On the third critical field in superconductivity

Bernard Helffer
Mathématiques -
Université Paris Sud- UMR CNRS 8628
supported by the PROGRAMME HPRN-CT-2002-00277
and the ESF programme SPECT.
(After S. Fournais and B. Helffer)

Luminy 2005

Main goals

Using recent results by the authors on the spectral asymptotics of the Neumann Laplacian with magnetic field, we give precise estimates on the critical field, H_{C_3}, describing the appearance of superconductivity in superconductors of type II. Furthermore, we prove that the local and global definitions of this field coincide. Near H_{C_3} only a small part, near the boundary points where the curvature is maximal, of the sample carries superconductivity. We give precise estimates on the size of this zone and decay estimates in both the normal (to the boundary) and parallel variables.

Setup and results for general domains

Our main motivation comes from superconductivity.

As appeared from the works of Bernoff-Sternberg [BeSt], Lu-Pan [LuPa1, LuPa2, LuPa3, LuPa4], and Helffer-Pan [HePa], the determination of the lowest eigenvalues of the magnetic Schrödinger operator is crucial

- for a detailed description of the nucleation of superconductivity (on the boundary) for superconductors of Type II

- for accurate estimates of the critical field H_{C_3}.

The determination of the complete asymptotics of the lowest eigenvalues of the Schrödinger operators was essentially achieved (except for exponentially small effects) in the two-dimensional case with the works of [HeMo2] and [FoHe2]. See also Bonnaillie, Bonnaillie-Dauge for the case with corner.

What remained to be determined was the corresponding asymptotics for the critical field.

We will actually obtain much more and clarify the links between the various definitions of critical fields considered in the mathematical or physical literature and supposed to define the right critical field.

Ginzburg-Landau functional

The Ginzburg-Landau functional is given by

$$\mathcal{E}_{\kappa,H}[\psi,\vec{A}] = \int_\Omega \Big\{ |p_{\kappa H\vec{A}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 + \kappa^2 H^2 |\operatorname{curl}\vec{A} - 1|^2 \Big\}\, dx\,,$$

with $(\psi,\vec{A}) \in W^{1,2}(\Omega;\mathbb{C}) \times W^{1,2}(\Omega;\mathbb{R}^2)$ and where $p_{\vec{A}} = (-i\nabla - \vec{A})$.

We fix the choice of gauge by imposing that

$$\operatorname{Div}\vec{A} = 0 \quad \text{in } \Omega\,,\ \vec{A}\cdot\nu = 0 \quad \text{on } \partial\Omega\,.$$

Minimizers $(\psi, \vec{A})$ of the functional satisfy the Ginzburg-Landau equations,

$$\left.\begin{aligned} p^2_{\kappa H \vec{A}}\psi &= \kappa^2(1-|\psi|^2)\psi \\ \text{curl}\,^2\vec{A} &= -\frac{i}{2\kappa H}(\overline{\psi}\nabla\psi - \psi\nabla\overline{\psi}) - |\psi|^2\vec{A} \end{aligned}\right\} \quad \text{in} \quad \Omega\,; \tag{1a}$$

$$\left.\begin{aligned} (p_{\kappa H \vec{A}}\psi)\cdot\nu &= 0 \\ \text{curl}\,\vec{A} - 1 &= 0 \end{aligned}\right\} \quad \text{on} \quad \partial\Omega\,. \tag{1b}$$

Here $\text{curl}\,(A_1, A_2) = \partial_{x_1}A_2 - \partial_{x_2}A_1$,

$$\text{curl}\,^2\vec{A} = (\partial_{x_2}(\,\text{curl}\,\vec{A}), -\partial_{x_1}(\,\text{curl}\,\vec{A}))\,.$$

Let $\vec{F}$ denote the vector potential generating the constant exterior magnetic field

$$\left.\begin{aligned} \text{Div}\,\vec{F} &= 0 \\ \text{curl}\,\vec{F} &= 1 \end{aligned}\right\} \quad \text{in}\,\Omega\,,\ \vec{F}\cdot\nu = 0 \quad \text{on}\,\partial\Omega\,.$$

The pair $(0, \vec{F})$ is called the Normal State.

