Q-process and genealogy

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1. **Constant size population models**
   - Introduction to the lookdown model
   - A constant size population model with non-fixation

2. **Branching population models**
   - Branching population models process with immigration
   - An inhomogeneous branching population with non-extinction
The Moran model (1958)

Let \{1, 2, \ldots, n\} be the individuals in a finite population of size \(n\).

1. Initially, each individual is either black or blue, and the \(k\) black types are randomly distributed among the \(n\) individuals.

2. For every ordered couple of individuals \((i, j)\), \(i\) reproduces over \(j\) at the times of independent Poisson processes with unit rate, and \(i\) gives its type to \(j\).

We are interested in the process \(X^n(t) = \sum_{1 \leq i \leq n} 1\{\text{individual } i\ \text{black at } t\}\)

Question: Does one type fixate? What about the distribution of this fixation time?
Figure: The blue type fixates the whole population
The lookdown particle system, [Donnelly Kurtz 96]

Figure: The black type fixates the whole population
A basic population model

In these two models, the Markov chain

\[ X^n(t) = \sum_{1 \leq i \leq n} 1_{\{\text{individual } i \text{ black at } t\}} \]

has the same law. From the second model:

\[ \lim_{t \to \infty} X^n(t) = n 1_{\{\text{individual } 1 \text{ black at } 0\}}. \]

Therefore,

\[ \lim_{t \to \infty} X^n(t) = \begin{cases} n & \text{with probability } k/n \\ 0 & \text{with probability } (n - k)/n \end{cases} \]
The modified lookdown particle system, [Donnelly Kurtz 99]

Figure: The black type (will) fixate the whole population
A basic population model

From the third model:

\[ \inf \{ t \geq 0, X^n(t) \in \{0, n\} \} = \sum_{i=L(0)}^{n} e_i, \]

where:

- \( L(0) = \inf\{i \geq 1, \{\text{black, blue}\} \subset \text{the types of } \{1, 2, \ldots, i\} \text{ at time 0} \} \) is a random variable with law:
  \[ \mathbb{P}(L(0) = \ell) = \frac{\binom{n-k}{\ell-1} k + \binom{k}{\ell-1} (n-k)}{\binom{n}{\ell} \ell}. \]

- the \((e_i, 2 \leq i \leq n)\) are independent exponential random variables with parameter \(2\binom{i}{2} = i(i-1).\)

We now switch to \textit{infinite} constant size population.
A more general process, [Bertoin Le Gall 03]

A pure jump Generalized Fleming Viot (GFV) process $X_t \in [0, 1]$ has generator:

$$f(x) \rightarrow x \int_{(0,1)} \nu(dy) [f(x(1-y) + y) - f(x)]$$

$$+(1-x) \int_{(0,1)} \nu(dy) [f(x(1-y)) - f(x)].$$

At rate $\nu(dy)$, a reproduction event with size $y$ affects the population, currently at state $x$.

- with probability $x$, black type reproduces.
- with probability $1-x$, blue type reproduces.
- the past population $x$ is rescaled by a factor $(1 - y)$. 

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Figure: The dynamic of the GFV process
Figure: The associated lookdown process
We define the first level at which the two types are encountered:

\[ L(t) = \inf\{i \geq 1, \{\text{black, blue}\} \subset \text{the types of } \{1, 2, \ldots, i\} \text{ at time } t \} \],

which forms a Markov chain in continuous time, started at \( \ell \) under \( \mathbb{P}_\ell \). Notice that

\[ \{T > t\} = \{L(t) < \infty\} \text{ a.s.} \]

Under the condition:

\[ \mathbb{P}_3(L(t) < \infty)/\mathbb{P}_2(L(t) < \infty) \to 0 \text{ as } t \to \infty, \quad (1) \]

the fixation time \( T = \inf\{t > 0, X_t \in \{0, 1\}\} \) is a.s. finite, and:

\[ \mathbb{P}(L(t) < \infty) = \sum_{k \geq 2} \mathbb{P}(L(s) = k)\mathbb{P}_k(L(t-s) < \infty) \]

\[ \sim \mathbb{P}(L(s) = 2)\mathbb{P}_2(L(t-s) < \infty) \text{ as } t \to \infty. \]
Figure: The lookdown process conditioned on non-fixation
Figure: The lookdown process conditioned on non-fixation
Set $r_2 = \int_{[0,1]} \nu(dy)y^2$. The process

$$h(t, X_t) = X_t(1 - X_t) e^{r_2 t}$$

$$= \frac{1}{2} \mathbb{P}(L(t) = 2 \mid X_{[0,t]}) e^{r_2 t}$$

defines a non-negative martingale. We therefore consistently define a new process by setting:

$$\mathbb{P}(X^h \in A) = \mathbb{E} \left( \frac{h(t, X_t)}{h(0, X_0)}, A \right)$$

for any event $A \in \sigma(X_s, s \leq t)$.

