FROM ALE TO ALF GRAVITATIONAL INSTANTONS

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Abstract

In this article, we give an analytic construction of ALF hyperkähler metrics on *smooth deformations* of the Kleinian singularity $\mathbb{C}^2/\mathcal{D}_k$, with $\mathcal{D}_k$ the binary dihedral group of order $4k$, $k \geq 2$. More precisely, we start from the ALE hyperkähler metrics constructed on these spaces by P.B. Kronheimer, and use analytic methods, e.g. resolution of a Monge-Ampère equation, to produce ALF hyperkähler metrics with the same associated Kähler classes.

Introduction

This article deals with an analytic construction of a certain class of examples of four dimensional *non-compact, complete, Ricci-flat manifolds*. One prominent feature of such spaces lies in their appearance as limit spaces, after rescaling, of families of compact Einstein 4-manifolds; this, among others, illustrates the interesting role played by non-compact complete Ricci-flat manifolds in Riemannian geometry in dimension 4.

Now, dimension 4 moreover allows one to specialise the question to *Ricci-flat Kähler*, and even to *hyperkähler*, non-compact, complete manifolds. If one adds furthermore a decay condition on the Riemannian curvature tensor, this leads to:

Definition 0.1 (Gravitational instantons) *Let (X, g, I, J, K) be a non-compact, complete, hyperkähler manifold of real dimension 4, and denote by ρ the distance to some fixed point $o \in X$. Then X is called* gravitational instanton *if its Riemannian curvature tensor Rm^g satisfies the following L^2 condition:*

$$\int_X |\operatorname{Rm}^g|^2 \, d\mu \quad \textit{is finite,} \quad \textit{where } d\mu = \frac{\rho^4}{\operatorname{Vol}_g(B_g(o,\rho))} \operatorname{vol}^g. \tag{1}$$

One can observe that this definition does not depend on the choice $o \in X$, as this choice does not affect the asymptotic behaviour of $d\mu$.

Besides this differential-geometric definition, gravitational instantons also appear as fundamental objects in theoretical physics – where Condition (1) is thought of as a "finite type action" assumption –, in fields such as Quantum Gravity [Haw] or String and M-Theories, see [CH, CK2] and references within.

Recall that hyperkähler metrics are Ricci-flat. The fundamental Bishop-Gromov theorem [GLP] thus implies that on gravitational instantons, ball volume grows at most with euclidean rate. In other words, the "ball volume growth ratio measure" $d\mu$ in Definition 0.1 is at least positively bounded below by vol^g.

If a bound $d\mu \leq C\,\mathrm{vol}^g$, $C > 0$, also holds, one deals with *Asymptotically Locally Euclidean ("ALE")* instantons. These hyperkähler manifolds are very well understood: they are completely classified, after [BKN] and [Kro2] (with recent extension [Şuv] and [Wri] to the Kähler Ricci-flat case), and their classification corresponds to an exhaustive construction by Kronheimer [Kro1] – we shall often refer to these spaces as *Kronheimer's instantons* for this reason. In a nutshell, the hyperkähler structures of these spaces are asymptotic to that of a quotient $\mathbb{R}^4/\Gamma$, with Γ a finite subgroup of $\mathrm{SU}(2) = \mathrm{Sp}(1)$; when moreover Γ is fixed, these spaces are all diffeomorphic to the *minimal resolution of the Kleinian singularity* $\mathbb{C}^2/\Gamma$.

Now, a result by Minerbe [Min1] – see also [CC] – states the following quantisation on gravitational instantons: *if the asymptotic ball volume growth is less than euclidean, i.e. quartic, it is at most cubic*; one jumps from a bound $d\mu \geq c\,\mathrm{vol}^g$ to a bound $d\mu \geq c(\rho + 1)\,\mathrm{vol}^g$. If an analogous reverse upper bound holds, one then speaks about *Asymptotically Locally Flat*, or *ALF*, gravitational instantons. Roughly speaking, half of these spaces are classified, by Minerbe again [Min3]; their geometry at infinity is that of a circle fibration over $\mathbb{R}^3$, and they are explicitly described by the so-called *Gibbons-Hawking ansatz.* This includes the prototypical Taub-NUT metric, living on $\mathbb{R}^4$ itself [EGH].

Results. — The only possibility left for the asymptotic geometry of the ALF gravitational instantons is that of a circle fibration over $\mathbb{R}^3/\pm$ [Min2]. This second family, not classified yet, includes Atiyah-Hitchin's "$\mathcal{D}_0$-instanton" [AH], as well as the "$\mathcal{D}_k$-families", $k \geq 2$, produced by Cherkis-Kapustin [CK1, CK2] and made more precise by Cherkis-Hitchin [CH]. To this regard, our main result consists in a construction of such spaces with independent methods (see Paragraph "*Comments*" below), and states as:

Theorem 0.2 *Let $\big(X, g, I_1^X, I_2^X, I_3^X\big)$ be an ALE gravitational instanton modelled on $\mathbb{R}^4/\mathcal{D}_k$, with $\mathcal{D}_k$ the binary dihedral group of order $4k$, $k \geq 2$, in the sense that the infinities of X and $\mathbb{R}^4/\mathcal{D}_k$ are diffeomorphic, and that the hyperkähler structure* *of X is asymptotic to that of $\mathbb{R}^4/\mathcal{D}_k$ via the diffeomorphism in play. Then there exists on X a family of ALF hyperkähler structures $(g_m, J^X_{1,m}, J^X_{2,m}, J^X_{3,m})_{m\in(0,\infty)}$, such that, for any fixed $m \in (0,\infty)$:*

1. *one can choose the diffeomorphism above so that g_m is asymptotic to the $\mathcal{D}_k$-quotient of the Taub-NUT metric $\mathbf{f}_m$ with fibres of length $\pi(2/m)^{1/2}$ at infinity;*

2. *the Kähler classes $[g_m(J^X_{j,m}\cdot,\cdot)]$, $j = 1,2,3$, are the same as those of the initial ALE hyperkähler structure, and moreover $\mathrm{vol}^{g_m} = \mathrm{vol}^{g_X}$;*

3. *the curvature tensor Rm^{g_m} has cubic decay.*

As is understood here, the Taub-NUT metric $\mathbf{f}_m$ is invariant under $\mathcal{D}_k$, and thus makes perfect sense on $\mathbb{R}^4/\mathcal{D}_k$. The asymptotics between the ALF metric g_m and $\mathbf{f}_m$ – a by-product of our construction – are as follows: if $R = d_{\mathbf{f}_m}(0,\cdot)$ ($0 \in \mathbb{R}^4/\mathcal{D}_k$), then $(g_m - \mathbf{f}_m)$ and $\nabla^{\mathbf{f}_m}(g_m - \mathbf{f}_m)$ are $O(R^{-2+\epsilon})$ for all $\epsilon > 0$.

Before discussing in more details on how Theorem 0.2 is proved, we shall underline that our construction heavily relies on the *computation of the asymptotics of the ALE instantons modelled on* $\mathbb{R}^4/\mathcal{D}_k$. More precisely, the construction of these spaces by Kronheimer allows one to write down these asymptotics as power series, the main term of which is the euclidean model $(\mathbf{e}, I_1, I_2, I_3)$, and this actually holds for any finite subgroup Γ of $\mathrm{SU}(2)$ alluded above. We describe in this article the first non-vanishing terms of those expansions:

Theorem 0.3 *Let $\big(X, g, I^X_1, I^X_2, I^X_3\big)$ be an ALE gravitational instanton modelled on $\mathbb{R}^4/\Gamma$. Then one can choose a diffeomorphism Φ between X minus a compact subset and $\mathbb{R}^4/\Gamma$ minus a ball such that:*

1. *$\Phi_* g_X - \mathbf{e} = h_X + O(r^{-6})$, $\Phi_* I^X_1 - I_1 = \iota^X_1 + O(r^{-6})$ and if $\omega^X_1 = g_X(I^X_1\cdot,\cdot)$ and $\omega^{\mathbf{e}}_1 = \mathbf{e}(I_1\cdot,\cdot)$, then $\Phi_*\omega^X_1 - \omega^{\mathbf{e}}_1 = \varpi^X_1 + O(r^{-6})$, where h_X, ι^X_1 and ϖ^X_1 admit explicit formulas and are $O(r^{-4})$; for instance $\varpi^X_1 = -\sum_{j=1}^3 c_j(X) dd^c_{I_j}(r^{-2})$ for some explicit constants $c_j(X)$.*

2. *when Γ is not a cyclic subgroup of $\mathrm{SU}(2)$, the $O(r^{-6})$ of the previous point can be replaced by $O(r^{-8})$.*

Here the O are understood in an asymptotically euclidean context: ε is $O(r^{-a})$ if for any $\ell \geq 0$, $\big|(\nabla^{\mathbf{e}})^\ell \varepsilon\big|_{\mathbf{e}} = O(r^{-a-\ell})$ near infinity.

Another crucial analytic tool in our construction is a *Calabi-Yau type theorem*, adapted to ALF geometry:

Theorem 0.4 *Let* $\alpha, \beta \in (0,1)$ *and let* (Y, g_Y, J_Y, ω_Y) *be an ALF Kähler 4-manifold of dihedral type of order* $(3, \alpha, \beta)$*. Let* f *a smooth function in* $C^{3,\alpha}_{\beta+2}(Y, g_Y)$*. Then there exists a smooth function* $\varphi \in C^{5,\alpha}_{\beta}(Y, g_Y)$ *such that* $\omega_Y + dd^c_{J_Y}\varphi$ *is Kähler, and verifying the Monge-Ampère equation:*

$$(\omega_Y + dd^c_{J_Y}\varphi)^2 = e^f \omega_Y^2. \tag{2}$$

Let us say at this stage that an ALF Kähler 4-manifold of dihedral type of order $(3, \alpha, \beta)$ is a complete non-compact Kähler manifold of real dimension 4, agreeing at infinity with a dihedral quotient of $\mathbb{C}^2 = (\mathbb{R}^4, I_1)$ with Taub-NUT metric, "up to order $(3, \alpha, \beta)$". The precise meaning of this assertion, and the definition of weighted Hölder spaces $C^{3,\alpha}_{\beta+2}$ and $C^{5,\alpha}_{\beta}$, are given below, when using Theorem 0.4.

Comments. — We should start with some words on previous constructions of ALF dihedral gravitational instantons. As mentioned above, $\mathcal{D}_k$ ALF instantons are known to exist since the works [CK1, CK2], where such spaces are produced as *moduli spaces of solutions to Nahm's equations or of singular monopoles*; they have moreover been described in an explicit manner in [CH], via *generalised Legendre transform* and *twistor theory*. Despite of this, and the fact that in both cases the underlying spaces are Kronheimer's instantons, due to the difference in the methods of construction, we were not able to show that these previous examples and our examples coincide. However, it seems highly probable that in fact, *both constructions produce the same families of ALF hyperkähler metrics*; more precisely, we believe that *these constructions are (almost) exhaustive, in the sense that any ALF dihedral gravitational instanton, except Atiyah-Hitchin's* $\mathcal{D}_0$*-instanton, fits into the produced examples* (up to a tri-holomorphic isometry): this folklore conjecture, establishing a strong link between ALE and ALF instantons, is the analogue of the classification of [Min3]. Such a classification seems nonetheless delicate to avoid in affirming that both constructions actually coincide, and we hope that some of the analytic aspects of our construction, and especially the ALF Calabi-Yau type theorem, could be of some help on that matter.

More closely to the statement of Theorem 0.2, notice that it is *not of a perturbative nature*: this corresponds to taking the parameter m in the whole range $(0, +\infty)$. The price to pay is somehow that so far, we do not control what happens when m goes to 0. We conjecture that the ALF hyperkähler structure converges back, in C^∞_{loc}-topology, to the initial ALE one, as is the case on $\mathbb{C}^2$; this question will be handled in a future article.

Now in Theorem 0.3, the existence of the first order variation terms h_X, ι_1^X and ϖ_1^X is of course not new, as they already appear along Kronheimer's construction [Kro1]. What is new though is their explicit determination, which we could only find in the simplest case of the Eguchi-Hanson space (see e.g. [Joy, p.153]), i.e.

when $\Gamma = \mathcal{A}_2 = \{\pm\,\mathrm{id}_{\mathbb{C}^2}\}$. Notice at this point that as suggested by the statement of Theorem 0.3, the shapes of h_X, ι_1^X and ϖ_1^X follow a general pattern which is only slightly affected by the order of the group Γ; up to a multiplicative constant, we can indeed compute them on the explicit Eguchi-Hanson example. We think moreover that Theorem 0.3 is of further interest; for instance, the order of precision it brings could be useful in more general gluing constructions.

To conclude, we comment briefly Theorem 0.4. This result comes within the general scope of generalising the celebrated Calabi-Yau theorem [Yau] to non-compact manifolds, initiated by Tian and Yau [TY1, TY2]. A statement similar to ours can be found in [Hei, Prop. 4.1], in a more general and abstract framework. One interest of our result, nonetheless, simply lies in the fact that although we ask more precise asymptotics on our data than Hein does, we get in compensation sharper asymptotics on our solution φ, which echo in the asymptotics of Theorem 0.2.

Organisation of the article. — This paper is divided into three parts, corresponding respectively to Theorems 0.2, 0.3 and 0.4, plus an appendix. Part 1 is devoted to the proof of Theorem 0.2. We first draw in section 1.1 a detailed program of construction of our hyperkähler ALF metrics, leading us to the expected result (Theorem 1.3). In section 1.2 are recalled essential facts on the Taub-NUT metric, seen as a Kähler metric on $\mathbb{C}^2$. The construction itself occupies section 1.3 and 1.4; it consists into a gluing of the "non-compact part" of the Taub-NUT metric with the "compact part" of the ALE metric of some ALE instanton, which we subsequently correct into a Ricci-flat metric thanks to Theorem 0.4. The concluding section 1.5, mainly computational, deals with the proof of two technical lemmas useful to our construction.

In Part 2, which is mostly independent of Part 1, after recalling some basic facts about Kronheimer's construction of ALE instantons, we state Theorem 2.1, which is a specified version of Theorem 0.3 – in particular we give the promised explicit formulas (section 2.1). We give further details on Kronheimer's construction and classification in section 2.2, where we also fix the diffeomorphism of Theorem 0.3. Then we compute the tensors h_X, ι_1^X and ϖ_1^X in section 2.3; using similar techniques, we show in the following section 2.4 that the precision of the asymptotics is automatically improved when Γ is binary dihedral, tetrahedral, octahedral or icosahedral. We develop in the last section 2.5 few informal digressive considerations on the approximation of complex structures of certain ALE instantons by the standard I_1, relied on links observed in the construction of Part 1.

Part 3 is devoted to the proof of Theorem 0.4, which uses a *continuity method.* The method is explained in section 3.1; the required analysis is done in the following section, and the conclusion is given in the last section.

Finally, the appendix gives a short account of a description of the Taub-NUT metric on $\mathbb{C}^2$ suggested by C. LeBrun [LeB].

Throughout all the article, $\mathbb{C}^2$ stands for $(\mathbb{R}^4, I_1)$ with I_1 the standard complex structure given by the coordinates $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$; we denote by I_2 and I_3 the other two standard complex structures on $\mathbb{R}^4 \cong \mathbb{H}$, given respectively by the coordinates $(x_1 + ix_3, x_4 + ix_2)$ and $(x_1 + ix_4, x_2 + ix_3)$.

1 Construction of ALF hyperkähler metrics

1.1 Strategy of construction

Outline of the strategy. — As described in [LeB] and as we shall see in next section, one can describe the Taub-NUT metric on $\mathbb{R}^4$ as a $\mathcal{D}_k$-invariant hyperkähler metric with volume form the standard euclidean one $\Omega_{\mathbf{e}}$, Kähler for the standard complex structure I_1, and compute a somehow explicit potential, φ say, for it.

Now, given one of Kronheimer ALE gravitational instantons $\big(X, g_X, I_1^X, I_2^X, I_3^X\big)$ modelled on $\mathbb{R}^4/\mathcal{D}_k$, we have a diffeomorphism Φ_X between infinities of X and $\mathbb{R}^4/\mathcal{D}_k$ such that $\Phi_{X*}g_X$ is asymptotic to the standard euclidean metric $\mathbf{e}$, and $\Phi_{X*}I_1^X$ is asymptotic to I_1. It is this way quite natural to try and take $dI_1^X d(\Phi_X^*\varphi)$ glued with g_X as an ALF metric on X, before we correct it into a hyperkähler metric. This naive idea works in a straightforward manner when $\big(X, I_1^X\big)$ is *a minimal resolution of* $(\mathbb{C}^2/\mathcal{D}_k, I_1)$ *and* Φ_X *the associated map.* However this fails in the general case, where $\big(X, I_1^X\big)$ is a *deformation of* $(\mathbb{C}^2/\mathcal{D}_k, I_1)$, without further precautions: the size of the Taub-NUT potential φ, roughly of order r^4 *as well as its euclidean derivatives*, together with the error term $\Phi_{X*}I_1^X - I_1$ on the complex structure, even make wrong the assertion that the rough candidate $dI_1^X d(\Phi_X^*\varphi)$ is positive – in the sense that $dI_1^X d(\Phi_X^*\varphi)(\cdot, I_1^X \cdot)$ is a metric – near the infinity of X.

Fortunately, up to choosing a different complex structure on X to work with, we can make the appropriate corrections on φ so as to get a good enough ALF metric on X to start with, and then run the same machinery as in the minimal resolution case, up to minor but yet technical adjustments, so as to end up with Theorem 0.2.

Detailed strategy, and involvements of Kronheimer's instantons asymptotics. — We shall now be more specific about the different steps involved in the program we are following throughout this part.

1. Let SO(3) act on the complex structures of X as follows: for $A = (a_{j\ell}) \in$ SO(3), define the triple $(AI^X)_.$ as

$$(AI^X)_. = \big((AI^X)_j\big)_{j=1,2,3} = \big(a_{j1}I_1^X + a_{j2}I_2^X + a_{j3}I_3^X\big)_{j=1,2,3};$$

then $\big(X, g_X, (AI^X)_1, (AI^X)_2, (AI^X)_3\big)$ is again hyperkähler, and is therefore an ALE gravitational instanton modelled on $\mathbb{R}^4/\mathcal{D}_k$.

2. With the model $\mathbb{R}^4/\mathcal{D}_k$ at infinity fixed, Kronheimer's instantons are parametrised [Kro1] by a triple $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathfrak{h} \otimes \mathbb{R}^3 - D$, where $\mathfrak{h}$ is a $(k+2)$-dimensional real vector space endowed with some scalar product $\langle \cdot, \cdot \rangle$, and D is a finite union of spaces $H \otimes \mathbb{R}^3$ with H a hyperplane in $\mathfrak{h}$ (as notation suggests, $\mathfrak{h}$ is a Lie algebra; we will be more specific about its interpretation in part 2). This parametrisation is compatible with the SO(3)-action of Point 1. in the sense that if ζ is the parameter associated to $\big(X, g_X, I_1^X, I_2^X, I_3^X\big)$, and if $\big(Y, g_Y, I_1^Y, I_2^Y, I_3^Y\big)$ is the instanton associated to $A\zeta$, defined by:

$$A\zeta = \big((A\zeta)_j\big)_{j=1,2,3} = \big(a_{j1}\zeta_1 + a_{j2}\zeta_2 + a_{j3}\zeta_3\big)_{j=1,2,3}, \tag{3}$$

then there is a tri-holomorphic isometry between $\big(Y, g_Y, I_1^Y, I_2^Y, I_3^Y\big)$ and $\big(X, g_X, (AI^X)_1, (AI^X)_2, (AI^X)_3\big)$: this is Lemma 2.3, stated and proved in Part 2. Defined this way, $A\zeta$ is of course still in $\mathfrak{h} \otimes \mathbb{R}^3 - D$; otherwise $A\zeta \in H \otimes \mathbb{R}^3$ for one of the hyperplanes H mentioned above, and thus $\zeta = A^t(A\zeta) \in H \otimes \mathbb{R}^3$, which would be absurd.

3. In general, one can take the diffeomorphism Φ_X between infinities of X and $\mathbb{R}^4/\mathcal{D}_k$ so that $\Phi_{X*}I_1^X - I_1 = O(r^{-4})$ with according decay on derivatives, which is not good enough for the construction we foresee. We can nonetheless improve the precision thanks to the following two lemmas:

   Lemma 1.1 *If $\xi \in \mathfrak{h} \otimes \mathbb{R}^3 - D$ is such that $|\xi_2|^2 - |\xi_3|^3 = \langle \xi_2, \xi_3 \rangle = 0$, and $\big(Y, g_Y, I_1^Y, I_2^Y, I_3^Y\big)$ is the associated ALE instanton, then one can choose Φ_Y such that there exists a diffeomorphism $\beth = \beth_\xi$ of $\mathbb{R}^4$ commuting with the action of $\mathcal{D}_k$, and such that:*

$$\Big|(\nabla^{\mathbf{e}})^\ell\big(\Phi_{Y*}I_1^Y - \beth^* I_1\big)\Big|_{\mathbf{e}} = O(r^{-8-\ell}) \quad \textit{for all } \ell \geq 0.$$

   Moreover, the shape of $\beth$ is given by: $\beth(z_1, z_2) = \big(1 + \frac{a}{\kappa + r^4}\big)(z_1, z_2)$, where $\kappa, a \in \mathbb{R}$, and (z_1, z_2) are the standard complex coordinates on $(\mathbb{C}^2, I_1)$, and $\big|(\nabla^{\mathbf{e}})^\ell\big(\Omega_{\mathbf{e}} - \beth^*\Omega_{\mathbf{e}}\big)\big|_{\mathbf{e}} = O(r^{-8-\ell})$ for all $\ell \geq 0$.

   Lemma 1.2 *For any $\zeta \in \mathfrak{h} \otimes \mathbb{R}^3$, there exists $A \in SO(3)$ such that $|(A\zeta)_2|^2 - |(A\zeta)_3|^2 = \langle (A\zeta)_2, (A\zeta)_3 \rangle = 0$.*

   Lemma 1.1, *which relies on our analysis of the asymptotics of Kronheimer's instantons*, is proved in section 1.3, assuming a general statement for these asymptotics that is seen in Part 2; Lemma 1.2, which is elementary, is proved at the end of this section.

4. Recall that ζ is the parameter of our given instanton X. We choose A as in Lemma 1.2, consider the instanton Y associated to $\xi = A\zeta \in \mathfrak{h} \otimes \mathbb{R}^3 - D$, and perform the gluing of the Kähler forms and correct a prototypical ALF metric into a hyperkähler metric, with the potential $\varphi^\flat := \beth^*\varphi$ instead of φ. Thanks to the better coincidence of the complex structures, the rough candidate $dI_1^Y d(\Phi_Y^* \varphi^\flat)$ is now positive at infinity, and actually also rather close to $\mathbf{f}^\flat := \beth^*\mathbf{f}$, with $\mathbf{f}$ the Taub-NUT metric on $\mathbb{R}^4$, Kähler for I_1.

   We should moreover specify here that the gluing also requires a precise description of the Kähler form $\omega_1^Y := g_Y\big(I_1^Y\cdot,\cdot\big)$, *which is again part of the analysis of the asymptotics of Kronheimer's ALE instantons.*

   We get this way after corrections a Ricci-flat, actually a hyperkähler, manifold $\big(Y, g'_Y, I_1^Y, J_2^Y, J_3^Y\big)$, with $\Phi_{Y*}g'_Y$ asymptotic to $\mathbf{f}^\flat$, and $\big[g'_Y(I_1^Y\cdot,\cdot)\big] = [g_Y(I_1^Y\cdot,\cdot)]$; the construction also gives $\big[g'_Y(J_j^Y\cdot,\cdot)\big] = [g_Y(I_j^Y\cdot,\cdot)]$, $j = 2, 3$.

5. We let $A^t = A^{-1}$ act back on the previous data to come back to X, and end up with a hyperkähler manifold $\big(X, g'_X, J_1^X, J_2^X, J_3^X\big)$, with $\big[g'_X(J_j^X\cdot,\cdot)\big] = [g_X(I_j^X\cdot,\cdot)]$, $j = 1, 2, 3$, and $\Phi_{X*}g'_X$ asymptotic to $\mathbf{f}^\flat$, provided that Φ_X is the composite of Φ_Y and the tri-holomorphic isometry of Point 2 .

We shall also add that we can play on the metric $\mathbf{f}$ in this construction. Indeed, $\mathbf{f}$ is invariant under some fixed circle action on $\mathbb{R}^4$, and the length for $\mathbf{f}$ of the fibres of this action tends to some constant $L > 0$ at infinity. We can make this length vary in the whole $(0, \infty)$ and keep the same volume form for $\mathbf{f}$; given $m \in (0, \infty)$ that we call the "*mass parameter*", we then denote by $\mathbf{f}_m$ the Taub-NUT metric giving length $L(m) = \pi\sqrt{\frac{2}{m}}$ to the fibres at infinity, and of volume form $\Omega_{\mathbf{e}}$ (the choice of the parameter m instead of L will become clear in next section).

We can then sum our construction up by the following statement, which is the main result of this part, and is a specified version of Theorem 0.2:

Theorem 1.3 *Consider an ALE gravitational instanton $\big(X, g_X, I_1^X, I_2^X, I_3^X\big)$ modelled on $\mathbb{R}^4/\mathcal{D}_k$. Then there exists a one-parameter family $\big(g'_{X,m}, J_{1,m}^X, J_{2,m}^X, J_{3,m}^X\big)$ of smooth hyperkähler metrics on X such that, for any fixed $m \in (0, \infty)$:*

- *the equality $\big[g'_{X,m}\big(J_{j,m}^X\cdot,\cdot\big)\big] = \big[g_X\big(I_j^X\cdot,\cdot\big)\big]$ of Kähler classes holds for $j = 1, 2, 3$;*

- *$g'_{X,m}$ and g_X have the same volume form;*

- *$g'_{X,m}$ is ALF in the sense that one has the asymptotics*

$$\Big|(\nabla^{\mathbf{f}_m^\flat})^\ell\big(\Phi_{X*}g'_{X,m} - \mathbf{f}_m^\flat\big)\Big|_{\mathbf{f}_m^\flat} = O\big(R^{-1-\delta}\big), \qquad \ell = 0, 1,$$

  for any $\delta \in (0, 1)$, and that $\mathrm{Rm}^{g'_{X,m}}$ has cubic decay at infinity.

Here R is a distance function for $\mathbf{f}_m^\flat$, and Φ_X is an ALE diffeomorphism between infinities of X and $\mathbb{R}^4/\mathcal{D}_k$, in the sense that $|\Phi_{X*}g_X - \mathbf{e}|_{\mathbf{e}}$ and $|\Phi_{X*}I_j^X - I_j|_{\mathbf{e}}$ are $O(r^{-4})$, with according decay on derivatives.

In this statement, $\mathbf{f}_m^\flat = \daleth^*\mathbf{f}_m$, where $\daleth = \daleth_{A\zeta}$ is given by Lemma 1.1, $\zeta \in \mathfrak{h}\otimes\mathbb{R}^3 - D$ is the parameter associated to $\big(X, g_X, I_1^X, I_2^X, I_3^X\big)$, and A is chosen as in Lemma 1.2. There might a slight ambiguity here, since different $A \in \mathrm{SO}(3)$ could do – namely, given ζ as in Lemma 1.2, there may be many A satisfying its conclusions; we will see however in Remarks 1.4 and 1.9 that $\daleth$ as we construct it is not affected by this choice.

Points 1. and 5. above do not need further developments. We postpone the triholomorphic isometry of Point 2. to Part 2, paragraph 2.2.1, as it is easier to tackle with a few further notions on Kronheimer's classification of ALE gravitational instantons. As for Point 3., as mentioned already, the proof of 1.1 is given in section 1.3 assuming results from Part 2; apart from the proof of Lemma 1.2 which we shall settle now, our main task in the current part is thus the gluing and the subsequent corrections stated in Point 4., to which we devote sections 1.3 and 1.4 below, after recalling a few useful facts on the Taub-NUT metric seen as a Kähler metric on $(\mathbb{C}^2, I_1)$ in next section.

Proof of Lemma 1.2. For $\zeta \in \mathfrak{h}\otimes\mathbb{R}^3$, define the matrix $Z(\zeta) = \big(\langle\zeta_j,\zeta_\ell\rangle\big)_{1\le j,\ell\le 3}$ of its scalar products. It is elementary matrix calculus to check that the $\mathrm{SO}(3)$-action defined by (3), and referred to in the statement of the lemma, translates into: $Z(A\zeta) = AZ(\zeta)A^t$.

Fixing ζ, we thus want to find $A \in \mathrm{SO}(3)$ such that $AZ(\zeta)A^t$ has shape:

$$\begin{pmatrix} \mu & * & * \\ * & \lambda & 0 \\ * & 0 & \lambda \end{pmatrix}. \tag{4}$$

Since $Z = Z(\zeta)$ is symmetric, there exists $O \in \mathrm{SO}(3)$ such that $OZO^t = \mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)$, and we now look for $Q \in \mathrm{SO}(3)$ such that $Q\,\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)Q^t$ has shape (4); setting then $A = QO$ leads us to the conclusion. If two of the λ_j are the same then we are done, up to letting act one of the permutation matrices $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{smallmatrix}\right)$. Up to this action again, we can therefore assume $\lambda_1 > \lambda_2 > \lambda_3$.

Setting

$$Q = \begin{pmatrix} \left(\frac{\lambda_1-\lambda_2}{\lambda_1-\lambda_3}\right)^{1/2} & 0 & \left(\frac{\lambda_2-\lambda_3}{\lambda_1-\lambda_3}\right)^{1/2} \\ 0 & 1 & 0 \\ -\left(\frac{\lambda_2-\lambda_3}{\lambda_1-\lambda_3}\right)^{1/2} & 0 & \left(\frac{\lambda_1-\lambda_2}{\lambda_1-\lambda_3}\right)^{1/2} \end{pmatrix},$$

a direct computation gives: $Q\,\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)Q^t = \left(\begin{smallmatrix}\lambda_1+\lambda_3-\lambda_2 & 0 & -\Lambda\\ 0 & \lambda_2 & 0\\ -\Lambda & 0 & \lambda_2\end{smallmatrix}\right)$, where $\Lambda = (\lambda_1-\lambda_2)^{1/2}(\lambda_2-\lambda_3)^{1/2}$. $\square$

Remark 1.4 *Our choice for Q is a little arbitrary; however,one can show that the only possibilities for writing $Q\,\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)Q^t$, that is, AZA^t, under shape* (4) *are the $\left(\begin{smallmatrix}\lambda_1+\lambda_3-\lambda_2 & \Lambda\cos\phi & \Lambda\sin\phi\\ \Lambda\cos\phi & \lambda_2 & 0\\ \Lambda\sin\phi & 0 & \lambda_2\end{smallmatrix}\right)$, $\phi\in\mathbb{R}$, and again $\lambda_1\geq\lambda_2\geq\lambda_3$.*

1.2 The Taub-NUT metric as a Kähler metric on $(\mathbb{C}^2, I_1)$

Before we proceed to the gluing of the Taub-NUT metric with the ALE metric of one of Kronheimer's instantons, we recall a few facts about this very Taub-NUT metric on $\mathbb{C}^2$, that will be used in the analytic upcoming sections 1.3 and 1.4. Our main reference here are [GH, LeB].

1.2.1 Gibbons-Hawking versus LeBrun ansätze

Gibbons-Hawking ansatz. — As recalled in the Introduction, the Taub-NUT metric on $\mathbb{R}^4$ is often described via the Gibbons-Hawking ansatz as follows: given $m\in(0,\infty)$, set

$$\mathbf{f}_m = V(dy_1^2+dy_2^2+dy_3^2)+V^{-1}\eta^2,$$

where (y_1,y_2,y_3) is a circle fibration of $\mathbb{R}^4\backslash\{0\}$ over $\mathbb{R}^3\backslash\{0\}$, V is the function $\frac{1+4mR}{2R}$ (harmonic in the y_j coordinates) with $R^2=y_1^2+y_2^2+y_3^2$, and where η is a connection 1-form for this fibration such that $d\eta = *_{\mathbb{R}^3}dV$. Thus defined, the metric $\mathbf{f}_m$ confers length $\pi\sqrt{2/m}$ to the fibres at infinity, and is hermitian for the almost-complex structures

$$J_a : \begin{cases} V dy_a \longmapsto \eta, \\ \quad dy_b \longmapsto dy_c, \end{cases}$$

with $(a,b,c)\in\big\{(1,2,3),(2,3,1),(3,1,2)\big\}$. These are in fact complex structures, verifying the quaternionic relations $J_aJ_bJ_c=-1$, for which $\mathbf{f}_m$ is Kähler, thanks to the harmonicity of V: $\mathbf{f}_m$ is thus hyperkähler. One checks moreover that this way, the metric $\mathbf{f}_m$ and the complex structures extend as such through $0\in\mathbb{R}^4$.

We now switch point of view to a description better adapted to our construction.

LeBrun's potential. — As depicted in [LeB] and reviewed in detail in Appendix A, one can give a somehow more concrete support of the description of $\mathbf{f}_m$, through which the complex structure J_1 mentioned above is the standard I_1 on $\mathbb{C}^2$, and

$\mathrm{vol}^{\mathbf{f}_m} = \Omega_{\mathbf{e}}$, the standard euclidean volume form. One starts with the following implicit formulas:

$$
\begin{aligned}
|z_1| &= e^{m(u^2-v^2)}u,\\
|z_2| &= e^{m(v^2-u^2)}v,
\end{aligned}
\tag{5}
$$

defining functions $u, v : \mathbb{C}^2 \to \mathbb{R}$, invariant under the circle action $e^{i\theta} \cdot (z_1, z_2) = (e^{i\theta}z_1, e^{-i\theta}z_2)$ which makes $\mathbb{S}^1$ as a subgroup of $\mathrm{SU}(2)$; notice the role of m in these formulas, which enlightens our choice of taking it as the parameter of the upcoming construction. One then sets $y_1 = \frac{1}{2}(u^2 - v^2)$, $y_2 + iy_3 = -iz_1z_2$, $R = \frac{1}{2}(u^2 + v^2) = \big(y_1^2 + y_2^2 + y_3^2\big)^{1/2}$; these are $\mathbb{S}^1$-invariant functions, making (y_1, y_2, y_3) as a principal-$\mathbb{S}^1$ fibration $\mathbb{C}^2 \to \mathbb{R}^3$ away from the origins. One finally defines:

$$
\varphi_m := \frac{1}{4}\big(u^2 + v^2 + m(u^4 + v^4)\big) = \frac{1}{2}\big(R + m(R^2 + y_1^2)\big).
\tag{6}
$$

One can then check – see Appendix A – that $dd^c_{I_1}\varphi_m$ is positive is the sense of I_1-hermitian 2-forms, and that $(dd^c_{I_1}\varphi_m)^2 = 2\Omega_{\mathbf{e}}$. If one sets moreover $V = \frac{1+4mR}{2R}$, and $\eta = I_1 V dy_1$, noticing by passing that η is then a connection 1-form for the fibration with $d\eta = *_{\mathbb{R}^3} dV$, one has: $\mathbf{f}_m := V(dy_1^2 + dy_2^2 + dy_3^2) + V^{-1}\eta^2 = (dd^c_{I_1}\varphi_m)(\cdot, I_1\cdot)$. This metric is well-defined at $0 \in \mathbb{C}^2$, as $(dd^c_{I_1}\varphi_m)(\cdot, I_1\cdot) = \mathbf{e}$ at that point.

The metric $\mathbf{f}_m$ is therefore Kähler for I_1 with volume form $\mathrm{vol}^{\mathbf{f}_m} = \Omega_{\mathbf{e}}$ on the whole $\mathbb{C}^2$; by the standard properties of Kähler metrics, it is thus Ricci-flat. One recovers a complete hyperkähler data after checking that the defining equations

$$
\mathbf{f}_m(J_j\cdot, \cdot) = \omega_j^{\mathbf{e}}, \qquad \text{where} \qquad \omega_j^{\mathbf{e}} = \mathbf{e}(I_j\cdot, \cdot), \qquad j = 2, 3,
\tag{7}
$$

with I_2, I_3 the other two standard complex structures on $\mathbb{R}^4 \cong \mathbb{H}$, give rise to integrable complex structures, verifying respectively $J_j : V dy_j \mapsto \eta$, $dy_k \mapsto dy_\ell$ for $(j, k, \ell) = (2, 3, 1), (3, 1, 2)$, as well as the quaternionic relations together with I_1.

Let us now give a look at the length of the $\mathbb{S}^1$-fibres at infinity. Consider the vector field $\xi := i\big(z_1\frac{\partial}{\partial z_1} - \overline{z_1}\frac{\partial}{\partial \overline{z_1}} - z_2\frac{\partial}{\partial z_2} + \overline{z_2}\frac{\partial}{\partial \overline{z_2}}\big)$ giving the infinitesimal action of $\mathbb{S}^1$. One has $dy_1(-I_1\xi) = V^{-1}$, thus $\eta(\xi) = 1$, and $dy_j(\xi) = 0$, $j = 1, 2, 3$; since R is $\mathbb{S}^1$-invariant, the length of the fibres is just $2\pi V^{-1/2}$, which tends to $\pi\sqrt{2/m}$.

Remark 1.5 *Even if we can let m vary, this description actually leads to essentially one metric; indeed, if κ_s is the dilation of factor $s > 0$ of $\mathbb{R}^4$, one gets with help of formulas* (5) *and* (6) *the following homogeneity property: $\kappa_s^*\mathbf{f}_m = s^2\mathbf{f}_{ms^2}$, which is of course coherent with the length of the fibres at infinity and the fact that* $\mathrm{vol}^{\mathbf{f}_m} = \mathrm{vol}^{\mathbf{f}_{ms^2}} = \Omega_{\mathbf{e}}$.

From now on, we see the mass parameter m as fixed, and we drop the indices m when there is no risk of confusion.

