, we have no proof that the candidate described by Figure 3 is a minimal 3-partition.

The case of the square: $k = 3$ continued

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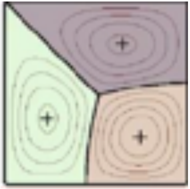
But if we assume that the minimal 3-partition has one critical point and has the symmetry, then numerical computations lead to Figure 3.
Numerics suggest more : the center of the square is the critical point of the partition.

This point of view is explored numerically by Bonnaillie-Helffer [BH] and theoretically by Noris-Terracini [NT].

Why this symmetry ?

The picture of Cybulski-Babin-Holst has another symmetry (with respect to the diagonal) and the same energy.



Actually there is a continuous family of candidates !!

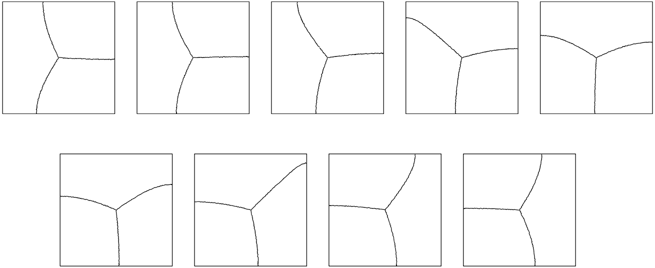


Figure: Continuous family of 3-partitions with the same energy.

This can be explained (Bonnaillie–Helffer–Hoffmann-Ostenhof) by the analysis of some Aharonov-Bohm spectrum !

Section 4: The Aharonov-Bohm Operator

Let us recall some definitions and results about the Aharonov-Bohm Hamiltonian (for short **AB**X-Hamiltonian) with a singularity at X introduced in [BHHO, HHOO] and motivated by the work of Berger-Rubinstein.
We denote by $X = (x_0, y_0)$ the coordinates of the pole and consider the magnetic potential with flux at X

$$\Phi = \pi$$

$$\mathbf{A}^X(x,y) = (A_1^X(x,y), A_2^X(x,y)) = \frac{1}{2}\left(-\frac{y-y_0}{r^2}, \frac{x-x_0}{r^2}\right). \quad (9)$$

We know that the magnetic field vanishes identically in $\dot{\Omega}_X$. The **AB**X-Hamiltonian is defined by considering the Friedrichs extension starting from $C_0^\infty(\dot{\Omega}_X)$ and the associated differential operator is

$$-\Delta_{\mathbf{A}^X} := (D_x - A_1^X)^2 + (D_y - A_2^X)^2 \text{ with } D_x = -i\partial_x \text{ and } D_y = -i\partial_y. \tag{10}$$

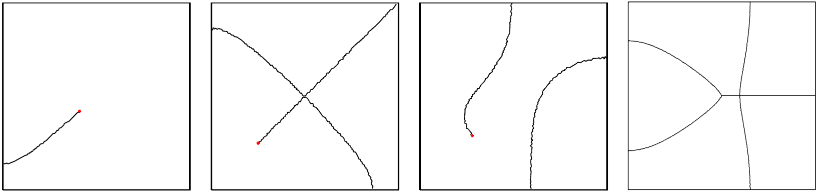
Let K_X be the antilinear operator

$$K_X = e^{i\theta_X}\,\Gamma\,,$$

with $(x-x_0)+i(y-y_0)=\sqrt{|x-x_0|^2+|y-y_0|^2}\,e^{i\theta_X}$, and where Γ is the complex conjugation operator $\Gamma u=\bar{u}$.
A function u is called K_X-real, if $K_X u = u$.
The operator $-\Delta_{\mathbf{A}^X}$ is preserving the K_X-real functions and we can consider a basis of K_X-real eigenfunctions.
Hence we only analyze the restriction of the $\mathbf{AB}X$-Hamiltonian to the K_X-real space $L^2_{K_X}$ where

$$L^2_{K_X}(\dot{\Omega}_X)=\{u\in L^2(\dot{\Omega}_X)\,,\ K_X\,u=u\}\,.$$

It was shown that the nodal set of such a K_X real eigenfunction has the same structure as the nodal set of an eigenfunction of the Laplacian except that an odd number of half-lines meet at X.



