On Nodal domains and spectral minimal partitions.

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(After B. Helffer, T. Hoffmann-Ostenhof, S. Terracini, G. Vial, V. Bonnaillie-Noël, G. Verzini)

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Abstract :

Given an open set Ω and a partition of Ω by k open sets ω_j, we can consider the quantity $\max_j \lambda(\omega_j)$ where $\lambda(\omega_j)$ is the ground state energy of the Dirichlet realization of the Laplacian in ω_j.If we denote by $\mathfrak{L}_k(\Omega)$ the infimum over all the k-partitions of $\max_j \lambda(\omega_j)$, a minimal k-partition is then a partition which realizes the infimum. Although the analysis is rather standard when $k = 2$ (we find the nodal domains of a second eigenfunction), the analysis of higher k's becomes non trivial and quite interesting. In this talk, we would like to discuss the properties of minimal spectral partitions, illustrate the difficulties by considering simple cases like the disc or the square ($k = 3$) and will also exhibit the possible role of the hexagone in the asymptotic behavior as $k \to +\infty$ of $\mathfrak{L}_k(\Omega)$. This work has started in collaboration with T. Hoffmann-Ostenhof and has been continued (published or in progress) in collaboration with (by alphabetic order) V. Bonnaillie-Noël, T. Hoffmann-Ostenhof, S. Terracini, G. Verzini, and G. Vial.

Main goal :

We only consider Laplacians operators in $2D$ bounded domains Ω. We would like to analyze the relations between the nodal domains of the eigenfunctions of the Dirichlet realization and the partitions of Ω by k open sets D_i which are minimal in the sense that the maximum over the k D_i's of the ground state energy of the Dirichlet realization of the Laplacian in D_i is minimal inside the class of all the partitions of Ω.

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We define for any function $u \in C_0^0(\overline{\Omega})$

$$N(u) = \overline{\{x \in \Omega \mid u(x) = 0\}} \tag{1}$$

and call the components of $\Omega \setminus N(u)$ the nodal domains of u.

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The number of nodal domains of such a function will be called $\mu(u)$.

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Sometimes (at least for some proofs) we have to relax this definition by considering measurable sets for the partitions.

We now introduce the notion of

Definition 2 : Spectral minimal partition sequence.

For any integer $k \geq 1$, and for $\mathcal{D}$ in $\mathfrak{O}_k$, we introduce

$$\Lambda(\mathcal{D}) = \max_i \lambda(D_i). \quad (3)$$

Then we define

$$\mathfrak{L}_k(\Omega) = \inf_{\mathcal{D} \in \mathfrak{O}_k} \Lambda(\mathcal{D}). \quad (4)$$

and say that

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If $k = 2$, it is rather well known (see [HeHO1] or [CTV3]) that

$$\mathfrak{L}_2(\Omega) = \lambda_2(\Omega)\,.$$

We discuss roughly the notion of regular and strong partition.

Definition 3 : Strong partition.

A partition $\mathcal{D} = \{D_i\}_{i=1}^{k}$ of Ω in $\mathfrak{O}_k$ is called strong if

$$\mathrm{Int}\,(\overline{\cup_i D_i}) \setminus \partial\Omega = \Omega\,. \tag{5}$$

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Attached to a strong partition, we associate a closed set in $\overline{\Omega}$:

Definition 4 : Boundary set of a partition.

$$N(\mathcal{D}) = \overline{\cup_i \left(\partial D_i \cap \Omega\right)}\,. \tag{6}$$

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In the case of a nodal partition the boundary set of the partition $N(\mathcal{D})$ is the nodal set.

This leads us to introduce the set $\mathcal{R}(\Omega)$ of the regular partitions through the properties of the associated boundary set.

Definition 6 : Regular boundary set.

(i) There are finitely many distinct $x_i \in \Omega \cap N$ and associated positive integers ν_i with $\nu_i \geq 2$ s. t. near each of the x_i, N is the union of $\nu_i(x_i)$ smooth curves with one end at x_i and s. t. in the complement of these points in Ω, N is locally diffeomorphic to a regular curve.