A minimizer (ψ, A) for which ψ never vanishes will be called SuperConducting State $=$ SCS.

In the other cases, one will speak about Mixed State=MS.

The general question is to determine the topology of the sets of (κ, H) corresponding to each of these three situations. One will also have to distinguish in the third case, between "surface" states =MSS and "interior" states MIS, the "surface" states living near the boundary.

The hope is to describe these a priori complicate sets by defining suitable critical fields $H_{c_j}(\kappa)$ $(j = 1, 2, 3)$ describing for a given κ the transition from SCS to MIS, then from MIS to MSS, and then from MSS to NS.

This appears to correspond grossomodo to the situation when κ is large. When κ is small, one is waiting for a direct transition from SCS to NS.

Existence of the upper critical field $H_{C3}(\kappa)$

It is known that, for given values of the parameters κ, H, the functional $\mathcal{E}$ has minimizers.

However, after some analysis of the functional, one finds (see [GiPh]) that given κ there exists $H(\kappa)$ such that if $H > H(\kappa)$ then $(0, \vec{F})$ is the only minimizer of $\mathcal{E}_{\kappa,H}$ (up to change of gauge).

Following Lu and Pan [LuPa1], we define

$$H_{C_3}(\kappa) = \inf\{H > 0 \ : \ (0, \vec{F}) \text{ minimizer of } \mathcal{E}_{\kappa,H}\} \,.$$

In the physical interpretation of a minimizer $(\psi, \vec{A})$, $|\psi(x)|^2$ is a density[1] measuring the behavior of the material near the point x. Therefore, $H_{C_3}(\kappa)$ is the value of the external magnetic field, H, at which the material loses its superconductivity completely.

[1] Note that ψ is NOT L^2-normalized

A central question in the mathematical treatment of Type II superconductors is to establish the asymptotic behavior of $H_{C_3}(\kappa)$ for large κ.

We will also discuss the relevance of this definition and describe how $H_{C_3}(\kappa)$ can be determined by the study of a linear problem.

Although not proved, this suggests that the transition from Normal State to Mixed Surface State when the external field is decreasing will occur by bifurcation.

Our first result is the following strengthening of a result in [HePa].

Theorem A

Suppose Ω is a bounded simply-connected domain in $\mathbb{R}^2$ with smooth boundary. Let $k_{\max}$ be the maximal curvature of $\partial\Omega$. Then

$$H_{C_3}(\kappa) = \frac{\kappa}{\Theta_0} + \frac{C_1}{\Theta_0^{\frac{3}{2}}} k_{\max} + \mathcal{O}(\kappa^{-\frac{1}{2}}) \;, \qquad (2)$$

where C_1, Θ_0 are universal constants.•

When Ω is a disc we get the improved estimate

$$H_{C_3}(\kappa) = \frac{\kappa}{\Theta_0} + \frac{C_1}{\Theta_0^{\frac{3}{2}}} k_{\max} + \mathcal{O}(\kappa^{-1}) \;. \qquad (3)$$

Remark

The constants Θ_0, C_1 are defined in terms of auxiliary spectral problems.

Remark

The improvement in (2) compared to He-Pan (which uses He-Morame) is in the estimate on the remainder ($\mathcal{O}(\kappa^{-\frac{1}{2}})$ instead of $\mathcal{O}(\kappa^{-\frac{1}{3}})$). The new result is optimal in the sense that the next term depends on detailed geometric properties of the boundary.

In order to expand H_{C_3} to higher orders we will impose some geometric condition on Ω (see later).

Our second result is a precise estimate on the size of the superconducting region in the case where H is close to, but below, H_{C_3}.

Localization at the boundary

From the work of Helffer-Morame [HeMo2] (improving Del Pino-Fellmer-Sternberg and Lu-Pan) (see also Helffer-Pan [HePa] for the non-linear case) we know that, when H is sufficiently closed to $H_{C3}(\kappa)$, minimizers of the Ginzburg-Landau functional are exponentially localized to a region near the boundary. This is called Surface Superconductivity.

Note that this localization leads to the proof of :

$$||\psi||_{L^2(\Omega)} \leq C\kappa^{-\frac{1}{4}}||\psi||_{L^4(\Omega)} , \qquad (4)$$

which is true for κ large enough.