The Taub-NUT metric and the action of the binary dihedral group on $\mathbb{C}^2$. — For $k \geq 2$, which we fix until the end of this part, the action of the *binary dihedral group $\mathcal{D}_k$ of order* $4k$ seen as a subgroup of $\mathrm{SU}(2) = \mathrm{Sp}(1)$ is generated by the matrices $\zeta_k := \begin{pmatrix} e^{i\pi/k} & 0 \\ 0 & e^{-i\pi/k} \end{pmatrix}$ and $\tau := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. One has: $\zeta_k^* y_j = y_j$, $j = 1, 2, 3$, and thus $\zeta_k^* R = R$, and $\zeta_k^* \eta = \eta$, whereas: $\tau^* y_j = -y_j$, $j = 1, 2, 3$, thus $\tau^* R = R$, and $\tau^* \eta = -\eta$. The Taub-NUT metric $\mathbf{f}$ is therefore $\mathcal{D}_k$-invariant, and descends smoothly to $(\mathbb{R}^4 \backslash \{0\})/\mathcal{D}_k$: this is the metric we are going to glue at infinity of $\mathcal{D}_k$-ALE instantons in the next section. Before though, we need a few more analytical tools for the Taub-NUT metric as we describe it here.

1.2.2 Orthonormal frames, covariant derivatives and curvature

In addition to the above relations between the vector field ξ, and the 1-forms η and dy_j, $j = 1, 2, 3$, one has that the data

$$(e_0, e_1, e_2, e_3) := \big(V^{1/2}\xi, -I_1 V^{1/2}\xi, V^{-1/2}\zeta, V^{-1/2} I_1 \zeta\big), \tag{8}$$

is the dual frame of the orthonormal frame of 1-forms

$$(e_0^*, e_1^*, e_2^*, e_3^*) := \big(V^{-1/2}\eta, V^{1/2}dy_1, V^{1/2}dy_2, V^{1/2}dy_3\big) \tag{9}$$

on $\mathbb{C}^2 \backslash \{0\}$, provided that the vector field ζ is defined by:

$$\zeta := \frac{1}{2iR}\Big(e^{4my_1}\big(z_2 \frac{\partial}{\partial \overline{z_1}} - \partial \overline{z_2} \frac{\partial}{\partial z_1}\big) + e^{-4my_1}\big(z_1 \frac{\partial}{\partial \overline{z_2}} - \partial \overline{z_1} \frac{\partial}{\partial z_2}\big)\Big), \tag{10}$$

see Appendix A; we keep the notations $(e_j)_{j=0,\ldots,3}$ and $(e_j^*)_{j=0,\ldots,3}$ throughout this part. An explicit computation made in Appendix A then gives the estimates

$$\big|(\nabla^{\mathbf{f}})^\ell e_j\big|_{\mathbf{f}} = O\big(R^{-1-\ell}\big) \quad \text{near infinity for all } \ell \geq 1 \text{ and } j = 0, \ldots, 3.$$

Consequently, for all $\ell \geq 0$, $\big|(\nabla^{\mathbf{f}})^\ell \operatorname{Rm}^{\mathbf{f}}\big|_{\mathbf{f}} = O\big(R^{-3-\ell}\big)$ – this justifies the terminology "*Asymptotically Locally Flat*" for $\mathbf{f}$; this estimate, done using the Gibbons-Hawking ansatz, can also be found e.g. in [Min1, §1.0.3].

We close this section by two further useful estimates, giving an idea of the geometric gap between $\mathbf{e}$ and $\mathbf{f}$: first, at the level of distance functions, rearranging formulas (5) gives: $R \leq 2r^2$, which is sharp is general; second, there exists $C = C(m) > 0$ such that outside the unit ball of $\mathbb{C}^2$, $C^{-1}r^{-2}\mathbf{e} \leq \mathbf{f} \leq Cr^2\mathbf{e}$, which, again, is sharp in general. Details are given in Section A.2, in Appendix A.

1.3 Gluing the Taub-NUT metric to an ALE metric

As is usual when gluing Kähler metrics, we shall work on potentials to glue the ALF model-metric to an ALE one. The previous section gives us the potential φ for the ALF metric (equation (6)); the following paragraph provides us a sharp enough potential for the ALE metric.

1.3.1 Approximation of the ALE Kähler form as a complex hessian

Asymptotics of the Kähler form and the complex structure. — In view of Step 3. and 4. of the program developed in section 1.1, since we are performing our gluing on some specific ALE instantons, we fix *for the rest of this part*

$$\xi \in \mathfrak{h} - D, \qquad \text{such that:} \quad |\xi_2|^2 - |\xi_3|^2 = \langle \xi_2, \xi_3 \rangle = 0, \tag{11}$$

and consider the associated ALE instanton $\big(Y, g_Y, I_1^Y, I_2^Y, I_3^Y\big)$. Lemma 1.1 gives an ALE diffeomorphism $\Phi_Y : Y \backslash K \to (\mathbb{R}^4 \backslash B)/\mathcal{D}_k$, where K is some compact subset of Y and B a ball in $\mathbb{R}^4$ centred at the origin; recall that by "ALE diffeomorphism" we mean that for all $\ell \geq 0$,

$$\big|(\nabla^{\mathbf{e}})^{\ell}(\Phi_{Y*} g_Y - \mathbf{e})\big|_{\mathbf{e}} = O(r^{-4-\ell}),$$

and likewise on the complex structures. Before using the more specific properties of Φ_Y at the level of complex structures, let us mention the following: we want to proceed to a gluing of Kähler metrics, and the convenient way of doing so is to glue *the Kähler forms, via their potentials*. We already have a candidate for the potential of an ALF metric at infinity at hand: as evoked, this would be $\Phi_{Y*}\varphi^{\flat}$ (see Point 4. in section 1.1). Conversely, we need to kill the ALE metric near infinity, and for this we want a sharp enough potential, in a sense that we make clear below, see Proposition 1.10. We thus need for this a sharp knowledge of *the Kähler form* $\omega_1^Y := g_Y(I_1^Y \cdot, \cdot)$, and since we are about to compute I_1^Y-complex hessians as well, we also need a precise description of the complex structure I_1^Y. These are given by the following, from which Lemma 1.1 actually follows as we shall see at the end of this section, with the same Φ_Y:

Lemma 1.6 *One can choose the ALE diffeomorphism Φ_Y such that*

$$\Phi_{Y*}\omega_1^Y = \omega_1^{\mathbf{e}} - c\big(|\xi_1|^2\theta_1 + \langle \xi_1, \xi_2 \rangle \theta_2 + \langle \xi_1, \xi_3 \rangle \theta_3\big) + O(r^{-8}) \tag{12}$$

where $c > 0$ is some universal constant, $\theta_j = \frac{1}{4} dd^c_{I_j}\big(r^{-2}\big)$, $j = 1, 2, 3$, on the one hand, and if ι_1^Y denotes $\Phi_{Y*}I_1^Y - I_1$, then it is given by:

$$\mathbf{e}(\iota_1^Y \cdot, \cdot) = -c\big(|\xi_2|^2 + |\xi_3|^2\big)\frac{rdr \cdot \alpha_1}{r^6} + O(r^{-8})$$

where c is the same constant as above and $\alpha_1 = I_1 rdr$, on the other hand.

We can moreover assume that $\Phi_{Y*}\Omega_Y = \Omega_{\mathbf{e}}$, where $\Omega_Y = \mathrm{vol}^{g_Y}$.

In this statement the error terms $O(r^{-8})$ are understood in the "euclidean way", namely for any $\ell \geq 0$, the ℓth $\nabla^{\mathbf{e}}$-derivatives of these tensors are $O(r^{-8-\ell})$. This lemma requires further notions on Kronheimer's construction, and is more precisely a direct application of Theorem 2.1 of Part 2 to $Y = X_\xi$ with ξ verifying (11). Notice however the error term order -8, whereas one would expect -6, if one thinks for instance about the Eguchi-Hanson metric ([Joy, Ex. 7.2.2]); this estimate is crucial in proving Lemma 1.1, and is specific to (groups containing) dihedral binary groups. Besides, the assertion on the volume forms is only needed in next paragraph.

Approximating ω_1^Y as an I_1^Y-complex hessian. — We shall see for now how Lemma 1.6 allows us to approximate the Kähler form ω_1^Y as an I_1^Y-complex hessian, with respect to the Taub-NUT metric pushed-forward to Y:

Proposition 1.7 *Take Φ_Y as in Lemmas 1.1 and 1.6, and denote by $\tilde{\mathbf{f}}$ a smooth extension of ${\Phi_Y}^*\mathbf{f}$ on Y. Then there exists a function Ψ on Y such that near infinity,*

$$\big|(\nabla^{\tilde{\mathbf{f}}})^\ell\big(\omega_1^Y - dd^c_{I_1^Y}\Psi\big)\big|_{\tilde{\mathbf{f}}} = O(R^{-2}), \qquad \ell = 0, 1, 2. \tag{13}$$

More precisely, Ψ can be decomposed as a sum ${\Phi_Y}^*\Psi_{\mathrm{euc}} + {\Phi_Y}^*\Psi_{\mathrm{mxd}}$, where on the one hand, $\Psi_{\mathrm{euc}} = O(r^2)$, $\big|\Psi_{\mathrm{euc}}\big|_{\mathbf{e}} = O(r)$, and

$$\big|(\nabla^{\mathbf{e}})^\ell\big(\omega_1^{\mathbf{e}} - c|\xi_1|^2\theta_1 - dd^c_{\Phi_{Y*}I_1^Y}\Psi_{\mathrm{euc}}\big)\big|_{\mathbf{e}} = O(r^{-8-\ell}) \qquad \text{for all } \ell \geq 0, \tag{14}$$

and on the other hand, Ψ_{mxd}, $\big|d\Psi_{\mathrm{mxd}}\big|_{\mathbf{f}} = O(R^{-1})$, and

$$\big|(\nabla^{\mathbf{f}})^\ell\big(-c(\langle\xi_1,\xi_2\rangle\theta_2 + \langle\xi_1,\xi_3\rangle\theta_3) - dd^c_{\Phi_{Y*}I_1^Y}\Psi_{\mathrm{mxd}}\big)\big|_{\mathbf{f}} = O(R^{-2}), \quad \ell = 0, 1, 2. \tag{15}$$

Proof. Notice that once the statement on Ψ_{euc} (the "euclidean component" of Ψ) and Ψ_{mxd} (the "mixed component") are known, estimates (13) follow at once by transposition to Y of estimates (14) and (15) and of the expansion of ω_1^Y stated in Lemma 1.6, keeping the following fact in mind:

Fact 1 *If α is a tensor of type $(2,0)$, $(1,1)$ or $(0,2)$ such that $\big|(\nabla^{\mathbf{e}})^\ell\alpha\big|_{\mathbf{e}} = O(r^{-2a-\ell})$, $a \geq 1$, $\ell = 0, 1, 2$, on $\mathbb{R}^4$, then $\big|(\nabla^{\mathbf{f}})^\ell\alpha\big|_{\mathbf{f}} = O(R^{1-a})$, $\ell = 0, 1, 2$.*

This fact takes into account estimates such as $R = O(r^2)$ and $C^{-1}r^{-2}\mathbf{e} \leq \mathbf{f} \leq Cr^2\mathbf{e}$ of Proposition A.9 at level $\ell = 0$, and follows from explcit computations using the formulas of Lemma A.11; these are given in Appendix A.

We hence come to the statements on Ψ_{euc} and Ψ_{mxd}. We consider before starting a large constant K such that the image of Φ_Y is contained in both $\{r \geq$

$K\} \subset \mathbb{R}^4/\mathcal{D}_k$ and $\{R \geq K\} \subset \mathbb{R}^4/\mathcal{D}_k$, and define a cut-off function $\chi : \mathbb{R} \to [0,1]$ such that:

$$\chi(t) = \begin{cases} 0 & \text{if} \quad t \leq K-1, \\ 1 & \text{if} \quad t \geq K, \end{cases}$$

which will be useful when defining functions to be pulled-back to Y via Φ_Y.

The euclidean component Ψ_{euc}. In an asymptotically euclidean setting, a natural first candidate for the potential of a Kähler form is $\frac{1}{4}r^2$. Now remember we are working with I_1^Y – or more exactly with $\Phi_{Y*}I_1^Y$, but we forget about the push-forward here for simplicity of notation; following Lemma 1.6, a straightforward computation gives, near infinity in $\mathbb{R}^4$:

$$\begin{aligned} dd^c_{I_1^Y}\Big(\frac{1}{4}r^2\Big) &= \frac{1}{2}d\big[(I_1+\iota_1^Y)rdr\big] = \frac{1}{2}d\big[\alpha_1 + c(|\xi_2|^2+|\xi_3|^2)r^{-4}\alpha_1 + O(r^{-7})\big] \\ &= \omega_1^{\mathbf{e}} - c(|\xi_2|^2+|\xi_3|^2)\theta_1 + O(r^{-8}), \end{aligned}$$

where the O are understood in the euclidean way. On the other hand observe that $I_1 d(r^{-2}) = -2r^{-4}\alpha_1$, and thus

$$dd^c_{I_1^Y}(r^{-2}) = d\big[(I_1+\iota_1^Y)d(r^{-2})\big] = d\big[-2r^{-4}\alpha_1 + O(r^{-7})\big] = 4\theta_1 + O(r^{-8}).$$

Now define

$$\Psi_{\mathrm{euc}} = \frac{1}{4}\chi(r)\big(r^2 + c(|\xi_2|^2+|\xi_3|^2-|\xi_1|^2)r^{-2}\big);$$

on $\mathbb{R}^4/\mathcal{D}_k$ (it is $\mathcal{D}_k$-invariant); it has support in the image of Φ_Y, has the growth stated in the lemma as well as its differential, and by the previous two estimates we get that $\omega_1^{\mathbf{e}} - c|\xi_1|^2\theta_1 - dd^c_{\Phi_{Y*}I_1^Y}\Psi_{\mathrm{euc}} = O(r^{-8})$ for $\mathbf{e}$ with according decay on the derivatives, as wanted.

The mixed component Ψ_{mxd}. The main reason why we could construct Ψ_{euc} such as to reach estimates (14) is essentially that θ_1 can be realised as an I_1-complex hessian, at least away from 0. Now realising θ_2 and θ_3 as I_1-complex hessians as well does not seem possible: see [Joy, p.202] on that matter. Nonetheless, θ_2 and θ_3 may not be so problematic when looked at via $\mathbf{f}$. We can indeed approximate them precisely enough with respect to this metric by the I_1 or I_1^Y-complex hessians of some well-chosen $\mathcal{D}_k$-invariant functions, provided that we partially leave the euclidean world and use also functions coming from Taub-NUT geometry, e.g. y_1 and R (hence the previous dichotomy "euclidean/mixed"):

Lemma 1.8 *Consider the complex valued function*

$$\psi_c := -2\frac{(y_2+iy_3)\sinh(4my_1)}{r^2R}$$

on $\mathbb{R}^4\backslash\{0\}$*. Then near infinity:*

1. $\left|(\nabla^{\mathbf{f}})^{\ell}\psi_c\right|_{\mathbf{f}} = O(R^{-1})$ *for* $\ell = 0, \dots, 4$*;*
2. $\left|(\nabla^{\mathbf{f}})^{\ell}\big(dd^c_{I_1}\psi_c - (\theta_2 + i\theta_3)\big)\right|_{\mathbf{f}} = O(R^{-2})$ *for* $\ell = 0, 1, 2$*, and these estimates hold for* I_1^Y *as well.*

The proof of this crucial lemma is essentially computational, which is why we postpone it to section 1.5. For now set $\psi_2 = \mathfrak{Re}(\psi_c)$ and $\psi_3 = \mathfrak{Im}(\psi_c)$, and define

$$\Psi_{\mathrm{mxd}} := -c\chi(R)(\langle\xi_1, \xi_2\rangle\psi_2 + \langle\xi_1, \xi_3\rangle\psi_3).$$

In view of Lemma 1.8, such a function, defined on the image of Φ_Y, verifies the growth assertions of Proposition 1.7, as well as the estimates (15): Proposition 1.7 is proved. $\square$

We are now in position to perform the gluing advertised in Point 4. of the program of section 1.1. This is done in next paragraph to which the reader may jump directly, since we conclude the current paragraph by the proof of Lemma 1.1, assuming Lemma 1.6 (and more precisely the assertion on I_1^Y in that statement).

Proof of Lemma 1.1 following Lemma 1.6. We fix Φ_Y as in Lemma 1.6; we work on $\mathbb{R}^4$, and to simplify notations we forget about the push-forwards by Φ_Y.

We are thus looking for a diffeomorphism $\daleth$ of $\mathbb{R}^4$ such that $\left|I_1^Y - \daleth^* I_1\right|_{\mathbf{e}} = O(r^{-8})$, with according decay on euclidean derivatives – until the end of this proof we forget about ALF geometry and stick to the euclidean setting; we will thus content ourselves with using O in this euclidean meaning. An explicit formula is given for $\daleth$ in the statement of Lemma 1.1, which is: $\daleth(z_1, z_2) \mapsto \big(1 + \frac{a}{\kappa + r^4}\big)(z_1, z_2)$ with (z_1, z_2) the standard complex coordinates on $(\mathbb{C}^2, I_1)$; since the value of κ does not affect asymptotic considerations – changing κ only contributes as a $O(r^{-8})$ –, we could thus, up to determining the value of the constant a, simply check that such a $\daleth$ meets our requirement, in light of the asymptotics for I_1^Y stated in Lemma 1.6.

We prefer nonetheless the following more constructive approach. If we are to look for some $\daleth$ as in the statement, we should certainly take it with shape $(z_1, z_2) \mapsto (z_1 + \varepsilon_1, z_2 + \varepsilon_2)$, with $\varepsilon_j = O(r^{-4})$, $j = 1, 2$. The condition $I_1^Y - \daleth^* I_1 = O(r^{-8})$ can be rewritten as a condition on ε_1: $I_1^Y \daleth^* dz_1 = \daleth^*(I_1 dz_1) + O(r^{-8}) = \daleth^*(-idz_1) + O(r^{-8})$, i.e. $I_1^Y(dz_1 + d\varepsilon_1) = -i(dz_1 + d\varepsilon_1) + O(r^{-8})$. Recall the writing $I_1^Y = I_1 + \iota_1^Y$; the previous condition hence gives us: $I_1 dz_1 + I_1 d\varepsilon_1 + \iota_1^Y dz_1 = -i(dz_1 + d\varepsilon_1) + O(r^{-8})$. Since $I_1 dz_1 = -idz_1$ and $I_1 d\varepsilon_1 = I_1(\partial\varepsilon_1 + \overline{\partial}\varepsilon_1) = i(-\partial\varepsilon_1 + \overline{\partial}\varepsilon_1)$ (with ∂ and $\overline{\partial}$ those attached to I_1), the final condition is: $2i\overline{\partial}\varepsilon_1 = -\iota_1^Y dz_1 + O(r^{-8}) = dz_1(\iota_1^Y \cdot) + O(r^{-8})$.

Set $\alpha_j = I_j r dr$, $j = 2, 3$; from Lemma 1.6, $rdr(\iota_1^Y \cdot) = \frac{c(|\xi_2|^2 + |\xi_3|^2)}{r^4}\alpha_1 + O(r^{-7})$, $\alpha_1(\iota_1^Y \cdot) = \frac{c(|\xi_2|^2 + |\xi_3|^2)}{r^4} r dr + O(r^{-7})$, $\alpha_2(\iota_1^Y \cdot), \alpha_3(\iota_1^Y \cdot) = O(r^{-7})$. Hence from the

equality
$$dz_1 = \frac{1}{r^2}\big[z_1(rdr + i\alpha_1) - \overline{z_2}(\alpha_2 + i\alpha_3)\big],$$
we have: $dz_1(\iota_1^Y\cdot) = \frac{c(|\xi_2|^2+|\xi_3|^2)}{r^6} z_1(\alpha_1 + irdr) + O(r^{-8}) = \frac{ic(|\xi_2|^2+|\xi_3|^2)}{r^6}\big(z_1^2 d\overline{z_1} + z_1z_2d\overline{z_2}\big) + O(r^{-8})$. At last we must thus solve
$$2\overline{\partial}\varepsilon_1 = \frac{c(|\xi_2|^2 + |\xi_3|^2)}{r^6}\big(z_1^2 d\overline{z_1} + z_1z_2 d\overline{z_2}\big) + O(r^{-8}).$$
One easily checks that $\varepsilon_1 = -\frac{c(|\xi_2|^2+|\xi_3|^2)z_1}{4r^4}$ is an exact solution. A similar analysis leads us to $\varepsilon_2 = -\frac{c(|\xi_2|^2+|\xi_3|^2)z_2}{4r^4}$, and one checks easily that this way, one has indeed $I_1^Y - \daleth^* I_1 = O(r^{-8})$. The last point to be dealt with is the singularity of the ε_j at 0; one can nonetheless take instead $\varepsilon_j = -\frac{c(|\xi_2|^2+|\xi_3|^2)z_j}{1+4r^4}$, or, better, $\varepsilon_j = \frac{c(|\xi_2|^2+|\xi_3|^2)z_j}{4(\kappa+r^4)}$ with $\kappa \geq 1$ large enough so that $\daleth = \mathrm{id}_{\mathbb{C}^2} + (\varepsilon_1, \varepsilon_2)$ is a diffeomorphism of $\mathbb{C}^2$; we leave it to the reader as an exercise to check that $\kappa = 20c(|\xi_2|^2 + |\xi_3|^2)$ is sufficient.

The estimate $\daleth^*\Omega_{\mathbf{e}} - \Omega_{\mathbf{e}} = O(r^{-8})$ amounts to seeing that $\mathfrak{Re}\big(\frac{\partial\varepsilon_1}{\partial z_1} + \frac{\partial\varepsilon_2}{\partial z_2}\big) = O(r^{-8})$: extend $id(z_1+\varepsilon_1)\wedge d(\overline{z_1}+\overline{\varepsilon_1})\wedge id(z_2+\varepsilon_2)\wedge d(\overline{z_2}+\overline{\varepsilon_2})$, and look at the linear terms in ε_1, ε_2. Since after multiplication by $a := -\frac{c(|\xi_2|^2+|\xi_3|^2)}{4}$ the error would again be $O(r^{-8})$, we can do this computation with $\frac{z_1}{r^4}$ and $\frac{z_2}{r^4}$ playing the respective roles of ε_1 and ε_2. Now $\frac{\partial}{\partial z_j}\big(\frac{z_j}{r^4}\big) = \frac{1}{r^4} - \frac{2|z_j|^2}{r^6}$, $j = 1, 2$. Since these are real, we only need to compute the sum $\frac{\partial}{\partial z_1}\big(\frac{z_1}{r^4}\big) + \frac{\partial}{\partial z_2}\big(\frac{z_2}{r^4}\big)$, which is $\frac{2}{r^4} - \frac{2|z_1|^2}{r^6} - \frac{2|z_2|^2}{r^6} = 0$.

The $\mathcal{D}_k$-invariance of $\daleth$ thus constituted is clear. □

Remark 1.9 *According to the preceding proof, $\daleth$ as we construct it depends only on $c(|\xi_2|^2 + |\xi_3|^2)$. If now ξ is chosen as an $A\zeta$, $A \in \mathrm{SO}(3)$, $\zeta \in \mathfrak{h} - D$, so as to satisfy condition* (11) *as is evoked in Point 3. in the program of section 1.1, by Remark 1.4, $|\xi_2|^2 = |\xi_3|^2$ does not depend on A, and has to be the middle eigenvalue of the matrix $(\langle\zeta_j, \zeta_\ell\rangle)$. Consequently, $\daleth = \daleth_{A\zeta}$ does not depend on $A \in \mathrm{SO}(3)$.*

1.3.2 The gluing

We keep the notations of the previous paragraph: $\big(Y, g_Y, (I_j^Y)_{j=1,2,3}\big)$ is a $\mathcal{D}_k$-ALE instanton with parameter ξ verifying (11), Φ_Y an asymptotic isometry between infinities of Y and $\mathbb{R}^4/\mathcal{D}_k$ fixed by Lemma 1.6, and $\daleth$ is given by Lemma 1.1 which we may also see as as diffeomorphism of $(\mathbb{R}^4\backslash\{0\})/\mathcal{D}_k$.

As alluded above, the form we want to glue $\omega_1^Y = g_Y(I_1^Y\cdot,\cdot)$ with at infinity is $dI_1^Y d\varphi^\flat$, where $\varphi^\flat = \daleth^*\varphi$, with $\varphi = \varphi_m$ is LeBrun's I_1-potential for $\mathbf{f}$ given by (6). We set likewise $\mathbf{f}^\flat = \daleth^*\mathbf{f}$, both on $\mathbb{R}^4$ and its quotient. Recall that $\Psi = \Psi_{\mathrm{euc}} + \Psi_{\mathrm{mxd}}$ is defined in Proposition 1.7 as an approximate I_1^Y-complex potential of ω_1^Y. Next proposition explains how to glue $dI_1^Y d\varphi^\flat$ to ω_1^Y, so as to obtain an ALF metric on Y at the end:

Proposition 1.10 *Take $K\geq 0$ so that the identification Φ_Y between infinities of $\mathbb{R}^4/\mathcal{D}_k$ and Y is defined on $\varphi\geq K$. Consider $r_0\gg 1$, $\beta\in(0,1]$ and set*

$$\Phi^\flat_m=\kappa\circ\big(\varphi^\flat+\Psi_{\mathrm{mxd}}-K\big)-\chi\big((r-r_0)^\beta\big)\widetilde{\Psi}_{\mathrm{euc}},$$

where $\kappa:\mathbb{R}\to\mathbb{R}$ is a convex function vanishing on $(-\infty,0]$ and equal to $\mathrm{id}_{\mathbb{R}}$ on $[1,\infty)$, χ is the cut-off function $\frac{d\kappa}{dt}$, and $\widetilde{\Psi}_{\mathrm{euc}}:=\chi(r-r_0)\Psi_{\mathrm{euc}}$. Then if the parameters K and r_0 (resp. β) are chosen large enough (resp. small enough), the symmetric 2-tensor g_m associated via I^Y_1 to the I^Y_1-(1,1)-form

$$\omega_m:=\omega^Y_1+dd^c_{I^Y_1}\Phi^\flat_m$$

is well-defined on the whole Y, is a Kähler metric for I^Y_1, is ALF in the sense that $\big|(\nabla^{\mathbf{f}^\flat})^\ell(g_m-\mathbf{f}^\flat)\big|_{\mathbf{f}^\flat}=O\big(R^{-2}\big)$ for $\ell=0,1,2$, and its volume form Ω_m verifies

$$\big|(\nabla^{\mathbf{f}^\flat})^\ell(\Omega_m-\Omega_Y)\big|_{\mathbf{f}^\flat}=O\big(R^{-2}\big)\tag{16}$$

for $\ell=0,1,2$, where Ω is the volume form of the ALE metric g_Y.

Proof. To begin with, we mention the following comparison between $\mathbf{f}$ and its correction $\mathbf{f}^\flat=\beth^*\mathbf{f}$, that we will keep in mind:

Lemma 1.11 *For $\ell=0,1,2$, we have: $\big|(\nabla^{\mathbf{f}})^\ell(\mathbf{f}^\flat-\mathbf{f})\big|_{\mathbf{f}}=O(R^{-1})$ on $\mathbb{R}^4$. Moreover $\beth^*R=R+O(R^{-1})$.*

The proof of this lemma is postponed to Section 1.5 (§1.5.2). For now, we first consider the closed I^Y_1-hermitian form $dd^c_{I^Y_1}\kappa\circ\big(\varphi^\flat+\Psi_{\mathrm{mxd}}-K\big)$ on Y. Even though K is not fixed yet, this form is equal to $dd^c_{I^Y_1}\big(\varphi^\flat+\Psi_{\mathrm{mxd}}\big)$ on $\{\varphi^\flat+\Psi_{\mathrm{mxd}}\geq K+1\}$ seen on Y via Φ_Y – this is possible for K large enough since $\varphi^\flat+\Psi_{\mathrm{mxd}}$ is proper on $\mathbb{R}^4$ as $\varphi^\flat\geq\beth^*R\sim R$ (by Lemma 1.11) and $\Psi_{\mathrm{mxd}}=O(R^{-1})$. Moreover κ is convex, and thus $dd^c_{I^Y_1}\big[\kappa\circ\big(\varphi^\flat+\Psi_{\mathrm{mxd}}-K\big)\big]$ is non-negative wherever $dd^c_{I^Y_1}\big(\varphi^\flat+\Psi_{\mathrm{mxd}}\big)$ is, which we claim is the case near infinity. Since indeed $\big|dd^c_{I^Y_1}\Psi_{\mathrm{mxd}}\big|_{\mathbf{f}}=O(R^{-1})$, our claim will be checked if we prove the estimate:

$$\Big|dd^c_{I^Y_1}\varphi^\flat-\frac{1}{2}\big[\mathbf{f}^\flat\big(I^Y_1\cdot,\cdot\big)-\mathbf{f}^\flat\big(\cdot,I^Y_1\cdot\big)\big]\Big|_{\mathbf{f}^\flat}=O(R^{-2}),\tag{17}$$

as $\varpi_{\mathbf{f}^\flat}:=\frac{1}{2}\big[\mathbf{f}^\flat\big(I^Y_1\cdot,\cdot\big)-\mathbf{f}^\flat\big(\cdot,I^Y_1\cdot\big)\big]$ is nothing but the I^Y_1-hermitian form associated to the I^Y_1-hermitian metric $\frac{1}{2}\big[\mathbf{f}^\flat+\mathbf{f}^\flat\big(I^Y_1\cdot,I^Y_1\cdot\big)\big]$ – notice $\varpi_{\mathbf{f}^\flat}$ is not closed in general. Pushing-forward by $\beth$, proving estimate (17) amounts to seeing that:

$$\Big|dd^c_{\beth_*I^Y_1}\varphi-\frac{1}{2}\big[\mathbf{f}\big(\beth_*I^Y_1\cdot,\cdot\big)-\mathbf{f}\big(\cdot,\beth_*I^Y_1\cdot\big)\big]\Big|_{\mathbf{f}^\flat}=O(R^{-2}).$$

Now $dd^c_{\daleth_* I_1^Y}\varphi = d\daleth_* I_1^Y d\varphi = \omega_{\mathbf{f}} + d\jmath d\varphi$, where $\omega_{\mathbf{f}} = \mathbf{f}(I_1\cdot,\cdot)$ and $\jmath = \daleth_* I_1^Y - I_1$. Let us estimate $d\jmath d\varphi$; by Lemma 1.1 and by the analogue Fact 1 in the proof of Proposition 1.7 for $(1,1)$-tensors, for all $\ell \geq 0$, $\big|(\nabla^{\mathbf{f}})^\ell \jmath\big|_{\mathbf{f}} = O(R^{-3})$, whereas $\big|(\nabla^{\mathbf{f}})^\ell \varphi\big|_{\mathbf{f}} = O(R^{2-\ell})$; therefore $\big|d\jmath d\varphi\big|_{\mathbf{f}} = O(R^{-2})$. On the other hand, still from $\daleth_* I_1^Y = I_1 + \jmath$, $\mathbf{f}\big(\daleth_* I_1^Y\cdot,\cdot\big) - \mathbf{f}\big(\cdot,\daleth_* I_1^Y\cdot\big) = 2\omega_{\mathbf{f}} + \mathbf{f}(\jmath\cdot,\cdot) - \mathbf{f}(\cdot,\jmath\cdot)$. The error term $\mathbf{f}(\jmath\cdot,\cdot) - \mathbf{f}(\cdot,\jmath\cdot)$ is controlled by $|\jmath|_{\mathbf{f}}$, which is $O(R^{-3})$. We have thus proved estimate (17). Thanks to the general formal formula

$$\nabla^{g+h}T = \nabla^g T + (g+h)^{-1} * \nabla^g h * T, \tag{18}$$

(see e.g. [GV, p.21]) for any metrics g and $g+h$ (h is thus seen here as a perturbation) and any tensor T, with Lemma 1.11 take $g = \mathbf{f}$, $g+h = \mathbf{f}^\flat$ and T the tensor in play, we prove with the same techniques an estimate similar to (17) up to order 2, that is:

$$\big|(\nabla^{\mathbf{f}^\flat})^\ell\big(dd^c_{\daleth_* I_1^Y}\varphi - \varpi_{\mathbf{f}^\flat}\big)\big|_{\mathbf{f}^\flat} = O(R^{-2}),$$

for $\ell = 1,2$. If therefore K is chosen large enough, and taking moreover the contribution of Ψ_{mxd} into account, $\omega_1^Y + dd^c_{I_1^Y}\big(\varphi^\flat + \Psi_{\mathrm{mxd}} - K\big)$ is well-defined and is an I_1^Y-Kähler form, and is equal to $\big(\omega_1^Y - dd^c_{I_1^Y}\Psi_{\mathrm{mxd}}\big) + \varpi_{\mathbf{f}^\flat}$ up to a $O(R^{-2})$ error at orders 0,1 and 2 for $\mathbf{f}^\flat$; *we fix such a K once for all.*

We now deal with the summand $-\chi\big((r-r_0)^\beta\big)\widetilde{\Psi}_{\mathrm{euc}}$ of $\Phi_m^\flat$, which is meant to kill the ALE part the Kähler form we reached, or equivalently of the I_1^Y-hermitian form $\big(\omega_1^Y - dd^c_{I_1^Y}\Psi_{\mathrm{mxd}}\big) + \varpi_{\mathbf{f}^\flat}$. As before, there are two issues here: the positivity of the resulting I_1^Y-(1,1) form on Y, and its asymptotics.

About the latter, since we are only looking at what happens near infinity, notice they are independent of r_0 and β. Indeed, for any value of these parameters, and provided that r_0 is chosen much larger than K, we have on $r \geq r_0 + 1$, by definition of $\Phi_m^\flat$,

$$\omega_1^Y + dd^c_{I_1^Y}\Phi_m^\flat = \big(\omega_1^Y - dd^c_{I_1^Y}\Psi\big) + dd^c_{\daleth_* I_1^Y}\varphi,$$

with Ψ that of Proposition 1.7; the parenthesis in the right-hand side is thus $O(R^{-2})$ for $\mathbf{f}$ by this proposition, and again this holds for $\mathbf{f}^\flat$ by Lemma 1.11. We have already dealt with the asymptotics of $dd^c_{\daleth_* I_1^Y}\varphi$ in the previous step, and know they verify the announced estimates, i.e. the metric associated to this Kähler form via I_1^Y differs from $\mathbf{f}^\flat$ up to order 2 by a $O(R^{-2})$ error.

We are therefore left with the positivity assertion, which has to be proved carefully since we essentially have to subtract a metric to another one, hence our use of the two parameters r_0 and β. This boils down to the following:

- take r_0 so that on $r \geq r_0$, $dd^c_{I_1^Y}\big(\varphi^\flat + \Psi_{\mathrm{mxd}} - K\big) \geq \frac{1}{2}\varpi_{\mathbf{f}^\flat}$,

- consider the remaining part $\omega_1^Y - dd^c_{I_1^Y}\big[\chi\big((r-r_0)^\beta\big)\widetilde{\Psi}_{\mathrm{euc}}\big]$, which can be rewritten as
$$\chi\big((r-r_0)^\beta\big)\big(\omega_1^Y - dd^c_{I_1^Y}\widetilde{\Psi}_{\mathrm{euc}}\big) + \big(1-\chi\big((r-r_0)^\beta\big)\big) + R_\beta,$$
where R_β vanishes outside of $\{r_0 \le r \le r_0+1\}$, and $|R_\beta|_{\mathbf{e}} \le C\beta$ for some constant $C = C(r_0)$ independent of β;

- consider $\chi\big((r-r_0)^\beta\big)\big(\omega_1^Y - dd^c_{I_1^Y}\widetilde{\Psi}_{\mathrm{euc}}\big)$: it is $O(r^{-4})$ for $\mathbf{e}$, that is $O(r^{-2})$ thus $O(R^{-1})$ for $\mathbf{f}$ or $\mathbf{f}^\flat$, and vanishes outside $\{r \ge r_0\}$; one can thus fix r_0 large enough so that this 2-form is $\ge -\frac{1}{6}\varpi_{\mathbf{f}^\flat}$ everywhere on Y;

- fix finally β so that$|R_\beta|_{\mathbf{e}}$ is small enough to say that $|R_\beta|_{\mathbf{f}^\flat} \le \frac{1}{6}\varpi_{\mathbf{f}^\flat}$ where it may not vanish, i.e. on $\{r_0 \le r \le r_0+1\}$. This way $\omega_1^Y - dd^c_{I_1^Y}\big[\chi\big((r - r_0)^\beta\big)\widetilde{\Psi}_{\mathrm{euc}}\big] \ge -\frac{1}{6}\varpi_{\mathbf{f}^\flat} - \frac{1}{6}\varpi_{\mathbf{f}^\flat} = -\frac{1}{3}\varpi_{\mathbf{f}^\flat}$, and therefore $\omega_1^Y + dd_{I_1^Y}\Phi_m^\flat \ge \frac{1}{2}\varpi_{\mathbf{f}^\flat} - \frac{1}{3}\varpi_{\mathbf{f}^\flat} = \frac{1}{2}\varpi_{\mathbf{f}^\flat} - \frac{1}{3}\varpi_{\mathbf{f}^\flat}$ on $\{r \ge r_0\}$, whereas it is equal to $\omega_1^Y + dd^c_{I_1^Y}\big(\varphi^\flat + \Psi_{\mathrm{mxd}} - K\big) \ge 0$ on $Y \backslash \{r \ge r_0\}$, hence the desired positivity assertion.