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(ii) $\partial\Omega \cap N$ consists of a (possibly empty) finite set of points z_i, s.t. at each z_i, ρ_i, with $\rho_i \geq 1$ lines hit the boundary. Moreover, $\forall z_i \in \partial\Omega$, then N is near z_i the union of ρ_i distinct smooth half-curves which hit z_i.

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(iii) N has the equal angle meeting property.

By equal angle meeting property, we mean that the half curves cross with equal angle at each critical point of N and also at the boundary together with the boundary.

To better describe the situation, we need some additional definitions.

- We say that D_i, D_j are neighbors or $D_i \sim D_j$, if $D_{i,j} := \mathrm{Int}\,(\overline{D_i \cup D_j}) \setminus \partial\Omega$ is connected.

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- We associate to each $\mathcal{D}$ a graph $G(\mathcal{D})$ by associating to each D_i a vertex and to each pair $D_i \sim D_j$ an edge.

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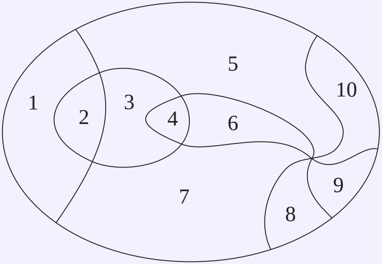
- We associate to each $\mathcal{D}$ a graph $G(\mathcal{D})$ by associating to each D_i a vertex and to each pair $D_i \sim D_j$ an edge.
- We will say that the graph is bipartite if it can be colored by two colors (two neighbours having two different colors).

Note that the graph of a nodal partition is always bipartite.

Let us give two examples of regular strong partitions.

Figure 1: An example of regular strong bipartite partition with associated graph.



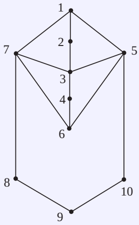
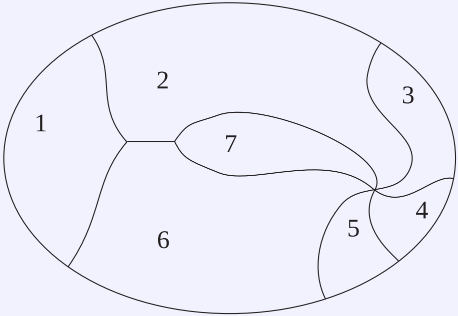
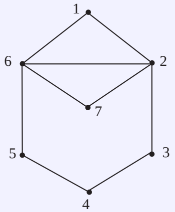


Figure 2: An example of regular strong non bipartite partition with associated graph.





It has been proved by Conti-Terracini-Verzini [CTV1, CTV2, CTV3] that

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This result is completed by (see Helffer–Hoffmann-Ostenhof–Terracini [HeHOTe]) :

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A natural question is whether a minimal partition is the nodal partition. Next theorem will give a simple criterion.

Theorem 9

A minimal partition whose graph is bipartite is a nodal partition.

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Hint : If D_i and D_j are neighbours, then, with $D_{ij} := \mathrm{Int}\,(\overline{D_i \cup D_j})$,

$$\lambda_2(D_{ij}) = \mathfrak{L}_k(\Omega)\ .$$

In other words, (D_i, D_j) is a minimal 2-partition in D_{ij}.

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Similar properties hold for k'-subpartitions $k' < k$.

A natural question is now to determine how general is the previous situation.

Surprisingly this only occurs in the so called Courant-sharp situation.
First recall that the Courant theorem says :

Theorem 10

Let $k \geq 1$, λ_k be the k-th eigenvalue and $E(\lambda_k)$ the eigenspace of $H(\Omega)$ associated to λ_k. Then, $\forall u \in E(\lambda_k) \setminus \{0\}$, $\mu(u) \leq k$.