Localization at the points of maximal curvature

The statement is that minimizers are also localized in the tangential variable to a small zone around the points of maximum curvature.

In order to give a precise statement let us first rapidly recall some Notations concerning the boundary .
Let $\gamma : \mathbb{R}/|\partial\Omega| \to \mathbb{R}^2$ a parametrization of $\partial\Omega$ with $|\gamma'(s)| = 1$. For $s \in \mathbb{R}/|\partial\Omega|$ $k(s)$ is the curvature of $\partial\Omega$ at the point $\gamma(s)$. Furthermore,

$$k_{\max} := \max_{s\in\mathbb{R}/|\partial\Omega|} k(s) \ , \ K(s) := k_{\max} - k(s) \ . \quad (5)$$

Furthermore, $t = t(x)$ measures the distance to the bndary

$$t(x) := \text{ dist } (x, \partial\Omega) \ .$$

Let $\nu(s)$ the interior normal vector to $\partial\Omega$ at $\gamma(s)$ and $\Phi : \mathbb{R}/|\partial\Omega| \times (0, t_0) \to \Omega$ by

$$\Phi(s, t) = \gamma(s) + t\nu(s) \ .$$

Then, for t_0 sufficiently small, Φ is a diffeo. with image $\{x \in \Omega \big| \text{ dist } (x, \partial\Omega) < t_0\}$, and $t(\Phi(s,t)) = t$. Thus, in a neighborhood of the boundary, the function $s = s(x)$ is defined by $(s(x), t(x)) = \Phi^{-1}(x)$.

Our estimate is an improvement of a similar estimate in [HePa] (see also [He-Mo]).

Theorem B : Tangential Agmon estimates (non-linear case)

Let Ω be a bounded simply-connected domain in $\mathbb{R}^2$ with smooth boundary. Let $(\psi, \vec{A}) = (\psi_{\kappa,H}, \vec{A}_{\kappa,H})$ be a family of minimizers of the Ginzburg-Landau functional depending on κ, H. We suppose that $H = H(\kappa)$ in such a way that $\rho := H_{C_3}(\kappa) - H$ satisfies $0 < \rho = o(1)$ as $\kappa \to \infty$. Then $\exists \alpha, C > 0$ such that if $\kappa > C$, then

$$\int_\Omega \chi_1^2(\kappa^{\frac{1}{4}} t) e^{2\alpha\sqrt{\kappa} K(s)} |\psi(x)|^2 \, dx \leq C e^{C\rho\sqrt{\kappa}} \int_\Omega |\psi(x)|^2 \, dx \,. \tag{6}$$

Here $K(s)$ is the function defined in (5):

$$K(s) := k_{\max} - k(s) \,.$$

Discussion of critical fields

Actually, we should define more than one critical field, instead of just H_{C_3}. We define an upper and a lower critical field, $\underline{H_{C_3}(\kappa)} \leq \overline{H_{C_3}(\kappa)}$, by

$$\begin{aligned}&\overline{H_{C_3}(\kappa)}\\&\quad = \inf\{H > 0 \ : \ \forall H' > H\,, (0, \vec{F})\\&\qquad\qquad \text{unique minimizer of } \mathcal{E}_{\kappa,H'}\}\,,\end{aligned}$$

$$\underline{H_{C_3}(\kappa)} = H_{C_3}(\kappa)\,. \tag{7}$$

The proof of Theorem A gives a lower bound to $\underline{H_{C_3}(\kappa)}$ and an upper bound to $\overline{H_{C_3}(\kappa)}$, so the expansion is valid for both fields.

The physical idea of a sharp value for the external magnetic field strength at which superconductivity disappears, requires the different definitions of the critical field to coincide. A more precise result will establish this identification under a (generically satisfied) geometric assumption on $\partial\Omega$.

Most works analyzing H_{C_3} relate (more or less implicitly) these global critical fields to local ones given purely in terms of spectral data of a magnetic Schrödinger operator, i.e. in terms of a linear problem.