The last part of the statements concerns volume forms, and is a direct consequence of the estimates on the metrics, after observing that (on $\mathbb{R}^4$, say; recall that $\Phi_{Y*}\Omega_Y = \Omega_{\mathbf{e}}$): $\mathrm{vol}^{\mathbf{f}^\flat} - \Omega_Y = \beth^* \mathrm{vol}^{\mathbf{f}} - \Omega_{\mathbf{e}} = \beth^*\Omega_{\mathbf{e}} - \Omega_{\mathbf{e}}$, which can be written as $\varepsilon\Omega_{\mathbf{e}}$ with $\big|(\nabla^{\mathbf{e}})^\ell\varepsilon\big|_{\mathbf{e}} = O(r^{-8})$, $\ell \ge 0$, by Lemma 1.1. This converts into $\big|(\nabla^{\mathbf{f}^\flat})^\ell\varepsilon\big|_{\mathbf{f}^\flat} = O(R^{-4})$, $\ell \ge 0$, which is better than wanted. □

1.4 Corrections on the glued metric

1.4.1 A Calabi-Yau type theorem

We want to correct our I_1^Y-Kähler metric g_m from Proposition 1.10 into a Ricci-flat Kähler metric. For this it is sufficient to correct it into an I_1^Y-Kähler metric with volume form Ω_Y, since this is the volume of the I_1^Y-Kähler metric g_Y – and, as is well-known, once the complex structure is fixed, the Ricci tensor of a Kähler metric depends only on its volume form. As suggested by the program ending to Theorem 1.3, at the level of I_1^Y-Kähler forms, we want to stay in the same class; in other words, we are looking for the I_1^Y-complex hessian of some function to be the desired correction.

The tool we are willing to use to determine this function is the ALF Calabi-Yau type theorem of the Introducion, which we recall now (we call the manifold in play $\mathcal{Y}$ for more genericity):

Theorem 1.12 *Let $\alpha, \beta \in (0,1)$ and let $(\mathcal{Y}, g_{\mathcal{Y}}, J_{\mathcal{Y}}, \omega_{\mathcal{Y}})$ an ALF Kähler 4-manifold of dihedral type of order $(3,\alpha,\beta)$. Let f a smooth function in $C^{3,\alpha}_{\beta+2}(\mathcal{Y}, g_{\mathcal{Y}})$. Then* *there exists a smooth function* $\varphi \in C^{5,\alpha}_{\beta}(\mathcal{Y}, g_{\mathcal{Y}})$ *such that* $\omega_{\mathcal{Y}} + dd^c_{J_{\mathcal{Y}}}\varphi$ *is Kähler, and* $(\omega_{\mathcal{Y}} + dd^c_{J_{\mathcal{Y}}}\varphi)^2 = e^f\omega_{\mathcal{Y}}^2$.

The weighted Hölder spaces of this statement follow a classical definition, and are the analogues of those defined in next paragraph for g_m on Y. Let us now make the following remark: since we want to construct a metric with volume form Ω_Y, this is tempting to take $f = \log\big(\frac{\Omega_Y}{\mathrm{vol}_{g_m}}\big)$ to apply Theorem 1.12. But so far we only control such an f up to two derivatives (see Proposition 1.10, estimates (16)).

The other issue is that $\mathcal{Y}$ being a "ALF Kähler manifold of dihedral type" means that outside a compact subset, $\mathcal{Y}$ is diffeomorphic to the complement of a ball in $\mathbb{R}^4/\mathcal{D}_k$, and that one can choose the diffeomorphism $\Phi_{\mathcal{Y}}$ between infinities of $\mathcal{Y}$ and $\mathbb{R}^4/\mathcal{D}_k$ such that for all $\ell = 0,\dots,3$, $\big|(\nabla^{g_{\mathcal{Y}}})^\ell\big(\Phi_{\mathcal{Y}*}g_{\mathcal{Y}} - \mathbf{f}\big)\big|_{g_{\mathcal{Y}}} = O(\rho^{-\beta-\ell})$, in addition with a similar statement of the α-Hölder derivative of $(\nabla^{g_{\mathcal{Y}}})^3(\Phi_{\mathcal{Y}*}g_{\mathcal{Y}} - \mathbf{f})$, and analogous statements on the complex structures $\Phi_{\mathcal{Y}*}J^{\mathcal{Y}}$ and I_1, up to order $(4,\alpha)$. Here again, a reading of Proposition 1.10 indicates us that the asymptotics at our disposal do not allow us to take $\Phi_{\mathcal{Y}} = \Phi_Y$.

We remedy to these technical problems as follows. First we correct g_m into an I_1^Y-Kähler metric with volume form Ω_Y – which is nothing but a *Ricci-flat* I_1^Y-Kähler metric – *outside a compact subset of* Y, which gives us an f with compact support; then we put this corrected metric into so-called *Bianchi gauge* with respect to ${\Phi_Y}^*\mathbf{f}^\flat$, which corresponds to correct Φ_Y itself so as to fit into the definition of an ALF Kähler manifold of dihedral type up to the desired order.

We conclude this paragraph by stressing that the proof of Theorem 1.12 requires a full part on its own; we thus refer the reader to Part 3 below for this.

1.4.2 Ricci-flatness outside a compact subset

To correct g_m into an I_1^Y-Kähler metric with volume form Ω_Y outside a compact subset of Y, we use the inverse function theorem on Monge-Ampère operators, between relevant Hölder spaces. Namely, we define on Y the following *weighted Hölder spaces*:

$$C^{\ell,\alpha}_{\delta}(Y, g_m) := \big\{f \in C^{\ell,\alpha}_{\mathrm{loc}} \big| \|f\|_{C^{\ell,\alpha}_{\delta}} < \infty\big\}, \tag{19}$$

for $\ell \in \mathbb{N}$, $\alpha \in (0,1]$, $\delta \in \mathbb{R}$, and where

$$\|f\|_{C^{\ell,\alpha}_{\delta}} := \big\|R^\delta f\big\|_{C^0} + \dots + \big\|R^{\delta+\ell}(\nabla^{g_m})^\ell f\big\|_{C^0} + \sup_{x\in Y}[R^\ell(\nabla^{g_m})^\ell f]^\alpha_\delta,$$

with

$$[u]^\alpha_\delta = \sup_{\substack{(x,y)\in Y,\\ d_{g_m}(x,y)<\mathrm{inj}_{g_m}}} \left|\max\big(R(x)^{\alpha+\delta}, R(y)^{\alpha+\delta}\big)\frac{u(x)-u(y)}{d_{g_m}(x,y)^\alpha}\right|_{g_m} \tag{20}$$

for u a $C^{\ell,\alpha}_{\mathrm{loc}}$ tensor ($u(x)-u(y)$ interpreted via parallel transport), with R a smooth positive extension of ${\Phi_Y}^*R$ on Y, and C^0-norms of the tensors computed with g_m.

We then state the following, indicating the type of functions which can help correcting ω_m in the sense raised above:

Proposition 1.13 *Fix $(\alpha_1,\delta_1)\in(0,1)^2$ such that $\alpha_1+\delta_1<1$, $\delta_1>\frac{7}{8}$. There exists a smooth function $\psi\in C^{2,\alpha_1}_{\delta_1-1}\cap C^{3,\alpha_1}_{\delta_1-2}$ such that $\omega_\psi:=\omega_m+dd^c_{I_1^Y}\psi$ is Kähler for I_1^Y, and such that $\frac{1}{2}\omega_\psi^2=\Omega_Y$ outside a compact set.*

Proof. Taking χ a cut-off function as in Proposition 1.10 and setting $\chi_{R_1}=\chi(R-R_1)$, we are done if we solve the problem $\big(\omega_m+dd^c_{I_1^Y}\psi\big)^2=(1-\chi_{R_1})\omega_m^2+2\chi_{R_1}\Omega_Y$ for R_1 large enough. This is manageable, with help of the inverse function theorem, since:

- $\omega_m^2-\big((1-\chi_{R_1})\omega_m^2+2\chi_{R_1}\Omega_Y\big)=\chi_{R_1}(\omega_m^2-2\Omega_Y)$, and $\Big\|\chi_{R_1}\frac{\omega_m^2-2\Omega_Y}{\Omega_Y}\Big\|_{C^{k,\alpha_1}_{\delta_1}}$ tends to 0 as R_1 goes to ∞ thanks to estimates (16) for $k=0,1$;
- the linearisation of the Monge-Ampère operators $C^{2+\varepsilon,\alpha_1}_{\delta_1-1-\varepsilon}\to C^{\varepsilon,\alpha_1}_{\delta_1+1-\varepsilon}$, $\varepsilon=0,1$, $\psi\mapsto(\omega_m+dd^c_{I_1^Y}\psi)^2/\omega_m^2$, at $\psi=0$, are the scalar Laplacians $\Delta_{g_m}:C^{2+\varepsilon,\alpha_1}_{\delta_1-1-\varepsilon}\to C^{\varepsilon,\alpha_1}_{\delta_1+1-\varepsilon}$. These are surjective, with kernel reduced to constant functions, according to the appendix of [BM], and using that (Y,g_m) is asymptotically a circle fibration over $\mathbb{R}^3/\pm$.

Once R_1 is chosen large enough to apply the inverse function theorem simultaneously, and once ψ is fixed in $C^{2,\alpha_1}_{\delta_1-1}\cap C^{3,\alpha_1}_{\delta_1-2}$ so that $\big(\omega_m+dd^c_{I_1^Y}\psi\big)^2=(1-\chi_{R_1})\omega_m^2+2\chi_{R_1}\Omega_Y$, the last point to be checked is the positivity of $\omega_\psi:=\omega_m+dd^c_{I_1^Y}\psi$. Since $dd^c_{I_1^Y}\psi=O(R^{-\delta_1})$, ω_ψ is asymptotic to ω_m, hence positive near infinity. Since its determinant $\frac{(1-\chi_{R_1})\omega_m^2+2\chi_{R_1}\Omega_Y}{\omega_m^2}$ relatively to ω_m never vanishes, it is positive on the whole Y. The smoothness of ψ is local. ☐

1.4.3 Bianchi gauge for ω_ψ

Motivation. — We are now willing to deduce regularity statements on g_ψ, using its Ricci-flatness near infinity. However this cannot be done immediately. The reason is that the Ricci-flatness condition is invariant under diffeomorphisms, and consequently *the linearisation of the Ricci tensor seen as an operator on metrics is not elliptic*, which is problematic when looking for regularity.

One can however bypass this difficulty by fixing *a gauge*, which infinitesimally corresponds to looking at metrics with good diffeomorphisms. We introduce the diffeomorphisms we shall work with in next paragraph; then the gauge is fixed, and regularity is deduced from this process (Propositions 1.16 and 1.17). Notice that the Ricci-flatness of g_ψ is an indispensable prerequisite in this procedure, since the gauge alone is not enough in general to obtain the regularity statement we are seeking here.

ALF diffeomorphisms of $\mathbb{C}^2$. — The class of diffeomorphisms we work with to perform our gauge enters into the following definition; we define the dual frames $(e_0^\flat,\dots,e_3^\flat)$ and $\big((e_0^*)^\flat,\dots,(e_3^*)^\flat\big)$ as the pull-backs by $\daleth$ of the frames (e_i) and (e_i^*) defined in section 1.2.2 by formulas (8), (9).

Definition 1.14 *Let $(\ell,\alpha)\in\mathbb{N}^*\times(0,1)$, and let $\nu>-1$. We denote by $\mathbf{Diff}_\nu^{\ell,\alpha}$ the class of diffeomorphisms ϕ of $\mathbb{C}^2$ such that:*

- *ϕ has regularity (ℓ,α);*
- *there exists a constant C such that for any $x\in\mathbb{C}^2$, $d_{\mathbf{f}^\flat}\big(x,\phi(x)\big)\le C\big(1+R(x)\big)^{-\nu}$;*
- *let $R_0\ge 1$ such that for any $x\in\{R\ge R_0\}$, $d_{\mathbf{f}^\flat}\big(0,\phi(x)\big)\ge 1$. Denote by $\gamma_x:[0,1]\to\mathbb{C}^2$ a minimising geodesic for $\mathbf{f}^\flat$ joining $\phi(x)$ to x and by p_{γ_x} the parallel transport along γ_x. Consider the maps $\phi_{ij}:\{R\ge R_0\}\to\mathbb{R}$ given by*

$$\phi_{ij}(x)=(e_i^\flat)^*\big((T_x\phi\circ p_{\gamma_x}(1)-\mathrm{id}_{\mathbb{C}^2})(e_j^\flat)\big),\qquad i,j=0,\dots,3,$$

  and extend them smoothly in $\{R\le R_0\}$. We then ask: $\phi_{ij}\in C^{\ell-1,\alpha}_{\nu+1}(\mathbb{C}^2,\mathbf{f}^\flat)$.

We endow $\mathbf{Diff}_\nu^{\ell,\alpha}$ with the natural topology.

We moreover denote by $\big(\mathbf{Diff}_\nu^{\ell,\alpha}\big)^{\mathcal{D}_k}$ the set of diffeomorphisms of $\mathbf{Diff}_\nu^{\ell,\alpha}$ commuting with the action of $\mathcal{D}_k$.

The Hölder spaces are those defined for $\mathbf{f}^\flat$ on $\mathbb{C}^2$, in the same way as those of defining equation (19). Notice that we authorise the distance between a point and its image to go to ∞; nonetheless, the sub-linear rate of blow-up we allow makes clear the existence of the R_0 of the third item .

Diffeomorphisms as Riemannian exponential maps. — We now "parametrise" our diffeomorphisms via vector fields:

Lemma 1.15 *There exists a neighbourhood $\mathscr{V}_\nu^{\ell,\alpha}$ of 0 in $C_\nu^{\ell,\alpha}(\mathbb{C}^2,\mathbf{f}^\flat)$ such that for any Z in that neighbourhood, the map $\phi_Z: x\longmapsto \exp_x^{\mathbf{f}^\flat}\big(Z(x)\big)$ is in $\mathbf{Diff}_\nu^{\ell,\alpha}$.*

The weighted spaces of vector fields are defined analogously to that of the previous paragraph, or equivalently: $Z\in C_\nu^{\ell,\alpha}(\mathbb{C}^2,\mathbf{f}^\flat)$ if and only if $Z\in C^{\ell,\alpha}_{\mathrm{loc}}$ and $\chi(R)(e_i^\flat)^*(Z)\in C_\nu^{\ell,\alpha}(\mathbb{C}^2,\mathbf{f}^\flat)$, $i=1,\dots,3$ (with χ a cut-off function as in 1.10). A similar statement with $\mathcal{D}_k$-invariant vector fields, and diffeomorphisms commuting with the action of $\mathcal{D}_k$ of course holds. We simply call $\big(\mathscr{V}_\nu^{\ell,\alpha}\big)^{\mathcal{D}_k}$ the neighbourhood of 0 in $\big(C_\nu^{\ell,\alpha}(\mathbb{C}^2,\mathbf{f}^\flat)\big)^{\mathcal{D}_k}$, the $\mathcal{D}_k$-invariant vector fields of $C_\nu^{\ell,\alpha}(\mathbb{C}^2,\mathbf{f}^\flat)$. Notice finally that for a genuine parametrisation, we would also need the surjectivity and the injectivity of $Z\mapsto\phi_Z$ from $\mathscr{V}_\nu^{\ell,\alpha}$ onto its image. We do not need however this degree of precision, since as seen in Proposition 1.16 below, it is enough for us to realise sufficiently many diffeomorphisms of $\mathbf{Diff}_\nu^{\ell,\alpha}$ under the shape ϕ_Z.

Proof. The regularity assertions are rather standard. We shall nonetheless pay a particular attention to the fact that we authorise vector fields blowing up at infinity, when verifying the injectivity of ϕ_Z for a given Z close to 0 in $C_\nu^{\ell,\alpha}$; the key is the decay of the derivatives of Z at infinity. Suppose $(\ell,\alpha)=(1,0)$ to fix ideas. For the injectivity of ϕ_Z with fixed $Z\in C_\nu^{1,0}$ and $\|Z\|_{C_\nu^{1,0}}\leq 1$ say, we claim that there exists a constant C independent of Z such that for any triple (x,y,z) such that $\phi_Z(x)=\phi_Z(y)=:z$,

$$d_{\mathbf{f}^\flat}(x,y)\leq C\big(1+R(z)\big)^{-4-3\nu}\|Z\|_{C_\nu^{1,0}}d_{\mathbf{f}^\flat}(x,y),$$

from which the injectivity of ϕ_Z follows at once provided $\|Z\|_{C_\nu^{1,0}}$ is small enough. We reach this claim thanks to the estimate $\big|\operatorname{Rm}^{\mathbf{f}^\flat}\big|=O(R^{-3})$, as follows. For x,y as in the claim, call respectively γ_x and γ_y the geodesics $t\mapsto\exp_x^{\mathbf{f}^\flat}\big(tZ(x)\big)$ and $t\mapsto\exp_y^{\mathbf{f}^\flat}\big(tZ(y)\big)$, and denote by p_{γ_x}, p_{γ_y} the attached parallel transports. Using [BK, Prop. 6.6], control first $d_{\mathbf{f}^\flat}(x,y)$ by $\big|p_{\gamma_x}(1)\big(Z(x)\big)-p_{\gamma_y}(1)\big(Z(y)\big)\big|_{\mathbf{f}^\flat}\big(1+R(z)\big)^{-3-2\nu}$. Then control $\big|p_{\gamma_x}(1)\big(Z(x)\big)-p_{\gamma_y}(1)\big(Z(y)\big)\big|_{\mathbf{f}^\flat}$ by $d_{\mathbf{f}^\flat}(x,y)\big(1+R(z)\big)^{-1-\nu}\|Z\|_{C_\nu^{1,0}}$; for this interpolate between γ_x and γ_y by $\gamma_s(t):=\exp_{\alpha(s)}^{\mathbf{f}^\flat}\big[tZ\big(\alpha(s)\big)\big]$, where α is a minimising geodesic for $\mathbf{f}^\flat$ joining x and y. This is where one uses the estimates on the derivatives of Z. □

The gauge. — Denote by $B^h=\delta^h+\frac{1}{2}\operatorname{tr}^h$ the Bianchi operator associated to any smooth metric h on $\mathbb{R}^4$. The gauge process now states as:

Proposition 1.16 *Let $(\alpha_2,\delta_2)\in(0,1)^2$ such that $\alpha_2<\alpha_1$, $\frac{7}{8}<\delta_2<\delta_1$, with (α_1,δ_1) fixed in Proposition 1.13. There exists a smooth diffeomorphism $\phi\in\big(\mathbf{Diff}_{\delta_2-1}^{1,\alpha_2}\big)^{\mathcal{D}_k}$ hence descending to $\mathbb{R}^4/\mathcal{D}_k$ such that*

$$B^{\phi^*\mathbf{f}^\flat}\big((\Phi_Y)_*g_\psi\big)=0$$

near infinity on $\mathbb{C}^2$, where g_ψ stands for the I_1^Y-Kähler metric associated to the Kähler form ω_ψ of Proposition 1.13. As a consequence, $\mathbf{f}^\flat-(\phi\circ\Phi_Y)_*g_\psi\in C_{\delta_2}^{1,\alpha_2}\big(X,\mathbf{f}^\flat\big)$.

Proof. Fix (α_2,δ_2) as in the statement, and consider the map

$$\begin{aligned}\Xi\,:\,(\mathscr{V}^{2,\alpha_2}_{\delta_2-1})^{\mathcal{D}_k}\times\mathbf{Met}^{1,\alpha_2}_{\delta_2}(\mathbf{f}^\flat)^{\mathcal{D}_k}\;&\longrightarrow\;C^{0,\alpha_2}_{\delta_2+1}(T^*\mathbb{C}^2,\mathbf{f}^\flat)^{\mathcal{D}_k}\\ (Z,g)\qquad\qquad\qquad&\longmapsto\;B^{\phi_Z^*\mathbf{f}^\flat}(g),\end{aligned}$$

where $\mathbf{Met}^{1,\alpha_2}_{\delta_2}(\mathbf{f}^\flat)^{\mathcal{D}_k}$ denotes the set of $\mathcal{D}_k$-invariant metrics g on $\mathbb{R}^4$ such that $g-\mathbf{f}^\flat\in C^\infty_{\delta_2}\big(X,\mathbf{f}^\flat\big)$ – we shall see further why we anticipate the action of $\mathcal{D}_k$ at this point of our discussion.

We would like to solve the equation

$$B^{\phi_Z^*\mathbf{f}^\flat}(g_\psi)=0,\qquad\text{i.e}\qquad\Xi(Z,g_\psi)=0,\tag{21}$$

and for this use the implicit function theorem near $(0,\mathbf{f}^\flat)$, since the differential of Ξ with respect to its first argument is $(\nabla^{\mathbf{f}^\flat})^*\nabla^{\mathbf{f}^\flat}$, which as we shall see enjoys isomorphism properties. Forgetting that g_ψ may not be defined via Φ_Y on the whole $\mathbb{C}^2$, if we are to do so nonetheless, we need to make g_ψ arbitrarily close to $\mathbf{f}^\flat$ in $C^\infty_{\delta_2}\big(X,\mathbf{f}^\flat\big)$. Since we only want equation (21) to be solved near infinity, instead of g_ψ we consider, for χ a cut-off function as in Proposition 1.10, the metric

$$g_{R_2}:=\chi(R-R_2)g_\psi+\big(1-\chi(R-R_2)\big)\mathbf{f}^\flat,$$

which makes sense via Φ_Y on the whole $\mathbb{C}^2$ provided R_2 is large enough; since g_ψ is $C^{1,\alpha_1}_{\delta_1}$ close to $\mathbf{f}^\flat$ at infinity, we have that $\|g_{R_2}-\mathbf{f}^\flat\|_{C^{1,\alpha_2}_{\delta_2}}(\mathbf{f}^\flat)$ goes to 0 when R_2 goes to ∞. We are thus left with checking the isomorphism assertion on

$$(\nabla^{\mathbf{f}^\flat})^*\nabla^{\mathbf{f}^\flat}=\frac{\partial\Xi}{\partial Z}\Big|_{(0,\mathbf{f}^\flat)}:(C^{2,\alpha_2}_{\delta_2-1})^{\mathcal{D}_k}\longrightarrow(C^{0,\alpha_2}_{\delta_2+1})^{\mathcal{D}_k}\cong\big(C^{0,\alpha_2}_{\delta_2+1}(T^*\mathbb{C}^2)\big)^{\mathcal{D}_k}$$

where the isomorphism on the right hand side is just the duality for $\mathbf{f}^\flat$. We shall moreover replace $\mathbf{f}^\flat$ by $\mathbf{f}$ (and the weighted spaces subsequently) since these are diffeomorphic to each other. Now surjectivity follows from that of $(\nabla^{\mathbf{f}})^*\nabla^{\mathbf{f}}$ between $C^{2,\alpha_2}_{\delta_2-1}(T\mathbb{C}^2,\mathbf{f})$ and $C^{0,\alpha_2}_{\delta_2+1}(T\mathbb{C}^2,\mathbf{f})$, which amounts by the theory of self-adjoint operators on weighted spaces to the injectivity of this operator on $C^{0,\alpha_2}_{3-(\delta_2+1)}=C^{0,\alpha_2}_{2-\delta_2}$. This injectivity is proved as follows: if $w\in C^{0,\alpha_2}_{2-\delta_2}$ is in the kernel of $(\nabla^{\mathbf{f}})^*\nabla^{\mathbf{f}}$, we get first by weighted elliptic estimates (see the techniques of [BM, App.]) that $w\in C^{2,\alpha_2}_{2-\delta_2}$. Then, an integration by parts yields:

$$0=\int_{\{R\le t\}}\langle w,(\nabla^{\mathbf{f}})^*\nabla^{\mathbf{f}}w\rangle_{\mathbf{f}}\,\mathrm{vol}^{\mathbf{f}}=\int_{\{R\le t\}}\big|\nabla^{\mathbf{f}}w\big|^2\,\mathrm{vol}^{\mathbf{f}}-\int_{\{t=R\}}w\odot\nabla^{\mathbf{f}}w\,\mathrm{vol}^{\mathbf{f}|_{\{t=R\}}},$$

and the boundary term is $O(t^{2-(2-\delta_2)-(3-\delta_2)})=O(t^{2\delta_2-3})$ (spheres of radius t have volume in t^2 in Taub-NUT geometry). Letting t go to ∞, we get $\nabla^{\mathbf{f}}w=0$, hence

$w = 0$ as w decays at infinity. Notice that this argument gives the injectivity of $(\nabla^{\mathbf{f}})^*\nabla^{\mathbf{f}} : C^{2,\alpha_2}_{\delta} \to C^{0,\alpha_2}_{\delta+2}$ for $\delta > \frac{1}{2}$, hence for $\delta > 0$, as $(\nabla^{\mathbf{f}})^*\nabla^{\mathbf{f}}$ has no critical weight in $(0,1)$

Now we are interested in the injectivity of $(\nabla^{\mathbf{f}})^*\nabla^{\mathbf{f}} : (C^{2,\alpha_2}_{\delta_2-1})^{\mathcal{D}_k} \to (C^{0,\alpha_2}_{\delta_2+1})^{\mathcal{D}_k}$; this is where the invariance under the action of $\mathcal{D}_k$ is needed. Let thus $v \in (C^{2,\alpha_2}_{\delta_2-1})^{\mathcal{D}_k}$ such that $(\nabla^{\mathbf{f}})^*\nabla^{\mathbf{f}} v = 0$. Forget momentarily about the $\mathcal{D}_k$-invariance of v, and write it $\sum_{i=0}^3 v^i e_i$, on $\{R \geq 1\}$, say, with (e_i) the frame defined by (8); each v^i is thus in $C^{2,\alpha_2}_{\delta_2-1}$ near infinity. Now since $(\nabla^{\mathbf{f}})^\ell e_i = O(R^{-1-\ell})$, $\ell = 1,2,3$ and because $(\nabla^{\mathbf{f}})^*\nabla^{\mathbf{f}}(v^i e_i)$ is equal to $(\Delta_{\mathbf{f}} v^i)e_i$ plus a linear combination of the $\nabla^{\mathbf{f}}_{e_j} v^i \nabla^{\mathbf{f}}_{e_k} e_i$ and the $v^i(\nabla^{\mathbf{f}})^2_{e_j,e_k} e_i$, we get that each v^i is the solution of a Dirichlet problem

$$\begin{cases} \Delta_{\mathbf{f}} v^i = w^i \in C^{0,\alpha_2}_{\delta_2+2} \quad \text{on } \{R \geq 1\}, \\ v^i|_{\{R=1\}} \in C^{2,\alpha_2}. \end{cases}$$

Recall that $\Delta_{\mathbf{f}} : C^{2,\alpha_2}_{\delta_2-1} \to C^{0,\alpha_2}_{\delta_2+1}$ is surjective, with kernel reduced to the constants (see the proof of Proposition 1.13), and that $\Delta_{\mathbf{f}} : C^{2,\alpha_2}_{\delta_2} \to C^{0,\alpha_2}_{\delta_2+2}$ is an isomorphism (see e.g. [BM, App.]); those properties transfer to Dirichlet problems to tell us that each v^i can be written as $c_i + u_i$, with c_i a constant and $u_i \in C^{2,\alpha_2}_{\delta_2}$. Thus v is asymptotic to $\sum_i c_i e_i$; but v is $\mathcal{D}_k$-invariant, whereas $\tau^* e_i = -e_i$ for any i. This forces the c_i to be 0, and as a result $v \in C^{2,\alpha_2}_{\delta_2}$. Finally, we know that $(\nabla^{\mathbf{f}})^*\nabla^{\mathbf{f}}$ is injective on this latter space, so $v = 0$; in other words, the action of $\mathcal{D}_k$ allows us to say that 0 is no more a critical weight for $(\nabla^{\mathbf{f}})^*\nabla^{\mathbf{f}}$ on vector fields.

The smoothness of Z, and therefore that of ϕ_Z, is purely local. $\square$

Regularity of g_ψ. — We conclude this paragraph by the following statement, which finally allows us to apply Theorem 1.12:

Proposition 1.17 *With the same notation as in Proposition 1.16,* $\mathbf{f}^\flat - (\phi \circ \Phi_Y)_* g_\psi \in C^{4,\alpha_2}_{8\delta_2-7}\big(X, \mathbf{f}^\flat\big)$ *near infinity, and in particular,* $\big|\operatorname{Rm}^{g_\psi}\big|_{g_\psi} = O(R^{-2-(8\delta_2-7)})$.

Proof. The assertion on the curvature of g_ψ directly follows from the estimate stated on $\varepsilon := g_\psi - \phi^*\mathbf{f}^\flat$ (or $\phi_*\varepsilon$), and the fact that $\big|\operatorname{Rm}^{\mathbf{f}^\flat}\big|_{\mathbf{f}^\flat} = O(R^{-3})$. For the regularity statement on ε, proceed as follows: set $\mathbf{F} = \phi^*\mathbf{f}^\flat$, and define the operator $\Phi^{\mathbf{F}}$ by:

$$\Phi^{\mathbf{F}}(h) = \operatorname{Ric}(h) + (\delta^h)^* B^{\mathbf{F}} h$$

on C^2_{loc} metrics. This way, $\Phi^{\mathbf{F}}(\mathbf{F}) = \Phi^{\mathbf{F}}(g_\psi) = 0$ near infinity. New $\Phi^{\mathbf{F}}$ is of order 2, hence schematically,

$$0 = \Phi^{\mathbf{F}}(\mathbf{F}) - \Phi^{\mathbf{F}}(g_\psi) = \big(d_{\mathbf{F}}\Phi^{\mathbf{F}}\big)(\varepsilon) + P(\varepsilon, \partial\varepsilon, \partial^2\varepsilon),$$

with $P(\varepsilon,\partial\varepsilon,\partial^2\varepsilon)$ an at-least-quadratic combination of ε, its first and its second derivatives, with coefficients depending on $\mathbf{F}$. Now, in local coordinates,

(22)
$$\begin{aligned}
\big(\Phi^{\mathbf{F}}(h)\big)_{ij} = &-\frac{1}{2}\mathbf{F}^{kr}(\partial_{ij}h_{kr}-\partial_{ir}h_{jk}-\partial_{jk}h_{ir}+\partial_{kr}h_{ij})\\
&+(\mathbf{F}^{kr}-h^{kr})\big(\partial_{ij}h_{kr}-\tfrac{1}{2}(\partial_{ir}h_{jk}-\partial_{jk}h_{ir})+\partial_{kr}h_{ij}\big)\\
&+\frac{1}{2}\big[(\partial_j h^{kr})(\partial_i h_{kr}+\partial_k h_{ir}-\partial_r h_{ik})-(\partial_i h^{kr})(\partial_j h_{kr}+\partial_k h_{jr}-\partial_r h_{jk})\big]\\
&-\big(\Gamma^\ell_{jk}(h)\Gamma^k_{i\ell}(h)-\Gamma^\ell_{ij}(h)\Gamma^k_{k\ell}(h)\big)\\
&+\mathrm{Sym}_{ij}\big[h_{j\ell}(\partial_i h^{\ell p})(\partial_k h_{rp}+\partial_r h_{pk}-\partial_p h_{rk})+h_{k\ell}(\partial_i h^{\ell p})(\partial_j h_{rp}+\partial_r h_{jp}-\partial_p h_{jr})\big]\\
&-\big[\mathbf{F}^{kr}(\tfrac{1}{2}\partial_\ell h_{kr}-\partial_k h_{\ell r})+\tfrac{1}{2}h_{kr}\partial_\ell\mathbf{F}^{kr}+\mathbf{F}^{kr}h_{\ell m}\Gamma^\ell_{kr}(h)+\mathbf{F}^{kr}h_{km}\Gamma^m_{\ell r}(h)\big].
\end{aligned}$$

The interest of this formula lies in the following: in $P\big(\varepsilon,\partial\varepsilon,\partial^2\varepsilon\big)$,

1. the only occurrence of the second derivatives of $\varepsilon = g_\psi - \mathbf{F}$, which we denote by $\partial^2\varepsilon$, in (22), is via a tensor of type $\varepsilon\odot\partial^2\varepsilon$, where $\odot$ is some algebraic operation with coefficients depending only on g_ψ and $\mathbf{F}$;

2. terms of type $\varepsilon\odot\varepsilon$ do not appear in (22), unless through a term of type $\varepsilon\odot\varepsilon\odot\partial\varepsilon$;

3. the algebraic coefficients are controlled (for $\mathbf{F}$ say) in C^{1,α_2}.

We sum these three points up by writing:

$$\frac{1}{2}\mathscr{L}_{\phi^*\mathbf{f}^\flat}\varepsilon+\varepsilon\star\partial^2\varepsilon = Q(\varepsilon,\partial\varepsilon) \tag{23}$$

where $\mathscr{L}_{\phi^*\mathbf{f}^\flat} = d_{\mathbf{F}}\Phi^{\mathbf{F}}$ is the Lichnerowicz laplacian of $\mathbf{F}=\phi^*\mathbf{f}^\flat$, the symbols $\star$ denote algebraic operations, and Q is at least quadratic in its arguments, and can be factorised by $\varepsilon\star\partial\varepsilon$. Since $\varepsilon\in C^{1,\alpha_2}_{\delta_2}$ – computed with respect to $\phi^*\mathbf{f}^\flat$ or g_ψ, which does not matter because of the size of the error term ε itself –, the right-hand-side of (23) is in $C^{0,\alpha_2}_{\delta_2}\big(X,\mathbf{f}^\flat\big)$. Again since $\varepsilon\in C^{1,\alpha_2}_{2\delta_2+1}$, the linear operator $\eta\to\frac{1}{2}\mathscr{L}_{\phi^*\mathbf{f}^\flat}\eta+\varepsilon\star\partial^2\eta$ is elliptic and one can draw for this operator weighted estimates similar to those for $\mathscr{L}_{\phi^*\mathbf{f}^\flat}$. From this we deduce that $\varepsilon\in C^{2,\alpha_2}_{2\delta_2-1}$; we conclude by repeating this argument twice, giving us first $\varepsilon\in C^{3,\alpha_2}_{2(2\delta_2-1)-1}=C^{3,\alpha_2}_{4\delta_2-3}$, and then $\varepsilon\in C^{4,\alpha_2}_{2(4\delta_2-3)-1}=C^{4,\alpha_2}_{8\delta_2-7}$. □

1.4.4 Conclusion: proof of Theorem 1.3

We have proved that $\mathbf{f}^\flat$ and $(\phi\circ\Phi_Y)_*g_\psi$ are $C^{4,\alpha}_\beta$, hence $C^{3,\alpha}_\beta$, close, provided that we take $\alpha=\alpha_2$ and $\beta=8\delta_2-7$ (which is positive since $\delta_2>\frac{7}{8}$), to fulfil completely the requirements of Theorem 1.12, we are only left with checking that $(\phi \circ \Phi_Y)_* I_1^Y$ is also $C^{3,\alpha}_\beta$ close to the complex structure $I_1^\flat := \beth^* I_1$.

The estimate $(\phi \circ \Phi_Y)_* I_1^Y - I_1^\flat \in C^0_\beta$ follows easily form the decomposition $(\Phi_Y)_* I_1^Y - \phi^* I_1^\flat = \big((\Phi_Y)_* I_1^Y - I_1\big) + (I_1 - I_1^\flat) + (I_1^\flat - \phi^* I_1^\flat)$, from the estimates $\big|(\Phi_Y)_* I_1^Y - I_1\big|_{\mathbf{e}} = O(r^{-4})$ and $\big|I_1 - I_1^\flat\big|_{\mathbf{e}} = O(r^{-4})$ converted into $\big|(\Phi_Y)_* I_1^Y - I_1\big|_{\mathbf{f}}, \big|I_1 - I_1^\flat\big|_{\mathbf{f}^\flat} = O(R^{-1})$, and $\big|I_1^\flat - \phi^* I_1^\flat\big|_{\mathbf{f}^\flat} = O(R^{-\delta_2})$ following from $\phi \in \mathbf{Diff}^{1,\alpha_2}_{\delta_2-1}$.

For higher order estimates, remember that g_ψ is Kähler for I_1^Y, and $\mathbf{f}^\flat$ for $I_1^\flat$. It is thus enough for instance to evaluate the successive $(\nabla^{\mathbf{f}^\flat})^\ell\big((\phi \circ \Phi_Y)_* I_1^Y\big)$. In view of formula (18), we thus write formally for $\ell = 1$

$$\nabla^{\mathbf{f}^\flat}\big((\phi\circ\Phi_Y)_* I_1^Y\big) = \underbrace{\nabla^{\phi_* g_\psi}\big((\phi \circ \Phi_Y)_* I_1^Y\big)}_{=0 \text{ since } g_\psi \text{ is } I_1^Y\text{-Kähler}} + (\mathbf{f}^\flat)^{-1} * \nabla^{\phi_* g_\psi}(\mathbf{f}^\flat - \phi_* g_\psi) * \big((\phi\circ\Phi_Y)_* I_1^Y\big),$$

which easily gives $\nabla^{\mathbf{f}^\flat}\big((\phi\circ\Phi_Y)_* I_1^Y\big) \in C^0_{\beta+1}$ in view of $(\mathbf{f}^\flat - \phi_* g_\psi) \in C^1_\beta$. For $\ell \geq 2$, simply use inductively formula (18), and the estimate $(\mathbf{f}^\flat - \phi_* g_\psi) \in C^{4,\alpha}_\beta$ – when reaching $\ell = 4$, we also get the right Hölder estimate from the iterated formula, and estimates such as $(\nabla^{\phi_* g_\psi})^4 \mathbf{f}^\flat \in C^{0,\alpha}_\beta$.

As sketched in the introduction of this section, we now apply Theorem 1.12, with $\big(\mathcal{Y}, g_{\mathcal{Y}}, J_{\mathcal{Y}}, \omega_{\mathcal{Y}}\big) = (Y, g_\psi, I_1^Y, \omega_\psi)$, and $f = \log\big(\frac{\Omega_Y}{\mathrm{vol}^{g_\psi}}\big)$, which is smooth and has compact support. This gives us an I_1^Y-metric $g_{RF,m}$ on Y, with volume form Ω_Y and which is thus Ricci-flat, and with Kähler form $\omega_\psi + dd^c_{I_1^Y}\varphi$ for some smooth $\varphi \in C^{5,\alpha}_\beta(Y, g_\psi)$ with β close to 1.