Then we say that

Definition 11

u is Courant-sharp if

$$u \in E(\lambda_k) \setminus \{0\} \quad \text{and} \quad \mu(u) = k \, .$$

For any integer $k \geq 1$, we denote by L_k the smallest eigenvalue whose eigenspace contains an eigenfunction with k nodal domains. We set $L_k = \infty$, if there are no eigenfunctions with k nodal domains.
In general, one can show, that

$$\lambda_k \leq \mathfrak{L}_k \leq L_k \ . \tag{7}$$

The last goal consists in giving the full picture of the equality cases :

Theorem 12

Suppose $\Omega \subset \mathbb{R}^2$ is regular.
If $\mathfrak{L}_k = L_k$ or $\mathfrak{L}_k = \lambda_k$ then

$$\lambda_k = \mathfrak{L}_k = L_k \ .$$

In addition, one can find in $E(\lambda_k)$ a Courant-sharp eigenfunction.

Except the Courant-sharp situation it is not easy to determine if a k-partition is minimal.

Proposition 13 : A nice property

Let $\mathcal{D} = (D_i)_i$ a minimal k-partition of Ω.
Let Ω' a connected open set such that

$$\cup_{i=1}^{k} D_i \subset \Omega' \subset \Omega \,.$$

If $G(\mathcal{D})$ is bipartite in Ω' then $\mathfrak{L}_k(\Omega) = \lambda_k(\Omega')$.

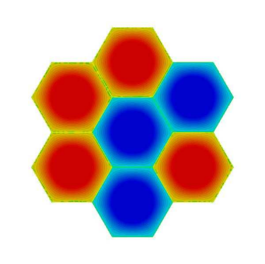
This gives a good test for controlling if a given partition is a good candidate for being a minimal partition.

Example

Take $k = 7$, D_i isometric to a given regular hexagon, $\Omega = \mathrm{Int}\,(\overline{\cup_i D_i})$, then, for any Ω' as above, it is evident that $\lambda_1(\mathrm{Hexa}) = \lambda_\ell(\Omega')$ for some $\ell \geq 7$, but one can verify numerically

$$\Lambda(\mathcal{D}) = \lambda_1(\mathrm{Hexa}) = \lambda_7(\Omega')\,.$$

Figure Hexa : 7 hexagons together minus 4 segments.



We conjecture [BHV] that this property is true for any k.

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This property is no more true for k large in the case of an equilateral triangle.

Using Theorem 12, it is now easier to analyze the situation for the disk or for rectangles (at least in the irrational case), since we have just to check for which eigenvalues one can find associated Courant-sharp eigenfunctions.

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The first remark is that all the eigenvalues are simple if a^2/b^2 is irrational. Assuming a^2/b^2 irrational, we can associate to each eigenvalue $\lambda_{m,n}$, an (essentially) unique eigenfunction $u_{m,n}$ such that $\mu(u_{m,n}) = nm$.
Given $k \in \mathbb{N}^*$, the lowest eigenvalue corresponding to k nodal domains is given by

$$L_k = \pi^2 \inf_{mn=k} (m^2/a^2 + n^2/b^2)\,.$$

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$$\frac{b}{a} \geq \sqrt{\frac{8}{3}}\,. \tag{8}$$

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When $1 \leq \frac{b}{a} < \sqrt{\frac{8}{3}}$, in particular, in the case of the square, it is not to difficult to see that $\mathfrak{L}_3$ is strictly less than L_3.

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When $1 \leq \frac{b}{a} < \sqrt{\frac{8}{3}}$, in particular, in the case of the square, it is not to difficult to see that $\mathfrak{L}_3$ is strictly less than L_3.

Restricting to the half-rectangle and assuming that there is a minimal partition which is symmetric with the horizontal or vertical symmetry axes of the square, it is natural to analyze the second eigenvalue (and corresponding nodal partition) of a family of Dirichlet-Neumann problems defined on an half-square.

Numerical computations by Bonnaillie-Noël–Vial give :

Figure 4 : Trace on the half-square of the candidate for the 3-partition of the square.

The complete structure is obtained from the half square by symmetry with respect to the horizontal axis.
http://www.bretagne.ens-cachan.fr/math/Simulations/MinimalPartitions/

Here we describe some unpublished results [HeHO4] on the possible "topological" types of 3-partitions.