Let, for $B \in \mathbb{R}_+$, the magnetic Neumann Laplacian $\mathcal{H}(B)$ be the self-adjoint operator (with Neumann boundary conditions) associated to the quadratic form

$$W^{1,2}(\Omega) \ni u \mapsto \int_\Omega |(-i\nabla - B\vec{F})u|^2 \, dx \ ,$$

We define $\lambda_1(B)$ as the lowest eigenvalue of $\mathcal{H}(B)$.

The local upper critical fields can now be defined :

$$\overline{H_{C_3}^{\mathrm{loc}}(\kappa)} = \inf\{H > 0 \ : \ \forall H' > H, \lambda_1(\kappa H') \geq \kappa^2\} \ ,$$
$$\underline{H_{C_3}^{\mathrm{loc}}(\kappa)} = \inf\{H > 0 \ : \ \lambda_1(\kappa H) \geq \kappa^2\} \ . \tag{8}$$

The difference between $\overline{H_{C_3}^{\mathrm{loc}}(\kappa)}$ and $\underline{H_{C_3}^{\mathrm{loc}}(\kappa)}$—and also between $\overline{H_{C_3}(\kappa)}$ and $\underline{H_{C_3}(\kappa)}$—can be retraced to the general non-existence of an inverse to the function $B \mapsto \lambda_1(B)$, i.e. to lack of strict monotonicity of λ_1.

The case of the disk

The detailed spectral analysis in Bauman-Phillips-Tang [BaPhTa] in the case where Ω is a disc does not exclude that, in this case, $\overline{H^{\rm loc}_{C_3}(\kappa)}$ and $\underline{H^{\rm loc}_{C_3}(\kappa)}$ differ even for large values of κ. They prove the estimate,

$$\left|\overline{H^{\rm loc}_{C_3}(\kappa)} - \underline{H^{\rm loc}_{C_3}(\kappa)}\right| \leq \frac{C}{\kappa}\,, \qquad \text{in the case of the disc.}$$

However, a more precise analysis (Fournais-He) in this special case shows that actually (for the disc) $\overline{H^{\rm loc}_{C_3}(\kappa)} = \underline{H^{\rm loc}_{C_3}(\kappa)}$ for sufficiently large values of κ.

Comparison Theorem C

Let Ω be a bounded simply-connected domain in $\mathbb{R}^2$ with smooth boundary and let $\kappa > 0$, then the following general relations hold

$$\overline{H_{C_3}(\kappa)} \geq \overline{H^{\rm loc}_{C_3}(\kappa)}\ , \tag{9}$$

$$\underline{H_{C_3}(\kappa)} \geq \underline{H^{\rm loc}_{C_3}(\kappa)}\ . \tag{10}$$

OPEN QUESTION

For general domains we do not know that the local fields $\underline{H^{\rm loc}_{C_3}(\kappa)}$ and $\overline{H^{\rm loc}_{C_3}(\kappa)}$ coincide.

The next theorem improves Theorem C and is typical of type II materials.

Identification Theorem D

Let Ω be a bounded simply-connected domain in $\mathbb{R}^2$ with smooth boundary. Then $\exists$ $\kappa_0 > 0$ such that, for $\kappa > \kappa_0$, we have

$$\overline{H_{C_3}(\kappa)} = \overline{H_{C_3}^{\mathrm{loc}}(\kappa)}\ .$$

Results for non-degenerate domains

In order to obtain more precise results, we need to impose geometric conditions on Ω.

Generic Assumption

The domain $\Omega \subset \mathbb{R}^2$ is bounded and simply-connected and has smooth boundary. Furthermore, $\exists$ a finite number of points $\{s_1, \ldots, s_N\} \in \mathbb{R}/|\partial\Omega|$ of maximal curvature and these maxima are non-degenerate.

Stronger Assumption

The domain $\Omega \subset \mathbb{R}^2$ is bounded and simply-connected and has smooth boundary. Furthermore, $\exists$ a unique point $s_0 \in \mathbb{R}/|\partial\Omega|$ of maximal curvature and this maximum is non-degenerate, in the sense that $k_2 := -k''(s_0) \neq 0$.

In Fournais-He [FoHe2] the asymptotics of $\lambda_1(B)$, for large B, was calculated.