We need two more complex structures for Theorem 1.3. Recall we have two more symplectic forms coming with the ALE hyperkähler structure $\big(Y, g_Y, I_1^Y, I_2^Y, I_3^Y\big)$, namely $\omega_2^Y := g_Y\big(I_2^Y\cdot,\cdot\big)$ and $\omega_3^Y := g_Y\big(I_3^Y\cdot,\cdot\big)$. We simply define J_2^Y and J_3^Y as the endomorphisms verifying $g_{RF,m}\big(J_2^Y\cdot,\cdot\big) = \omega_2^Y$ and $g_{RF,m}\big(J_3^Y\cdot,\cdot\big) = \omega_3^Y$; one then checks these are almost complex structures, satisfying the quaternionic relations with I_1^Y, using $(\omega_2^Y)^2 = (\omega_3^Y)^2 = 2\,\mathrm{vol}^{g_{RF,m}}$ and that the $I_1^Y - (1,1)$ part of ω_2^Y and ω_Y^3 is 0. To check J_2^Y and J_3^Y are integrable, use moreover that the holomorphic symplectic 2-form $\omega_2^Y + i\omega_3^Y$, whose $g_{RF,m}$-norm is constant, is $g_{RF,m}$-parallel.

The cubic decay of $\mathrm{Rm}^{g_{RF,m}}$ comes as follows: first, an over-quadratic decay is easily deduced from $(g_\psi - g_{RF,m}) \in C^2_{\beta+2}\big(Y, g_\psi\big)$ and $\mathrm{Rm}^{g_\psi} = O(R^{-2-\beta})$ (Proposition 1.17). Then a result of Minerbe [Min2, Cor. A.2] (see also [CC]) asserts that we automatically end up with a cubic rate decay of the curvature. □

1.5 Verification of the technical Lemmas 1.8 and 1.11

We conclude this part by the left-over proofs of Lemmas 1.8 and 1.11, both useful in the gluing performed in section 1.3. Recall that on the one hand, Lemma 1.8 is about verifying the asymptotics at different orders of a function ψ_c, the hessian of which is meant to approximate the 2-form $\theta_2 + i\theta_3$ in the Taub-NUT framework, although such an approximation is likely to be vain in the euclidean setting; and that on the other hand, Lemma 1.11 consists in saying that even though $\mathbf{f}^\flat = \beth^*\mathbf{f}$, with $\beth$ a diffeomorphism of $\mathbb{R}^4$ better adapted to the euclidean scope, the transition between $\mathbf{f}$ and $\mathbf{f}^\flat$ is relatively harmless.

1.5.1 Proof of Lemma 1.8

Asymptotics of ψ_c and its successive derivatives. We first look at the first point of the statement of Lemma 1.8. Since ψ_c is $\mathbb{S}^1$-invariant when looked at on $\mathbb{C}^2$ (recall that the $\mathbb{S}^1$-action on $\mathbb{C}^2$ is given by $\alpha\cdot(z_1,z_2) = (e^{i\alpha}z_1, e^{-i\alpha}z_2)$), or in other words is a function of y_1, y_2, y_3 (recall in particular that $2r^2 = R\cosh(4my_1) + y_1\sinh(4my_1)$, following formulas (5) and the definitions of y_1 and R given in paragraph 1.2.2), we have: $d\psi_c = \frac{\partial\psi_c}{\partial y_1}dy_1 + \frac{\partial\psi_c}{\partial y_2}dy_2 + \frac{\partial\psi_c}{\partial y_3}dy_3$, and one can see as well the partial derivatives $\frac{\partial\psi_c}{\partial y_\ell}$ as functions of the y_j only. If we thus prove here that for any p, q, $s \geq 0$ such that $p+q+s \leq 4$,

$$\frac{\partial^{p+q+s}\psi_c}{\partial y_1^p\partial y_2^q\partial y_3^s} = O(R^{-1-q-s}), \tag{24}$$

we will get the desired estimates, since we moreover know that $\left|(\nabla^{\mathbf{f}})^\ell dy_j\right|_{\mathbf{f}} = O(R^{-1-\ell})$ for all $\ell \geq 1$ and $j = 1,2,3$.

The estimate (24) at order 0 is immediate, since $\sinh(4my_1) = O(R^{-1}r^2)$ – this follows from the identity $2r^2 = R\cosh(4my_1) + y_1\sinh(4my_1)$. What is thus clearly to be seen is that each time we differentiate with respect to y_2 or y_3, we win an R^{-1}, and each time we differentiate with respect to y_1, we lose nothing. Let us see how it goes at order 1, that is when $p+q+s=1$. If $p=1$ and $q=s=0$, then (near infinity, where $\chi(R)\equiv 1$):

$$\frac{\partial\psi_c}{\partial y_1} = -4(y_2+iy_3)\Big(\frac{4m\cosh(4my_1)}{2Rr^2} - \frac{y_1\sinh(4my_1)}{2r^2R^3} - \frac{\sinh(4my_1)}{4r^4R}\frac{\partial(2r^2)}{\partial y_1}\Big)$$

and $\frac{\partial(2r^2)}{\partial y_1} = 2V\big(y_1\cosh(4my_1) + R\sinh(4my_1)\big)$ (recall that $V = \frac{1+4mR}{2R}$), so that, after simplifying:

$$\frac{\partial\psi_c}{\partial y_1} = -4(y_2+iy_3)\Big(\frac{1}{r^4} - \frac{1}{R^3} + \frac{1}{4r^4R}\Big),$$

and this is $O(R^{-1})$, since $r^{-2} = O(R^{-1})$ (as $R = O(r^2)$).

If $q=1$ and $p=s=0$, then

$$\frac{\partial\psi_c}{\partial y_2} = -2\frac{\sinh(4my_1)}{r^2R} - 2(y_2+iy_3)\sinh(4my_1)\Big(\frac{y_2}{r^2R^3} + \frac{y_2\cosh(4my_1)}{2r^4R^2}\Big),$$

since $\frac{\partial(2r^2)}{\partial y_2} = \frac{y_2}{R}\cosh(4my_1)$. As $\sinh(4my_1)$ and $\cosh(4my_1)$ are $O(r^2R^{-1})$, we end up with $\frac{\partial\psi_c}{\partial y_2} = O\big(r^2/(R^2r^2)\big) + O\big(R\cdot r^2/R\cdot(r^{-2}R^{-2} + r^2/R\cdot r^{-4}R^{-1})\big) = O(R^{-2})$. The case $s=1$ and $p=s=0$, i.e. the estimate on $\frac{\partial\psi_c}{\partial y_3}$, is done by substituting y_3 to y_2.

In a nutshell, we win one order each time we differentiate y_2, y_3, R and r^2 with respect to y_2 or y_3, which moreover kills functions of y_1 such as $\sinh(4my_1)$; we win one order as well when differentiating y_2, y_3 and R with respect to y_1, but this does not hold any more for r^2 or functions like $\sinh(4my_1)$. More formally, using explicit formulas for the $\frac{\partial(2r^2)}{\partial y_j}$, $j=1,2,3$, we can easily prove by induction that for any p,q,s there exists a polynomial $Q_{p,q,s}$ of total degree $\leq (1+p+q+s)$ in its first two variables, and $2+3p+2(q+s)$ in total, such that:

$$\frac{\partial^{p+q+s}\psi_c}{\partial y_1^p\partial y_2^q\partial y_3^s} = \frac{Q_{p,q,s}\big(Re^{\pm 4my_1}, y_1e^{\pm 4my_1}, R, y_1, y_2, y_3\big)}{(2r^2)^{1+p+q+s}R^{2(1+p+q+s)}}$$

– for instance, $Q_{1,0,0}\big(Re^{\pm 4my_1}, y_1e^{\pm 4my_1}, R, y_1, y_2, y_3\big) = 4(y_2+iy_3)\big[\big(R\cosh(4my_1) + y_1\sinh(4my_1)\big)^2 - R^2 - 4R^3\big]$. If now $P(\xi_1,\xi_2,\eta_1,\dots,\eta_4) = \xi_1^{a_1}\xi_2^{a_2}\eta_1^{b_1}\cdots\eta_4^{b_4}$ is one of the monomials appearing in $Q_{p,q,s}$ and $a := a_1+a_2$, $b := b_1+\cdots+b_4$ so that $a\leq 2(1+p+q+s)$ and $a+b\leq 2+3p+2(q+s)$, since $Re^{\pm 4my_1}, y_1e^{\pm 4my_1} = O(r^2)$, we get that:

$$\frac{P\big(Re^{\pm 4my_1}, y_1e^{\pm 4my_1}, R, y_1, y_2, y_3\big)}{(r^2)^{1+p+q+s}R^{2+2(p+q+s)}} = O\bigg(\frac{(r^2)^aR^b}{(r^2)^{1+p+q+s}R^{2(1+p+q+s)}}\bigg),$$

and this is $O\big(r^{2a-2(1+p+q+s)}R^{b-2(1+p+q+s)}\big)$; since $a\leq 1+p+q+s$ and $r^{-2} = O(R^{-1})$, this is finally $O\big(R^{a+b-3(1+p+q+s)}\big)$, which in turn is $O(R^{-(1+q+s)})$ since $a+b\leq 2+3p+2(q+s)$. Therefore $\frac{\partial^{p+q+s}\psi_c}{\partial y_1^p\partial y_2^q\partial y_3^s} = O(R^{-(1+q+s)})$, and this settles the proof of point 1. of the statement.

Asymptotics of $\theta_2+i\theta_3$, and comparison with $dd^c_{I_1}\psi_c$ and $dd^c_{I_1^Y}\psi_c$. We thus come now to point 2. of this statement. We do it for $\ell=0$; it will become clear from this that the subsequent estimates could be dealt with in an analogous way. Our strategy for proving the desired estimate is the following: first we restrict ourselves to $dd^c_{I_1}\psi_c$; next we decompose $dd^c_{I_1}\psi_c - (\theta_2+i\theta_3)$ into its $dy_1\wedge\eta$-component and its $dy_1\wedge\eta$-free component; we then observe that the $dy_1\wedge\eta$-free components of both $dd^c_{I_1}\psi_c$ and $(\theta_2+i\theta_3)$ have already the size we want, whereas we need to look at the $dy_1\wedge\eta$-component of *the very difference* $\big[dd^c_{I_1}\psi_c - (\theta_2+i\theta_3)\big]$ to reach the desired estimate. We conclude by collecting together these estimates, and settling the case of the error term $d(I_1^Y - I_1)d\psi_c$.

Since ψ_c is $\mathbb{S}^1$-invariant,

$$\begin{aligned}dd^c_{I_1}\psi_c =&V^{-1}\Big(\frac{\partial^2\psi_c}{\partial y_1^2}-V^{-1}\frac{\partial V}{\partial y_1}\frac{\partial\psi_c}{\partial y_1}\Big)dy_1\wedge\eta+\Big(\frac{\partial^2\psi_c}{\partial y_2^2}+\frac{\partial^2\psi_c}{\partial y_3^2}+V^{-1}\frac{\partial V}{\partial y_1}\frac{\partial\psi_c}{\partial y_1}\Big)dy_2\wedge dy_3\\ &+V^{-1}\Big(\frac{\partial^2\psi_c}{\partial y_1\partial y_2}-V^{-1}\frac{\partial V}{\partial y_2}\frac{\partial\psi_c}{\partial y_1}\Big)(dy_2\wedge\eta-Vdy_3\wedge dy_1)\\ &+V^{-1}\Big(\frac{\partial^2\psi_c}{\partial y_1\partial y_3}-V^{-1}\frac{\partial V}{\partial y_3}\frac{\partial\psi_c}{\partial y_1}\Big)(dy_3\wedge\eta-Vdy_1\wedge dy_2),\end{aligned}$$

and since $(\xi,-I_1V\xi,\zeta,I_1\zeta)$ is the dual frame of (η,dy_1,dy_2,dy_3) and $(\theta_2+i\theta_3)$ is $(1,1)$ for I_1,

$$\begin{aligned}\theta_2+i\theta_3 =&V(\theta_2+i\theta_3)(\xi,I_1\xi)dy_1\wedge\eta+(\theta_2+i\theta_3)(\zeta,I_1\zeta)dy_2\wedge dy_3\\ &+(\theta_2+i\theta_3)(\xi,I_1\zeta)(Vdy_1\wedge dy_2-dy_3\wedge\eta)\\ &+(\theta_2+i\theta_3)(\xi,\zeta)(Vdy_3\wedge dy_1-dy_2\wedge\eta).\end{aligned}\tag{25}$$

We already know that (on $R\geq K$), $\frac{\partial\psi_c}{\partial y_1}=-4(y_2+iy_3)\big(\frac{m}{r^4}-\frac{1}{R^3}+\frac{1}{4r^4R}\big)$, thus (recall that $\frac{\partial(2r^2)}{\partial y_1}=V(|z_1|^2-|z_2|^2)$):

$$\frac{\partial^2\psi_c}{\partial y_1^2}=-4(y_2+iy_3)\Big(-\frac{2mV(|z_1|^2-|z_2|^2)}{r^6}+\frac{3y_1}{R^5}-\frac{y_1}{4r^4R^3}-\frac{V(|z_1|^2-|z_2|^2)}{4r^6R}\Big),$$

the main term of which is $\frac{8mV(y_2+iy_3)(|z_1|^2-|z_2|^2)}{r^6}$, in the sense that it is $O(R^{-1})$, whereas the other summands are $O(R^{-2})$. Moreover, from the estimates of Point 1. and the fact that $\frac{\partial V}{\partial y_j}=O(R^{-2})$, $j=1,2,3$, we get that:

$$dd^c_{I_1}\psi_c=\frac{8mV(y_2+iy_3)(|z_1|^2-|z_2|^2)}{r^6}dy_1\wedge\eta+O(R^{-2}).$$

when estimated with respect to $\mathbf{f}$.

Now recall that $\alpha_j=I_jrdr$, $j=1,2,3$, and observe that:

$$\begin{aligned}\theta_2+i\theta_3&=\frac{rdr\wedge\alpha_2-\alpha_3\wedge\alpha_1+irdr\wedge\alpha_3-i\alpha_1\wedge\alpha_2}{r^6}=\frac{(rdr-i\alpha_1)\wedge(\alpha_2+i\alpha_3)}{r^6}\\ &=\frac{(z_1d\overline{z_1}+z_2d\overline{z_2})\wedge(-z_2dz_1+z_1dz_2)}{r^6}\\ &=\frac{z_1z_2(dz_1\wedge d\overline{z_1}-dz_2\wedge d\overline{z_2})+z_2^2dz_1\wedge d\overline{z_2}-z_1^2dz_1\wedge d\overline{z_2}}{r^6}=\frac{\vartheta\wedge\phi}{r^6},\end{aligned}$$

if we set $\vartheta = z_1 d\overline{z_1} + z_2 d\overline{z_2}$ and $\phi = -z_2 dz_1 + z_1 dz_2$. Direct computations – use e.g. (10) – give:

$$\begin{aligned} \vartheta(\xi) &= -(|z_1|^2 - |z_2|^2), & \vartheta(\zeta) &= \frac{2z_1z_2}{iR}\cosh(4my_1), \\ \phi(\xi) &= -2iz_1z_2, & \phi(\zeta) &= -\frac{y_1}{2iR}. \end{aligned}$$

In particular, $\vartheta(\xi) = O(r^2)$, $\vartheta(\zeta) = O(r^2R^{-1})$, $\phi(\xi) = O(R)$ and $\phi(\zeta) = O(1)$. Moreover, since ϑ (resp. ϕ) is (0,1) (resp. (1,0)) for I_1, $(\theta_2 + i\theta_3)(\xi, I_1\xi) = -\frac{2i}{r^6}\vartheta(\xi)\phi(\xi) = \frac{8mz_1z_2(|z_1|^2-|z_2|^2)}{r^6}$. Therefore, from (25) and since ϑ (resp. ϕ) has type (0,1) (resp. (1,0)) for I_1, using $r^{-2} = O(R^{-1})$ when necessary, we get:

$$\theta_2 + i\theta_3 = \frac{8mVz_1z_2(|z_1|^2 - |z_2|^2)}{r^6} dy_1 \wedge \eta + O(R^{-2}).$$

with respect to $\mathbf{f}$. Since $y_2 + iy_3 = -iz_1z_2$, we thus have $\big|dd^c_{I_1}\psi_c - (\theta_2 + i\theta_3)\big|_{\mathbf{f}} = O(R^{-2})$.

We set $\iota_1^Y := I_1^Y - I_1$, and conclude with an estimate on $\big|d(I_1^Y - I_1)d\psi_c\big|_{\mathbf{f}} = |d\iota_1^Y d\psi_c|_{\mathbf{f}}$, which is controlled by $|\iota_1^Y|_{\mathbf{f}}|\nabla^{\mathbf{f}}d\psi_c|_{\mathbf{f}} + |\nabla^{\mathbf{f}}\iota_1^Y|_{\mathbf{f}}|d\psi_c|_{\mathbf{f}}$. But $|\iota_1^Y|_{\mathbf{f}}$ and $|\nabla^{\mathbf{f}}\iota_1^Y|_{\mathbf{f}}$ are $O(r^{-2})$ hence $O(R^{-1})$ (see e.g. the proof of Proposition 1.10), and $|d\psi_c|_{\mathbf{f}}$ and $|\nabla^{\mathbf{f}}d\psi_c|_{\mathbf{f}}$ are $O(R^{-1})$ as well from Point 1., and as a result $\big|d(I_1^Y - I_1)d\psi_c\big|_{\mathbf{f}} = O(R^{-2})$.

This settles the case $\ell = 0$ of the statement. Cases $\ell = 1$ and 2 are done in the same way, noticing in particular that when letting $\nabla^{\mathbf{f}}$ act on the $(\nabla^{\mathbf{f}})^j\psi_c$ or the $(\nabla^{\mathbf{f}})^j\iota_1^Y$, we keep the same order of precision. □

Remark 1.18 *The function ψ_c is not so small with respect to $\mathbf{e}$, at least at positive orders; for instance, the best we seem able to do on its differential is $|d\psi_c|_{\mathbf{e}} = O(rR^{-1})$.*

1.5.2 Comparison between $\mathbf{f}$ and $\mathbf{f}^\flat$: proof of Lemma 1.11

Before comparing the metrics, and for this the 1-forms $dy_j^\flat := \beth^* dy_j$, $j = 1, 2, 3$, and $\eta^\flat := \beth^*\eta$ to their natural ("unflat") analogues, we shall compare the $y_j^\flat := \beth^* y_j$ to the y_j, $j = 1, 2, 3$:

Lemma 1.19 *We have: $y_j^\flat - y_j = O(R^{-1})$, $j = 1, 2, 3$. Consequently if $R^\flat := \beth^* R$, then: $R^\flat - R = O(R^{-1})$.*

Proof of Lemma 1.19 – estimates on $(y_2^\flat - y_2)$ and $(y_3^\flat - y_3)$. Since $y_2 = \frac{1}{2i}\big(z_1z_2 - \overline{z_1z_2}\big)$, and $\beth(z_1, z_2) = (\alpha z_1, \alpha z_2)$ with $\alpha = 1 + \frac{a}{\kappa + r^4}$ by Lemma 1.1, it is clear that $y_2^\flat = \alpha^2 y_2 = y_2 + O(y_2 r^{-4})$, that is: $y_2^\flat - y_2 = O(Rr^{-4})$, and this is $O(R^{-1})$ – recall that $R = O(r^2)$.

Similarly, $y_3 = -\frac{1}{2}\big(z_1 z_2 + \overline{z_1 z_2}\big)$, thus $y_3^\flat - y_3 = y_3(\alpha^2-1)$, which is $O(R^{-1})$.

Estimate on $(y_1^\flat - y_1)$. The case of $y_1^\flat$ is slightly more subtle, and for this we shall use the very definition of y_1. We *fix* $(z_1, z_2) \in \mathbb{C}^2$. Since $\beth^* z_1 = \alpha z_1$, $\beth^* z_2 = \alpha z_2$, if one sets $u^\flat = \beth^* u$ and $v^\flat = \beth^* v$, LeBrun's formulas (5) become:

$$\alpha^2|z_1|^2 = e^{2m[(u^\flat)^2-(v^\flat)^2]}(u^\flat)^2, \qquad (26)$$
$$\alpha^2|z_2|^2 = e^{2m[(v^\flat)^2-(u^\flat)^2]}(v^\flat)^2,$$

which we rewrite as:

$$|z_1|^2 = e^{2m\alpha^2[(u^\flat/\alpha)^2-(v^\flat/\alpha)^2]}(u^\flat/\alpha)^2,$$
$$|z_2|^2 = e^{2m\alpha^2[(v^\flat/\alpha)^2-(u^\flat/\alpha)^2]}(v^\flat/\alpha)^2.$$

These are precisely the equations verified by $u_{m\alpha^2}$ and $v_{m\alpha^2}$ instead of $\frac{u^\flat}{\alpha}$ and $\frac{v^\flat}{\alpha}$; by uniqueness of the solutions when $|z_1|$ and $|z_2|$ are fixed, $\frac{u^\flat}{\alpha} = u_{m\alpha^2}$ and $\frac{v^\flat}{\alpha} = v_{m\alpha^2}$, that is: $u^\flat = \alpha u_{m\alpha^2}$ and $v^\flat = \alpha v_{m\alpha^2}$, and consequently $y_1^\flat = \frac{1}{2}[(u^\flat)^2 - (v^\flat)^2] = \frac{\alpha^2}{2}(u_{m\alpha^2}^2 - v_{m\alpha^2}^2) = \alpha^2 y_{1,m\alpha^2}$.

Now still with (z_1, z_2) fixed, differentiating LeBrun's equations *with respect to the mass parameter*, μ say, since we also see m as fixed, and rearranging them gives:

$$\frac{\partial y_{1,\mu}}{\partial \mu} = -\frac{4R_\mu y_{1,\mu}}{1+4\mu R_\mu};$$

in particular $y_{1,\mu}$ is a non-increasing (resp. non-decreasing) function of μ on $\{|z_1| \leq |z_2|\}$ (resp. on $\{|z_2| \leq |z_1|\}$).

Since $\alpha \geq 1$, we have for instance on $\{|z_1| \leq |z_2|\}$ the estimate:

$$0 \leq y_{1,m} - y_{1,m\alpha^2} = \int_m^{m\alpha^2} \frac{4R_\mu y_{1,\mu}}{1+4\mu R_\mu} d\mu \leq y_{1,m} \int_m^{m\alpha^2} \frac{d\mu}{\mu} = 2y_{1,m}\log\alpha,$$

and similarly $0 \leq y_{1,m\alpha^2} - y_{1,m} \leq -2y_{1,m}\log\alpha$ on $\{|z_2| \leq |z_1|\}$. Since in both cases $\log\alpha = O(r^{-4}) = O(R^{-2})$, we have:

$$y_{1,m\alpha^2} - y_{1,m} = O(y_{1,m}R^{-2}) = O(R^{-1}).$$

Therefore $y_1^\flat - y_1 = \alpha^2(y_{1,m\alpha^2} - y_1) + (\alpha^2-1)y_1 = O(R^{-1})$ as claimed, since $\alpha - 1 = O(r^{-4}) = O(R^{-2})$ and in particular $\alpha \sim 1$ near infinity.

The estimate $R^\flat - R = O(R^{-1})$ comes as follows: $(R^\flat - R)(R^\flat + R) = (R^\flat)^2 - R^2 = (y_1^\flat)^2 - y_1^2 + (y_2^\flat)^2 - y_2^2 + (y_3^\flat)^2 - y_3^2 = O(1)$ from the previous estimates, and thus $R^\flat - R = O\big(\frac{1}{R^\flat + R}\big)$, which in particular is $O(R^{-1})$. ■

Estimates on the $dy_j^\flat - dy_j$, $j = 1, 2, 3$, and $\eta^\flat - \eta$. We come back to the proof of Lemma 1.11 itself, and start with analysing the transition involved by $\beth$ at the level of 1-forms. We adopt by places the following elementary strategy to evaluate the gap between our fundamental 1-forms and their pull-backs by $\beth$: for γ one of the dy_j or η, we write

$$\gamma^\flat = \gamma^\flat(\xi)\eta + V^{-1}\gamma^\flat(-I_1\xi)dy_1 + \gamma^\flat(\zeta)dy_2 + \gamma^\flat(I_1\zeta)dy_3,$$

and then evaluate the difference $\gamma^\flat(\xi) - \gamma(\xi)$, and the subsequent ones. We start with the easy cases of dy_2 and dy_3; for more concision, we use the complex expression $\gamma = dy_2 + idy_3$.

Keep the notation $\beth(z_1, z_2) = (\alpha z_1, \alpha z_2)$; then $\beth^*(dy_2 + idy_3) = d\big(\alpha^2(y_2 + iy_3)\big) = \alpha^2(dy_2 + idy_3) + (y_2 + iy_3)d(\alpha^2)$. Since $\alpha = 1 + O\big(r^{-4}\big)$, we focus on $d(\alpha^2)$, or rather on $d\alpha$. As α is invariant under the usual action of $\mathbb{S}^1$, we already know that $d\alpha(\xi) = 0$. Moreover,

$$d\alpha = -2a\frac{r^2 d(r^2)}{(\kappa + r^4)^2}, \tag{27}$$

which we keep under this shape since $d(r^2) = \overline{z_1}dz_1 + z_1 d\overline{z_1} + \overline{z_2}dz_2 + z_2 d\overline{z_2}$ is easy to evaluate against $I_1\xi$, ζ and $I_1\zeta$. As a matter of fact, all computations done:

$$\begin{aligned} &d\alpha(-I_1\xi) = -4a\frac{|z_1|^4 - |z_2|^4}{(\kappa + r^4)^2}, \qquad d\alpha(\zeta) = \frac{-8ar^2}{(\kappa + r^4)^2}\frac{y_2\cosh(4my_1)}{R},\\ &d\alpha(I_1\zeta) = \frac{-8ar^2}{(\kappa + r^4)^2}\frac{y_3\cosh(4my_1)}{R}. \end{aligned} \tag{28}$$

In particular, $d\alpha(-I_1\xi) = O(r^{-4}) = O(R^{-2})$, and $d\alpha(\zeta) = O(R^{-1}r^{-4})$ and $d\alpha(I_1\zeta) = O(R^{-1}r^{-4})$, which are $O(R^{-3})$. Since $\alpha \sim 1$ and $y_2 + iy_3 = O(R)$, we end up with $(dy_2^\flat + idy_3^\flat)(-I_1\zeta) = O(R^{-2})$, $(dy_2^\flat + idy_3^\flat)(\zeta) = 1 + O(R^{-1})$ and $(dy_2^\flat + idy_3^{I_1\flat})(\zeta) = i + O(R^{-1})$. In other words,

$$\big|(dy_2^\flat + idy_3^\flat) - (dy_2 + idy_3)\big|_{\mathbf{f}} = O(R^{-1}).$$

In a way similar to what is done above on $y_1^\flat - y_1$, the estimate on $dy_1^\flat - dy_1$ requires little extra care. First, likewise y_1, $y_1^\flat$ is invariant under the action of $\mathbb{S}^1$, since $\beth$ commutes to this action; therefore $dy_1^\flat(\xi) = 0$. Next, pulling-back Formula 52 for dy_1 (proof of Proposition A.9 below) by $\beth$ gives:

$$dy_1^\flat = \frac{1}{4mR^\flat}\big(e^{-4my_1^\flat}d(\alpha^2|z_1|^2) - e^{4my_1^\flat}d(\alpha^2|z_2|^2)\big)$$

When evaluating $dy_1^\flat$, we decompose the term $e^{-4my_1^\flat}d(\alpha^2|z_1|^2) - e^{4my_1^\flat}d(\alpha^2|z_2|^2)$ into $\sigma := \alpha^2\big(e^{-4my_1^\flat}d(|z_1|^2) - e^{4my_1^\flat}d(|z_2|^2)\big)$ and $\rho := \big(e^{-4my_1^\flat}|z_1|^2 - e^{4my_1^\flat}|z_2|^2\big)d(\alpha^2) = \alpha^{-2}\big((u^\flat)^2 - (v^\flat)^2\big)d(\alpha^2) = 4\alpha^{-1}y_1^\flat d\alpha$.

Now $\sigma(-I_1\xi) = 2\alpha^2\big(|z_1|^2e^{-4my_1^\flat} + |z_2|^2e^{4my_1^\flat}\big) = 4R^\flat$, and by (28), $\rho(-I_1\xi) = 4\alpha^{-1}y_1^\flat d\alpha(-I_1\xi) = -16a\alpha^{-1}y_1^\flat\frac{|z_1|^4-|z_2|^4}{(\kappa+r^4)^2}$; this way:

$$dy_1^\flat(-I_1\xi) = (V^\flat)^{-1} - 8a\alpha^{-1}\frac{y_1^\flat}{1+4mR^\flat}\frac{|z_1|^4-|z_2|^4}{(\kappa+r^4)^2}, \tag{29}$$

where $V^\flat = \daleth^*V = \frac{1+4mR^\flat}{2R^\flat}$. Since the last summand is $O(r^{-4})$ and thus $O(R^{-2})$, and $(V^\flat)^{-1} - V^{-1} = \frac{2R^\flat}{1+4mR^\flat} - \frac{2R}{1+4mR} = 2\frac{R^\flat-R}{(1+4mR^\flat)(1+4mR)} = O(R^{-3})$, we have $dy_1^\flat(-I_1\xi) = V + O(R^{-2})$.

Moreover $\sigma(\zeta) = \frac{\alpha^2}{2iR}\big(e^{4m(y_1-y_1^\flat)}(z_1z_2-\overline{z_1z_2})-e^{-4m(y_1-y_1^\flat)}(z_1z_2-\overline{z_1z_2})\big) = \alpha^2\frac{y_2}{R}\sinh[4m(y_1-y_1^\flat)]$, and $\rho(\zeta) = 4\alpha^{-1}y_1^\flat d\alpha(\zeta) = -32a\alpha^{-1}\frac{r^2}{(\kappa+r^4)^2}\frac{y_1^\flat y_2\cosh(4my_1)}{R}$ by (28). Thus

$$dy_1^\flat(\zeta) = \alpha^2\frac{y_2}{2R(1+4mR^\flat)}\sinh[4m(y_1-y_1^\flat)] - 16a\alpha^{-1}\frac{r^2}{(\kappa+r^4)^2}\frac{y_1^\flat y_2\cosh(4my_1)}{R(1+4mR^\flat)}; \tag{30}$$

since $y_1 - y_1^\flat = O(R^{-1})$, the first summand is $O(R^{-2})$, whereas since $\cosh(4my_1) = O(r^2R^{-1})$, the second summand is $O(R^{-1}r^{-4})$, that is $O(R^{-3})$, and as a result $dy_1^\flat(\zeta) = O(R^{-2})$. Similarly $dy_1^\flat(I_1\zeta) = O(R^{-2})$ (just replace y_2 by y_3 in the last equality above).

Estimate on $\eta^\flat$. We conclude our estimate of $|\mathbf{f}^\flat - \mathbf{f}|_{\mathbf{f}}$ by the estimate on $\eta^\flat$. We start with a formula for $\eta^\flat$; since on $\{z_1\neq 0\}$, $\frac{d(\daleth^*\overline{z_1})}{\daleth^*\overline{z_1}} - \frac{d(\daleth^*z_1)}{\daleth^*z_1} = \frac{d(\alpha^2\overline{z_1})}{\alpha^2\overline{z_1}} - \frac{d(\alpha^2z_1)}{\alpha^2z_1} = \frac{d\overline{z_1}}{\overline{z_1}} + \frac{d(\alpha^2)}{\alpha^2} - \frac{dz_1}{z_1} - \frac{d(\alpha^2)}{\alpha^2} = \frac{d\overline{z_1}}{\overline{z_1}} - \frac{dz_1}{z_1}$, and similarly $\daleth^*\big(\frac{d\overline{z_2}}{\overline{z_2}} - \frac{dz_2}{z_2}\big) = \frac{d\overline{z_2}}{\overline{z_2}} - \frac{dz_2}{z_2}$ on $\{z_2\neq 0\}$, we have on $\{z_1z_2\neq 0\}$, according to the identity $\eta^\flat = \frac{i}{4R}\big[u^2\big(\frac{d\overline{z_1}}{\overline{z_1}} - \frac{dz_1}{z_1}\big) - v^2\big(\frac{d\overline{z_2}}{\overline{z_2}} - \frac{dz_2}{z_2}\big)\big]$ ([Auv, Lemma 1.6]):

$$\eta^\flat = \frac{i}{4R^\flat}\Big[(u^\flat)^2\Big(\frac{d\overline{z_1}}{\overline{z_1}} - \frac{dz_1}{z_1}\Big) - (v^\flat)^2\Big(\frac{d\overline{z_2}}{\overline{z_2}} - \frac{dz_2}{z_2}\Big)\Big]$$

From this we compute $\eta^\flat(\xi) = 1$ and $\eta^\flat(-I_1\xi) = 0$. We also compute $\eta^\flat(\zeta)$ as follows:

$$\begin{aligned}\eta^\flat(\zeta) =& \frac{i}{4R^\flat}\frac{1}{2iR}\Big[(u^\flat)^2e^{4my_1}\Big(\frac{z_2}{\overline{z_1}} - \frac{\overline{z_2}}{z_1}\Big) - (v^\flat)^2e^{-4my_1}\Big(\frac{z_1}{\overline{z_2}} - \frac{\overline{z_1}}{z_2}\Big)\Big]\\ =& \frac{i}{4R^\flat}\frac{\alpha^2}{2iR}\Big[(u^\flat)^2e^{4my_1}\frac{z_1z_2-\overline{z_1z_2}}{\alpha^2|z_1|^2} - (v^\flat)^2e^{-4my_1}\frac{z_1z_2-\overline{z_1z_2}}{\alpha^2|z_2|^2}\Big]\\ =& \frac{i\alpha^2y_2}{2R^\flat R}\sinh\big[4m(y_1-y_1^\flat)\big],\end{aligned}$$

since from the pulled-back LeBrun's equations (26), $\frac{(u^\flat)^2}{\alpha^2|z_1|^2} = e^{-4my_1^\flat}$ and $\frac{(v^\flat)^2}{\alpha^2|z_2|^2} = e^{4my_1^\flat}$. Similarly $\eta^\flat(I_1\zeta) = \frac{i\alpha^2y_3}{2R^\flat R}\sinh[4m(y_1-y_1^\flat)]$, and since $(y_1-y_1^\flat) = O(R^{-1})$, both $\eta^\flat(\zeta)$ and $\eta^\flat(I_1\zeta)$ are $O(R^{-2})$. Gathering those estimates, we get that

$$|\eta^\flat - \eta|_{\mathbf{f}} = O(R^{-2}),$$

which is better than needed.

Recall that $\mathbf{f} = V(dy_1^2 + dy_2^2 + dy_3^2) + V^{-1}\eta$; since $V^{-1} - (V^\flat)^{-1}$, and similarly $V - V^\flat$, are $O(R^{-3})$, in view on the estimates we have just proved on the $dy_j - dy_j^\flat$ and $\eta^\flat - \eta$, we have:

$$|\mathbf{f}^\flat - \mathbf{f}|_{\mathbf{f}} = O(R^{-1}).$$

Estimate on $\nabla^{\mathbf{f}}(\mathbf{f} - \mathbf{f}^\flat)$. We now prove that $\left|\nabla^{\mathbf{f}}(\mathbf{f} - \mathbf{f}^\flat)\right|_{\mathbf{f}} = O(R^{-1})$, which is the same as proving that $\left|\nabla^{\mathbf{f}}\mathbf{f}^\flat\right|_{\mathbf{f}} = O(R^{-1})$. In view of the previous estimates on $V - V^\flat$, $V^{-1} - (V^\flat)^{-1}$, on the $dy_j - dy_j^\flat$ and on $\eta - \eta^\flat$, and since the $\nabla^{\mathbf{f}} dy_j$ and $\nabla^{\mathbf{f}}\eta$ are $O(R^{-2})$ for $\mathbf{f}$, it will be sufficient for our purpose to see that the $\nabla^{\mathbf{f}}(dy_j - dy_j^\flat)$ and $\nabla^{\mathbf{f}}(\eta - \eta^\flat)$ are $O(R^{-1})$ for $\mathbf{f}$.

We start with $\nabla^{\mathbf{f}}(dy_2 - dy_2^\flat)$ and $\nabla^{\mathbf{f}}(dy_3 - dy_3^\flat)$. We have $d(y_2 + iy_3) - d(y_2^\flat + iy_3^\flat) = (\alpha^2 - 1)d(y_2 + iy_3) + 2(y_2 + iy_3)\alpha d\alpha$, we know that $\alpha - 1 = O(r^{-4}) = O(R^{-2})$, and we actually proved that $|d\alpha|_{\mathbf{f}} = O(r^{-4}) = O(R^{-2})$. Similarly, we will be done if we prove that $|\nabla^{\mathbf{f}} d\alpha|_{\mathbf{f}}$ is still $O(r^{-4})$.

Since α is $\mathbb{S}^1$-invariant, $d\alpha = \frac{\partial\alpha}{\partial y_1}dy_1 + \frac{\partial\alpha}{\partial y_2}dy_2 + \frac{\partial\alpha}{\partial y_3}dy_3$; the $\frac{\partial\alpha}{\partial y_j}$ are $\mathbb{S}^1$-invariant as well, and thus $\nabla^{\mathbf{f}} d\alpha = \sum_{j,\ell=1}^3 \frac{\partial^2\alpha}{\partial y_j \partial y_\ell} dy_j \otimes dy_\ell + \sum_{j=1}^3 \frac{\partial\alpha}{\partial y_j}\nabla^{\mathbf{f}} dy_j$. The last summand is $O(R^{-2}r^{-4})$, since the $\frac{\partial\alpha}{\partial y_j}$ are $O(r^{-4})$ and the $|\nabla^{\mathbf{f}} dy_j|_{\mathbf{f}}$ are $O(R^{-2})$; we thus focus on the hessian $\sum_{j,\ell=1}^3 \frac{\partial^2\alpha}{\partial y_j \partial y_\ell} dy_j \otimes dy_\ell$, and all we need to prove is $\frac{\partial^2\alpha}{\partial y_j \partial y_\ell} = O(r^{-4})$ (actually, $O(R^{-2})$) for all j, ℓ. Now in terms of the y_j variables,

$$\alpha = 1 + \frac{a}{\kappa + \big((y_1^2 + y_2^2 + y_3^2)^{1/2}\cosh(4my_1) + y_1\sinh(4my_1)\big)^2},$$

and using that $e^{4m|y_1|} = O(Rr^{-2})$, proving that $\frac{\partial^2\alpha}{\partial y_j \partial y_\ell} = O\big((R\cosh(4my_1) + y_1\sinh(4my_1))^{-2}\big) = O(r^{-4})$ for all j, ℓ amounts to an easy exercise. This settles the cases of $\nabla^{\mathbf{f}}(dy_2 - dy_2^\flat)$ and $\nabla^{\mathbf{f}}(dy_3 - dy_3^\flat)$.