For a "real" groundstate (one pole), one can prove [HHOO] that the nodal set consists of one line joining the pole and the boundary.

Extension to many poles

First we can extend our construction of an Aharonov-Bohm Hamiltonian in the case of a configuration with ℓ distinct points $X_1, \dots, X_\ell$ (putting a flux π at each of these points). We can just take as magnetic potential

$$\mathbf{A}^{\mathbf{X}} = \sum_{j=1}^{\ell} \mathbf{A}^{X_j} \,,$$

where $\mathbf{X} = (X_1, \dots, X_\ell)$.

We can also construct (see [HHOO]) the antilinear operator $K_{\mathbf{X}}$, where θ_X is replaced by a multivalued-function $\phi_{\mathbf{X}}$ such that $d\phi_X = 2\mathbf{A}^{\mathbf{X}}$ and $e^{i\phi_{\mathbf{X}}}$ is univalued and C^∞.

We can then consider the real subspace of the $K_{\mathbf{X}}$-real functions in $L^2_{K_{\mathbf{X}}}(\dot{\Omega}_{\mathbf{X}})$. It has been shown in [HHOO] (see in addition [1]) that the $K_{\mathbf{X}}$-real eigenfunctions have a regular nodal set (like the eigenfunctions of the Dirichlet Laplacian) with the exception that at each singular point X_j ($j = 1, \ldots, \ell$) an odd number of half-lines should meet.

We denote by $L_k(\dot{\Omega}_{\mathbf{X}})$ the lowest eigenvalue (if any) such that there exists a $K_{\mathbf{X}}$-real eigenfunction with k nodal domains.

Section 5: A magnetic characterization of a minimal partition

We now discuss the following theorem.

Theorem 4

Let Ω be simply connected. Then

$$\mathfrak{L}_k(\Omega) = \inf_{\ell \in \mathbb{N}} \inf_{X_1, \ldots, X_\ell} L_k(\dot{\Omega}_{\mathbf{X}}) \,.$$

Let us present a few examples illustrating the theorem. When $k = 2$, there is no need to consider punctured Ω's. The infimum is obtained for $\ell = 0$.

When $k = 3$, it is possible to show that it is enough, to minimize over $\ell = 0$, $\ell = 1$ and $\ell = 2$.

In the case of the disk and the square, it is proven that the infimum cannot be for $\ell = 0$ and we conjecture that the infimum is for $\ell = 1$ and attained for the punctured domain at the center.

Let us give a sketch of the proof.
Considering a minimal k-partition $\mathcal{D} = (D_1, \dots, D_k)$, we know that it has a regular representative and we denote by $X^{odd}(\mathcal{D}) := (X_1, \dots, X_\ell)$ the critical points of the partition corresponding to an odd number of meeting half-lines.
Then the guess is that $\mathfrak{L}_k(\Omega) = \lambda_k(\dot{\Omega}_{\mathbf{X}})$ (Courant sharp situation).
One point to observe is that we have proven in [HHOT1] the existence of a family u_i such that u_i is a groundstate of $H(D_i)$ and $u_i - u_j$ is a second eigenfunction of $H(D_{ij})$ when $D_i \sim D_j$.

Then we find a sequence $\epsilon_i(x)$ of $\mathbb{S}^1$-valued functions, where ϵ_i is a suitable[1] square root of $e^{i\phi_{\mathbf{X}}}$ in D_i, such that $\sum_i \epsilon_i(x)u_i(x)$ is an eigenfunction of the **ABX**-Hamiltonian associated with the eigenvalue $\mathfrak{L}_k$.

Conversely, any family of nodal domains of an Aharonov-Bohm operator on $\dot{\Omega}_{\mathbf{X}}$ corresponding to L_k gives a k-partition.

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[1] Note that by construction the D_i's never contain any pole.