We can also prove that (under above Generic Assumption) $\lambda_1 : [B_0, \infty) \to [\lambda_1(B_0), \infty)$ is bijective for B_0 sufficiently large.

THIS IS NOT A TRIVIAL RESULT OBTAINED BY A SIMPLE MONOTONICITY ARGUMENT.

Proposition E

Suppose Ω satisfies Generic Assumption. Then $\exists \kappa_0$ such that, if $\kappa \geq \kappa_0$, then the equation for H:

$$\lambda_1(\kappa H) = \kappa^2 \,, \tag{11}$$

has a unique solution $H(\kappa)$.

So for large κ, the upper and lower local fields coincide.

So for $\kappa \geq \kappa_0$, the local critical field $H^{\rm loc}_{C_3}(\kappa)$ can be defined as the solution of

$$\lambda_1(\kappa H^{\rm loc}_{C_3}(\kappa)) = \kappa^2 \; . \tag{12}$$

We can calculate the asymptotics of $H^{\rm loc}_{C_3}(\kappa)$ (based on the asymptotics of $\lambda_1(B)$ from [FoHe2]). The result is that this solution $H^{\rm loc}_{C_3}(\kappa)$ has the formal asymptotic expansion

$$H_{\rm formal} = \frac{\kappa}{\Theta_0}\Big(1 + \frac{C_1 k_{\rm max}}{\sqrt{\Theta_0 \kappa}} - C_1\sqrt{\frac{3k_2}{2}}\kappa^{-\frac{3}{2}} + \kappa^{-\frac{7}{4}} \sum_{j=0}^{\infty} \eta_j \kappa^{-\frac{j}{4}}\Big) \;, \tag{13}$$

as $\kappa \rightarrow +\infty$. Here $k_2 = \max -k''(s_j)$.
The coefficients $\eta_j \in \mathbb{R}$ are computable recursively. The expression for $H_{\rm formal}$ is to be understood as an asymptotic series.

So we can identify the lower and upper local fields and therefore find the following result.

Theorem F

Suppose Ω is either the disc or that it satisfies Generic Assumption. Then $\exists \kappa_0 > 0$ such that, when $\kappa > \kappa_0$, then

$$H_{C_3}^{\mathrm{loc}}(\kappa) = \underline{H_{C_3}(\kappa)} = \overline{H_{C_3}(\kappa)}\ . \tag{14}$$

Proof

The case of the disc follows from Theorems C and D. For the non-degenerate case—i.e. under Generic Assumption— Theorem F follows from combining Proposition E with Theorems C and D.

Remark

Under Generic Assumption, the known asymptotics of $H_{C_3}^{\mathrm{loc}}(\kappa)$ can, of course, be combined with Theorem F to find the leading order terms of the expansion of $H_{C_3}(\kappa)$ for κ large.

Some walk inside the proofs

Another caracterization of the local critical fields

In addition to the (global) critical fields $\overline{H_{C_3}(\kappa)}$ and $\underline{H_{C_3}(\kappa)}$, we have also defined local fields.

These local fields can also be obtained by considering the values where the normal solution[2] $(0, \vec{F})$ is a not unstable local minimum of $\mathcal{E}_{\kappa,H}$, i.e.

$$\overline{H_{C_3}^{\mathrm{loc}}(\kappa)} = \inf\{H > 0 \ : \ \forall H' > H, \mathrm{Hess}\mathcal{E}_{\kappa,H'}\big|_{(0,\vec{F})} \geq 0\} \ ,$$

$$\underline{H_{C_3}^{\mathrm{loc}}(\kappa)} = \inf\{H > 0 \ : \ \mathrm{Hess}\mathcal{E}_{\kappa,H}\big|_{(0,\vec{F})} \geq 0\} \ .$$

[2]Remember that $(0, \vec{F})$ is always a stationary point of the Ginzburg-Landau functional $\mathcal{E}_{\kappa,H}$.