Since our treatment of $dy_1 - dy_1^\flat$ is a little less conventional, we shall see now how goes that of $\nabla^{\mathbf{f}}(dy_1 - dy_1^\flat)$. According to formulas (29) and (30) and the previous estimates on the derivatives of r^2, it is enough to see that $dy_1^\flat = O(1)$ and $dR^\flat = O(1)$, which are known for the previous step, giving in particular $d\sinh[4m(y_1 - y_1^\flat)] = \cosh[4m(y_1 - y_1^\flat)]d(y_1 - y_1^\flat)$, which is $O(R^{-1})$ (actually $O(R^{-2})$) for $\mathbf{f}$ since $\cosh[4m(y_1 - y_1^\flat)] \sim 1$ and $|d(y_1 - y_1^\flat)|_{\mathbf{f}} = O(R^{-2})$.

The treatment of $\eta^\flat$ is similar.

We prove finally that $\left|(\nabla^{\mathbf{f}})^2(\mathbf{f} - \mathbf{f}^\flat)\right|_{\mathbf{f}} = O(R^{-1})$ with the same techniques. $\square$

2 Asymptotics of ALE hyperkähler metrics

We prove in this part an explicit version of Theorem 0.3; we indeed compute explicitly the first non-vanishing perturbative terms of the hyperkähler data of the ALE gravitational instantons seen as deformations of Kleinian singularities. This gives in particular the asymptotics stated in the previous part, Lemma 1.6, which are crucial in our construction of ALF metrics, as mentioned already.

2.1 Kronheimer's ALE instantons

2.1.1 Basic facts and notations

We introduce a few notions about the ALE gravitational instantons constructed by Kronheimer in [Kro1] – and which is exhaustive in the sense that any ALE gravitational instanton is isomorphic to one of Kronheimer's list –, so as to state properly the main result of this part, i.e. Theorem 2.1 of next paragraph, dealing with precise asymptotics of those asymptotically euclidean spaces.

Finite subgroups of $\mathrm{SU}(2)$*, and McKay correspondence.* — The classification of the finite subgroup of $\mathrm{SU}(2)$ is well-known: up to conjugacy, in addition to the binary dihedral groups $\mathcal{D}_k$ used in Part 1, one has the cyclic groups of order $k \geq 2$, generated by $\begin{pmatrix} e^{2i\pi/k} & 0 \\ 0 & e^{-2i\pi/k} \end{pmatrix}$, on the one hand, and the binary tetrahedral, octahedral and icosahedral groups of respective orders 24, 48 and 120, which admit more complicated generators – all we need to notice for further purpose is that they respectively contain $\mathcal{D}_2$, $\mathcal{D}_3$ and $\mathcal{D}_5$ (among others) as subgroups. When no specification is needed, we shall adopt the notation Γ for any fixed group among these finite subgroups of $\mathrm{SU}(2)$.

ALE instantons modelled on $\mathbb{R}^4/\Gamma$. — Kronheimer's construction now consists in producing asymptotically euclidean hyperkähler metrics on *smooth deformations* of the Kleinian singularity $\mathbb{C}^2/\Gamma$, which are diffeomorphic to the *minimal resolution of* $\mathbb{C}^2/\Gamma$. More precisely, the hyperkähler manifolds Kronheimer produces are parametrised as follow: since Γ is a finite subgroup of $\mathrm{SU}(2)$, McKay's correspondence [McK] associates a simple Lie algebra, $\mathfrak{g}_\Gamma$ say, to this group; for instance, the Lie algebra associated to $\mathcal{D}_k$ is $\mathfrak{so}(2k+4)$, (also referred to as D_{k+2} – we prefer the $\mathfrak{so}$ notation which is less confusing when working with binary dihedral groups!). Pick a (real) Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_\Gamma$. Then:

For any $\zeta \in \mathfrak{h} \otimes \mathbb{R}^3$ outside a codimension 3 set D, there exists an ALE gravitational instanton $\left(X_\zeta, g_\zeta, I_1^\zeta, I_2^\zeta, I_3^\zeta\right)$ modelled on $\mathbb{R}^4/\Gamma$ at infinity in the sense that there exists a diffeomorphisms Φ_ζ between infinities of X_ζ and $\mathbb{R}^4/\Gamma$ such that: $\Phi_{\zeta *} g_\zeta - \mathbf{e} = O(r^{-4})$, $\Phi_{\zeta *} I_j^\zeta - I_j = O(r^{-4})$, $j = 1, 2, 3$.

The O are here understood in the asymptotically euclidean setting, i.e. $\varepsilon = O(r^{-a})$ means: for all $\ell \geq 0$, $\left|(\nabla^{\mathbf{e}})^{\ell}\varepsilon\right| = O(r^{-a-\ell})$; since we remain in this setting until the end of this part, we shall keep this convention throughout the following sections 2.3 and 2.4.

2.1.2 Asymptotics of ALE instantons: statement of the theorem

Up to a judicious choice of the ALE diffeomorphism Φ_ζ, which actually is part of Kronheimer's construction, one can be more accurate about the $O(r^{-4})$-error term evoked above. This is the purpose of the main result of this part:

Theorem 2.1 *Given $\zeta \in \mathfrak{h} \otimes \mathbb{R}^3 - D$, one can choose the diffeomorphism Φ_ζ between infinities of X_ζ and $\mathbb{R}^4/\Gamma$ such that ${\Phi_\zeta}_* g_\zeta - \mathbf{e} = h_\zeta + O(r^{-6})$, ${\Phi_\zeta}_* I_1^\zeta - I_1 = \iota_1^\zeta + O(r^{-6})$ and if $\omega_1^\zeta := g_\zeta(I_1^\zeta \cdot, \cdot)$, then ${\Phi_\zeta}_* \omega_1^\zeta - \omega_1^{\mathbf{e}} = \varpi_1^\zeta + O(r^{-6})$, where:*

(31)
$$\begin{aligned} h_\zeta = &- \|\Gamma\| \sum_{(j,k,\ell)\in\mathfrak{A}_3} |\zeta_j|^2 \frac{(rdr)^2 + \alpha_j^2 - \alpha_k^2 - \alpha_\ell^2}{r^6} - \|\Gamma\| \langle \zeta_1, \zeta_2 \rangle \frac{\alpha_1 \cdot \alpha_2 - rdr \cdot \alpha_3}{r^6} \\ &- \|\Gamma\| \langle \zeta_1, \zeta_3 \rangle \frac{\alpha_1 \cdot \alpha_3 + rdr \cdot \alpha_2}{r^6} - \|\Gamma\| \langle \zeta_2, \zeta_3 \rangle \frac{\alpha_2 \cdot \alpha_3 - rdr \cdot \alpha_1}{r^6}, \end{aligned}$$

where ι_1^ζ is given via the coupling:

$$\begin{aligned} \mathbf{e}(\iota_1^\zeta \cdot, \cdot) = &\|\Gamma\| (|\zeta_3|^2 - |\zeta_2|^2) \frac{\alpha_2 \cdot \alpha_3}{r^6} - \|\Gamma\| (|\zeta_3|^2 + |\zeta_2|^2) \frac{rdr \cdot \alpha_1}{r^6} \\ &- \|\Gamma\| \langle \zeta_2, \zeta_3 \rangle \frac{(rdr)^2 + \alpha_3^2 - \alpha_1^2 - \alpha_2^2}{r^6}, \end{aligned} \tag{32}$$

and

$$\varpi_1^\zeta = -\|\Gamma\| |\zeta_1|^2 \theta_1 - \|\Gamma\| \langle \zeta_1, \zeta_2 \rangle \theta_2 - \|\Gamma\| \langle \zeta_1, \zeta_3 \rangle \theta_3, \tag{33}$$

with $\|\Gamma\| = \mathbf{c}|\Gamma|$ for a universal constant $\mathbf{c} > 0$. Moreover, ${\Phi_\zeta}_* \mathrm{vol}^{g_\zeta} = \Omega_{\mathbf{e}}$, and if Γ is binary dihedral, tetrahedral, octahedral or icosahedral, the error term can be taken of size $O(r^{-8})$.

Recall the notations $\alpha_j = I_1 rdr$, $j = 1, 2, 3$, and $\theta_a = \frac{rdr \wedge \alpha_a - \alpha_b \wedge \alpha_c}{r^6}$, $(a, b, c) \in \{(1,2,3), (2,3,1), (3,1,2)\}$. The scalar product on $\mathfrak{h}$ used in this statement is the one induced by the Killing form.

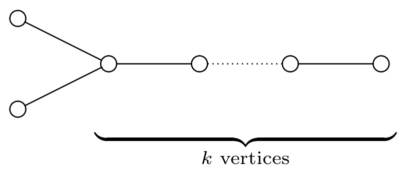
The rest of this part is devoted to the proof of this result. In next section we specify the meaning of the space of parameters $\mathfrak{h} - D$; in particular we see how $\mathfrak{h}$ is identified to the degree 2 homology of our Kronheimer's instantons, which is helpful in computing the constant $\mathbf{c}$ of the statement, as well as the coefficients appearing in formulas (31)-(33). We also fix the choice of the diffeomorphisms Φ_ζ, and check their properties on volume forms (Lemma 2.5). The explicit determination of h_ζ, ι_1^ζ and ϖ_1^ζ is the purpose of sections 2.3 and 2.4.

2.2 Precisions on Kronheimer's construction

2.2.1 The degree 2 homology/cohomology

The "forbidden set" D. — We keep the notation Γ for one of the subgroups of $\mathrm{SU}(2)$ mentioned in the previous section. We saw that Kronheimer's ALE instantons asymptotic to $\mathbb{R}^4/\Gamma$ are parametrised by a triple $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathfrak{h} \otimes \mathbb{R}^3 - D$, with $\mathfrak{h}$ a real Cartan subalgebra of the Lie algebra associated to Γ by McKay correspondence; for instance, if $\Gamma = \mathcal{D}_k$, $k \geq 2$, then one can take $\mathfrak{h}$ as the Cartan subalgebra of $\mathfrak{so}(2k+4)$ constituted by matrices of shape $\mathrm{diag}(\lambda_1, \dots, \lambda_{k+2}, -\lambda_1, \dots, -\lambda_{k+2})$. We shall first be more specific about the "forbidden set" D; according to [Kro1, Cor. 2.10], it is the union of codimension 3 subspaces $D_\theta \otimes \mathbb{R}^3$ over a positive root system of $\mathfrak{h}$, with D_θ the kernels of the concerned roots; as such, it thus has codimension 3 in $\mathfrak{h}$.

Topology of X_ζ. — Recall the notation $\big(X_\zeta, g_\zeta, I_1^\zeta, I_2^\zeta, I_3^\zeta\big)$ for the hyperkähler manifold of admissible parameter ζ – this is actually also defined as a hyperkähler orbifold if $\zeta \in D$. Those spaces are diffeomorphic to the minimal resolution of $\mathbb{C}^2/\Gamma$ (for I_1, say) [Kro1, Cor. 3.12]; as such they are simply connected and, again when $\Gamma = \mathcal{D}_k$, their rank 2 topology is given by the diagram:



(which is nothing but the Dynkin diagram associated to $\mathfrak{so}(2k+4)$), where each vertex represents the class of a sphere of -2 self-intersection, and where two vertices are linked by an edge if and only if the corresponding spheres intersect, in which case they intersect normally at one point.

Furthermore, there is an identification between $H^2(X_\zeta, \mathbb{R})$ and $\mathfrak{h}$ such that:

- the cohomology class of the Kähler form $\omega_j^\zeta := g_\zeta\big(I_j^\zeta \cdot, \cdot\big)$ is ζ_j, $j = 1, 2, 3$;
- $H_2(X_\zeta, \mathbb{Z})$ is identified with the root lattice of $\mathfrak{h}$; more precisely, given simple roots of $\mathfrak{h}$ and the corresponding basis of $H_2(X_\zeta, \mathbb{Z})$, the intersection matrix of this basis is exactly the opposite of the Cartan matrix of the simple roots,

see [Kro1, p.678]; in the case $\Gamma = \mathcal{D}_k$, $k \geq 2$, this matrix is thus:

$$
{\scriptstyle k+2}\left[\;\underbrace{\begin{pmatrix}
2 & 0 & -1 & 0 & \cdots & 0 \\
0 & 2 & -1 & 0 & & \vdots \\
-1 & -1 & 2 & -1 & \ddots & \vdots \\
0 & 0 & -1 & \ddots & \ddots & 0 \\
\vdots & & \ddots & \ddots & \ddots & -1 \\
0 & \cdots & \cdots & 0 & -1 & 2
\end{pmatrix}}_{k+2}\right. .
$$

From the latter, we deduce the following lemma, identifying cup-product on H^2 and the scalar product on $\mathfrak{h}$ induced by the Killing form, up to signs:

Lemma 2.2 *Consider $\alpha, \beta \in H^2(X_\zeta, \mathbb{R})$, such that α or β has compact support. Then $\alpha \cup \beta = \int_{X_\zeta} \alpha \wedge \beta = -\langle \alpha, \beta \rangle$, where the latter is computed with seeing α and β in $\mathfrak{h}$ via the above identification between $H^2(X_\zeta, \mathbb{R})$ and $\mathfrak{h}$.*

Proof. We do it for $\Gamma = \mathcal{D}_k$, $k \geq 2$. By Poincaré duality, the computation of $\alpha \cup \beta$ amounts to that of intersection numbers for a basis of $H_2(X_\zeta, \mathbb{Z})$. But through the identification between $H_2(X_\zeta, \mathbb{Z})$ and the root lattice of $\mathfrak{h}$ above, the matrix of intersection numbers on the one hand and that of scalar products of the corresponding basis (or dually, of the simple roots) are the same up to signs. $\square$

Period matrix. — For $\zeta \in \mathfrak{h} - D$, consider as above a basis Σ_j, $j = 1, \dots, r$ say, of $H_2(X_\zeta, \mathbb{Z})$; from the previous paragraph, the *period matrix*

$$
P(\zeta) = \big(P_{j\ell}(\zeta)\big)_{\substack{1 \leq j \leq 3 \\ 1 \leq \ell \leq r}} := \left(\int_{\Sigma_\ell} \omega_j^\zeta\right)_{\substack{1 \leq j \leq 3 \\ 1 \leq \ell \leq r}}
$$

can be computed thanks to the identities $[\omega_j^\zeta] = \zeta_j$. One easily sees that these $P(\zeta) = P(\xi)$ if and only if $\zeta = \xi$. With this formalism Kronheimer's classification [Kro2, Thm. 1.3] can be stated as : *two ALE gravitational instantons are isomorphic as hyperkähler manifolds if and only if they have the same period matrix.* From this we deduce (see also [BR, p.8, (4)]):

Lemma 2.3 *Let $\zeta \in \mathfrak{h} - D$, and let $A \in SO(3)$ act on ζ and the complex structures I_j^ζ as in section 1.1. Then there exists a tri-holomorphic isometry between $\big(X_\zeta, g_\zeta, (AI^\zeta)_1, (AI^\zeta)_2, (AI^\zeta)_3\big)$ and $\big(X_{A\zeta}, g_{A\zeta}, I_1^{A\zeta}, I_2^{A\zeta}, I_3^{A\zeta}\big)$.*

Proof. Just check that in both cases, the period matrix is $AP(\zeta)$, and apply Kronheimer's classification theorem. $\square$

2.2.2 Analytic expansions.

Choice of the chart at infinity. — Consider a parameter $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathfrak{h} \otimes \mathbb{R}^3$ and set

$$\zeta' = (0, \zeta_2, \zeta_3), \qquad \zeta'' = (0, 0, \zeta_3)$$

– we will keep these notations below. As described in [Kro1, p.677], there exist proper continuous maps:

$$\begin{array}{rcl} \lambda_1^{\zeta} : \big(X_\zeta, g_\zeta, I_1^\zeta, I_2^\zeta, I_3^\zeta\big) & \longrightarrow & \big(X_{\zeta'}, g_{\zeta'}, I_1^{\zeta'}, I_2^{\zeta'}, I_3^{\zeta'}\big), \\ \lambda_2^{\zeta'} : \big(X_{\zeta'}, g_{\zeta'}, I_1^{\zeta'}, I_2^{\zeta'}, I_3^{\zeta'}\big) & \longrightarrow & \big(X_{\zeta''}, g_{\zeta''}, I_1^{\zeta''}, I_2^{\zeta''}, I_3^{\zeta''}\big), \\ \lambda_3^{\zeta''} : \big(X_{\zeta''}, g_{\zeta''}, I_1^{\zeta''}, I_2^{\zeta''}, I_3^{\zeta''}\big) & \longrightarrow & (\mathbb{R}^4/\Gamma, \mathbf{e}, I_1, I_2, I_3), \end{array}$$

which are diffeomorphisms (at least) on $(\lambda_3^{\zeta''} \circ \lambda_2^{\zeta'} \circ \lambda_1^{\zeta})^{-1}(\{0\})$, $(\lambda_3^{\zeta''} \circ \lambda_2^{\zeta'})^{-1}(\{0\})$, and $(\lambda_3^{\zeta''})^{-1}(\{0\})$ respectively. As soon as $\zeta'' \notin D$ (resp. $\zeta', \zeta \notin D$), $\lambda_3^{\zeta''}$ (resp. $\lambda_2^{\zeta'}, \lambda_1^{\zeta}$) is a resolution of singularities for the third (resp. the second, the first) pair of complex structures; in particular, if $\zeta' \notin D$ (resp. if $\zeta \notin D$), then $\lambda_2^{\zeta'}$ (resp. λ_1^{ζ}) is smooth, and holomorphic for the appropriate pair of complex structures.

To get a "coordinate chart" on X_ζ (or rather, to view objects on $\mathbb{R}^4/\Gamma$), one sets:

$$F_\zeta : (\lambda_3^{\zeta''} \circ \lambda_2^{\zeta'} \circ \lambda_1^{\zeta})^{-1} : (\mathbb{R}^4 \backslash \{0\})/\Gamma \longrightarrow X_\zeta$$

(beware this is not exactly the same order of composition as Kronheimer's "coordinate chart", but this is not a problem by symmetry).

"Homogeneity" and consequences. — We shall see that the F_ζ are going be the Φ_ζ of Theorem 2.1. For now, according to Proposition 3.14 in [Kro1] and its proof, we have for any ζ the converging expansion

$${F_\zeta}^* g_\zeta = \mathbf{e} + \sum_{j=2}^{\infty} h_\zeta^{(j)},$$

with $h_\zeta^{(j)}$ a homogeneous polynomial of degree j in ζ with coefficients homogeneous symmetric 2-tensors on $\mathbb{R}^4/\Gamma$ – more precisely, if κ_s is the dilation $x \mapsto sx$ for any positive s, $\kappa_s^* h_\zeta^{(j)} = s^{-2(j-1)} h_\zeta^{(j)}$. We will thus be concerned with determining explicitly the term $h_\zeta^{(2)}$, and moreover show that when Γ is binary dihedral then $h_\zeta^{(3)} = 0$. For now, observe that Kronheimer's arguments, consisting in analyticity and homogeneity properties of his construction, can also be used to give the existence of analogous expansions of other tensors such as the complex structures, and therefore the Kähler forms, or the volume forms as well. We can write for example

$$(34) \qquad {F_\zeta}^* I_1^\zeta = I_1 + \sum_{j=1}^{\infty} \iota_{1,j}^\zeta,$$

where $\iota_{1,j}^{\zeta}$ is a homogeneous polynomial of degree j in ζ with coefficients (1,1)-tensors, satisfying $\kappa_s^*\iota_{1,j}^{\zeta} = s^{-2j}\iota_{1,j}^{\zeta}$ (and again, the lower-order term $\iota_{1,1}^{\zeta}$ vanishes, but we will find this fact again below).

2.2.3 Minimal resolutions, invariance of the holomorphic symplectic structure.

We know that as soon as $\zeta \notin D$, $\lambda_1^{\zeta} : \big(X_{\zeta}, I_1^{\zeta}\big) \to \big(X_{\zeta'}, I_1^{\zeta'}\big)$ is a minimal resolution, and a similar statement holds for $\lambda_2^{\zeta'} : \big(X_{\zeta'}, I_2^{\zeta'}\big) \to \big(X_{\zeta''}, I_2^{\zeta''}\big)$ and $\lambda_3^{\zeta''} : \big(X_{\zeta''}, I_3^{\zeta''}\big) \to (\mathbb{R}^4/\Gamma, I_3)$ whenever $\zeta' \notin D$ or $\zeta'' \notin D$, respectively ([Kro1, p.675]).

As seen already, those maps can happen to be smooth – for instance λ_1^{ζ} is, when $\zeta, \zeta' \notin D$; we are then only left with their holomorphicity property. This can be used nevertheless with their asymptotic preserving of the hyperkähler structure, to see that they do preserve the appropriate holomorphic symplectic structure:

Lemma 2.4 *Fix $\zeta \in \mathfrak{h}\otimes\mathbb{R}^3$, and assume that $\zeta'' \notin D$. Then the map $\lambda_3^{\zeta''}$ verifies: $(\lambda_3^{\zeta''})^*(\omega_1^{\mathbf{e}} + i\omega_2^{\mathbf{e}}) = \omega_1^{\zeta''} + i\omega_2^{\zeta''}$.*

Similarly, if $\zeta', \zeta'' \notin D$, then $(\lambda_2^{\zeta'})^*(\omega_3^{\zeta''} + i\omega_1^{\zeta''}) = \omega_3^{\zeta'} + i\omega_1^{\zeta'}$; if $\zeta, \zeta' \notin D$, then $(\lambda_1^{\zeta})^*(\omega_2^{\zeta'} + i\omega_3^{\zeta'}) = \omega_2^{\zeta} + i\omega_3^{\zeta}$.

Proof. The assertion on $\lambda_3^{\zeta''}$ is actually classical, and can be settled in the following elementary way. Call θ the 2-form $(\lambda_3^{\zeta''})_*(\omega_1^{\zeta''} + i\omega_2^{\zeta''})$, well-defined on $(\mathbb{R}^4\backslash\{0\})/\Gamma$, pulled-back to $\mathbb{R}^4\backslash\{0\}$. Since $\lambda_3^{\zeta''}$ is holomorphic for the pair $\big(I_3^{\zeta''}, I_3\big)$ and $\omega_1^{\zeta''} + i\omega_2^{\zeta''}$ is a holomorphic (2,0)-form for $I_3^{\zeta''}$, θ is a holomorphic (2,0)-form for I_3, and can thus be written as $f(\omega_1^{\mathbf{e}} + i\omega_2^{\mathbf{e}})$, where f is thus holomorphic for I_3 on $\mathbb{R}^4\backslash\{0\}$. By Hartogs' lemma it can be extended to the whole $\mathbb{R}^4$; however, since $(\lambda_3^{\zeta''})_*\omega_j^{\zeta''} = {F_{\zeta''}}^*\omega_j^{\zeta''} \sim \omega_j^{\mathbf{e}}$ near infinity on $\mathbb{R}^4/\Gamma$, $j = 1, 2$, which can be seen as a consequence of the power series expansions analogous to (34) for Kähler forms, we get that f tends to 1 at infinity. It is therefore constant, equal to 1, which exactly means that $(\lambda_3^{\zeta''})^*(\omega_1^{\mathbf{e}} + i\omega_2^{\mathbf{e}}) = \omega_1^{\zeta''} + i\omega_2^{\zeta''}$.

We deal with the assertion on $\lambda_2^{\zeta'}$ in a somehow similar way. Since $\zeta', \zeta'' \notin D$, $\lambda_2^{\zeta'}$ is a global diffeomorphism between the smooth $X_{\zeta'}$ and $X_{\zeta''}$, holomorphic for the pair $\big(I_2^{\zeta'}, I_2^{\zeta''}\big)$; since $\omega_3^{\zeta'} + i\omega_1^{\zeta'}$ trivialises $K_{(X_{\zeta'}, I_2^{\zeta'})}$ and is a (2,0)-holomorphic form for $I_2^{\zeta'}$, $(\lambda_2^{\zeta'})^*(\omega_3^{\zeta''} + i\omega_1^{\zeta''})$ can be written as $f(\omega_3^{\zeta'} + i\omega_1^{\zeta'})$ with f a holomorphic function on $\big(X_{\zeta'}, I_2^{\zeta'}\big)$. Again f tends to 1 near the infinity of $X_{\zeta'}$, since there $(\lambda_2^{\zeta'})^*\omega_j^{\zeta''} \sim \omega_j^{\zeta'}$, $j = 1, 3$. Moreover $\omega_3^{\zeta''} + i\omega_1^{\zeta''}$ never vanishes on $X_{\zeta''}$, and so neither does f on $X_{\zeta''}$. We collect those observations by saying that $\log(|f|^2)$ is a $g_{\zeta'}$-harmonic function on $X_{\zeta'}$ tending to zero at infinity, and thus identically vanishing. Since f is holomorphic, it is not hard seeing that it is therefore constant, thus $f \equiv 1$, or in other words: $(\lambda_2^{\zeta'})^*(\omega_3^{\zeta''} + i\omega_1^{\zeta''}) = \omega_3^{\zeta'} + i\omega_1^{\zeta'}$.

The assertion on λ_1^ζ is done in the exact same way. $\square$

An easy but fundamental consequence of the construction of F_ζ via the λ_j^ζ and the previous lemma is the invariance of the volume form, which we state for ζ corresponding to smooth X_ζ so as to avoid useless technicalities:

Lemma 2.5 *The volume form $F_\zeta{}^* \operatorname{vol}^{g_\zeta}$ does not depend on $\zeta \in \mathfrak{h} \otimes \mathbb{R}^3 - D$, and is equal to the standard $\Omega_{\mathbf{e}}$.*

Proof. Notice first that once we know that $F_\zeta{}^* \operatorname{vol}^{g_\zeta}$ does not depend on ζ, the equality $F_\zeta{}^* \operatorname{vol}^{g_\zeta} = \Omega_{\mathbf{e}}$ is a direct consequence of the expansion of $F_\zeta{}^* \operatorname{vol}^{g_\zeta}$ as a power series of ζ, the constant term of which is $\Omega_{\mathbf{e}}$. To prove that $F_\zeta{}^* \operatorname{vol}^{g_\zeta}$ is independent of ζ, we proceed within three steps, considering first ζ'', and then ζ' and ζ. Even if $\zeta \notin D$, ζ' or ζ'' might lie in D; we can however assume this is not the case without loss of generality, since $F_\zeta{}^* \operatorname{vol}^{g_\zeta}$ can be written as a power series of ζ. Now from the hyperkähler data $\big(X_{\zeta''}, g_{\zeta''}, I_1^{\zeta''}, I_2^{\zeta''}, I_3^{\zeta''}\big)$, we know that $\operatorname{vol}^{g_{\zeta''}} = \frac{1}{2}(\omega_1^{\zeta''})^2$. Since $F_\zeta{}^*(\omega_1^{\zeta''}) = \omega_1^{\mathbf{e}}$ (the standard Kähler form on $\mathbb{C}^2$), we get that $F_{\zeta''}{}^* \operatorname{vol}^{g_{\zeta''}} = \Omega_{\mathbf{e}}$.

Consider now $X_{\zeta'}$; we know that $\omega_3^{\zeta'}$ is "preserved" by $\lambda_2^{\zeta'}$, and therefore:

$$F_{\zeta'}{}^* \operatorname{vol}^{g_{\zeta'}} = \frac{1}{2} F_{\zeta'}{}^*(\omega_3^{\zeta'})^2 = \frac{1}{2} F_{\zeta''}{}^*(\lambda_2^{\zeta'})_*(\omega_3^{\zeta'})^2 = \frac{1}{2} F_{\zeta''}{}^*(\omega_3^{\zeta''})^2 = F_{\zeta''}{}^* \operatorname{vol}^{g_{\zeta''}},$$

the last equality coming from the fact that $\omega_3^{\zeta''}$ is one of the Kähler forms of the hyperkähler structure $(g_{\zeta''}, I_1^{\zeta''}, I_2^{\zeta''}, I_3^{\zeta''})$.

To conclude, we notice that ω_3^ζ is preserved by λ_1^ζ i.e. $\omega_3^\zeta = (\lambda_1^\zeta)^* \omega_3^{\zeta'}$, and thus

$$F_\zeta{}^* \operatorname{vol}^{g_\zeta} = \frac{1}{2} F_\zeta{}^*(\omega_3^\zeta)^2 = \frac{1}{2} F_{\zeta'}{}^*(\lambda_1^\zeta)_*(\omega_3^\zeta)^2 = \frac{1}{2} F_{\zeta'}{}^*(\omega_3^{\zeta'})^2 = F_{\zeta'}{}^* \operatorname{vol}^{g_{\zeta'}};$$

here we could also have used the forms ω_2^ζ and $\omega_2^{\zeta'}$. To make a long story short, the reason for the volume form invariance is that at each step of the composition of the λ_i^ζ, at least one Kähler form is preserved. $\square$

2.3 Explicit determination of h_ζ

2.3.1 Verifying a gauge

We shall now work more precisely on the first possibly non-vanishing term of the expansion of $F_{t\zeta}{}^* g_{t\zeta}$, $t \in \mathbb{R}$, ζ fixed; this allows us to *redefine* $h_\zeta^{(2)}$ as follows:

Definition 2.6 *Fix $\zeta \in \mathfrak{h} \otimes \mathbb{R}^3$, and set on $\mathbb{R}^4 \backslash \{0\}$:*

$$h_\zeta := \frac{1}{2} \frac{d^2}{dt^2}\bigg|_{t=0} F_{t\zeta}{}^* g_{t\zeta}, \tag{35}$$

which is then $O(r^{-4})$, with $\nabla^{\mathbf{e}}$-ℓth derivatives $O(r^{-4-\ell})$, near both 0 and infinity, and verifies:

$$F_\zeta{}^* g_\zeta = \mathbf{e} + h_\zeta + \varepsilon_\zeta,$$

with $(\nabla^{\mathbf{e}})^\ell \varepsilon_\zeta = O(r^{-6-\ell})$. More precisely, h_ζ is a homogeneous polynomial of degree 2 in ζ, with coefficients symmetric 2-tensors homogeneous of degree 2 in the sense that $\kappa_s^* h_\zeta = s^{-2} h_\zeta$, where κ_s is the dilation $x \mapsto sx$ of $\mathbb{R}^4 \backslash \{0\}$ for any $s > 0$; as for ε_ζ, it is a sum of terms of degree at least 3 in ζ.

As indicated by the title of this section, given an admissible ζ, we want to analyse h_ζ, which is the first (a priori, possibly) non-vanishing term in the expansion of g_ζ (from now on, for the sake of simplicity, we forget about the F_ζ – we will be more accurate about this abuse of notation whenever needed). There already exists a rather powerful theory of deformations of Kähler-Einstein metrics; see in particular [Bes, ch.12] for an overview on that subject. Nonetheless, because of the diffeomorphisms action in general, much of the theory is configured so as to work once a gauge is fixed, precisely killing the ambiguity coming from the diffeomorphisms.

The following proposition asserts that the h_ζ are indeed in some gauge, making us able of further considerations – just as is done in paragraph 1.4.3. Let us specify though that in determining explicitly h_ζ, we will be more concerned with other specific properties of that tensor, namely with its inductive decomposition into hermitian and skew-hermitian parts with respect to I_1, I_2 and I_3. As we shall see though, the gauge and the decomposition are rather intricate with one another; seeing the verification of the gauge as a guiding thread, we state:

Proposition 2.7 *Fix $\zeta \in \mathfrak{h} \otimes \mathbb{R}^3$. Then the lower order term h_ζ of the deformation g_ζ of $\mathbf{e}$ on $\mathbb{R}^4 \backslash \{0\}$ is in Bianchi gauge with respect to $\mathbf{e}$, and more precisely:*

$$\mathrm{tr}^{\mathbf{e}}(h_\zeta) = 0 \qquad \textit{and} \qquad \delta^{\mathbf{e}} h_\zeta = 0.$$

Moreover, the I_1-skew-hermitian part of h_ζ is $h_{\zeta'}$, the I_2-skew-hermitian part of $h_{\zeta'}$ is $h_{\zeta''}$, and $h_{\zeta''}$ is I_3-hermitian, while the I_1-hermitian part of h_ζ, the I_2-hermitian part of $h_{\zeta'}$ and $h_{\zeta''}$ give rise to closed forms, that is:

$$d\big(h_\zeta(I_1\cdot,\cdot) - h_\zeta(\cdot, I_1\cdot)\big) = d\big(h_{\zeta'}(I_2\cdot,\cdot) - h_{\zeta'}(\cdot, I_2\cdot)\big) = d\big(h_{\zeta''}(I_3\cdot,\cdot)\big) = 0 \ \textit{on}\ \mathbb{R}^4 \backslash \{0\}.$$

Remark 2.8 *We took the liberty of possibly having ζ in D since these statements are made on $\mathbb{R}^4\backslash\{0\}$. More precisely, even if X_ζ is not smooth, its orbifold singularities lie above $0 \in \mathbb{R}^4$ via F_ζ, and h_ζ is smooth on the regular part of X_ζ, i.e. $(F_\zeta)^* h_\zeta$ is smooth on $\mathbb{R}^4\backslash\{0\}$.*

Proof. Let us deal first with the assertion on $\operatorname{tr}^{\mathbf{e}}(h_\zeta)$. At any point of $(\mathbb{R}^4\backslash\{0\})/\Gamma$, for any t:

$$\mathrm{vol}^{g_{t\zeta}} = \det{}^{\mathbf{e}}(g_{t\zeta})\Omega_{\mathbf{e}} = \det{}^{\mathbf{e}}\big(\mathbf{e} + t^2 h_\zeta + O(t^3)\big)\Omega_{\mathbf{e}} = \big(1 + t^2 \operatorname{tr}^{\mathbf{e}}(h_\zeta) + O(t^3)\big)\Omega_{\mathbf{e}}.$$

But we saw in Lemma 2.5 that for all t, $\mathrm{vol}^{g_{t\zeta}} = \Omega_{\mathbf{e}}$; consequently, $\operatorname{tr}^{\mathbf{e}}(h_\zeta) = 0$.

We now deal with the divergence assertion. As for the previous lemma, we proceed inductively on the shape of ζ; the hermitian/skew-hermitian decomposition as well as the closedness property will come out along the different steps of the induction. For this we assume that $\zeta' = (0, \zeta_2, \zeta_3)$ and $\zeta'' = (0, 0, \zeta_3)$ are as well out of the “forbidden set” D. Again, since h_ζ can be written as a sum of quadratic polynomials of ζ times symmetric 2-forms independent of ζ, this assumption does not actually lead to a loss of generality.

Step 1: $\delta^{\mathbf{e}} h_{\zeta''} = 0$. We hence start with $\zeta'' = (0,0,\zeta_3)$. Since I_3 is parallel for $\mathbf{e}$, we have that $d^{*_{\mathbf{e}}}[h_{\zeta''}(\cdot, I_3\cdot)] = (\delta^{\mathbf{e}} h_{\zeta''})(I_3\cdot)$; indeed, given any local $\mathbf{e}$-orthonormal frame $(e_j)_{j=1,\dots,4}$:
(36)

$$d^{*_{\mathbf{e}}}[h_{\zeta''}(\cdot, I_3\cdot)] = -\sum_{j=1}^{4} e_j \lrcorner \big[\nabla^{\mathbf{e}}_{e_j}\big(h_{\zeta''}(\cdot, I_3\cdot)\big)\big] \quad \text{and} \quad \delta^{\mathbf{e}} h_{\zeta''} = -\sum_{j=1}^{4} (\nabla^{\mathbf{e}}_{e_j} h_{\zeta''})(e_j, \cdot),$$

see for instance [Biq, 1.2.11] for the first equality, and [Biq, 1.2.13] for the second one. Moreover $h_{\zeta''}$ is clearly I_3-hermitian, since the $g_{t\zeta''}$ are, which is straightforward from the holomorphicity of the $\lambda_3^{t\zeta''}$ for the pairs $\big(I_3^{t\zeta''}, I_3\big)$; $h_{\zeta''}(\cdot, I_3\cdot)$ is therefore a (1,1)-form for I_3. It is furthermore closed, since the $g_{t\zeta''}(\cdot, I_3\cdot)$ are. We can now use the Kähler identity “$d^* = [\Lambda, d^c]$” with the structure $(\mathbf{e}, I_3)$ and write:

$$d^{*_{\mathbf{e}}}\big(h_{\zeta''}(\cdot, I_3\cdot)\big) = [\Lambda_{\omega_3^{\mathbf{e}}}, d^c_{I_3}]\big(h_{\zeta''}(\cdot, I_3\cdot)\big).$$

But $\Lambda_{\omega_3^{\mathbf{e}}}\big(h_{\zeta''}(\cdot, I_3\cdot)\big) = -\frac{1}{2}\operatorname{tr}^{\mathbf{e}}(h_{\zeta''}) = 0$, and since $h_{\zeta''}(\cdot, I_3\cdot)$ is I_3-hermitian and closed, $d^c_{I_3}\big(h_{\zeta''}(\cdot, I_3\cdot)\big) = d\big(h_{\zeta''}(\cdot, I_3\cdot)\big) = 0$, hence the result.

Step 2: $\delta^{\mathbf{e}} h_{\zeta'} = 0$. We go on our induction and analyse $h_{\zeta'}$, where we recall the notation $\zeta' = (0, \zeta_2, \zeta_3)$. We proceed through the following lines:

(i) we come back momentarily to $h_{\zeta''}$ and prove it is I_2-skew-hermitian;

(ii) we prove that the I_2-skew-hermitian part of $h_{\zeta'}$ is $h_{\zeta''}$, which is known to be divergence-free for $\mathbf{e}$;

(iii) we conclude by proving that the I_2-hermitian part of $h_{\zeta'}$ is $\mathbf{e}$-divergence-free as well.