Section 6: Asymptotics of the energy for minimal k-partitions for k large.

We recall results of [HHOT]. Faber-Krahn implies :

$$\mathfrak{L}_k(\Omega) \geq k\lambda(\mathrm{Disk}_1)A(\Omega)^{-1}\,.$$

Using the hexagonal tiling, it is easy to see that:

$$\limsup_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k} \leq \lambda(\mathrm{Hexa}_1)A(\Omega)^{-1}\,.$$

The hexagonal conjecture (Van den Berg, Caffarelli-Lin [CL], Bourdin- Bucur-Oudet [BBO], Bonnaillie-Helffer-Vial [BHV]) is

$$\lim_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k} = \lambda(\mathrm{Hexa}_1)A(\Omega)^{-1}\,.$$

New 2013!

There is a recent improvment (asymptotically) of the lower bound by J. Bourgain [Bo] (see also the very recent Steinerberger [St]). One ingredient is a refinement of the Faber-Krahn inequality:

Lemma by Hansen-Nadirashvili

For a nonempty simply connected bounded domain $\Omega \subset \mathbb{R}^2$, we have

$$A(\Omega)\,\lambda(\Omega) \geq \left(1 + \frac{1}{250}(1 - \frac{r_i(\Omega)}{r_0(\Omega)})^2\right)\lambda(\mathrm{Disk}_1)\,,$$

with $r_0(\Omega)$ the radius of the disk of same area as Ω and $r_i(\Omega)$ the inradius of Ω.

Actually as corrected in a new version (20 August of 2013), one needs a modified version for treating non simply connected domains.

The other very tricky idea is to use quantitatively that all the open

The inequality obtained by Bourgain is the following (see (26) in his note) as $k \to +\infty$, is that for any $\delta \in (0, \delta_0)$

$$\frac{\mathfrak{L}_k(\Omega)}{k} \geq (1 + o(1))\lambda(\mathrm{Disk}_1)A(\Omega)^{-1} \times b(\delta) \qquad (11)$$

where

$$b(\delta) := (1 + 250\delta^{-3})\left(\frac{\pi}{\sqrt{12}}(1-\delta)^{-2} + 250\delta^{-3}\right)^{-1}.$$

and $\delta_0 \in (0,1)$ is computed with the help of the packing condition. This condition reads

$$\frac{\delta_0^3}{250} = \left(\frac{1-\delta_0}{p}\right)^2 - 1\,,$$

where p is a packing constant determined by Blind ($p \sim 0.743$).

But for $\delta > 0$ small enough, we get $b(\delta) > 1$ (as a consequence of $\frac{\pi}{\sqrt{12}} < 1$), hence Bourgain has improved what was obtained via Faber-Krahn.

As also observed by Steinerberger, one gets

$$\frac{\lambda(\mathrm{Hexa}_1)}{\lambda(\mathrm{Disk}_1)} \geq \sup_{\delta \in (0,\delta_0)} b(\delta) > 1\,,$$

which gives a limit for any improvement of the estimate.

In any case, we have

$$\liminf_{k \to +\infty} \frac{\mathfrak{L}_k(\Omega)}{k} \geq \lambda(\mathrm{Disk}_1) A(\Omega)^{-1} \times \sup_{\delta \in (0,\delta_0)} b(\delta) \qquad (12)$$

Coming back to Pleijel, we can rewrite the proof by saying that this is simply the inequality

$$4\pi \geq A(\Omega) \liminf_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k} \times \limsup_{n\to+\infty} \frac{N(\phi_n)}{n}\,, \tag{13}$$

together with (12).
Actually, we have

$$4\pi \geq A(\Omega) \liminf \frac{L_k(\Omega)}{k} \times \limsup \frac{N(\phi_n)}{n}\,.$$

which implies the previous one by (8).