This immediately results of the observation that the Hessian, $\mathrm{Hess}\mathcal{E}_{\kappa,H}$, at the normal solution is given by

$$\mathrm{Hess}\mathcal{E}_{\kappa,H}\big|_{(0,\vec{F})}[\phi,\vec{a}]$$
$$= \int_\Omega |(-i\nabla - \kappa H\vec{F})\phi|^2 - \kappa^2|\phi|^2 + (\kappa H)^2|\ \mathsf{curl}\ \vec{a}|^2\,dx\ .$$

Let us sketch how we get the general comparison between the local and global fields given in Theorem C.

About the proof of Theorem C

We first prove (9). Suppose $H > \overline{H_{C_3}(\kappa)}$. Then $(0, \vec{F})$ is the only minimizer of $\mathcal{E}_{\kappa,H}$. In particular, for all $\phi, \vec{A}$,

$$\mathcal{E}_{\kappa,H}[\phi, \vec{F} + \vec{A}] \geq \mathcal{E}_{\kappa,H}[0, \vec{F}] = 0$$

This implies that $\mathrm{Hess}\mathcal{E}_{\kappa,H}\big|_{(0,\vec{F})} \geq 0$. Since $H > \overline{H_{C_3}(\kappa)}$ was arbitrary, we get (9).

Next we prove (10). Suppose $H < \underline{H_{C_3}^{\rm loc}(\kappa)}$. Then $\lambda_1(\kappa H) < \kappa^2$. Let ψ be a ground state for $\mathcal{H}(\kappa H)$. We use, for $\eta > 0$, the pair $(\eta\psi, \vec{F})$ as a trial state in $\mathcal{E}_{\kappa,H}$,

$$\mathcal{E}_{\kappa,H}[\eta\psi, \vec{F}] = (\lambda_1(\kappa H) - \kappa^2)\eta^2 \|\psi\|_{L^2(\Omega)}^2 + \frac{\kappa^2}{2}\eta^4 \|\psi\|_{L^4(\Omega)}^4 .$$

Since $\lambda_1(\kappa H) - \kappa^2 < 0$, we get $\mathcal{E}_{\kappa,H}[\eta\psi, \vec{F}] < 0$ for η sufficiently small (using that $W^{1,2}(\Omega) \subset L^4(\Omega)$).

Thus $(0, \vec{F})$ is not a minimizer for $\mathcal{E}_{\kappa,H}$. Since $H < \underline{H_{C_3}^{\rm loc}(\kappa)}$ was arbitrary, this proves (10) and therefore finishes the proof of the lemma.

Around the proof of Theorem D

The proof is by contradiction.

If there exists a sequence $\kappa = \kappa_n$ s.t. $\kappa_n \to +\infty$ and $\overline{H_{C3}(\kappa)} > \overline{H_{C3}^{loc}(\kappa)}$, we can find H in $]\overline{H_{C3}(\kappa)}\ ,\ \overline{H_{C3}^{loc}(\kappa)}[$ and a pair of minimizers $(\psi, \vec{A})$ with ψ non trivial,

$$\lambda_1(\kappa H \vec{F}) \geq \kappa^2\,,$$

and

$$\mathcal{E}(\psi, \vec{A}) \leq 0\ .$$

This leads to

$$0 < \Delta := \kappa^2 ||\psi||_2^2 - Q_{\kappa H \vec{A}}[\psi] = \kappa^2 ||\psi||_4^4\ ,$$

where $Q_{\kappa H \vec{A}}[\psi]$ is the energy of ψ.
The last equality is a consequence of the first G-L equation.

Combining with (4), this gives

$$||\psi||_2 \leq C\kappa^{-\frac{3}{4}}\Delta^{\frac{1}{4}}\ .$$

By comparison of the quadratic forms Q respectively associated with $\vec{A}$ et $\vec{F}$, we get, with $\vec{a} = \vec{A} - \vec{F}$:

$$\Delta \leq \left[\kappa^2 - (1-\rho)\lambda_1(\kappa H\vec{F})\right]\|\psi\|_2^2 + \rho^{-1}(\kappa H)^2 \int_\Omega |\vec{a}\psi|^2\, dx\ , \tag{15}$$

for all $0 < \rho < 1$.