We tackle Point (i). Recall that the map $\lambda_2^{\zeta'}: X_{\zeta'} \to X_{\zeta''}$ is holomorphic for the pair $\big(I_2^{\zeta'}, I_2^{\zeta''}\big)$; since we forget about $F_{\zeta'}$ and $F_{\zeta''}$, this amounts to writing $I_2^{\zeta'} = I_2^{\zeta''}$. Recall that in the same way as for the metric, the complex structures admit an analytic expansion, which can be written as a power series of ζ with coefficients homogeneous (1,1)-tensors on $(\mathbb{R}^4\backslash\{0\})/\Gamma$. *We assume momentarily that the first order variation vanishes*, and we thus write $I_2^{\zeta''} = I_2 + \iota_2^{\zeta''} + \epsilon_2^{\zeta''}$, where $\iota_2^{\zeta''} = \frac{1}{2}\frac{d^2}{dt^2}\big|_{t=0} I_2^{t\zeta''}$, is $O(r^{-4})$ (with according decay on derivatives), and $(\nabla^{\mathbf{e}})^\ell \epsilon_2^{\zeta''} = O(r^{-6-\ell})$ for all $\ell \geq 0$.

Now $\iota_2^{\zeta''}$ splits into an $\mathbf{e}$-symmetric part and an $\mathbf{e}$-anti-symmetric part. But according to [Bes, 12.96], to the anti-symmetric part, $(\iota_2^{\zeta''})^a$ say, corresponds an I_2-holomorphic $(2,0)$-form θ via the coupling $\mathbf{e}\big(\cdot, (\iota_2^{\zeta'})^a \cdot\big) = \theta$; this we get by considering the second order variation of the Kähler-Einstein deformation $\big(g_{t\zeta''}, I_2^{t\zeta''}\big)$, satisfying the gauge $\operatorname{tr}^{\mathbf{e}}(h_{\zeta''}) = \delta^{\mathbf{e}} h_{\zeta''} = 0$, and observing that all the statements are *local*. We can lift θ on $\mathbb{R}^4\backslash\{0\}$, and then write $\theta = f dw_1 \wedge dw_2$, where w_1 and w_2 are the standard I_2-holomorphic coordinates $x_1 + ix_3$ and $x_4 + ix_2$, and f is thus I_2-holomorphic with decay r^{-4} at infinity. By Hartogs' lemma we can extend f through 0; we thus have an entire function on $(\mathbb{R}^4, I_2)$, decaying at infinity: the only possibility is $f \equiv 0$, and therefore $(\iota_2^{\zeta''})^a = 0$, or: $\iota_2^{\zeta''}$ is $\mathbf{e}$-symmetric.

Here we would like to follow [Bes, 12.96] again, to see for example that $\iota_2^{\zeta''}$ then corresponds to the I_2-skew-hermitian part of $h_{\zeta''}$, via the coupling $\omega_2^{\mathbf{e}}(\cdot, \iota_2^{\zeta''}\cdot)$ – this latter (2,0)-tensor being clearly I_2-skew-hermitian, because $\omega_2^{\mathbf{e}}$ is I_2-hermitian, and since for all t, $-1 = (I_2^{t\zeta''})^2 = I_2^2 + t^2(I_2\iota_2^{\zeta''} + \iota_2^{\zeta''} I_2) + O(t^3)$, thus $I_2\iota_2^{\zeta''} = -\iota_2^{\zeta''} I_2$. Since in our situation, $\omega_2^{\zeta''}$ does not vary, we could also expect from [Bes, 12.95] that the I_2-hermitian part of $h_{\zeta''}$ vanishes. Nonetheless some of the quoted arguments are of global nature, and one should check they can be adapted to our framework. This can be bypassed however by a rather simple computation, which we quote here: for any t,

$$g_{t\zeta''} = \begin{cases} \omega_2^{t\zeta''}(\cdot, I_2^{t\zeta''}\cdot) = \omega_2^{\mathbf{e}}(\cdot, I_2\cdot) + t^2\omega_2^{\mathbf{e}}(\cdot, \iota_2^{\zeta''}\cdot) + O(t^3) \quad \text{since} \quad \omega_2^{t\zeta''} = \omega_2^{\mathbf{e}} \\ \mathbf{e} + t^2 h_{\zeta''} + O(t^3), \end{cases}$$

and thus $h_{\zeta''} = \omega_2(\cdot, \iota_2^{\zeta''}\cdot)$ which is I_2-skew-hermitian, as announced.

We now claim that the I_2-skew-hermitian part of $h_{\zeta'}$ is nothing but $h_{\zeta''}$, which is Point (ii) of the current step. Indeed, since for all t, $I_2^{t\zeta'}=I_2^{t\zeta''}$ (consider $\lambda_2^{t\zeta'}$),

$$\begin{aligned}0=&g_{t\zeta'}(I_2^{t\zeta'}\cdot,I_2^{t\zeta'}\cdot)-g_{t\zeta'}=g_{t\zeta'}(I_2^{t\zeta''}\cdot,I_2^{t\zeta''}\cdot)-g_{t\zeta'}\\=&\mathbf{e}(I_2^{t\zeta''}\cdot,I_2^{t\zeta''}\cdot)+t^2h_{\zeta'}(I_2^{t\zeta''}\cdot,I_2^{t\zeta''}\cdot)-\mathbf{e}-t^2h_{\zeta'}+O(t^3)\\=&\underbrace{\mathbf{e}(I_2\cdot,I_2\cdot)}_{=\mathbf{e}}+t^2\mathbf{e}(I_2\cdot,\iota_2^{\zeta''}\cdot)+t^2\mathbf{e}(\iota_2^{\zeta''}\cdot,I_2\cdot)+t^2h_{\zeta'}(I_2\cdot,I_2\cdot)-\mathbf{e}-t^2h_{\zeta'}+O(t^3),\end{aligned}$$

and thus $h_{\zeta'}-h_{\zeta'}(I_2\cdot,I_2\cdot)=\mathbf{e}(I_2\cdot,\iota_2^{\zeta''}\cdot)+\mathbf{e}(\iota_2^{\zeta''}\cdot,I_2\cdot)$. We know that $\mathbf{e}(I_2\cdot,\iota_2^{\zeta'}\cdot)=\omega_2(\cdot,\iota_2^{\zeta''}\cdot)=h_{\zeta''}$. To conclude, use that $\mathbf{e}$ and $\mathbf{e}(\cdot,\iota_2^{\zeta''})$ are both symmetric, that $I_2\iota_2^{\zeta''}=-\iota_2^{\zeta''}I_2$, and that $\mathbf{e}$ is I_2-hermitian to see that for all X, Y,

$$\mathbf{e}(\iota_2^{\zeta''}X,I_2Y)=\mathbf{e}(I_2Y,\iota_2^{\zeta'}X)=-\mathbf{e}(Y,I_2\iota_2^{\zeta''}X)=\mathbf{e}(Y,\iota_2^{\zeta''}I_2X)=\mathbf{e}(I_2X,\iota_2^{\zeta''}Y),$$

i.e. $\mathbf{e}(\iota_2^{\zeta''}\cdot,I_2\cdot)=\mathbf{e}(I_2\cdot,\iota_2^{\zeta''}\cdot)=h_{\zeta''}$. We have proved that

$$\frac{1}{2}\big(h_{\zeta'}-h_{\zeta'}(I_2\cdot,I_2\cdot)\big)=h_{\zeta''},$$

as claimed. Since $\delta^{\mathbf{e}}h_{\zeta''}=0$, to see that $\delta^{\mathbf{e}}h_{\zeta'}=0$, we are only left with checking this identity on the I_2-hermitian part of $h_{\zeta'}$, which is Point (iii) of the current induction step.

For this, let us call φ this tensor twisted by I_2, namely $\varphi=\frac{1}{2}\big(h_{\zeta'}(I_2\cdot,\cdot)-h_{\zeta'}(\cdot,I_2\cdot)\big)$. As above, we want to see that $d^{*_{\mathbf{e}}}\varphi=0$. This is clearly an I_2-hermitian 2-form, that is an I_2-(1,1)-form. It is moreover trace-free with respect to $\mathbf{e}$, since $h_{\zeta'}$ is. If we check it is closed then we are done, using the Kähler identity $[\Lambda_{\omega_2},d^c_{I_2}]$. For this, we use an expansion of $\omega_2^{\zeta'}$: for all t,

$$\begin{aligned}\omega_2^{t\zeta'}=&\frac{1}{2}\big(g_{t\zeta'}(I_2^{t\zeta'}\cdot,\cdot)-g_{t\zeta'}(\cdot,I_2^{t\zeta'}\cdot)\big)=\frac{1}{2}\big(g_{t\zeta'}(I_2^{t\zeta''}\cdot,\cdot)-g_{t\zeta'}(\cdot,I_2^{t\zeta''}\cdot)\big)\\=&\frac{1}{2}\big(\mathbf{e}(I_2\cdot,\cdot)+t^2\mathbf{e}(\iota_2^{\zeta''}\cdot,\cdot)+t^2h_{\zeta'}(I_2\cdot,\cdot)\\&\quad-\mathbf{e}(\cdot,I_2\cdot)-t^2\mathbf{e}(\cdot,\iota_2^{\zeta''}\cdot)-t^2h_{\zeta'}(\cdot,I_2\cdot)\big)+O(t^3)\\=&\omega_2^{\mathbf{e}}+t^2\varphi+O(t^3),\ \text{since } \mathbf{e}(I_2\cdot,\cdot)=-\mathbf{e}(\cdot,I_2\cdot)=\omega_2^{\mathbf{e}}\text{ and }\mathbf{e}(\cdot,\iota_2^{\zeta''}\cdot)=\mathbf{e}(\iota_2^{\zeta''}\cdot,\cdot);\end{aligned}$$

this expansion can be differentiated term by term, so that $t^2d\varphi+O(t^3)=0$, hence $d\varphi=0$, as wanted.

Step 3: $\delta^{\mathbf{e}}h_\zeta=0$*.* We now analyse h_ζ. All the techniques to pass from $h_{\zeta''}$ to $h_{\zeta'}$ can actually be used again, and bring us to the desired conclusion:

1. we first observe that $I_1^\zeta = I_1^{\zeta'}$, and we define $\iota_1^{\zeta'} = \frac{d^2}{dt^2}\big|_{t=0} I_1^{t\zeta'}$ which we assume again to be the possibly lower-order non-vanishing variation of $I_1^{t\zeta'}$; then $(\iota_1^{\zeta'})^a = 0$, since otherwise we would have a non-trivial entire function on $\mathbb{C}^2$ going to 0 at infinity;

2. since $\omega_1^{\zeta'} = \omega_1^{\zeta''} = \omega_1^{\mathbf{e}}$, we get that $h_{\zeta'}$ is I_1-skew-hermitian, given by $\iota_1^{\zeta'}$ via the identity $h_{\zeta'} = \omega_1^{\mathbf{e}}(\cdot, \iota_1^{\zeta'} \cdot)$, and that the I_1-skew-hermitian component of h_ζ coincides with $h_{\zeta'}$, the $\delta^{\mathbf{e}}$ of which vanishes; we are thus left with the I_1-hermitian component of h_ζ;

3. this component is $\mathbf{e}$-trace-free (h_ζ is), and gives rises to an I_1-hermitian 2-form ψ, which is closed since the $\omega_1^{t\zeta}$ are; the Kähler identity $[\Lambda_{\omega_1^{\mathbf{e}}}, d^c_{I_1}] = d^{*_\mathbf{e}}$ then leads us to $d^{*_\mathbf{e}}\psi = 0$, which is equivalent to:

$$\delta^{\mathbf{e}}\big(I_1\text{-hermitian component of } h_\zeta\big) = 0.$$

To finish this proof, we justify our assumption of the vanishing of the first order variation of the complex structures. For instance, let us not assume that $\iota := \frac{d}{dt}\big|_{t=0} I_2^{t\zeta''}$ is a priori vanishing. Then it is defined on $\mathbb{R}^4\backslash\{0\}$, and is $O(r^{-2})$. Now as above, since $0 = \frac{d}{dt}\big|_{t=0} g_{t\zeta''}$ has vanishing trace and divergence for $\mathbf{e}$, the $\mathbf{e}$-antisymmetric part of ι has to vanish since it gives rise to a holomorphic function on $(\mathbb{R}^4, I_2)$ decaying at infinity. And we see as above that $\omega_2^{\mathbf{e}}(\cdot, \iota\cdot) = \frac{d}{dt}\big|_{t=0} g_{t\zeta''} = 0$, and thus $\iota = 0$. Similarly, the arguments for $\iota_1^{\zeta'}$ apply to $\frac{d}{dt}\big|_{t=0} I_1^{t\zeta'}$ with $\frac{d}{dt}\big|_{t=0} g_{t\zeta'} = 0$ instead of $h_{\zeta'}$, so that $\frac{d}{dt}\big|_{t=0} I_1^{t\zeta'} = 0$. $\square$

Remark 2.9 *By contrast with what is usually done, we used properties already known of $h_{\zeta'}$ and $h_{\zeta''}$, conjugated to properties of mappings between X_ζ, $X_{\zeta'}$ and $X_{\zeta''}$ to show that indeed, our first order deformations were in gauge, which is also retroactively used in some places, e.g. in killing tensors like $(\iota_2^{\zeta''})^a$.*

2.3.2 Lower order variation of the Kähler forms: general shape

As seen when proving that the gauge was verified, given $\zeta \in \mathfrak{h} \otimes \mathbb{R}^3$, $h_{\zeta''}$ is I_3-hermitian, the I_2-skew-hermitian part of $h_{\zeta'}$ is $h_{\zeta''}$, and the I_1-skew-hermitian part of $h_{\zeta'}$ is $h_{\zeta''}$. In order to determine h_ζ completely, we are thus left with working on the respective I_3, I_2 and I_1-hermitian components of $h_{\zeta''}$, $h_{\zeta'}$ and h_ζ, or equivalently on the respectively I_3, I_2 and I_1-(1,1) forms

$$\varpi_3^{\zeta''} := h_{\zeta''}(I_3\cdot, \cdot), \qquad \varpi_2^{\zeta'} := \frac{1}{2}\big(h_{\zeta'}(I_2\cdot, \cdot) - h_{\zeta'}(\cdot, I_2\cdot)\big)$$
$$\text{and} \quad \varpi_1^{\zeta} := \frac{1}{2}\big(h_\zeta(I_1\cdot, \cdot) - h_\zeta(\cdot, I_1\cdot)\big).$$

We interpret these forms as the first (possibly) non-vanishing variation term of $\omega_3^{\zeta''}$, $\omega_2^{\zeta'}$ and ω_1^{ζ}; as such and as seen above, these are closed forms. More precisely, they follow a general common pattern:

Proposition 2.10 *There exist real numbers $a_{1j}(\zeta)$, $a_{2j}(\zeta')$, $a_{3j}(\zeta'')$, $j = 1, 2, 3$, such that:*

$$\varpi_3^{\zeta''} = a_{31}(\zeta'')\theta_1 + a_{32}(\zeta'')\theta_2 + a_{33}(\zeta'')\theta_3, \quad \varpi_2^{\zeta'} = a_{21}(\zeta')\theta_1 + a_{22}(\zeta')\theta_2 + a_{23}(\zeta')\theta_3,$$
$$\textit{and} \quad \varpi_1^{\zeta} = a_{11}(\zeta)\theta_1 + a_{12}(\zeta)\theta_2 + a_{13}(\zeta)\theta_3,$$

where we recall the notations:

$$\theta_1 = \frac{rdr \wedge \alpha_1 - \alpha_2 \wedge \alpha_3}{r^6}, \quad \theta_2 = \frac{rdr \wedge \alpha_2 - \alpha_3 \wedge \alpha_1}{r^6}, \quad \theta_3 = \frac{rdr \wedge \alpha_3 - \alpha_1 \wedge \alpha_2}{r^6}.$$

Proof. We do it for ϖ_1^{ζ}, as it will be clear that the arguments would apply similarly to $\varpi_2^{\zeta'}$ and $\varpi_3^{\zeta''}$; we work on $\mathbb{R}^4 \backslash \{0\}$. As ϖ_1^{ζ} is of type (1,1) for I_1, it is at any point a linear combination of $rdr \wedge \alpha_1$, $\alpha_2 \wedge \alpha_3$, $rdr \wedge \alpha_2 - \alpha_3 \wedge \alpha_1$ and $rdr \wedge \alpha_3 - \alpha_1 \wedge \alpha_2$.

The symmetric tensor $\frac{1}{2}\big(h_\zeta + h_\zeta(I_1\cdot, I_1\cdot)\big)$ corresponding to ϖ_1^{ζ} is moreover trace-free for $\mathbf{e}$, which translates into $\varpi_1^{\zeta} \wedge \omega_1^{\mathbf{e}} = 0$. Since $\omega_1 = \frac{rdr\wedge\alpha_1 + \alpha_2\wedge\alpha_3}{r^2}$, we have $(rdr \wedge \alpha_2 - \alpha_3 \wedge \alpha_1) \wedge \omega_1^{\mathbf{e}} = (rdr \wedge \alpha_3 - \alpha_1 \wedge \alpha_2) \wedge \omega_1^{\mathbf{e}} = 0$, whereas $rdr \wedge \alpha_1 \wedge \omega_1^{\mathbf{e}} = \alpha_2 \wedge \alpha_3 \wedge \omega_1^{\mathbf{e}}$. As a consequence, the pointwise coefficient of $\alpha_2 \wedge \alpha_3$ is the opposite of that of $\alpha_2 \wedge \alpha_3$. To sum up, since the θ_j are $O(r^{-4})$ with corresponding decay (or growth, near 0) of their derivatives, which are precisely the orders of ϖ_1^{ζ}, we know that:

$$\varpi_1^{\zeta} = f\theta_1 + g\theta_2 + h\theta_3,$$

for three bounded functions f, g, h, with euclidean ℓ-th order derivatives of order $O(r^{-\ell})$, near 0 and infinity. We can be more precise here: from the properties of analytic expansions in play discussed in paragraph 2.2.2, we have that $\kappa_s^*\varpi_1^{\zeta} = s^{-2}\varpi_1^{\zeta}$, where κ_s is the dilation of factor $s > 0$ on $\mathbb{R}^4$. But we exactly have $\kappa_s^*\theta_j = s^{-2}\theta_j$, $j = 1, 2, 3$; therefore, f, g, h are functions on the sphere $\mathbb{S}^3$. Notice that from this point, we also know that ϖ_1^{ζ} is anti-self-dual (for $\mathbf{e}$), since the θ_j's are. Therefore ϖ_1^{ζ} is $\mathbf{e}$-harmonic on $\mathbb{R}^4 \backslash \{0\}$, which is the same as: $(\nabla^{\mathbf{e}})^*(\nabla^{\mathbf{e}})\varpi_1^{\zeta} = 0$. On the other hand, the θ_j are harmonic as well: they are anti-self-dual, and closed, since

$$\theta_j = \frac{1}{4} dd^c_{I_j}\Big(\frac{1}{r^2}\Big), \qquad j = 1, 2, 3. \tag{37}$$

Putting those facts together and setting $e_j = I_j \frac{1}{r} x_i \frac{\partial}{\partial x_i}$, $j = 1, 2, 3$ – forget about formulas (8) – so that $rdr(e_j) = 0$ and $\alpha_k(e_j) = r\delta_{jk}$, $j, k = 1, 2, 3$, we get that:

$$\Delta_{\mathbf{e}}(f\theta_1) = \frac{1}{r^2}(\Delta_{\mathbb{S}^3} f)\theta_1 - 2\sum_{k=1}^{3}(e_k \cdot f)\nabla^{\mathbf{e}}_{e_k}\theta_1$$

The $\nabla^{\mathbf{e}}_{e_k}\theta_1$ are easy to compute: since $e_k \cdot r = 0$, $\nabla^{\mathbf{e}}_{e_k}\theta_1 = \frac{1}{r^6}\nabla^{\mathbf{e}}_{e_k}(rdr \wedge \alpha_1 - \alpha_2 \wedge \alpha_3)$. Moreover since the I_j are parallel, we just have to compute $\nabla^{\mathbf{e}}_{e_k} rdr$; since $\nabla^{\mathbf{e}}(rdr) = \mathbf{e}$, $\nabla^{\mathbf{e}}_{e_k}(rdr) = \mathbf{e}(e_k, \cdot) = \frac{1}{r}\alpha_k$. Therefore $\nabla^{\mathbf{e}}_{e_1}\theta_1 = 0$, $\nabla^{\mathbf{e}}_{e_2}\theta_1 = \frac{2}{r}\theta_3$ and $\nabla^{\mathbf{e}}_{e_3}\theta_1 = -\frac{2}{r}\theta_2$. Thus $\Delta(f\theta_1) = \frac{1}{r^2}(\Delta_{\mathbb{S}^3} f)\theta_1 - \frac{2}{r}(e_2 \cdot f)\theta_3 + \frac{2}{r}(e_3 \cdot f)\theta_2$. A circular permutation on the indices gives as well $\Delta(g\theta_2) = \frac{1}{r^2}(\Delta_{\mathbb{S}^3} g)\theta_2 - \frac{2}{r}(e_3 \cdot g)\theta_1 + \frac{2}{r}(e_1 \cdot g)\theta_3$ and $\Delta(h\theta_3) = \frac{1}{r^2}(\Delta_{\mathbb{S}^3} h)\theta_3 - \frac{2}{r}(e_1 \cdot h)\theta_2 + \frac{2}{r}(e_2 \cdot h)\theta_1$. Since the θ_j are linearly independent, $\Delta\varpi_1^\zeta = 0$ translates into:

$$\Delta_{\mathbb{S}^3} f - 4(e_3 \cdot g - e_2 \cdot h) = \Delta_{\mathbb{S}^3} g - 4(e_1 \cdot h - e_3 \cdot f) = \Delta_{\mathbb{S}^3} h - 4(e_2 \cdot f - e_1 \cdot g) = 0. \tag{38}$$

On the other hand, $d\varpi_1^\zeta = 0$ is equivalent to $e_1 \cdot f + e_2 \cdot g + e_3 \cdot h = e_2 \cdot f - e_1 \cdot g = e_3 \cdot g - e_2 \cdot h = e_1 \cdot h - e_3 \cdot f = 0$; the latter three equalities, plugged into equations (38), exactly give $\Delta_{\mathbb{S}^3} f = \Delta_{\mathbb{S}^3} g = \Delta_{\mathbb{S}^3} h = 0$, hence: f, g and h are constant. $\square$

Remark 2.11 *We have not used the Γ-invariance of the tensors here; nonetheless, since the θ_j are $\mathrm{SU}(2)$-invariant, which comes from the identities $\theta_j = dd^c_{I_j}(\frac{1}{r^2})$, this does not give us any further information.*

2.3.3 Lower order variation of the Kähler forms: determination of the coefficients

We know form the formal expansion of g_ζ (or those of $g_{\zeta'}$ and $g_{\zeta''}$) that the a_{jk} coefficients of Proposition 2.10 are quadratic homogeneous polynomials in their arguments. Their explicit form is given as follows.

Proposition 2.12 *With the same notations as in Proposition 2.10,*

$$\begin{array}{lll} a_{31}(\zeta'') = 0, & a_{32}(\zeta'') = 0, & a_{33}(\zeta'') = -\,\|\Gamma\||\zeta_3|^2, \\ a_{21}(\zeta') = 0, & a_{22}(\zeta') = -\,\|\Gamma\||\zeta_2|^2, & a_{23}(\zeta') = -\,\|\Gamma\|\langle\zeta_2, \zeta_3\rangle, \\ a_{11}(\zeta) = -\|\Gamma\||\zeta_1|^2, & a_{12}(\zeta) = -\,\|\Gamma\|\langle\zeta_1, \zeta_2\rangle, & a_{13}(\zeta) = -\,\|\Gamma\|\langle\zeta_1, \zeta_3\rangle. \end{array}$$

where $\|\Gamma\| := \frac{|\Gamma|}{2\,\mathrm{Vol}(\mathbb{B}^4)} = \frac{|\Gamma|}{\pi^2}$.

Proof. We shall first prove the assertion on the $a_{3j}(\zeta'')$, and then apply the same techniques to determine the $a_{2j}(\zeta')$ – the $a_{1j}(\zeta)$ being dealt with in a similar way. One more time we can assume that ζ is chosen in $\mathfrak{h} - D$, and so that $\zeta', \zeta'' \notin D$.

The coefficient $a_{33}(\zeta'')$. To begin with, set $a = a_{31}(\zeta'')$, $b = a_{32}(\zeta'')$ and $c = a_{33}(\zeta'')$. We consider on $X_{\zeta''}$ (which is smooth by our assumption $\zeta'' \notin D$) a closed form λ with compact support representing ζ_3 by Poincaré duality; this is possible since minimal resolutions of $\mathbb{C}^2/\Gamma$ have compactly supported cohomology [Joy, Thm 8.4.3], and $X_{\zeta''}$ is diffeomorphic to such a resolution (this is actually a minimal resolution of $(\mathbb{C}^2/\Gamma, I_3)$, but we will not use this fact). Next, consider a smooth cut-off function χ, vanishing on $(-\infty,1]$, equal to 1 on $[2,+\infty)$. From the equality $\omega_3^{\mathbf{e}} = \frac{1}{2}dd^c_{I_3}(r^2)$, and from formulas (37), we have that

$$\varepsilon := \omega_3^{\zeta''} - \lambda - d\Big[\frac{1}{4}I_3 d\big(\chi(r)r^2\big) + \frac{1}{4}(aI_1+bI_2+cI_3)d\big(\chi(r)r^{-2}\big)\Big]$$

is well-defined on $X_{\zeta'}$, has cohomology class 0, and is $O(r^{-6})$ at infinity, with appropriate decay on its derivatives; here we write r instead of $(\lambda_3^{\zeta''})^*r$. As we need it further, we shall also see now that ε admits a primitive which decays at infinity.

From [Joy, Thm 8.4.1], ε can indeed be written as $h + d\beta + d^{*g_{\zeta''}}\gamma$, where h is in $C^\infty_3(X_\zeta,\Lambda^2)$ and is $g_{\zeta''}$-harmonic, and β and γ are in $C^\infty_2(X_\zeta,\Lambda^2)$; we used here classical notations for weighted spaces: for example, $\beta = O(r^{-2})$, $\nabla^{\mathbf{e}}\beta = O(r^{-3})$, and so on. The harmonic form h is actually decaying fast enough so that we can say it is closed and co-closed: write (all the operations and tensors are computed with respect to $g_{\zeta''}$) for all r

$$0 = \int_{\mathbb{B}_{X_{\zeta''}}(r)} (h,\Delta h)\,\mathrm{vol} = \int_{\mathbb{B}_{X_{\zeta''}}(r)} \big(|dh|^2+|d^*h|^2\big)\,\mathrm{vol} + \int_{\mathbb{S}_{X_{\zeta''}}(r)} (h\odot dh + h\odot d^*h)\,\mathrm{vol},$$

where $\mathbb{B}_{X_{\zeta''}}(r) = (\lambda_3^{\zeta''})^{-1}\big(\mathbb{B}^4(r)/\Gamma\big)$, and $\mathbb{S}_{X_{\zeta''}}(r)$ is its boundary. From what precedes, the boundary integral is easily seen to be $O(r^{3-3-4}) = O(r^{-4})$, and thus $dh = d^*h = 0$. Hence $0 = d\varepsilon = dd^*\gamma$; an integration by parts similar to the previous one, but with boundary term of size $O(r^{-2})$, leads us to $d^*\gamma = 0$, and thus $\varepsilon = h + d\beta$. According to [Joy, thm 8.4.1] again, $\mathcal{H}^2(X_{\zeta''}) \to H^2(X_{\zeta''})$, $h \mapsto [h]$ is an isomorphism; now here $[h] = [\varepsilon - d\beta] = 0$. Therefore $h = 0$, and $\varepsilon = d\beta$, with $\beta = O(r^{-2})$.

Write $\mathbb{B}(r)$ for $\mathbb{B}_{X_{\zeta''}}(r)$ to simplify notations; we shall now compute the integrals $\int_{\mathbb{B}(r)}(\omega_3^{\zeta''})^2$ in two different ways. First, recall that $(\omega_3^{\zeta''})^2 = 2\,\mathrm{vol}^{\mathbf{e}}$, and thus[1] $\int_{\mathbb{B}(r)}(\omega_3^{\zeta''})^2 = \frac{2r^4}{|\Gamma|}\mathrm{Vol}\,(\mathbb{B}^4)$. On the other hand, since $\omega_3^{\zeta''} = \lambda + d\varphi + \varepsilon$, with $\varphi = \frac{1}{4}I_3 d\big(\chi(r)r^2\big) + \frac{1}{4}(aI_1+bI_2+cI_3)d\big(\chi(r)r^{-2}\big)$, we have:

$$\begin{aligned}\int_{\mathbb{B}(r)}(\omega_3^{\zeta''})^2 = \int_{\mathbb{B}(r)}\lambda^2 + 2\int_{\mathbb{B}^4(r)/\Gamma}\lambda\wedge d\varphi + 2\int_{\mathbb{B}(r)}\lambda\wedge\varepsilon \\ + \int_{\mathbb{B}(r)}(d\varphi)^2 + 2\int_{\mathbb{B}(r)}\varepsilon\wedge d\varphi + \int_{\mathbb{B}^4(r)/\Gamma}\varepsilon^2.\end{aligned} \tag{39}$$

[1]Indeed, $\int_{\mathbb{B}(r)}(\omega_3^{\zeta''})^2$ is the limit as s goes to 0 of $\int_{\mathbb{B}(r)-\mathbb{B}(s)}(\omega_3^{\zeta''})^2$, since as an s-tubular neighbourhood of $E := (\lambda_3^{\zeta''})^{-1}\big(\{0\}\big)$ which is of real dimension 2, $\mathbb{B}(s)$ has its volume tending to 0 when s goes to 0. Now we can also see $\int_{\mathbb{B}(r)-\mathbb{B}(s)}(\omega_3^{\zeta''})^2$ on $\mathbb{R}^4/\Gamma$ via $\lambda_3^{\zeta''}$ which is diffeomorphic away from E, and since $(\lambda_3^{\zeta''})_*(\omega_3^{\zeta''})^2 = 2\Omega_{\mathbf{e}}$, $\int_{\mathbb{B}(r)-\mathbb{B}(s)}(\omega_3^{\zeta''})^2$ is twice the euclidean volume of the annulus of radii s and r in $\mathbb{R}^4/\Gamma$, hence the result when $s\to 0$.

Let us analyse those summands separately.

For r large enough, $\int_{\mathbb{B}(r)} \lambda^2 = \int_{X_{\zeta''}} \lambda^2 = \lambda \cup \lambda = -|\zeta_3|^2$, by Lemma 2.2, and the fact that $[\lambda] = [\omega_3^{\zeta''}] = (\zeta'')_3 = \zeta_3$.

The integral $\int_{\mathbb{B}(r)} \lambda \wedge d\varphi$ equals $\int_{\mathbb{S}(r)} \lambda \wedge \varphi$ by Stokes' theorem, where $\mathbb{S}(r)$ stands for $\mathbb{S}_{X_{\zeta''}}(r)$, and this vanishes for r large enough; similarly, $\int_{\mathbb{B}(r)} \lambda \wedge \varepsilon = \int_{\mathbb{B}(r)} \lambda \wedge d\beta = \int_{\mathbb{S}(r)} \lambda \wedge \beta = 0$ for r large enough.

We now come to $\int_{\mathbb{B}(r)} (d\varphi)^2$. By Stokes, this is equal to $\int_{\mathbb{S}(r)} d\varphi \wedge \varphi$, which we view back on $\mathbb{R}^4/\Gamma$ via $\lambda_3^{\zeta''}$. For $r \geq 2$, the integrand is

$$(\omega_3^{\mathbf{e}} + a\theta_1 + b\theta_2 + c\theta_3) \wedge \Big[\frac{1}{2}\alpha_3 - \frac{1}{2}\big(a\frac{\alpha_1}{r^4} + b\frac{\alpha_2}{r^4} + c\frac{\alpha_3}{r^4}\big)\Big]$$
$$= \frac{1}{2r^2}\Big(1 - \frac{c}{r^4}\Big)\alpha_1 \wedge \alpha_2 \wedge \alpha_3 + O(r^{-7}),$$

since $\omega_3^{\mathbf{e}} \wedge \alpha_3 = \frac{1}{r^2}\alpha_1 \wedge \alpha_2 \wedge \alpha_3$, $\omega_3^{\mathbf{e}} \wedge \alpha_1 = \frac{1}{r^2} r dr \wedge \alpha_3 \wedge \alpha_1 = 0$ and $\omega_3^{\mathbf{e}} \wedge \alpha_2 = \frac{1}{r^2} r dr \wedge \alpha_3 \wedge \alpha_2 = 0$ on $\mathbb{S}^3(r)/\Gamma$; $\theta_3 \wedge \alpha_3 = -\frac{\alpha_1 \wedge \alpha_2 \wedge \alpha_3}{r^6}$, $\theta_1 \wedge \alpha_3 = \frac{rdr \wedge \alpha_1 \wedge \alpha_3}{r^6} = 0$, $\theta_2 \wedge \alpha_3 = \frac{rdr \wedge \alpha_2 \wedge \alpha_3}{r^6} = 0$ on $\mathbb{S}^3(r)/\Gamma$; and $\theta_j \wedge \alpha_k = O(r^{-7})$ for $j, k = 1, 2, 3$. Observe that $\int_{\mathbb{S}^3(r)/\Gamma} \alpha_1 \wedge \alpha_2 \wedge \alpha_3 = 4r^2 \operatorname{Vol}\big(\mathbb{B}^4(r)\big)/|\Gamma|$ – for instance, compute $\int_{\mathbb{B}^4(r)} (\omega_3^{\mathbf{e}})^2 = \frac{1}{2}\int_{\mathbb{B}^4(r)} \omega_3^{\mathbf{e}} \wedge d\alpha_3$ by Stokes; we thus end up with:

$$\int_{\mathbb{B}(r)} (d\varphi)^2 = 2\big(1 - \frac{c}{r^4}\big)\frac{\operatorname{Vol}\big(\mathbb{B}^4(r)\big)}{|\Gamma|} + O(r^{-4}) = \frac{2(r^4 - c)}{|\Gamma|}\operatorname{Vol}\big(\mathbb{B}^4\big) + O(r^{-4}).$$

We conclude by the last two summands of (39). On the one hand, $\int_{\mathbb{B}(r)} \varepsilon \wedge d\varphi = \int_{\mathbb{S}(r)} \varepsilon \wedge \varphi = O(r^{3-6+1}) = O(r^{-2})$, since $\varepsilon = O(r^{-6})$ and $\varphi = O(r)$. On the other hand, $\int_{\mathbb{B}(r)} \varepsilon^2 = \int_{\mathbb{B}(r)} \varepsilon \wedge d\beta = \int_{\mathbb{S}(r)} \varepsilon \wedge \beta$: this is $O(r^{3-6-2}) = O(r^{-5})$ (and this is actually the only place where we need an estimate on the decay of a primitive of ε).

Collecting the different estimates, for r going to ∞ we have:

$$\frac{2r^4}{|\Gamma|}\operatorname{Vol}\big(\mathbb{B}^4\big) = -|\zeta_3|^2 + \frac{2\big(r^4 - c\big)}{|\Gamma|}\operatorname{Vol}\big(\mathbb{B}^4\big) + O\big(r^{-2}\big),$$

hence: $c = -\frac{|\Gamma|}{2\operatorname{Vol}(\mathbb{B}^4)}|\zeta_3|^2$.

The coefficients $a_{31}(\zeta'')$ and $a_{31}(\zeta'')$. We use the same techniques to see that $a_{31}(\zeta'')$ and $a_{32}(\zeta'')$, used above as a and b in φ, vanish. Recall for example that $\omega_1^{\zeta''} = \omega_1^{\mathbf{e}}$; therefore, $\mu := \omega_1^{\zeta''} - d\big[\frac{1}{4}I_1 d\big(\chi(r)r^2\big)\big]$ has compact support in $X_{\zeta''}$, and cohomology class $[\omega_1^{\zeta''}] = (\zeta'')_1 = 0$.

We can compute $\int_{\mathbb{B}(r)} \omega_3^{\zeta''} \wedge \omega_1^{\zeta''}$ in two different ways. First, this is 0 for all r, since $\omega_1^{\zeta''}$ and $\omega_3^{\zeta''}$ are Kähler forms for the same hyperkähler metric and anti-commuting complex structures, and thus $\omega_3^{\zeta''} \wedge \omega_1^{\zeta''} \equiv 0$. Secondly, using the sums $\omega_3^{\zeta''} = \lambda + d\varphi + \varepsilon$ and $\omega_1^{\zeta''} = \mu + d\psi$, with $\psi = \frac{1}{4} I_1 d\big(\chi(r) r^2\big)$, we write for all r:

$$\begin{aligned} \int_{\mathbb{B}(r)} \omega_3^{\zeta''} \wedge \omega_1^{\zeta''} = \int_{\mathbb{B}(r)} \lambda \wedge \mu + \int_{\mathbb{B}(r)} d\varphi \wedge \mu + \int_{\mathbb{B}(r)} \varepsilon \wedge \mu \\ + \int_{\mathbb{B}(r)} \lambda \wedge d\psi + \int_{\mathbb{B}(r)} d\varphi \wedge d\psi + \int_{\mathbb{B}(r)} \varepsilon \wedge d\psi. \end{aligned} \tag{40}$$

We briefly examine each summand. The first integral, $\int_{\mathbb{B}(r)} \lambda \wedge \mu$, is equal to $\int_{X_{\zeta''}} \lambda \wedge \mu$ for r large enough, and this is 0, since $[\mu] = 0$. The second integral can be rewritten as $\int_{\mathbb{S}(r)} d\varphi \wedge \lambda$, which vanishes for r large enough; the same is true for the third (since $\varepsilon = d\beta$) and fourth integrals. As for the sixth integral, it can be written as $\int_{\mathbb{S}(r)} \varepsilon \wedge \psi$, and this is $O(r^{3-6+1}) = O(r^{-2})$.