The uncertainty principle by S. Steinerberger

To explain this principle, we associate to a partition Ω_i of Ω

$$D(\Omega_i) = 1 - \frac{\min_j A(\Omega_j)}{A(\Omega_i)}\,,$$

and

$$\mathcal{A}(\Omega) = \inf_B \frac{A(\Omega \triangle B)}{A(\Omega)}\,,$$

where the infimum is over the balls of same area. Steinerberger's principle reads as follows

Theorem

There exists a universal constant $c > 0$, and $N_0(\Omega)$ such that, if the cardinal N of the partition $\geq N_0$, then

$$\sum_i (D(\Omega_i) + \mathcal{A}(\Omega_i)) \frac{A(\Omega_i)}{A(\Omega)} \geq c\,. \tag{14}$$

Application to equipartitions of energy λ

Let us show how we recover a lower bound for $\liminf (\mathfrak{L}_k(\Omega)/k)$.
We consider a k-equipartition of energy λ. The uncertainty principle says that its is enough to consider two cases.
We first assume that

$$\sum_i D(\Omega_i) \frac{A(\Omega_i)}{A(\Omega)} \geq \frac{c}{2} .$$

We can rewrite this inequality in the form:

$$k \inf_j A(\Omega_j) \leq (1 - \frac{c}{2}) A(\Omega) .$$

After implementation of Faber-Krahn, we obtain

$$\frac{k}{\lambda} \lambda(Disk_1) \leq (1 - \frac{c}{2}) A(\Omega) . \tag{15}$$

We now assume that

$$\sum_i \mathcal{A}(\Omega_i)\frac{A(\Omega_i)}{A(\Omega)} \geq \frac{c}{2}\,.$$

This assumption implies

$$A\left(\cup_{\{\mathcal{A}(\Omega_i)\geq \frac{c}{6}\}}\Omega_i\right) \geq \frac{c}{6}A(\Omega)\,. \tag{16}$$

The role of $\mathcal{A}$ can be understood in the following inequality due to [**?**]:
$\exists C > 0$ such that $\forall \omega$

$$A(\omega)\lambda(\omega) - \lambda(Disk_1) \geq C\mathcal{A}(\omega)^2 A(\omega)\,. \tag{17}$$

We apply this inequality with $\omega = \Omega_i$.
This reads

$$A(\Omega_i)\lambda - \lambda(Disk_1) \geq C\mathcal{A}(\Omega_i)^2 A(\Omega_i)\,.$$

Hence we get for any i such that $\mathcal{A}(\Omega_i) \geq \frac{c}{6}$, to

$$\lambda(Disk_1)(1 + \frac{Cc^2}{36}) \leq A(\Omega_i)\lambda\,. \tag{18}$$

which is an improvement of Faber-Krahn for these Ω_i.
Summing over i and using the information (16) leads to

$$\frac{k}{\lambda}\lambda(Disk_1) \leq (1 + \frac{Cc^2}{36})^{-1} A(\Omega) \left(1 + (1 - \frac{c}{6})\frac{Cc^2}{36}\right)$$

and finally to

$$\frac{k}{\lambda}\lambda(Disk_1) \leq \left(1 - \frac{Cc^3}{216 + 6Cc^2}\right) A(\Omega) \tag{19}$$

Putting (15) and (19) together, we obtain that for k large enough the k-partition satisfies

$$\frac{k}{\lambda}\lambda(Disk_1) \leq \max\left((1-\frac{c}{2}),(1-\frac{Cc^3}{216+6Cc^2})\right)A(\Omega)\,. \qquad (20)$$

If we apply this to minimal partitions, this reads

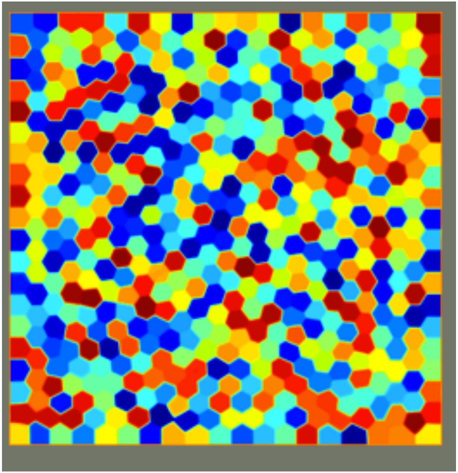
$$\lambda(Disk_1) \leq \max\left((1-\frac{c}{2}),(1-\frac{Cc^3}{216+6Cc^2})\right)A(\Omega)\liminf\frac{\mathfrak{L}_k(\Omega)}{k}\,. \qquad (21)$$

One recovers Bourgain's improvement (12) with a different constant.