Note that by the regularity of the system Curl-Div, combined with the Sobolev's injection theorem, we get

$$\|\vec{a}\|_4 \leq C_1\|\vec{a}\|_{W^{1,2}} \leq C_2\|\ \text{curl}\ \ \vec{a}\|_2\ .$$

Now Δ is also controlling $\| \text{ curl } \vec{a}\|_2^2$, so we get :

$$(\kappa H)^2 \|\vec{a}\|_4^2 \leq C\,\Delta\ .$$

Combining all these inequalities leads to :

$$\begin{aligned} 0 < \Delta \leq & \\ & \leq \left[\kappa^2 - (1-\rho)\lambda_1(\kappa H\vec{F})\right] \|\psi\|_2^2 + \rho^{-1}(\kappa H)^2 \|\vec{a}\|_4^2 \|\psi\|_4^2 \\ & \leq \left[\kappa^2 - \lambda_1(\kappa H\vec{F})\right] \|\psi\|_2^2 \\ & \qquad + C\rho\lambda_1(\kappa H)\Delta^{\frac{1}{2}}\kappa^{-\frac{3}{2}} + C\rho^{-1}\Delta^{\frac{3}{2}}\kappa^{-1}\ . \end{aligned}$$

Chosing $\rho = \sqrt{\Delta}\kappa^{-\frac{3}{4}}$, and using the rough upper bound $\lambda_1(\kappa H\,\vec{F}) < C\kappa^2$, we find

$$0 < \Delta \leq \left[\kappa^2 - \lambda_1(\kappa H)\right] \|\psi\|_2^2 + C\Delta\kappa^{-\frac{1}{4}}\ .$$

This leads to a contradiction for $\kappa = \kappa_n$ large enough

Perspectives

This is far to be the end of the story. Here are some additional questions :

1. One can instead consider the more physical functional :

   $$\mathcal{E}_{\kappa,H}[\psi,\vec{A}] = \int_\Omega \left\{ |p_{\kappa H \vec{A}}\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 + \kappa^2 H^2 \int_{\mathbb{R}^2} |\text{ curl } \vec{A} - 1|^2 \right\} dx \ ,$$

   The difference is that the last integration is over $\mathbb{R}^2$! This is particularly important if Ω is not simply connected !

2. What is going on in Dimension 3 ?
   Results by Pan, Helffer-Morame, Fournais-Helffer.

3. Is there a good definition of the Second Critical Field ?

   For an analysis near this field, see Pan, Almog and Sandier-Serfaty.

Note that this is below the second critical field that starts the beautiful analysis of E. Sandier and S. Serfaty.

Note also that other conditions than Neumann could be interesting.

References

[Ag] S. Agmon : *Lectures on exponential decay of solutions of second order elliptic equations.* Math. Notes, T. 29, Princeton University Press (1982).

[BaPhTa] P. Bauman, D. Phillips, and Q. Tang : Stable nucleation for the Ginzburg-Landau system with an applied magnetic field. Arch. Rational Mech. Anal. 142, p. 1-43 (1998).

[BeSt] A. Bernoff and P. Sternberg : Onset of superconductivity in decreasing fields for general domains. J. Math. Phys. 39, p. 1272-1284 (1998).

[BoHe] C. Bolley and B. Helffer : An application of semi-classical analysis to the asymptotic study of the supercooling field of a superconducting material. Ann. Inst. H. Poincaré (Section Physique Théorique) 58 (2), p. 169-233 (1993).

[Bon1] V. Bonnaillie : Analyse mathématique de la supraconductivité dans un domaine à coins : méthodes semi-classiques et numériques. Thèse de Doctorat, Université Paris 11 (2003).

[Bon2] V. Bonnaillie : On the fundamental state for a Schrödinger operator with magnetic fields in domains with corners. Asymptotic Anal. 41 (3-4), p. 215-258, (2005).

[BonDa] V. Bonnaillie and M. Dauge : Asymptotics for the fundamental state of the Schrödinger operator with magnetic field near a corner. In preparation, (2004).

[CFKS] H.L. Cycon, R.G. Froese, W. Kirsch, and B. Simon : *Schrödinger Operators.* Springer-Verlag, Berlin 1987.

[DaHe] M. Dauge and B. Helffer : Eigenvalues variation I, Neumann problem for Sturm-Liouville operators. J. Differential Equations 104 (2), p. 243-262 (1993).