Proposition 14

Let Ω be simply-connected and consider a minimal 3-partition $\mathcal{D} = (D_1, D_2, D_3)$ associated to $\mathfrak{L}_3$ and suppose that $\lambda_3 < \mathfrak{L}_3$. Let $X(\mathcal{D})$ the singular points of $N(\mathcal{D}) \cap \Omega$ and $Y(\mathcal{D}) = N(\mathcal{D}) \cap \partial\Omega$. Then there are three cases.

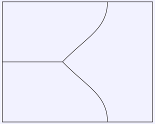
(a)

$X(\mathcal{D})$ consists of one point x with a meeting of three half-lines ($\nu(x) = 3$) and $Y(\mathcal{D})$ consists of

- either three y_1, y_2, y_3 points with $\rho(y_1) = \rho(y_2) = \rho(y_3) = 1$,
- or two points y_1, y_2 with $\rho(y_1) = 2,\ \rho(y_2) = 1$,
- or one point y with $\rho(y) = 3$.

Here, for $y \in \partial\Omega$, $\rho(y)$ is the number of half-lines ending at y.

Type (a)



(b)

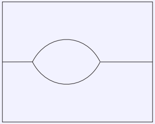
$X(\mathcal{D})$ consists of two distinct points x_1, x_2 so that
$\nu(x_1) = \nu(x_2) = 3$.
$Y(\mathcal{D})$ consists
either of two points y_1, y_2 such that

$$\rho(y_1) + \rho(y_2) = 2$$

or of one point y with

$$\rho(y) = 2 \ .$$

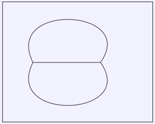
Type (b)



(c)

$X(\mathcal{D})$ consists again of two distinct points x_1, x_2 with
$\nu(x_1) = \nu(x_2) = 3$,
but $Y(\mathcal{D}) = \emptyset$.

Type (c)

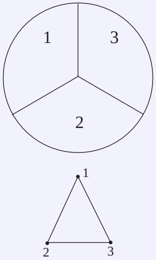


The proof of Proposition 14 relies essentially on Euler formula.

This leads (with some success) to analyze the minimal partition with some topological type. If in addition, we introduce some symmetries, this leads to guess some candidates for minimal partitions.

In the case of the disk, we have no proof that the minimal partition is the "Mercedes star". But if we assume that the minimal 3-partition is of type (a), then a double covering argument shows that it is indeed the Mercedes star.

Figure 5 : The logo Mercedes and associated graph



In the case of the square, we have no proof that the candidate described by Figure 4 is the minimal partition.

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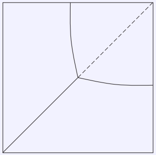
But if we assume that the minimal partition is of type (a) and has the symmetry, then numerical computations lead to the Figure 4. Numerics suggest more : the center of the square is the critical point of the partition.

Once this property is accepted, a double covering argument shows that this is the projection of a nodal partition on the covering

One can also try to look for a minimal partition having the symmetry with respect to the diagonal.

Figure 6 : Another candidate



THIS LEADS TO THE SAME VALUE OF $\Lambda(\mathcal{D})$.

So this strongly suggests that there is a continuous family of minimal 3-partitions of the square.
This can be explained by a double covering argument, which is analogous to the argument of isospectrality of Jakobson-Levitin-Nadirashvili-Polterovich [JLNP] and Levitin-Parnovski-Polterovich [LPP].

This is an alternative approach to the double covering approach.