Other remarks

- There are various controls of the validity of the conjecture using numerics directly or indirectly on theoretical consequences of this conjecture [BHV]. For example, taking as Ω a connected union of k hexagons of area 1 and putting poles at all the vertices of these hexagones in Ω, the Aharonov-Bohm operator should have $\lambda(\mathrm{Hexa1})$ as k-th eigenvalue.
- There is a corresponding (proved by Hales [Ha]) conjecture for k- partitions of equal area and minimal length called the honeycomb conjecture.
- There is a stronger conjecture (see [CL]) corresponding to the averaged sum (in the definition of $\mathfrak{L}_k(\Omega)$) instead of the max. The lower bound by Faber-Krahn is OK. It is unknown if it can be improved like for $\mathfrak{L}_k(\Omega)$.

Hexagonal conjecture.



This was computed for the torus by Bourdin-Bucur-Oudet [BBO] (for the sum).

Section 7: Asymptotics of the length for minimal k-partitions for k large (after Bérard-Helffer).

This work was inspired by papers of Brüning-Gromes [BrGr], Brüning [Br], Dong [Dong], Savo [Sa1] ...

Of course the hexagonal conjecture leads to a natural conjecture for the length of a minimal partition. If we define the length as:

$$P(\mathcal{D}) := \frac{1}{2}\sum_{i=1}^{k} \ell(\partial D_i)\,,$$

the "hexagonal conjecture" for the length will be

$$\lim_{k\to+\infty} (P(\mathcal{D}_k)/\sqrt{k}) = \frac{1}{2}\ell(\mathrm{Hexa}_1)\sqrt{A(\Omega)}\,, \tag{22}$$

where $\ell(\mathrm{Hexa}_1)$ is the length of the boundary of the hexagon of area 1:

$$\ell(\mathrm{Hexa}_1) = 2\sqrt{2\sqrt{3}}\,.$$

Implementing results of Hales [Ha] obtained in his proof of the honeycomb conjecture, we get the following asymptotic inequality for the length of a minimal k-partition:

$$\liminf_{k\to+\infty} \frac{P(\mathcal{D}_k)}{\sqrt{k}} \geq (12)^{\frac{1}{4}} \left((\pi \mathbf{j}^2)/\lambda(\mathrm{Hexa}_1)\right)^{\frac{1}{2}} A(\Omega)^{\frac{1}{2}} . \qquad (23)$$

Observing that $\left((\pi \mathbf{j}^2)/\lambda(\mathrm{Hexa}_1)\right)^{\frac{1}{2}} \sim 0,989$, we see that the right-hand side of (23) is very close to what would be the hexagonal conjecture for the length.

A universal lower bound for the length of equipartitions

For a domain Ω such that $\chi(\Omega) \geq 0$, one can actually obtain a universal estimate for the length of a regular spectral k-equipartition $\mathcal{D}_k$ which is independent of the energy.

$$P(\mathcal{D}_k) + \frac{1}{2}\ell(\partial\Omega) \geq k^{\frac{1}{2}}\, 12^{\frac{1}{8}}\, (\frac{\pi}{4})^{\frac{1}{4}}\, A(\Omega)^{\frac{1}{2}}\,. \tag{24}$$

Asymptotically this inequality is weaker than (23) but is universal and independent of the asymptotics of the energy. It is interesting to compare this lower bound for the spectral k-equipartitions with the universal lower bound for the equal area k-partitions

$$P(\mathcal{D}_k) + \frac{1}{2}\ell(\partial\Omega) \geq k^{\frac{1}{2}}\, 12^{\frac{1}{8}}\ A(\Omega)^{\frac{1}{2}}\,, \tag{25}$$

Hales results

The following statement is a particular case of Theorem 1-B established by T.C. Hales [Ha] in his proof of Lord Kelvin's honeycomb conjecture.