**Theorem (H. 12)**

*If the condition (1) holds, then:*

$$\mathbb{P}(X \in A \mid T > t) \to \mathbb{P}(X^h \in A), \text{ as } t \to \infty.$$
The dynamic of the process $X^h$

**Proposition (H. 12)**

*The generator of $X^h$ is:*

$$f(x) \rightarrow \int_{(0,1]} \nu(dy)y(1-y)\{f(x(1-y)+y)-f(x)\}$$

$$+ \int_{(0,1]} \nu(dy)y(1-y)\{f(x(1-y))-f(x)\}$$

$$+x \int_{(0,1]} \nu(dy)(1-y)^2\{f(x(1-y)+y)-f(x)\}$$

$$+(1-x) \int_{(0,1]} \nu(dy)(1-y)^2\{f(x(1-y))-f(x)\}$$
Outline

1 Constant size population models
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A continuous state branching process, [Jirina58, Lamperti67]

- A (critical) pure jump continuous state branching process (CB) $Y_t \in [0, \infty)$ has generator:
  \[
  f(x) \rightarrow x \int_{(0,\infty)} \pi(dy) \left[ f(x + y) - f(x) - yf'(x) \right]
  \]

- At rate $x\pi(dy)$, a reproduction event with size $y$ affects a population with size $x$; $y$ is simply added to the size of the (non-constrained) population.

- Due to the branching property, a family of CB processes $(Y_t(x), t \geq 0, x \geq 0)$ starting from $x \geq 0$ may be constructed.
Figure: The pure jump CB process, and its ratio process $Y_t(x)/Y_t(1)$
First similarities between the ratio process of a CB and a GFV

- The ratio $Y_t(x)/Y_t(1) \in [0, 1]$ is a pure jump martingale, with absorbing points $\{0, 1\}$.
- At the time of a reproduction event, the father is chosen to be blue or black according to the respective proportions of blue and black types.
How to cook up from the ratio process a GFV process?

1. A reproduction of size $y$ arises with rate $z \pi(dy)$ in the total population with size $z$.

2. Therefore, for the ratio, a reproduction event of size $r$ arises at rate

$$z \Phi_z^*(\pi)(dr) \text{ with } \phi_z(y) = \frac{y}{y + z} \in (0, 1).$$

3. There exists $\lambda : (0, \infty) \to (0, \infty)$ and a measure $\nu$ on $(0, 1)$ such that:

$$\Phi_z^*(\pi)(dr) = \lambda(z) \nu(dr),$$

iff $\pi$ belongs to the family of stable measures, $\pi(dy) = y^{-1-\alpha}dy$ for some $0 < \alpha < 2$, in which case $\nu(dr) = r^{-2} \text{Beta}(2 - \alpha, \alpha)(dr)$ and $\lambda(z) = z^{-\alpha}$, see [Birkner & al 05].
A (pure jump) CB process with immigration \( Y_t \in [0, \infty] \) -called CBI process- has generator:

\[
f(x) \rightarrow x \int_{(0,\infty)} \pi(dy) \left[ f(x + y) - f(x) - yf'(x) \right] + \int_{(0,\infty)} \pi^0(dy) \left[ f(x + y) - f(x) \right]
\]

At constant rate \( \pi^0(dy) \), independently of the population size, additional immigration events with size \( y \) affects the population.

Once again, a family \( (Y_t(x), t \geq 0, x \geq 0) \) of such CBI processes can be constructed, \( (Y_t(0), t \geq 0) \) counts the immigrants.
A GFV process with immigration, [Foucart 11]

- A (pure jump) GFV process with immigration $X_t \in [0, 1]$ -called a GFVI process- has generator:

$$f(x) \rightarrow x \int_{(0,1]} \nu(dy) [f(x(1-y) + y) - f(x)]$$

$$+ (1-x) \int_{(0,1]} \nu(dy) [f(x(1-y)) - f(x)]$$

$$+ \int_{(0,1]} \nu^0(dy) [f(x(1-y) + y) - f(x)].$$

- At constant rate $\nu^0(dy)$, independently of $x$, an immigration event with size $y$ affects the population.
Let $Y$ be a CBI with reproduction and immigration measures:

$$\pi(dy) = y^{-1-\alpha}dy \text{ and } \pi^0(dy) = y^{-\alpha}dy$$

for $1 < \alpha < 2$. We also set:

$$C(t) = \int_0^t ds \ Y_s(1)^{1-\alpha}.$$ 

**Theorem (Foucart, H., 12)**

The process:

$$\left( \frac{Y_{C^{-1}(t)}(x)}{Y_{C^{-1}(t)}(1)}, \ t \geq 0 \right)$$

is a GFVI starting at $x$, with reproduction and immigration measures:

$$\nu(dr) = r^{-2} \text{Beta}(2 - \alpha, \alpha)(dr) \text{ and } \nu^0(dr) = r^{-1} \text{Beta}(2 - \alpha, \alpha - 1)(dr).$$
Another presentation of branching processes