[DiSj] M. Dimassi and J. Sjöstrand : Spectral Asymptotics in the semi-classical limit. London Mathematical Society. Lecture Note Series 268. Cambridge University Press (1999).

[FoHe1] S. Fournais and B. Helffer : Energy asymptotics for type II superconductors. Preprint 2004. To appear in Calc. Var. and PDE.

[FoHe2] S. Fournais and B. Helffer : Accurate eigenvalue asymptotics for the magnetic Neumann Laplacian. Preprint 2004. To appear in Annales de l'Institut Fourier.

[FoHe3] S. Fournais and B. Helffer : On the third critical field in Ginzburg-Landau thory. Preprint 2005.

[FoHe4] S. Fournais and B. Helffer : On the Ginzburg-Landau critical field in three dimensions. Preprint 2005.

[GiPh] T. Giorgi and D. Phillips : The breakdown of superconductivity due to strong fields for the Ginzburg-Landau model SIAM J. Math. Anal. 30 (1999), no. 2, 341–359 (electronic).

[Hel] B. Helffer : *Introduction to the semiclassical analysis for the Schrödinger operator and applications.* Springer lecture Notes in Math. 1336 (1988).

[HeMo1] B. Helffer and A. Mohamed : Semiclassical analysis for the ground state energy of a Schrödinger operator with magnetic wells. J. Funct. Anal. 138 (1), p. 40-81 (1996).

[HeMo2] B. Helffer and A. Morame : Magnetic bottles in connection with superconductivity. J. Funct. Anal. 185 (2), p. 604-680 (2001).

[HeMo3] B. Helffer and A. Morame : Magnetic bottles for the Neumann problem : curvature effect in the case of dimension 3 (General case).

Ann. Sci. Ecole Norm. Sup. 37, p. 105-170 (2004).

[HePa] B. Helffer and X. Pan : Upper critical field and location of surface nucleation of superconductivity. Ann. Inst. H. Poincaré (Section Analyse non linéaire) 20 (1), p. 145-181 (2003).

[HeSj] B. Helffer and J. Sjöstrand : Multiple wells in the semiclassical limit I. Comm. Partial Differential Equations 9 (4), p. 337-408 (1984).

[LuPa1] K. Lu and X-B. Pan : Estimates of the upper critical field for the Ginzburg-Landau equations of superconductivity. Physica D 127, p. 73-104 (1999).

[LuPa2] K. Lu and X-B. Pan : Eigenvalue problems of Ginzburg-Landau operator in bounded domains. J. Math. Phys. 40 (6), p. 2647-2670, June 1999.

[LuPa3] K. Lu and X-B. Pan : Gauge invariant eigenvalue problems on $\mathbb{R}^2$ and $\mathbb{R}^2_+$. Trans. Amer. Math. Soc. 352 (3), p. 1247-1276 (2000).

[LuPa4] K. Lu and X-B. Pan : Surface nucleation of superconductivity in 3-dimension. J. of Differential Equations 168 (2), p. 386-452 (2000).

[Pan] X-B. Pan : Surface superconductivity in applied magnetic fields above H_{C_3} Comm. Math. Phys. 228, p. 327-370 (2002).

[PiFeSt] M. del Pino, P.L. Felmer, and P. Sternberg : Boundary concentration for eigenvalue problems related to the onset of superconductivity. Comm. Math. Phys. 210, p. 413-446 (2000).

[SaSe] E. Sandier, S. Serfaty : Important series of contributions....

[S-JSaTh] D. Saint-James, G. Sarma, E.J. Thomas :

Type II Superconductivity. Pergamon, Oxford 1969.

[St] P. Sternberg : On the Normal/Superconducting Phase Transition in the Presence of Large Magnetic Fields. In Connectivity and Superconductivity, J. Berger and J. Rubinstein Editors. Lect. Notes in Physics 63, p. 188-199 (1999).

[TiTi] D. R. Tilley and J. Tilley: *Superfluidity and superconductivity.* 3rd edition. Institute of Physics Publishing, Bristol and Philadelphia 1990.

[Ti] M. Tinkham, *Introduction to Superconductivity.* McGraw-Hill Inc., New York, 1975.