Hales theorem

Let Ω be a relatively compact open set in $\mathbb{R}^2$, and let $\mathcal{D} = \{D_i\}$ be a regular finite partition of Ω. Then,

$$P(\mathcal{D}) + \frac{1}{2}\ell(\partial\Omega) \geq (12)^{\frac{1}{4}} \sum_{i=1}^{\sharp(\mathcal{D})} \min\left(1, A(D_i)\right) . \tag{26}$$

Corollary

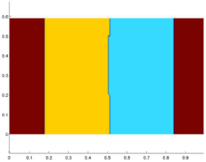
$$P(\mathcal{D}) + \frac{1}{2}\ell(\partial\Omega) \geq (12)^{\frac{1}{4}} \left(\min_i A(D_i)\right)^{\frac{1}{2}} \sharp(\mathcal{D}) . \tag{27}$$

Section 8: Minimal partitions for the torus

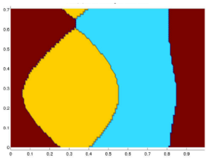
For the strongly anisotropic torus Helffer–Hoffmann-Ostenhof have shown that minimal k-partitions are equal cylinders whose section are small circles.

C. Lena has explored numerically the situation when varying the anisotropy of the torus. In particular, for the square torus he can exhibit in the case $k = 3$ and 5 surprising candidates.

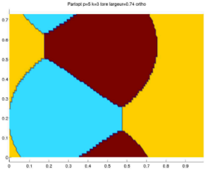
3-partitions



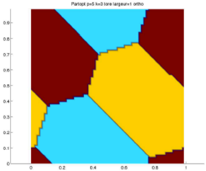
b=0.6



b=0.715



b=0.74

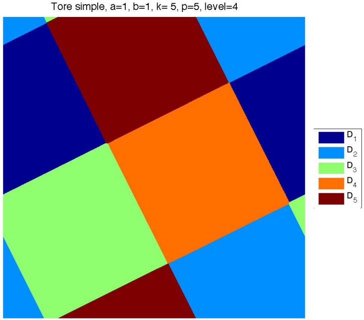


b=1

A conjecture is that, when reducing the anisotropy, the number of critical points increase from 0 to the maximal (even) number 6. At the moment, the numerical computations are not accurate enough to give evidence of this conjecture.

It is not excluded that there is a direct transition between the situation without critical points and the situation with 6 critical points. This second point of view is supported by the analysis of the limiting case $b = 1/\sqrt{2}$.

5-partitions



The candidate consists of five equal squares. This is the projection of the nodal partition of an explicit eigenfunction defined on the square with anti-periodic conditions. This eigenfunction is Courant-sharp in this restricted sense.

Other remarks

- The result by C. Lena [Le1] :
  In the case of ℓ poles, the map $\mathbb{R}^{2\ell} \ni \mathbf{X} \mapsto \lambda_k(\dot{\Omega}_{\mathbf{X}})$ is continuous.
  This result contains the continuity at the coelescing of poles or the continuity when a pole touches the boundary (see also [NNT].
- For one pole, where at least three lines of the nodal set meet, then the corresponding eigenvalue, if simple, is critical as function of the pole. See Noris - Terracini [NT] and Noris-Nys-Terracini [NNT]) or Lena.
- In the case of one pole, the continuity at the boundary implies the existence of a local extremum inside Ω.
- One can expect a magnetic characterization for minimal partitions of a surface. This should be based on the construction of a canonical Aharonov-Bohm operator on the punctured surface.

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B. Helffer: Département de Mathématiques, Bat. 425, Université Paris-Sud 11, 91 405 Orsay Cedex, France.

email: Bernard.Helffer@math.u-psud.fr