- Let $Y_t$ be our branching process.
  \[ \mathbb{E}_x (e^{-\lambda Y_t}) = e^{-x u_t^\lambda}, \quad t \geq 0, \]
- where $u_t$ satisfies:
  \[ u_t^\lambda + \int_0^t ds \psi(u_{t-s}^\lambda) = \lambda, \]
- with the branching mechanism:
  \[ \psi(\lambda) = \int_{(0,\infty)} [e^{-\lambda y} - 1 + \lambda y] \pi(dy) \]

**Figure:** A (discrete) branching process, with one (blue) type
Another presentation of branching processes

Let $Y_t$ be a branching process.

$$
\mathbb{E}_x(e^{-\lambda Y_t}) = e^{-xu_t^\lambda}, \ t \geq 0,
$$

where $u_t$ satisfies:

$$
u_t^\lambda + \int_0^t ds \psi(u_{t-s}^\lambda) = \lambda,
$$

with the branching mechanism:

$$
\psi(\lambda) = \int_{(0,\infty)} \left[ e^{-\lambda y} - 1 + \lambda y \right] \pi(dy) \ + \ \alpha \lambda + \beta \lambda^2
$$

Figure: A (discrete) branching process, with one (blue) type
Let $Y_t$ be an homogeneous measure-valued branching process.

$$
\mathbb{E}_{\delta_x} \left( e^{-Y_t(f)} \right) = e^{-u^f_t(x)}, \ t \geq 0,
$$

where $u_t$ satisfies:

$$
u^f_t(x) + \mathbb{E}_x \left( \int_0^t ds \ \psi(u^f_{t-s}(Z_s)) \right) = \mathbb{E}_x(f(Z_t))
$$

with the branching mechanism:

$$
\psi(\lambda) = \int_{(0,\infty)} \left[ e^{-\lambda y} - 1 + \lambda y \right] \pi(dy) \\
+ \alpha \lambda + \beta \lambda^2
$$
Measure valued branching processes, [Dawson, Dynkin]

- Let $Y_t$ be an inhomogeneous measure-valued branching process.

$$
\mathbb{E}_{\delta_x}(e^{-Y_t(f)}) = e^{-u_t^f(x)}, \quad t \geq 0,
$$

- where $u_t$ satisfies:

$$
u_t^f(x) + \mathbb{E}_x \left( \int_0^t ds \, \psi(Z_s, u_{t-s}^f(Z_s)) \right) = \mathbb{E}_x(f(Z_t))$$

- with the branching mechanism:

$$
\psi(z, \lambda) = \alpha(z)\lambda + \beta(z)\lambda^2
$$

**Figure:** A (discrete) branching process, with black and blue types
Williams decomposition under $\mathbb{N}_x$

- $\mathbb{N}_x$ denotes the canonical measure = “law” of the process started at an infinitesimal individual at $x$.

- We assume that the height $H_{\text{max}} = \inf\{t \geq 0, Y_t = 0\} \in [0, \infty]$ is a.e. finite:
  \[ \mathbb{N}_x(H_{\text{max}} = \infty) = 0. \]

- We define $P_x^{(h)}$ by its Radon-Nikodym derivative w.r.t. $P_x$ on $\mathcal{D}_t$, $0 \leq t \leq h$:
  \[ \frac{\partial_h \nu_{h-t}(Z_t)}{\partial_h \nu_h(x)} e^{-\int_0^t ds \partial \lambda \psi(Z_s, \nu_{h-s}(Z_s))}, \]

with $\nu_h(x) := \mathbb{N}_x(H_{\text{max}} > h)$.

Figure: A branching process decomposed into a trunk and subtrees
Williams decomposition under $\mathbb{N}_x$

- Conditionally on $(Z_s, 0 \leq s < h)$ with law $\mathbb{P}_x^{(h)}$, we define a Poisson point measure $\sum_{i \in J} \delta_{(s_i, Y_i)}(ds, dY)$ with intensity

$$1_{\{0 \leq s < h, H_{\max}(Y) + s \leq h\}} ds \cdot 2\beta(Z_s)\mathbb{N}_{Z_s}(dY).$$

- Denote by $\mathbb{N}_x^{(h)}$ the law of $(\sum_{i \in J} Y_i^{(t-s_i)+}, 0 \leq t < h)$.

**Theorem (Delmas, H., 12)**

The following desintegration of the canonical measure holds:

$$\mathbb{N}_x = \int_{h > 0} dh \ |\partial_h \nu_h(x)| \mathbb{N}_x^{(h)}.$$
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A Williams decomposition for spatially dependent superprocesses.
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