We are left with the fifth summand of (40), which we rewrite as $\int_{\mathbb{S}(r)} d\varphi \wedge \psi$, and view back on $\mathbb{R}^4/\Gamma$. For $r \geq 2$, the integrand is

$$(\omega_3^{\mathbf{e}} + a\theta_1 + b\theta_2 + c\theta_3) \wedge \frac{1}{2}\alpha_1 = -a \frac{\alpha_1 \wedge \alpha_2 \wedge \alpha_3}{2r^6}$$

on $\mathbb{S}^3(r)$, since there $\omega_3^{\mathbf{e}} \wedge \alpha_1 = \theta_2 \wedge \alpha_1 = \theta_3 \wedge \alpha_1 = 0$. As a consequence, $\int_{\mathbb{B}(r)} d\varphi \wedge d\psi = -a \operatorname{Vol}(\mathbb{B}^4)/|\Gamma|$.

This tells us that the right-hand-side of (40) is $-a \operatorname{Vol}(\mathbb{B}^4)/|\Gamma|$ when we let r go to ∞; as the left-hand-side is always 0, $a_{31}(\zeta'') = a = 0$. One proves that $a_{32}(\zeta'') = 0$ in the same way.

The $a_{2j}(\zeta')$ coefficients. Let us come now to the coeficients involved by $\varpi_2^{\zeta'}$. Since $I_2^{\zeta'} = I_2^{\zeta''}$ and since we have the equality of $I_2^{\zeta'}$-holomorphic (2,0)-forms $\omega_3^{\zeta''} + i\omega_1^{\zeta''} = \omega_3^{\zeta'} + i\omega_1^{\zeta'}$, we have the writing:

$$\omega_1^{\zeta'} = \mu + d\psi \qquad \text{and} \qquad \omega_3^{\zeta'} = \lambda + d\varphi + \varepsilon;$$

taking ν a compactly supported closed 2-form representing $\zeta_2 = [\omega_2^{\zeta'}]$ (since $(\zeta')_2 = \zeta_2$), we can write, for the same reasons as those invoked when proving the analogous formula for $\omega_3^{\zeta''}$:

$$\omega_2^{\zeta'} = \nu + d\xi + \eta$$

with $\xi = \frac{1}{4}\big[I_2 d\big(\chi(r) r^2\big) + (aI_1 + bI_2 + cI_3) d\big(\chi(r) r^{-2}\big)\big]$ where this time, $a = a_{21}(\zeta')$, $b = a_{22}(\zeta')$, $c = a_{23}(\zeta')$, and with $\eta \in C_6^\infty(X_{\zeta'}, \Lambda^2)$ of the form $d\gamma$ for some $\gamma \in C_2^\infty(X_{\zeta'}, \Lambda^2)$.

Exactly as in what precedes, we get from computing respectively $\int_{\mathbb{B}(r)}(\omega_2^{\zeta'})^2$ and $\int_{\mathbb{B}(r)}\omega_2^{\zeta'}\wedge\omega_1^{\zeta'}$, where now $\mathbb{B}(r)=(\lambda_3^{\zeta''}\circ\lambda_2^{\zeta'})^{-1}\big(\mathbb{B}^4(r)/\Gamma\big)$, that $a_{22}(\zeta')=b=-\frac{|\Gamma|}{2\operatorname{Vol}(\mathbb{B}^4)}|\zeta_2|^2$ and $a_{21}(\zeta')=a=0$. Now though it goes through similar lines, the computation of $\int_{\mathbb{B}(r)}\omega_2^{\zeta'}\wedge\omega_3^{\zeta'}$ is slightly new. Indeed, $\omega_2^{\zeta'}\wedge\omega_3^{\zeta'}=0$ identically, whereas:

$$\begin{aligned}\int_{\mathbb{B}(r)}\omega_2^{\zeta'}\wedge\omega_3^{\zeta'}=&\int_{\mathbb{B}(r)}\nu\wedge\lambda+\int_{\mathbb{B}(r)}\nu\wedge d\varphi+\int_{\mathbb{B}(r)}\nu\wedge\varepsilon\\&+\int_{\mathbb{B}(r)}d\xi\wedge\lambda+\int_{\mathbb{B}(r)}d\xi\wedge d\varphi+\int_{\mathbb{B}(r)}d\xi\wedge\varepsilon\\&+\int_{\mathbb{B}(r)}\eta\wedge\lambda+\int_{\mathbb{B}(r)}\eta\wedge d\varphi+\int_{\mathbb{B}(r)}\eta\wedge\varepsilon.\end{aligned}\tag{41}$$

The first integral of the right-hand-side equals $\int_{X_{\zeta'}}\nu\wedge\lambda=-\langle[\nu],[\lambda]\rangle=-\langle\zeta_2,\zeta_3\rangle$ for r large enough. For r large enough again, the second, the third, the fourth and the seventh integrals vanish (use Stokes' theorem). Further uses of Stokes' theorem and the estimates $\varphi,\xi=O(r)$, $\varepsilon,\eta=O(r^{-6})$ and $\beta,\gamma=O(r^{-2})$ (with $\varepsilon=d\beta$, $\eta=d\gamma$) give us that $\int_{\mathbb{B}(r)}d\xi\wedge\varepsilon$ and $\int_{\mathbb{B}(r)}\eta\wedge d\varphi$ are $O(r^{-2})$, and that $\int_{\mathbb{B}(r)}\eta\wedge\varepsilon=O(r^{-5})$.

We hence are left with computing the contribution of the fifth summand, namely $\int_{\mathbb{B}(r)}d\xi\wedge d\varphi$, in the right-hand-side of (41). This can be rewritten as $\int_{\mathbb{S}(r)}\xi\wedge d\varphi$; seen on $\mathbb{R}^4/\Gamma$, the integrand is:

$$\frac{1}{2}\Big[\alpha_2-\frac{1}{r^4}\big(a_{22}(\zeta')\alpha_2+c\alpha_3\big)\Big]\wedge\big(\omega_3^{\mathbf{e}}+a_{33}(\zeta')\theta_3\big).$$

All computations done, this can be rewritten on $\mathbb{S}^3(r)/\Gamma$ as: $-c\frac{\alpha_1\wedge\alpha_2\wedge\alpha_3}{2r^6}$. Thus $\int_{\mathbb{B}(r)}d\xi\wedge d\varphi=-\frac{2c\operatorname{Vol}(\mathbb{B}^4)}{|\Gamma|}+O(r^{-4})$ (recall that $\int_{\mathbb{S}^3(r)/\Gamma}\alpha_1\wedge\alpha_2\wedge\alpha_3=\frac{4r^6\operatorname{Vol}(\mathbb{B}^4)}{|\Gamma|}$).

Collecting the estimates of the last two paragraphs and letting r go to ∞, from (41) we get: $0=-\langle\zeta_2,\zeta_3\rangle-\frac{2c\operatorname{Vol}(\mathbb{B}^4)}{|\Gamma|}$, that is: $c=-\frac{|\Gamma|}{2\operatorname{Vol}(\mathbb{B}^4)}\langle\zeta_2,\zeta_3\rangle$.

The $a_{1j}(\zeta)$ coefficients. After noticing that $\omega_2^{\zeta}+i\omega_3^{\zeta}=\omega_2^{\zeta'}+i\omega_3^{\zeta'}$, and that ω_1^{ζ} can be written as a sum

$$\mu+d\psi+\varsigma,$$

where μ still has compact support but this time has class $[\omega_1^{\zeta}]=\zeta_1$, ψ is the 1-form $\frac{1}{4}\big[I_1d\big(\chi(r)r^2\big)+(a_{11}(\zeta)I_1+a_{12}(\zeta)I_2+a_{13}(\zeta)I_3)d\big(\chi(r)r^{-2}\big)\big]$, and $\varsigma\in C_6^{\infty}(X_{\zeta},\Lambda^2)$

can be written $d\sigma$ with $\sigma \in C_2^\infty(X_\zeta, \Lambda^2)$, we get that

$$a_{11}(\zeta) = -\frac{|\Gamma|}{2\operatorname{Vol}(\mathbb{B}^4)}|\zeta_1|^2, \qquad a_{12}(\zeta) = -\frac{|\Gamma|}{2\operatorname{Vol}(\mathbb{B}^4)}\langle\zeta_1,\zeta_2\rangle$$
$$\text{and} \quad a_{13}(\zeta) = -\frac{|\Gamma|}{2\operatorname{Vol}(\mathbb{B}^4)}\langle\zeta_1,\zeta_3\rangle,$$

from respective computations of $\int_{\mathbb{B}(r)}(\omega_1^{\zeta'})^2$, $\int_{\mathbb{B}(r)}\omega_1^{\zeta'}\wedge\omega_2^{\zeta'}$ and $\int_{\mathbb{B}(r)}\omega_1^{\zeta'}\wedge\omega_3^{\zeta'}$, where here $\mathbb{B}(r)$ is seen in X_ζ, following the same lines as above. □

2.3.4 Conclusion: proof of Theorem 2.1 (general Γ)

Let us sum the situation up. If we take $\Phi_\zeta = F_\zeta^{-1} : X_\zeta\backslash F_\zeta^{-1}(\{0\}) \to (\mathbb{R}^4\backslash\{0\})/\Gamma$ and keep the notations introduced in this section, we have $\Phi_{\zeta *}g_\zeta = \mathbf{e}+h_\zeta+O(r^{-6})$, $\Phi_{\zeta *}I_1^\zeta = I_1 + \iota_1^\zeta + O(r^{-6})$ and $\Phi_{\zeta *}\omega_1^\zeta = \omega_1^{\mathbf{e}} + \varpi_1^\zeta + O(r^{-6})$. The I_1-hermitian component of h_ζ is $\varpi_1^\zeta(\cdot, I_1\cdot)$, which we know, and its I_1-skew-hermitian component is $h_{\zeta'}$. Now the I_2-hermitian component of $h_{\zeta'}$ is $\varpi_2^{\zeta'}(\cdot, I_2\cdot)$, which we also know, and its I_2-skew-hermitian component is $h_{\zeta''}$. Finally, $h_{\zeta''}$ is I_3-hermitian, equal to $\varpi_3^{\zeta''}(\cdot, I_3\cdot)$, which we know as well. In a nutshell, we are able to write down explicitly h_ζ from Propositions 2.10 and 2.12:

$$\begin{aligned} h_\zeta =& \varpi_3^{\zeta''}(\cdot, I_3\cdot) + \varpi_2^{\zeta'}(\cdot, I_2\cdot) + \varpi_1^{\zeta}(\cdot, I_1\cdot) \\ =& -\|\Gamma\|\Big(\sum_{j=1}^3 |\zeta_j|^2\theta_j(\cdot, I_j\cdot) + \sum_{1\le j<k\le 3}\langle\zeta_j,\zeta_j\rangle\theta_j(\cdot, I_j\cdot)\Big), \end{aligned}$$

which gives exactly formula (31), with $\mathbf{c} = \frac{1}{2\operatorname{Vol}(\mathbb{B}^4)} = \pi^{-2}$.

From this and the formula for ϖ_1^ζ proved in 2.12 – which gives formula (33) of Theorem 2.1 –, we deduce the expected formula for ι_1^ζ. We know indeed that $\iota_1^\zeta = \iota_1^{\zeta'}$, and that $h_{\zeta'} = \omega_1^{\mathbf{e}}(\cdot, \iota_1^{\zeta'}\cdot)$ and $\iota_1^{\zeta'}$ is $\mathbf{e}$-symmetric, hence:

$$\begin{aligned} \mathbf{e}(\iota_1^\zeta\cdot,\cdot) =& \mathbf{e}(\cdot, \iota_1^{\zeta'}\cdot) = -\omega_1^{\mathbf{e}}(I_1\cdot, \iota_1^{\zeta'}\cdot) = -h_{\zeta'}(I_1\cdot,\cdot) \\ =& \|\Gamma\|\big(|\zeta_3|^2\theta_3(I_1\cdot, I_3\cdot) + |\zeta_2|^2\theta_2(I_1\cdot, I_2\cdot) + \langle\zeta_2,\zeta_3\rangle\theta_3(I_1\cdot, I_2\cdot)\big) \\ =& \|\Gamma\|\big(|\zeta_3|^2\theta_3(\cdot, I_2\cdot) - |\zeta_2|^2\theta_2(\cdot, I_3\cdot) - \langle\zeta_2,\zeta_3\rangle\theta_3(\cdot, I_3\cdot)\big) \\ =& \|\Gamma\||\zeta_3|^2\frac{\alpha_2\cdot\alpha_3 - rdr\cdot\alpha_1}{r^6} - \|\Gamma\||\zeta_2|^2\frac{\alpha_2\cdot\alpha_3 + rdr\cdot\alpha_1}{r^6} \\ &- \|\Gamma\|\langle\zeta_2,\zeta_3\rangle\frac{(rdr)^2 + \alpha_3^2 - \alpha_1^2 - \alpha_2^2}{r^6}, \end{aligned}$$

of which formula (32) is just a rewriting.

2.4 Vanishing of the third order terms when Γ is not cyclic

We shall see in this section that in the expansion $g_\zeta = \mathbf{e} + h_\zeta + \sum_{j=3}^{\infty} h_\zeta^{(j)}$, *if* Γ *is one of the* $\mathcal{D}_k$*,* $k \geq 2$*, or contains one of these as is the case when* Γ *is binary tetrahedral, octahedral or icosahedral*, then the third order term $h_\zeta^{(3)}$ vanishes, and that this holds as well for complex structures and Kähler forms. Keeping working with the diffeomorphisms F_ζ of the previous section even if we omit them to simplify notations, we claim:

Proposition 2.13 *Suppose* Γ *contains* $\mathcal{D}_k$*,* $k \geq 2$*, as a subgroup. Then* $g_\zeta = \mathbf{e} + h_\zeta + O(r^{-8})$*,* $I_1^\zeta = I_1 + \iota_1^\zeta + O(r^{-8})$*,* $\omega_1^\zeta = \omega_1 + \varpi_1^\zeta + O(r^{-8})$*, where by* $O(r^{-8})$ *we mean tensors whose* ℓ*th-order derivatives (for* $\nabla^{\mathbf{e}}$*) are* $O(r^{-8-\ell})$*.*

Proof. We shall first see that, for a general Γ, the crucial considerations made in section 2.3 on the second order term h_ζ of the expansion of g_ζ still hold for $h_\zeta^{(3)}$; first recall that $h_\zeta^{(3)}$ is a homogeneous polynomial of ζ of order 3, with coefficients $O(r^{-6})$ symmetric 2-tensors, with according decay on the derivatives, and those coefficients are *independent of* ζ. We start with claiming that:

$$\operatorname{tr}^{\mathbf{e}}(h_\zeta^{(3)}) = 0 \qquad \text{and} \qquad \delta^{\mathbf{e}} h_\zeta^{(3)} = 0.$$

Indeed, for the trace assertion, once $\zeta \in \mathfrak{h} \otimes \mathbb{R}^3 - D$ is fixed, one has for all t:

$$\begin{aligned}\Omega_{\mathbf{e}} = \operatorname{vol}^{g_{t\zeta}} &= \det{}^{\mathbf{e}}\big(\mathbf{e} + t^2 h_\zeta + t^3 h_\zeta^{(3)} + O(t^4)\big)\Omega_{\mathbf{e}}\\ &= \big(1 + t^2 \operatorname{tr}^{\mathbf{e}}(h_\zeta) + t^3 \operatorname{tr}^{\mathbf{e}}(h_\zeta^{(3)}) + O(t^4)\big)\Omega_{\mathbf{e}},\end{aligned}$$

since the higher order contributions of $t^2 h_\zeta$ are included in the $O(t^4)$, hence $\operatorname{tr}^{\mathbf{e}}(h_\zeta^{(3)}) = 0$.

We thus notice that $h_\zeta^{(3)}$ shares this property with h_ζ because the non-linear contributions of the $h_{t\zeta}$, which are of order at least 4 in t, do not interfere with the linear contribution of $h_{t\zeta}^{(3)}$. We thus generalize this observation to prove that $h_\zeta^{(3)}$ shares other properties with h_ζ, and to start with, that $\delta^{\mathbf{e}} h_\zeta^{(3)} = 0$, as promised. Again we proceed within three steps, considering first $\zeta'' = (0, 0, \zeta_3)$, and then $\zeta' = (0, \zeta_2, \zeta_3)$ and $\zeta = (\zeta_1, \zeta_2, \zeta_3)$.

The case of $h_{\zeta''}^{(3)}$ is immediate, and merely amounts to the fact that it is an I_3-hermitian tensor (the $g_{t\zeta''}$ are) with vanishing trace for $\mathbf{e}$, used with the Kähler identity $[\Lambda_{\omega_3}, d_{I_3}^c] = d^{*_{\mathbf{e}}}$ applied to $h_{\zeta''}^{(3)}(I_3\cdot, \cdot)$.

For the case of $h_{\zeta'}^{(3)}$, remember the following: we first saw that the second order variation of $I_2^{\zeta'} = I_2^{\zeta''}$ was $\mathbf{e}$-symmetric; this still holds for the third order term, since the only I_2-entire function on $\mathbb{C}^2$ decaying (like r^{-6}) at infinity is trivial. Then we identified the I_2-skew-hermitian part of $h_{\zeta'}$ with $h_{\zeta''}$; again, this holds for $h^{(3)}_{\zeta'}$ with $h^{(3)}_{\zeta''}$ (and the latter is indeed I_2-skew-hermitian). This amounts to looking at the term of order 3 in t of:

- the expansion of $g_{t\zeta''} = \omega_2^{t\zeta''}(\cdot, I_2^{t\zeta''}\cdot)$ to see that $h^{(3)}_{\zeta''}$ is indeed I_2-skew-hermitian (recall $\omega_2^{t\zeta''} = \omega_2$ for all t);
- the expansion of $g_{t\zeta'}(I_2^{t\zeta'}\cdot, I_2^{t\zeta'}\cdot) - g_{t\zeta'}$ to see that $\frac{1}{2}\big(h^{(3)}_{\zeta'} + h^{(3)}_{\zeta'}(I_2\cdot, I_2\cdot)\big) = h^{(3)}_{\zeta''}$.

We concluded by using the usual Kähler identity (for I_2) on the $\mathbf{e}$-trace-free I_2-(1,1) form $\frac{1}{2}\big(h_{\zeta'}(I_2\cdot,\cdot) - h_{\zeta'}(\cdot, I_2\cdot)\big)$, after seeing it was closed; we can do the same on its analogue $\frac{1}{2}\big(h^{(3)}_{\zeta'}(I_2\cdot,\cdot) - h^{(3)}_{\zeta'}(\cdot, I_2\cdot)\big)$, which is also an $\mathbf{e}$-trace-free I_2-(1,1) form, and is closed as seen when looking at the third order in t of the expansion of $\omega_2^{t\zeta'} = \frac{1}{2}\big(g_{t\zeta'}(I_2^{t\zeta'}\cdot,\cdot) - g_{t\zeta'}(I_2^{t\zeta'}\cdot, I_2\cdot)\big)$.

One deals with h_ζ in analogous way. In particular, we get in passing that the third order variation of $I_1^\zeta = I_1^{\zeta'}$, $\jmath_1^\zeta$ say, is $\mathbf{e}$-symmetric and anti-commutes to I_1, that the I_1-skew-hermitian part of $h^{(3)}_\zeta$ is $h^{(3)}_{\zeta'}$, related to $\jmath_1^\zeta$ by $h^{(3)}_{\zeta'} = \omega_1^{\mathbf{e}}(\cdot, \jmath_1^\zeta\cdot)$, and that its I_1-hermitian part gives rise to an $\mathbf{e}$-trace-free closed I_1-(1,1) form.

Running backward this description, we will thus be done if we show that the third order variations of the Kähler forms vanish when Γ contains a binary dihedral group. In general though, we know these are $O(r^{-6})$ near 0 and infinity with corresponding decay on their derivatives, that they are of type (1,1) for one of the I_j and trace-free; they are thus $*_{\mathbf{e}}$-anti-self-dual, and therefore can be written as $f\theta_1 + g\theta_2 + h\theta_3$, where this time, $r^2 f$, $r^2 g$ and $r^2 h$ depend only of the spherical coordinate of their argument. Our form are moreover closed, hence in particular harmonic; using again that the Laplace-Beltrami operator and the rough laplacian coincide on $(\mathbb{R}^4, \mathbf{e})$, and that the θ_j are harmonic, we have this time:

$$\Delta_{\mathbf{e}}(f\theta_1) = (\Delta_{\mathbf{e}} f)\theta_1 - 2\sum_{k=0}^{3}(e_k\cdot f)\nabla^{\mathbf{e}}_{e_k}\theta_1,$$

with $e_0 = \frac{x_j}{r}\frac{\partial}{\partial x_j}$. We set $\tilde{f} = r^2 f$; this is a function on $\mathbb{S}^3$, and $e_0\cdot f = e_0\cdot(r^{-2}\tilde{f}) = e_0\cdot(r^{-2})\tilde{f} = -2r^{-3}\tilde{f} = -2r^{-1}f$. Since on functions, $\Delta_{\mathbf{e}} = -\frac{1}{r^3}\partial_r\big(r^3\partial_r\,\cdot\big) + \frac{1}{r^2}\Delta_{\mathbb{S}^3}$, one has:

$$\Delta_{\mathbf{e}} f = \Delta_{\mathbf{e}}(r^{-2}\tilde{f}) = -\frac{1}{r^3}\partial_r\big(r^3\partial_r(r^{-2})\big)\tilde{f} + \frac{1}{r^4}\Delta_{\mathbb{S}^3}\tilde{f} = \frac{1}{r^4}\Delta_{\mathbb{S}^3}\tilde{f},$$

since $\partial_r\big(r^3\partial_r(r^{-2})\big) = 0$ (r^{-2} is the Green function on $\mathbb{R}^4$).

Moreover, $\nabla^{\mathbf{e}}_{e_0}\theta_1 = \partial_r(r^{-6})(rdr\wedge\alpha_1 - \alpha_2\wedge\alpha_3) + r^{-6}\nabla^{\mathbf{e}}_{e_0}(rdr\wedge\alpha_1 - \alpha_2\wedge\alpha_3) = -\frac{6}{r}\theta_1 + \frac{2}{r}\theta_1 = -\frac{4}{r}\theta_1$. We recall that $\nabla^{\mathbf{e}}_{e_1}\theta_1 = 0$, $\nabla^{\mathbf{e}}_{e_2}\theta_1 = \frac{2}{r}\theta_3$ and $\nabla^{\mathbf{e}}_{e_3}\theta_1 = -\frac{2}{r}\theta_2$, therefore:

$$\Delta_{\mathbf{e}}(f\theta_1) = \frac{1}{r^4}\big(\Delta_{\mathbb{S}^3}\tilde{f} - 16\tilde{f}\big)\theta_1 - \frac{2}{r^3}\big((e_2\cdot\tilde{f})\theta_3 - (e_3\cdot\tilde{f})\theta_2\big).$$

Writing the analogous equations on $\tilde g = r^2 g$, $\tilde h = r^2 h$, the equation $\Delta_{\mathbf{e}}(f\theta_1 + g\theta_2 + h\theta_3) = 0$ is equivalent to the system:

$$\begin{aligned} \Delta_{\mathbb{S}^3}\tilde f - 16\tilde f - 4(e_3\cdot\tilde g) + 4(e_2\cdot\tilde h) &= 0,\\ \Delta_{\mathbb{S}^3}\tilde g - 16\tilde g - 4(e_1\cdot\tilde h) + 4(e_3\cdot\tilde f) &= 0,\\ \Delta_{\mathbb{S}^3}\tilde h - 16\tilde h - 4(e_1\cdot\tilde f) + 4(e_3\cdot\tilde g) &= 0. \end{aligned} \tag{42}$$

Now the closure assertion on $f\theta_1+g\theta_2+h\theta_3$ is equivalent to $(e_1\cdot f)+(e_2\cdot g)+(e_3\cdot h) = (e_0\cdot f)-(e_3\cdot g)+(e_2\cdot h) = (e_0\cdot g)-(e_1\cdot h)+(e_3\cdot f) = (e_0\cdot h)-(e_2\cdot f)+(e_1\cdot g) = 0$.

Since $e_0\cdot u = -\frac{2}{r^3}\tilde u$ and $e_k\cdot u = \frac{1}{r^2}e_k\cdot\tilde u$ for $u = f,g,h$ and $k = 1,2,3$, we deduce from the latter equalities and the system (42) the equations:

$$\Delta_{\mathbb{S}^3}\tilde f - 8\tilde f = \Delta_{\mathbb{S}^3}\tilde g - 8\tilde g = \Delta_{\mathbb{S}^3}\tilde h - 8\tilde h = 0.$$

Setting $\hat f = r^2\tilde f$ and likewise for $\hat g$ and $\hat h$, we get that $\hat f$, $\hat g$, $\hat h$ are harmonic (on the whole $\mathbb{R}^4$) and homogeneous of degree 2. This is not hard seeing that they are thus linear combinations of the $x_1^2 - x_j^2$, $j = 2,3,4$, and the x_jx_k, $1\le j<k\le 4$.

The θ_j are Γ-invariant; f, g and h, and consequently $\hat f$, $\hat g$ and $\hat h$, must thus be as well. But *if Γ contains a binary dihedral group as a subgroup, then there is no non-trivial linear combination of the above polynomials which is Γ-invariant.* We use first the τ-invariance; if indeed $\mathcal{D}_k < \Gamma$ for some $k\ge 2$ and $u = \sum_{j=1}^3 a_j(x_1^2 - x_j^2) + \sum_{1\le j<\ell\le 4} a_{j\ell}x_jx_\ell$ is Γ-invariant, then $2u = u + \tau^* u = a_2(x_1^2 - x_2^2 + x_3^2 - x_4^2) + a_3(x_1^2 - x_3^2 + x_3^2 - x_1^2) + a_4(x_1^2 - x_4^2 + x_3^2 - x_2^2) + a_{12}(x_1x_2 + x_3x_4) + a_{13}(x_1x_3 - x_3x_1) + a_{14}(x_1x_4 - x_3x_2) + a_{23}(x_2x_3 - x_4x_1) + a_{24}(x_2x_4 - x_4x_2) + a_{34}(x_3x_4 + x_1x_2)$, that is: u has shape $a(x_1^2 - x_2^2 + x_3^2 - x_4^2) + 2b(x_1x_2 + x_3x_4) + 2c(x_1x_4 - x_3x_2)$, i.e. $a\mathfrak{Re}(z_1^2 + z_2^2) + b\mathfrak{Im}(z_1^2+z_2^2) + c\mathfrak{Im}(\overline{z_1}z_2)$, $a,b,c\in\mathbb{R}$, in complex notations. We now use the ζ_k-action and write:

$$\begin{aligned} ku = \sum_{\ell=0}^k \zeta_k^* u &= \sum_{\ell=0}^k a(\zeta_k^\ell)^*\mathfrak{Re}(z_1^2+z_2^2) + b(\zeta_k^\ell)^*\mathfrak{Im}(z_1^2+z_2^2) + c(\zeta_k^\ell)^*\mathfrak{Im}(\overline{z_1}z_2)\\ &= \mathfrak{Re}\left(\sum_{\ell=0}^k a(e^{2i\ell\pi/k}z_1^2 + e^{-2i\ell\pi/k}z_2^2)\right)\\ &\quad + \mathfrak{Im}\left(\sum_{\ell=0}^k b(e^{2i\ell\pi/k}z_1^2 + e^{-2i\ell\pi/k}z_2^2) + ce^{-2i\ell\pi/k}\overline{z_1}z_2\right)\\ &= 0, \end{aligned}$$

since $e^{2i\pi/k}\neq 1$ ($k\ge 2$). In particular, the third order variation term of ω_1^ζ vanishes; in other words, $\omega_1^\zeta = \omega_1 + \varpi_1^\zeta + O(r^{-8})$.

Since moreover $\jmath_1^\zeta$ is determined by $h_{\zeta'}^{(3)}$ which is also 0, this third order variation of the first complex structure vanishes as well, or: $I_1^\zeta = I_1 + \iota_1^\zeta + O(r^{-8})$. $\square$

This completes the proof of Theorem 2.1. Notice however that in view of the previous two sections, we could also have given similar statements on the second and third complex structures and Kähler forms of X_ζ. We chose to focus and the first ones since this is what is needed in our construction of Part 1, see in particular Lemma 1.6, which is just a specialization of Theorem 2.1: take $\zeta = \xi$ verifying condition (11), and $\Phi_Y = \Phi_\zeta$.

Nonetheless, the asymptotics of the second and third complex Kähler forms are available via Proposition 2.12, from which the asymptotics of the corresponding complex structures easily follow, since the asymptotics of the metric are known.

2.5 Comments on Lemma 1.1

2.5.1 The condition (11)

The first comment we want to make about Lemma 1.1 concerns the reason why we state it under the condition (11), which we can recall as $(|\zeta_2|^2 - |\zeta_3|^2) + 2i\langle\zeta_2, \zeta_3\rangle = 0$ (if one takes ζ instead of ξ as parameter).

One could instead try to generalize the proof we give in section 1.3 with help of the asymptotics given by Theorem 2.1, with ζ a generic element of $\mathfrak{h} \otimes \mathbb{R}^3 - D$. This is formally possible, but leads to include terms such as $\frac{1}{r^2 z_1}$, $\frac{1}{r^2 \overline{z_1}}$, $\frac{1}{r^2 z_2}$, $\frac{1}{r^2 \overline{z_2}}$ in the correction terms ε_1 and ε_2 of that proof, which is obviously not compatible with the requirement that $\daleth$ is a diffeomorphism of $\mathbb{R}^4$.

In others words, $(|\zeta_2|^2 - |\zeta_3|^2) + 2i\langle\zeta_2, \zeta_3\rangle$ appears as an obstruction for I_1^ζ to be approximated to higher orders by I_1, even with some liberty on the diffeomorphism between infinities of X_ζ and $\mathbb{R}^4/\Gamma$, which reveals some link between the parametrisation of the X_ζ and the general problem of the approximation of their complex structures.

2.5.2 Links with the parametrisation

Conversely we interpret of Lemma 1.1 as follows: when $\Gamma = \mathcal{D}_k$ – this would be true also in the tetrahedral, octahedral and icosahedral cases – and $(|\zeta_2|^2 - |\zeta_3|^2) + 2i\langle\zeta_2, \zeta_3\rangle = 0$, then the complex structure I_1^ζ can be viewed as approximating the standard complex structure I_1 with precision twice that of the general case, i.e. with an error $O(r^{-8})$ instead of $O(r^{-4})$, up to an adjustment of the ALE diffeomorphism given in Kronheimer's contruction. Now $(|\zeta_2|^2 - |\zeta_3|^2) + 2i\langle\zeta_2, \zeta_3\rangle = \langle\zeta_2 + i\zeta_3, \zeta_2 + i\zeta_3\rangle$, and this precisely the coefficient a_k in the equation of X_ζ seen as a submanifold of $\mathbb{C}^3$, which is

$$u^2 + v^2 w + w^{k+1} = a_0 + a_1 w + \cdots + a_k w^k + bv \tag{43}$$

(a_j being given by symmetric functions of the $(k+2)$ first diagonal values of $\zeta_2 + i\zeta_3 \in \mathfrak{h}_{\mathbb{C}}$ or $\mathfrak{h}_{\mathbb{C}}/(\text{Weyl group})$ of degree $(k+2-j)$, and b by their Pfaffian).

Denote by $X_{\mathcal{D}_k}$ the orbifold defined in $\mathbb{C}^3$ by the equation $u^2+v^2w+w^{k+1}=0$, i.e. equation (43) with $a_0 = \cdots = a_k = b = 0$. This is identified to $\mathbb{C}^2/\mathcal{D}_k$ via the map $(z_1, z_2) \mapsto (u, v, w) := \big(\frac{1}{2}(z_1^{2k+1}z_2 - z_2^{2k+1}z_1), \frac{i}{2}(z_1^{2k} + z_2^{2k}), z_1^2 z_2^2\big)$. This suggests that (u, v, w) in equation (43) should somehow have respective degrees $2k+2$, $2k$ and 4 in the z_1, z_2 variables, and this equation remains homogeneous if we give formal degree 2 to ζ. When $a_k = 0$, the right-hand-side member of (43) is therefore formally conferred "pure" degree at most $4k-4$, instead of $4k$.

We suspect that this corresponds to the improvement by four orders in the approximation of I_1^ζ by I_1 in the sense of Lemma 1.1. It would thus be of interest to draw a rigorous picture out of these informal considerations, establishing a more direct link of the kind suggested here between the parameter ζ and the associated complex structures, without passing by the analysis of g_ζ, which we unfortunately have not been able to so far.

3 Proof of Theorem 0.4

We shall use for proving Theorem 0.4 a classical method in the study of Monge-Ampère equations: the *continuity method*. This method was suggested by E. Calabi for the solution of the complex Monge-Ampère equation on compact Kähler manifolds. Since the successful use by Yau [Yau] of this method, it has been adapted to different non-compact settings; let us quote here the version by Joyce [Joy, ch. 8] for ALE manifolds, which greatly inspired ours. We also refer the reader to Tian and Yau's seminal works [TY1, TY2], which pioneered the research on generalising Calabi-Yau theorem to non-compact manifolds. We shall mention a result by Hein [Hei, Prop. 4.1] too, very similar to ours if taking Hein's parameter β equal to 3, but dealing with less precise asymptotics.

We hence start this part by describing the method, and follow by the analytic work (in particular, a priori estimates) it requires.

3.1 The continuity method.

The principle of this method is not to solve directly the Monge-Ampère equation (2), but to solve a one-parameter family of such equations in which the right-hand side term interpolates between ω_Y^2 to $e^f \omega_Y^2$. Concretely, we consider the equations

$$\big(\omega_Y + i\partial\overline{\partial}\varphi_t\big)^m = e^{tf}\omega_Y^m \tag{E_t}$$

for $t \in [0,1]$. Now set $S := \{t \in [0,1] \,|\, (E_t)$ has a unique solution $\varphi_t \in C^{5,\alpha}_\beta\}$; the weighted space involved in this definition is defined on Y in total analogy with those of §1.4.2 (replace g_m by g_Y, R by ρ, and so on).

Up to the uniqueness of the solution 0 of (E_0), it is obvious that $0 \in S$. We shall then prove:

1. the set S is closed (section 3.2);
2. the set S is open (section 3.3).

Theorem 0.4 then follows from an immediate connectedness argument.

In order to prove the openness of S, one observes that the linearisation of the Monge-Ampère operator is (nearly) a Laplacian, close enough to Δ_{g_Y}; this allows one to use the isomorphisms induced between some Hölder spaces, plus ellipticity of such an operator. This is done in section 3.3.

On the other hand, in order to prove the closedness of S (the hard part), we need some compactness properties for solutions of a family of (E_t), and this we get by establishing a priori estimates on such solutions. This is done in paragraph 3.2.5.

The easiest part is the uniqueness of the solution of any (E_t), and we shall deal with it now as a warm-up.

Lemma 3.1 *Let $t \in [0,1]$, and let φ_1, φ_2 be two solutions of (E_t). Assume that φ_1 and $\varphi_2 \in C^\infty_\beta$. Then $\varphi_1 = \varphi_2$.*

Proof. This is essentially the same argument as that used to prove the uniqueness of g_{RF} above, namely if for some t, φ_1 and φ_2 are C^2_β solutions of (E_t), then $\omega_{\varphi_1} = \omega_Y + i\partial\overline{\partial}\varphi_1$ is a Kähler form equivalent to ω_Y, and denoting by Δ_1 the Laplacian of its associated metric, one gets that $\Delta_1(\varphi_1 - \varphi_2)$ has constant sign. Since $\varphi_1 - \varphi_2$ tends to 0 at infinity, we thus have $\varphi_1 = \varphi_2$. □

3.2 Closedness of S: a priori estimates.

3.2.1 C^0 estimates.

We are proving in this paragraph an a priori estimate on the C^0 norm of a solution of a Monge-Ampère equation on Y.

The estimates can already be deduced from [Hei, Prop. 4.1]; we give our own proof though, because *weighted C^0 estimates* will be established below (§3.2.3) in a very similar but slightly more complicated way.

The techniques used here are quite close to the ones used by Yau in the compact case, namely a recursive use of integration by parts giving a variation of Moser's iteration adapted to the Monge-Ampère equation, and their adaptation by Joyce for his version of the Calabi-Yau theorem on ALE manifolds. Nonetheless, because ALF geometry is quite different from ALE geometry, we write in detail the manipulations that need to be adapted to our framework. In particular, because of different measures on both sides of a Sobolev inequality (Lemma 3.3) we make a crucial use of, our system of weights in the following integrals is not the same as Joyce's.

We start with the following :

Lemma 3.2 *Let $f \in C^0_{\beta+2}$, $\beta \in (0,1)$, and $\varphi \in C^3_\gamma$, $0 < \gamma \leq \beta$, such that $\big(\omega_Y + i\partial\overline{\partial}\varphi\big)^2 = e^f \omega_Y^2$. Then for any $p > \gamma^{-1}$, $p > 2$,*

$$\int_Y \big|\partial|\varphi|^{p/2}\big|^2 \omega_Y^2 \leq \frac{p^2}{2(p-1)} \int_Y |\varphi|^{p-2}\varphi(e^f - 1)\omega_Y^2,$$

where both integrals are finite.

Notice that in this lemma f plays the role of tf in our problem; this is not an issue, as a $C^0_{\beta+2}$ bound on f gives uniform $C^0_{\beta+2}$ bounds on the tf.