One considers the Aharonov-Bohm Laplacian in the square minus its center $\dot{\Omega} = \Omega \setminus \{0\}$, with the singularity of the potential at the center and normalized flux $\frac{1}{2}$.
The magnetic potential takes the form

$$\mathbf{A}(x, y) = (A_1, A_2) = \alpha \left(-\frac{y}{r^2}, \frac{x}{r^2} \right) \ . \tag{9}$$

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We know that the magnetic field vanishes and in any cutted domain (such that it becomes simply connected) one has

$$\mathbf{A}(x,y) = \alpha\, d\theta\,, \tag{10}$$

where

$$z = x + iy = r \exp i\theta\ . \tag{11}$$

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$$\mathbf{A}(x,y) = \alpha \, d\theta \,, \tag{10}$$

where

$$z = x + iy = r \exp i\theta \ . \tag{11}$$

Then the flux condition reads

$$\alpha = \frac{1}{2} \ . \tag{12}$$

So the Aharonov-Bohm operator in any open set $\Omega \subset \mathbb{R}^2 \setminus \{0\}$ is the Friedrichs extension starting from $C_0^\infty(\Omega)$ and the associated differential operator is

$$-\Delta_{\mathbf{A}} := (D_x - A_1)^2 + (D_y - A_2)^2 \,. \tag{13}$$

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In the case of the square, the operator commutes with the $\frac{\pi}{2}$ rotation.
In the case of rectangles, it commutes with the symmetries with respect to the main axis but these symmetries should be quantized by antilinear operators,

$$\Sigma_1 u(x,y) = \overline{u(-x,y)} \,.$$

and

$$\Sigma_2 u(x,y) = \overline{u(x,-y)} \,.$$

This operator is preserving "real" functions in the following sense. Following [HOOO], we will say that a function u is K-real, if it satisfies

$$Ku = u\,, \tag{14}$$

where K is an anti-linear operator in the form

$$K = \exp i\theta\,\Gamma\;, \tag{15}$$

where

$$\Gamma u = \bar{u}\;. \tag{16}$$

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The fact that $(-\Delta_{\mathbf{A}})$ preserves K-real eigenfunctions is an immediate consequence of

$$K \circ (-\Delta_{\mathbf{A}}) = (-\Delta_{\mathbf{A}}) \circ K\,. \tag{17}$$

As observed in [HOOO], it is easy to find a basis of K-real eigenfunctions. These eigenfunctions (which can be identified to real antisymmetric eigenfunctions of the Laplacian on a suitable double covering of the square) have a nice nodal structure

- which is locally the same inside the pointed square as the real eigenfunctions of the Laplacian,
- with the specific property that the number of lines arriving at the origine should be odd.

More generally a path of index one around the origine should always meet an odd number of nodal lines.

Lemma 15

The multiplicity of any eigenvalue is at least 2.

Proposition 16

The following problems have the same eigenvalues :

- The Dirichlet problem for the Bohm-Aharonov operator on the pointed square.
- The Dirichlet-Neumann problem on the upper-half square.
- The Dirichlet-Neumann problem on the left-half square.
- The Dirichlet-Neumann problem on the upper diagonal-half square.

Remarks

- The guess is that any nodal partition of a third K-real eigenfunction gives a minimal 3-partition.
- In the case of the general rectangle, Proposition 16 holds true except the last item but this is no more related to the 3-partition problem.

All the results or observations around the square and the rectangle arise from discussions, preliminary manuscripts written by or in collaboration with V. Bonnaillie-N, T. Hoffmann-Ostenhof, S. Terracini, G. Verzini or G. Vial.

We mention two conjectures. The first one is that

Conjecture 1

The limit of $\mathfrak{L}_k(\Omega)/k$ as $k \to +\infty$ exists.

The second one is that this limit is more explicitly given by

Conjecture 2

$$|\Omega| \lim_{k \to +\infty} \frac{\mathfrak{L}_k(\Omega)}{k} = \lambda_1(Hexa_1)\ .$$

This last conjecture says in particular that the limit is independent of Ω if Ω is a regular domain.

It is easy to show the upper bound in Conjecture 2 and Faber-Krahn gives a weaker lower bound

$$|\Omega|\mathfrak{L}_k(\Omega)/k \geq \lambda_1(D_1) \,,$$

where D_1 is the disk of area 1.

Of course the optimality of the regular hexagonal tiling appears in various contexts in Physics. But we have at the moment no idea of any approach for proving this in our context.

As mentioned after Proposition 13, we have explored in [BHV] numerically why this conjecture looks numerically reasonable.

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