Proof of Lemma 3.2. First, we set $T = \omega_Y + \omega_\varphi$, with $\omega_\varphi = \omega_Y + i\partial\overline{\partial}\varphi$. Picking $p > \gamma^{-1}$ and $p > 2$, we claim that

$$\int_Y d\big(\varphi|\varphi|^{p-2} d^c\varphi \wedge T\big) = 0.$$

Indeed, for any real number R big enough, if B_R denotes the set $\{\rho < R\}$ and S_R its boundary, Stokes' theorem asserts that

$$\int_{B_R} d\big(\varphi|\varphi|^{p-2} d^c\varphi \wedge T\big) = \int_{S_R} \varphi|\varphi|^{p-2} d^c\varphi \wedge T.$$

Now since the volume of S_R according to g is $O(R^2)$, since T is bounded with respect to g_Y and since $\varphi|\varphi|^{p-2}d^c\varphi = O(R^{-p\gamma-1})$ on S_R, we get that the right-hand-side term is $O(R^{1-p\gamma})$, and hence goes to 0 as R goes to ∞ by our choice of $p > \gamma^{-1}$. A little computation yields

$$\frac{1}{2} d\big(\varphi|\varphi|^{p-2} d^c\varphi \wedge T\big) = \varphi|\varphi|^{p-2} i\partial\overline{\partial}\varphi \wedge T + (p-1)|\varphi|^{p-2} i\partial\varphi \wedge \overline{\partial}\varphi \wedge T$$

because T is closed. But on the one hand, $i\partial\overline{\partial}\varphi \wedge T = (\omega_\varphi - \omega_Y) \wedge T = (e^f - 1)\omega_Y^2$ and on the other hand $|\varphi|^{p-2} i\partial\varphi \wedge \overline{\partial}\varphi = \frac{4}{p^2} i\partial\big(|\varphi|^{p/2}\big) \wedge \overline{\partial}\big(|\varphi|^{p/2}\big)$, hence

$$\int_Y i\partial\big(|\varphi|^{p/2}\big) \wedge \overline{\partial}\big(|\varphi|^{p/2}\big) \wedge T = \frac{p^2}{2(p-1)} \int_Y \varphi|\varphi|^{p-2}(1 - e^f)\omega_Y^2.$$

Finally, observe that $\frac{i\alpha\wedge\overline{\alpha}\wedge\omega_\varphi}{\omega_Y^2} = \frac{1}{2}e^f|\alpha|^2_{\omega_\varphi} \geq 0$, and that $i\alpha\wedge\overline{\alpha}\wedge\omega_Y = \frac{1}{2}|\alpha|^2_{g_Y}\omega_Y^2$ for any (1,0)-form α, and conclude by setting $\alpha = \partial\big(|\varphi|^{p/2}\big)$. The finiteness of the integrals in play merely comes from our choice of p. □

Next, we want to use the inequality we have just proved with some Sobolev embedding to estimate recursively $\|\varphi\|_{L^{p\varepsilon}_{d\lambda}}$ in terms of $\|\varphi\|_{L^p_{d\lambda}}$, with some fixed $\varepsilon > 1$, and some positive measure $d\lambda$. The Sobolev embedding states as :

Lemma 3.3 *There exists a constant C_S such that for any $u \in H^1_{\mathrm{loc}}$ (hence in L^4_{loc}) such that the $\int_Y |u|^4\rho^{-1}\operatorname{vol}^{g_Y}$ is finite, one has*

$$\Big(\int_Y |u|^4\rho^{-1}\operatorname{vol}^{g_Y}\Big)^{1/4} \leq C_S^{1/2}\Big(\int_Y |du|^2_g\operatorname{vol}^{g_Y}\Big)^{1/2}. \tag{44}$$

This inequality can be compared to [Hei, Prop. 3.2]; we postpone its proof at the end of the present paragraph.

First we initiate the induction by estimating $\|\varphi\|_{L^{p_0}_{d\lambda}}$ (or $\|\varphi\|_{L^{p_0\varepsilon}_{d\lambda}}$) for some p_0 independent of φ. It will be relevant to take $\varepsilon = 2$, and $d\lambda = \rho^{-1}\operatorname{vol}^{g_Y}$ in the following. Our initial estimation states:

Lemma 3.4 *Fix some $p_0 > 2$, $p_0 > \gamma^{-1}$. Then under the assumptions of Lemma 3.2 there exists C only depending on β, $\|f\|_{C^0_{\beta+2}}$, p_0 and g such that $\|\varphi\|_{L^{p_0\varepsilon}_{d\lambda}} \leq C$.*

Proof. Apply inequality (44) to $u = |\varphi|^{p_0/2}$ to get:

$$\Big(\int_Y |\varphi|^{2p_0}\rho^{-1}\operatorname{vol}^g\Big)^{1/2} \leq C_S\int_Y \big|d|\varphi|^{p_0/2}\big|^2_g\operatorname{vol}^g.$$

By Lemma 3.2, we have

$$\Big(\int_Y |\varphi|^{2p_0}\rho^{-1}\operatorname{vol}^{g_Y}\Big)^{1/2} \leq \frac{p_0^2C_S}{2(p_0-1)}\int_Y |\varphi|^{p_0-1}\big|e^f-1\big|\operatorname{vol}^{g_Y}.$$

Write $\big|e^f-1\big| = \big|e^f-1\big|^{a+b}$ with $a = \frac{p_0-1}{2(\beta+2)p_0}$ and $b = 1-a$, and apply Hölder inequality to the right-hand-side term of the inequality above with exponents $s = \frac{2p_0}{p_0-1}$ and $t = \big(1-\frac{1}{s}\big)^{-1} = \frac{2p_0}{1+p_0}$. This yields:

$$\begin{aligned}\Big(\int_Y |\varphi|^{2p_0}\rho^{-1}\operatorname{vol}^{g_Y}\Big)^{1/2} &\leq \frac{p_0^2C_S}{2(p_0-1)}\Big(\int_Y |\varphi|^{s(p_0-1)}\big|e^f-1\big|^{as}\Big)^{1/s}\Big(\int_Y \big|e^f-1\big|^{bt}\Big)^{1/t}\\ &= \frac{p_0^2C_S}{2(p_0-1)}\Big(\int_Y |\varphi|^{2p_0}\big|e^f-1\big|^{\frac{1}{(\beta+2)}}\Big)^{1/s}\Big(\int_Y \big|e^f-1\big|^{bt}\Big)^{1/t}.\end{aligned}$$

Noticing that $|e^f-1| = O(\rho^{-(\beta+2)})$ (and more precisely that at any point, $|e^f - 1| \leq e^{\|f\|_{C^0}}\|f\|_{C^0_{\beta+2}}\rho^{-(\beta+2)}$), we get that $\int_Y |\varphi|^{2p_0}|e^f-1|^{\frac{1}{(\beta+2)}} \leq C\int_Y |\varphi|^{2p_0}\rho^{-1}$ for some C depending only on the parameters announced. Moreover, we have that $(\beta+2)bt = \frac{2p_0}{1+p_0}\big((\beta+2) - \frac{p_0-1}{2p_0}\big) > 3$ because $p_0 > \gamma^{-1} \geq \beta^{-1}$, so that $\int_X \left|e^f-1\right|^{bt}$ is finite and equal to some constant, K^t say, also independent of φ. So far, we obtain that:

$$\|\varphi\|^{p_0}_{L^{2p_0}_{d\lambda}} \leq \frac{p_0^2 C C_S}{2(p_0-1)} K \|\varphi\|^{p_0-1}_{L^{2p_0}_{d\lambda}},$$

and the conclusion follows. □

We fix now $p_0 = \frac{2}{\gamma}$. Using the same techniques, we can prove a recursive control on $\|\varphi\|_{L^{2p}_{d\lambda}}$ from $\|\varphi\|_{L^{p}_{d\lambda}}$:

Lemma 3.5 *Under the assumptions of Lemma 3.2, there exists a constant C_1 only depending on β, γ, $\|f\|_{C^0_{\beta+2}}$ and g_Y such that for any $p \geq p_0$, $\|\varphi\|^p_{L^{2p}_{d\lambda}} \leq C_1 p \|\varphi\|^{p-1}_{L^p_{d\lambda}}$.*

Proof. The idea is the same as for the previous lemma: apply inequality (44) to $u = |\varphi|^{p/2}$, Lemma 3.2 and Hölder inequality (with well-chosen exponents). The first two give, for $p > \gamma^{-1}$, $p > 2$:

$$\Big(\int_Y |\varphi|^{2p}\rho^{-1}\,\mathrm{vol}^{g_Y}\Big)^{1/2} \leq \frac{p^2 C_S}{2(p-1)}\int_Y |\varphi|^{p-1}\big|e^f-1\big|\,\mathrm{vol}^{g_Y}.$$

Apply now Hölder inequality with exponents $s = \frac{p}{p-1}$, $t = p$ and weights $a = \frac{(p-1)}{(\beta+2)p} = \frac{1}{(\beta+2)s}$ and $b = 1-a$ to write:

$$\begin{aligned}&\Big(\int_Y |\varphi|^{2p}\rho^{-1}\,\mathrm{vol}^{g_Y}\Big)^{1/2}\\ &\qquad\leq \frac{p^2 C_S}{2(p-1)}\Big(\int_Y |\varphi|^p\big|e^f-1\big|^{\frac{1}{(\beta+2)}}\,\mathrm{vol}^{g_Y}\Big)^{(p-1)/p}\Big(\int_Y \big|e^f-1\big|^{bp}\,\mathrm{vol}^{g_Y}\Big)^{1/p}.\end{aligned}$$

But $|e^f-1|^{\frac{1}{(\beta+2)}} \leq C\rho^{-1}$ for some constant depending only on f. Moreover, $(\beta+2)pb = p(\beta+1)+1$, which is > 3 as soon as $p > \frac{2}{\beta+1}$; this is actually automatic since $p > \gamma^{-1} \geq \beta^{-1}$ and $\beta \in (0,1)$. Since furthermore $b = b(p)$ tends to $\frac{\beta+1}{\beta+2} := b_\infty > 0$ when p goes to ∞, $\big(\int_Y |e^f-1|^{bp}\,\mathrm{vol}^g\big)^{1/p}$ tends to $\sup_Y |e^f-1|^{b_\infty}$ when p goes to ∞, so we can claim: for all $p \geq p_0$, $\big(\int_Y |e^f-1|^{bp}\,\mathrm{vol}^{g_Y}\big)^{1/p} \leq K$ for some K depending only on f (and which we could evaluate in terms of $\|f\|_{C^0_{\beta+2}}$

only). Finally, we get that for all $p \geq p_0$:

$$\Big(\int_Y |\varphi|^{2p}\rho^{-1}\,\mathrm{vol}^{g_Y}\Big)^{1/2} \leq \frac{p^2 C_S C^{(p-1)/p} K}{2(p-1)}\Big(\int_Y |\varphi|^p \rho^{-1}\,\mathrm{vol}^{g_Y}\Big)^{(p-1)/p}$$
$$\leq C_1 p \Big(\int_Y |\varphi|^p\rho^{-1}\,\mathrm{vol}^{g_Y}\Big)^{(p-1)/p}$$

with $C_1 = (1+C)KC_S$ say, which only depends on the parameters announced. $\square$

We fix now $p_0 = 2\gamma^{-1}$ in Lemma 3.4.

Lemma 3.6 *Under the assumptions of Lemma 3.2, there exist two constants Q_0 and C_2 depending only on β, γ, $\|f\|_{C^0_{\beta+2}}$ and g_Y such that for any $q \geq 2p_0$, $\|\varphi\|_{L^q_{d\lambda}} \leq Q_0(C_2 q)^{-2/q}$.*

Letting q go to ∞, we get the C^0 a priori estimate for φ we are seeking :

Proposition 3.7 *Under the assumptions of Lemma 3.2, there exists a constant $Q_0 = Q_0(\beta,\gamma,\|f\|_{C^0_{\beta+2}}, g_Y)$ such that $\|\varphi\|_{C^0} \leq Q_0$.*

We conclude this paragraph by the proof of Lemma 3.3:

Proof of Lemma 3.3: We shall prove that there exist two constants C_1 and C_2 such that for any u as in the statement of the lemma,

$$\Big(\int_Y |u|^4\rho^{-1}\,\mathrm{vol}^{g_Y}\Big)^{1/4} \leq C_1\Big(\int_Y |du|^2_{g_Y}\,\mathrm{vol}^{g_Y} + \int_Y |u|^2\rho^{-2}\,\mathrm{vol}^{g_Y}\Big)^{1/2} \tag{45}$$

and

$$\int_Y |u|^2\rho^{-2}\,\mathrm{vol}^{g_Y} \leq C_2\int_Y |du|^2_{g_Y}\,\mathrm{vol}^{g_Y};$$

the Lemma will then follow at once. Let us start by the latter inequality. We prove it first for u with have compact support in $\{\rho \geq \rho_0\}$, where ρ_0 is chosen so that we are in the part of Y diffeomorphic to a neighbourhood of infinity in $R^4/\mathcal{D}_k$. This way, we can replace Y by a two-sheeted cover $\hat{Y}$ that is the total space of a circle fibration ϖ over $\mathbb{R}^3$ minus a ball, and replace g_Y and h and u by there pull-backs. Decompose u as $u_\perp + u_0$, with $u_0(x)$ the mean value of u along the fibre of ϖ passing by x. This makes u_0 a compactly supported function on $\mathbb{R}^3\backslash B$ (B the unit ball). Take spherical coordinates (ρ,θ,ϕ) on $\mathbb{R}^3$. Then

$$0 = \int_{\rho\geq 1}\partial_\rho(u_0^2\rho)d\rho\,\mathrm{vol}^{S^2} = 2\int_{\rho\geq 1} u_0\partial_\rho(u_0)\rho d\rho\,\mathrm{vol}^{S^2} + \int_{\rho\geq 1} u_0^2 d\rho\,\mathrm{vol}^{S^2}$$

which we rewrite as

$$\begin{aligned}\int_{\rho\geq 1} u_0^2\rho^{-2}\operatorname{vol}^{\mathbb{R}^3} &= -2\int_{\rho\geq 1} u_0\partial_\rho(u_0)\rho^{-1}\operatorname{vol}^{\mathbb{R}^3}\\ &\leq 2\Big(\int_{\rho\geq 1} u_0^2\rho^{-2}\operatorname{vol}^{\mathbb{R}^3}\Big)^{1/2}\Big(\int_{\rho\geq 1}\partial_\rho(u_0)^2\operatorname{vol}^{\mathbb{R}^3}\Big)^{1/2}\end{aligned}$$

(Cauchy-Schwarz inequality), i.e. $\int_{\rho=1}^\infty u_0^2\rho^{-2}\operatorname{vol}^{\mathbb{R}^3}\leq 4\int_{\rho=1}^\infty |du_0|^2\operatorname{vol}^{\mathbb{R}^3}$.

Now for the component $u_\perp$, after pulling it back on $\hat{Y}$, one has $\int_{\hat{Y}} u_\perp^2\rho^{-2}\operatorname{vol}^h = \int_{\rho\geq\rho_0}\rho^{-2}\operatorname{vol}^{\mathbb{R}^3}\int_{\text{fibre}} u_\perp^2\mu$. Since $u_\perp$ has zero mean along the fibres, $\int_{\text{fibre}} u_\perp^2\mu \leq c\int_{\text{fibre}} d|u_\perp|_h^2\mu$ for some c independent of the fibre, and thus $\int_{\hat{Y}} u_\perp^2\rho^{-2}\operatorname{vol}^h \leq c\int_{\hat{Y}} |du_\perp|_h^2\rho^{-2}\operatorname{vol}^h$; we even have a ρ^{-2} factor in the RHS, which we can loosely get rid of. Adding these estimations on separate components, we get that $\int_{\hat{Y}} u^2\rho^{-2}\operatorname{vol}^h\leq C\int_{\hat{Y}}|du|_h^2\operatorname{vol}^h$, which easily transposes on Y with g_Y. Now, to extend this Hardy type inequality to a general u as in the statement, one may proceed as in [Auv, §1.4.1]; the only point to be noticed is that the only integrable constant on Y for $\rho^{-1}\operatorname{vol}^{g_Y}$ is 0 (this replaces a zero mean assumption).

We now prove the Sobolev type inequality (45). Since such an inequality is known on compact manifolds, we can assume that u has compact support on $\{\rho\geq\rho_0\}$. We pull-back u to $\hat{Y}$ once more, and split it again into $u_0+u_\perp$. We easily get the inequality on $u_\perp$ (and even a much better one) by using the standard Sobolev embedding on the circle. Now for the component $u_\perp$, we write $\mathbb{R}^3\backslash B=\bigcup_{\ell\geq 0}\mathcal{A}_\ell$, where $\mathcal{A}_\ell$ is the annulus $\{2^\ell\leq\rho\leq 2^{\ell+1}\}$. Denote by $\kappa_\ell:\mathcal{A}_1\to\mathcal{A}_\ell$ the homothety of factor 2^ℓ, and write:

$$\begin{aligned}\int_{\mathbb{R}^3\backslash B}|u_0|^4\operatorname{vol}^{\mathbb{R}^3} &= \sum_{\ell=0}^\infty\int_{\mathcal{A}_\ell}|u_0|^4\operatorname{vol}^{\mathbb{R}^3}\sim\sum_{\ell=0}^\infty(2^\ell)^2\int_{\mathcal{A}_1}|\kappa_\ell^*u_0|^4\operatorname{vol}^{\mathbb{R}^3}\\ &\leq\sum_{\ell=0}^\infty(2^\ell)^2c\Big[\Big(\int_{\mathcal{A}_1}|d(\kappa_\ell^*u_0)|^2\operatorname{vol}^{\mathbb{R}^3}\Big)^2+\Big(\int_{\mathcal{A}_1}|\kappa_\ell^*u_0|^2\operatorname{vol}^{\mathbb{R}^3}\Big)^2\Big]\\ &\sim c\sum_{\ell=0}^\infty\Big[\Big(\int_{\mathcal{A}_\ell}|du_0|^2\operatorname{vol}^{\mathbb{R}^3}\Big)^2+\Big(\int_{\mathcal{A}_\ell}|u_0|^2\rho^{-2}\operatorname{vol}^{\mathbb{R}^3}\Big)^2\Big]\\ &= c\Big[\Big(\int_{\mathbb{R}^3\backslash B}|u_0|^2\operatorname{vol}^{\mathbb{R}^3}\Big)^2+\Big(\int_{\mathbb{R}^3\backslash B}|du_0|^2\rho^{-2}\operatorname{vol}^{\mathbb{R}^3}\Big)^2\Big].\end{aligned}$$

The norms here are taken for $g_{\mathbb{R}^3}$. We used the Sobolev embedding $L^{1,2}(\mathcal{A}_1)\to L^4(\mathcal{A}_1)$ between the first and the second lines, and denoted its norm (to the power 4) by c. $\square$

3.2.2 Unweighted second order and third order estimates.

The technique we use here is really the same as that used by Joyce in the ALE case, which is essentially the same as in the compact case. It is based on the following observation.

Proposition 3.8 *Let $f \in C^0_{\beta+2} \cap C^2$, $\beta \in (0,1)$, and $\varphi \in C^3_\gamma \cap C^4_{\mathrm{loc}}$, $0 < \gamma \leq \beta$, such that $\big(\omega_Y + i\partial\overline{\partial}\varphi\big)^2 = e^f \omega_Y^2$. Denote by F the function $\log(4 - \Delta\varphi) - \kappa\varphi$ on Y where κ is any constant and Δ the Laplacian of g_Y, and by Δ' the Laplacian operator with respect to ω_φ. Then there exists a constant C depending only on $\|\operatorname{Rm}^{\omega_Y}\|_{C^0}$ such that*

$$\Delta' F \leq (4-\Delta\varphi)^{-1}\|\Delta f\|_{C^0} + \kappa(2 - \operatorname{tr}^{\omega_\varphi}\omega) + C\operatorname{tr}^{\omega_\varphi}\omega_Y.$$

We do not prove this proposition, because (up to some minor changes, like 4, which is the real dimension of Y, instead of its complex dimension because we use $i\partial\overline{\partial}$ instead of dd^c) it all comes from a local formula, which is proved in [Yau], [Aub] or [Joy]. Nevertheless, because the computation of this formula can be considered as a tour-de-force, we quote it now: in the conditions of the proposition, if g' is the metric $\omega_\varphi(\cdot, J_Y\cdot)$, in local holomorphic coordinates and with Einstein's summation convention, one has

$$\begin{aligned}\Delta'(\Delta\varphi) = &-2\Delta f + 4g^{\alpha\bar\lambda}g'^{\mu\bar\beta}g'^{\gamma\bar\nu}\nabla_{\alpha\bar\beta\gamma}\varphi\nabla_{\bar\lambda\mu\bar\nu}\varphi \\ &+ 4g'^{\alpha\bar\beta}g^{\gamma\bar\delta}\big((\operatorname{Rm}^{\omega_Y})^{\bar\varepsilon}{}_{\bar\delta\gamma\bar\beta}\nabla_{\alpha\bar\varepsilon}\varphi - (\operatorname{Rm}^{\omega_Y})^{\bar\varepsilon}{}_{\bar\beta\alpha\bar\delta}\nabla_{\gamma\bar\varepsilon}\varphi\big).\end{aligned} \tag{46}$$

Here and in what follows, ∇ is the Levi-Civita connection of g_Y. We bring also the precision that $4 - \Delta\varphi$ is always positive: take the trace with respect to ω_Y of ω_φ, which is automatically positive as we saw when proving the uniqueness of the solution of (E_t)), and even $\geq 4e^{f/2}$, so that its inverse is a priori bounded above.

Furthermore, it is not hard to deduce from Proposition 3.8 a second order estimate on φ:

Corollary 3.9 *Under the assumptions of Proposition 3.8, there exists a constant $Q \geq 0$ depending only on β, γ, $\|f\|_{C^0_{\beta+2}}$, $\|\Delta f\|_{C^0}$, g, m and $\|\operatorname{Rm}^{\omega_Y}\|_{C^0}$ such that $\Delta\varphi \geq -Q$.*

Proof. Fix $\kappa = C + 1$ in Proposition 3.8. Two situations can occur: the function F of this proposition achieves its supremum or not. If not, since decay conditions imply that it tends to $2\log 2$ at infinity, we get that $4 - \Delta\varphi \leq 4e^{\kappa\varphi}$, and conclude with Proposition 3.7.

If now F achieves its supremum, at a point p say, we have that $\Delta' F(p) \geq 0$, and a little computation using this inequality shows that at p, $\operatorname{tr}^{\omega'}\omega_Y \leq C' := 4\kappa + \frac{1}{4}e^{\|f\|_{C^0}/2}\|\Delta f\|_{C^0}$. From this we get $F(p) \leq \|f\|_{C^0} + \log(2C') + \kappa\|\varphi\|_{C_0}$, and since $F \leq F(p)$ on Y, the conclusion follows, noticing that C' and $\|\varphi\|_{C_0}$ only depend on the parameters announced in Propositions 3.7 and 3.8. Details can be found in [Joy]. $\square$

Using now the positivity condition on ω_φ and the upper and lower estimates on $\Delta\varphi$, one gets :

Proposition 3.10 *Let $f \in C^0_{\beta+2} \cap C^2$, $\beta \in (0,1)$, and $\varphi \in C^3_\gamma \cap C^4_{\mathrm{loc}}$, $0 < \gamma \leq \beta$, such that $\left(\omega_Y + i\partial\overline{\partial}\varphi\right)^2 = e^f \omega_Y^2$. Then there exist some constants Q_1 and Q_2 depending only on β, γ, $\|f\|_{C^0_{\beta+2}}$, $\|\Delta f\|_{C^0}$, g_Y, and $\|\operatorname{Rm}^{\omega_Y}\|_{C^0}$ such that*

$$\left\|i\partial\overline{\partial}\varphi\right\|_{C^0} \leq Q_1 \quad \textit{and} \quad Q_2^{-1}\omega_Y \leq \omega_\varphi \leq Q_2\omega_Y.$$

Finally, we give a third order estimate :

Proposition 3.11 *Let $f \in C^0_{\beta+2} \cap C^3$, $\beta \in (0,1)$, and $\varphi \in C^3_\gamma \cap C^5_{\mathrm{loc}}$, $0 < \gamma \leq \beta$, such that $\left(\omega_Y + i\partial\overline{\partial}\varphi\right)^2 = e^f \omega_Y^2$. Then there exists a constant Q_3 depending only on β, γ, $\|f\|_{C^0_{\beta+2}}$, $\|f\|_{C^3}$, g, and $\|\operatorname{Rm}^{\omega_Y}\|_{C^1}$ such that $\left\|\nabla i\partial\overline{\partial}\varphi\right\|_{C^0} \leq Q_3$.*

Proof. Here again we do not give the complete proof because it is very similar to the one in [Joy]. We only say a few words about the main ingredients. Set S so that $4S^2 = \left\|\nabla i\partial\overline{\partial}\varphi\right\|^2_{C^0}$, with ∇ the Levi-Civita connection of g_Y. Formula (46) tells us that $\Delta'(\Delta\varphi) \geq cS^2 - C$ for some constants $c > 0$ and C depending only on the parameters announced. Moreover, a hard but local computation shows that $\Delta'(S^2)$ is equal to a nonpositive quantity plus a linear term (with coefficients that are polynomials in e^f, ω_φ, $\nabla^2 f$, $\operatorname{Rm}^{\omega_Y}$) in $\nabla i\partial\overline{\partial}\varphi$ and a quadratic term (with coefficients that are polynomials in e^f, ω_φ, $\nabla^3 f$, ∇R) in $\nabla i\partial\overline{\partial}\varphi$. This we can sum up by saying there exists a constant C' depending only on the parameters announced such that $\Delta'(S^2) \leq C'(S^2+S)$. Now those considerations give $\Delta'(S^2 - 2c^{-1}CC'\Delta\varphi) \leq -C\left(S - \frac{1}{2}\right)^2 + 2c^{-1}CC' + \frac{1}{4}C$; one concludes according to $(S^2 - 2c^{-1}CC'\Delta\varphi)$ achieves its supremum at some point (and using its Δ' is ≥ 0 at this point) or not (and using decay conditions). $\square$

3.2.3 C^0_δ estimates.

We now prove an estimate on a weighted C^0-norm of a solution of a complex Monge-Ampère equation on Y.

For this, we take a point of view close to that of paragraph 3.2.1 but we estimate $\|\varphi\rho^\delta\|_{C^0}$ (i.e. $\|\varphi\|_{C^0_\delta}$), $\delta \in (0,\gamma)$, instead of $\|\varphi\|_{C^0}$ with the same iteration. Of course, putting a weight $\rho^{p\delta}$ makes the integrations by parts a bit more complicated, but the whole spirit of the proof remains the same. Here again we write the computations in detail since they differ at some points from Joyce's.

To begin with, we state the weighted integration by parts formula we are using in our iteration scheme:

Lemma 3.12 *Let $f \in C^0_{\beta+2}$, $\beta \in (0,1)$, and $\varphi \in C^3_\gamma$, $0 < \gamma \leq \beta$, such that $\big(\omega_Y + i\partial\overline{\partial}\varphi\big)^2 = e^f\omega_Y^2$. Set $T = \omega_Y + \omega_\varphi$. Then for any $p > 2$, q such that $p\gamma - q > 1$,*

$$\int_Y \big|\partial\big(|\varphi|^{p/2}\rho^{q/2}\big)\big|^2\omega_Y^2 \leq \frac{p^2}{2(p-1)}\int_Y \rho^q|\varphi|^{p-2}\varphi(e^f-1)\omega_Y^m$$
$$+\frac{q}{2(p-1)}\int_Y |\varphi|^p\rho^{q-2}\big[(p+q-2)i\partial\rho\wedge\overline{\partial}\rho - (p-2)\rho i\partial\overline{\partial}\rho\big]\wedge T,$$

where those integrals are finite.

Proof. As in [Joy], we say that $\int_Y d\big[p^2\varphi|\varphi|^{p-2}\rho^q d^c\varphi\wedge T + (p-2)q|\varphi|^p\rho^{q-1}\wedge T\big] = 0$, because of the same Stokes' argument as in the proof of Lemma 3.2. Now, a direct computation yields

$$\begin{aligned}
&d\big[p^2\varphi|\varphi|^{p-2}\rho^q d^c\varphi\wedge T + (p-2)q|\varphi|^p\rho^{q-1}\wedge T\big]\\
&\quad= p^2(p-1)|\varphi|^{p-2}\rho^q d\varphi\wedge d^c\varphi\wedge T + p^2|\varphi|^{p-2}\rho^q dd^c\varphi\wedge T\\
&\qquad + 2pq(p-1)\varphi|\varphi|^{p-2}\rho^{q-1}d^c\varphi\wedge T + q(q-1)(p-2)|\varphi|^p\rho^{q-2}d\rho\wedge d^c\rho\wedge T\\
&\qquad + (p-2)q|\varphi|^p\rho^{q-1}dd^c\rho\wedge T
\end{aligned}$$

(notice that $d\varphi\wedge d^c\rho\wedge T = d\rho\wedge d^c\varphi\wedge T$ since T is of bidegree $(1,1)$). On the other hand,

$$\begin{aligned}
d\big(|\varphi|^{p/2}\rho^{q/2}\big)\wedge d^c\big(|\varphi|^{p/2}\rho^{q/2}\big)\wedge T =&\frac{p^2}{4}|\varphi|^{p-2}\rho^q d\varphi\wedge d^c\varphi + \frac{q^2}{4}|\varphi|^p\rho^{q-2}d\rho\wedge d^c\rho\\
&+\frac{pq}{2}|\varphi|^{p-2}\varphi\rho^{q-1}d\rho\wedge d^c\varphi\wedge T.
\end{aligned}$$

Comparing those identities one can write

$$\begin{aligned}
&d\big[p^2\varphi|\varphi|^{p-2}\rho^q d^c\varphi\wedge T + (p-2)q|\varphi|^p\rho^{q-1}\wedge T\big]\\
&\qquad= 4(p-1)d\big(|\varphi|^{p/2}\rho^{q/2}\big)\wedge d^c\big(|\varphi|^{p/2}\rho^{q/2}\big)\wedge T + p^2\varphi|\varphi|^{p-2}\rho^q dd^c\varphi\wedge T\\
&\qquad\quad + q|\varphi|^p\rho^{q-2}\big[(p-2)\rho dd^c\rho\wedge T - (p+q-2)d\rho\wedge d^c\rho\wedge T\big].
\end{aligned}$$

Then conclude after dividing this by 2, integrating both sides over Y and noticing that $i\partial\overline{\partial}\varphi\wedge T = (e^f-1)\omega^2$ and $T \geq \omega_Y$. Checking that all resulting integrals are finite is straightforward with the assumption $p\gamma - q > 1$. ☐

Now fix $\delta \in (0,\gamma)$; we take q in the latter lemma as $p\delta$, so the condition $p\gamma - q > 1$ becomes $p > \frac{1}{\gamma-\delta}$. The following lemma initiates our sequence of recursive controls:

Lemma 3.13 *Fix $p_0' > 2$, $p_0' > \frac{1}{\gamma-\delta}$, $p_0' \geq p_0 = \frac{2}{\gamma}$. Under the assumptions of Proposition 3.10, there exists a constant C_0 depending only on p_0', β, γ, δ, $\|f\|_{C^0_{\beta+2}}$, $\|f\|_{C^2}$, g, and $\|\operatorname{Rm}^{\omega_Y}\|_{C^0}$ such that $\|\varphi\rho^\delta\|_{L^{p_0'}_{d\lambda}} \leq C_0$.*

We recall that $d\lambda$ is the measure $\rho^{-1}\operatorname{vol}^{g_Y}$.

Proof. From Lemma 3.12 and Proposition 3.10, and also the facts that $d\rho = O(1)$ and $dd^c\rho = O(\rho^{-1})$, we get two constants c_1, c_2 depending only on the parameters announced such that for every $p > \frac{1}{\gamma-\delta}$,

(47)
$$\int_Y \left|\partial\left(|\varphi|^{p/2}\rho^{q/2}\right)\right|^2 \operatorname{vol}^{g_Y} \leq c_1 p\int_Y |\varphi|^{p-1}\rho^{p\delta-(\beta+2)}\operatorname{vol}^{g_Y} + c_2 p\int_Y |\varphi|^p\rho^{p\delta-2}\operatorname{vol}^{g_Y}.$$

We now play the same game of Hölder inequalities as in paragraph 3.2.1 for the first summand of the right-hand side. Indeed,

$$\int_Y |\varphi|^{p-1}\rho^{p\delta-(\beta+2)}\operatorname{vol}^{g_Y} \leq \Big(\int_Y \rho^{bt}\operatorname{vol}^{g_Y}\Big)^{1/t}\Big(\int_Y (|\varphi|\rho^\delta)^{2p}\rho^{-1}\operatorname{vol}^{g_Y}\Big)^{(p-1)/2p}.$$

with $t = \left(1-\frac{p-1}{2p}\right)^{-1} = \frac{2p}{p+1}$ and $b = \frac{p-1}{2p} + \delta - (\beta+2)$. Notice that the condition $bt < -3$, which ensures the convergence of $\int_Y \rho^{bt}\operatorname{vol}^{g_Y}$, is equivalent to $p > \frac{1}{\beta-\delta}$, and is thus automatically verified for the p we work with. Furthermore, when p goes to ∞, bt tends to $-3+2(\delta-\beta)$, which is < -3; the quantity $\big(\int_Y \rho^{bt}\operatorname{vol}^{g_Y}\big)^{1/t}$ is thus bounded above by some constant C_1 depending only on our parameters for the considered p, if we choose them away from $p_0 = \frac{2}{\gamma}$; the choice $p \geq p_0'$ is convenient.

On the other hand, we have to argue in a slightly different way for the summand $\int_Y |\varphi|^p\rho^{p\delta-2}\operatorname{vol}^{g_Y}$ of (47). Let us write

$$\int_Y |\varphi|^p\rho^{p\delta-2}\operatorname{vol}^{g_Y} \leq \Big(\int_Y |\varphi|^r\rho^{-1}\operatorname{vol}^{g_Y}\Big)^{1/t}\Big(\int_Y (|\varphi|\rho^\delta)^{2p}\rho^{-1}\operatorname{vol}^{g}\Big)^{1/s},$$

where $\frac{1}{s}+\frac{1}{t} = 1$, $\frac{2p\delta-1}{s} - \frac{1}{t} = p\delta - 2$ and $\frac{2p}{s}+\frac{q}{t} = p$. This gives $s = \frac{2p\delta}{p\delta-1} := s(p)$ (well-defined and > 2 as soon as $p > \frac{1}{\delta}$, and tends to 2 as p goes to ∞), $t = \frac{2p\delta}{p\delta+1}$ (which tends to 2) and $r = \frac{t}{\delta} = \frac{2p}{p\delta+1}$ (which tends to $\frac{2}{\delta}$). Now the condition $r\gamma > 2$ ensures the convergence of $\int_Y |\varphi|^r\rho^{-1}\operatorname{vol}^{g_Y}$, and this condition turns out to be equivalent to $p > \frac{1}{\gamma-\delta}$, which we assume.

Moreover, when p ranges over $[p_0', \infty)$ for $p_0' > \frac{1}{\gamma-\delta}$ (e.g. $p_0' = \frac{2}{\gamma-\delta}$), r ranges over $[r_0, \frac{2}{\delta})$, with $r_0 = \frac{2p_0'}{p_0'\delta+1} > 2$. From Lemma 3.6, $\big(\int_Y |\varphi|^r\rho^{-1}\operatorname{vol}^{g_Y}\big)^{1/t}$ (converges and) is bounded by some constant C_2. This constant C_2 depends only on the parameters announced. Finally, we can sum all this up saying that for $p \geq p_0'$,

$$\int_Y \big|\partial\big(|\varphi|^{p/2}\rho^{q/2}\big)\big|^2 \operatorname{vol}^{g_Y} \leq c_1 C_1 p \|\varphi\rho^\delta\|^{p-1}_{L^{2p}_{d\lambda}} + c_2 C_2 p \|\varphi\rho^\delta\|^{2p/s(p)}_{L^{2p}_{d\lambda}}.$$

Take $p = p_0'$, and apply inequality (44) to the LHS, with $u = |\varphi|^{p/2}\rho^{q/2}$; this yields :

$$\|\varphi\rho^\delta\|^{p_0'}_{L^{2p_0'}_{d\lambda}} \leq C_S c_1 C_1 p_0' \|\varphi\rho^\delta\|^{p_0'-1}_{L^{2p_0'}_{d\lambda}} + C_S c_2 C_2 p_0' \|\varphi\rho^\delta\|^{2p_0'/s(p_0')}_{L^{2p_0'}_{d\lambda}},$$

and one concludes noticing that $p_0' > p_0' - 1$ and $p_0' > 2p_0'/s(p_0')$, as $s(p_0') > 2$. $\square$

We fix $p_0' = \frac{2}{\gamma-\delta}$, so that it verifies all the assumptions of the latter lemma. As in paragraph 3.2.1, the last step makes the transition from $\|\varphi\rho^\delta\|_{L^p_{d\lambda}}$ to $\|\varphi\rho^\delta\|_{L^{2p}_{d\lambda}}$ for big enough p:

Lemma 3.14 *Under the assumptions of Proposition 3.10, there exists constants C_1 and C_2 only depending on β, γ, δ, $\|f\|_{C^0_{\beta+2}}$, $\|f\|_{C^2}$, g, m and $\|R^\omega\|_{C^0}$ such that for all $p \geq p_0' = \frac{2}{\gamma-\delta}$, $\|\varphi\rho^\delta\|^p_{L^{2p}_{d\lambda}} \leq C_1 p \|\varphi\rho^\delta\|^{p-1}_{L^p_{d\lambda}} + C_2 p \|\varphi\rho^\delta\|^p_{L^p_{d\lambda}}$.*

Proof. We saw in the beginning of the proof of Lemma 3.13 that there exist c_1 and c_2 depending only on the parameters announced such that for all $p > \frac{1}{\gamma-\delta}$

$$\int_Y \big|\partial\big(|\varphi|^{p/2}\rho^{q/2}\big)\big|^2 \operatorname{vol}^{g_Y} \leq c_1 p \int_Y |\varphi|^{p-1}\rho^{p\delta-(\beta+2)} \operatorname{vol}^{g_Y} + c_2 p \int_Y |\varphi|^p \rho^{p\delta-2} \operatorname{vol}^{g_Y}.$$

We deal again with the two summands of the right-hand side separately. For the concerned p, $\int_Y |\varphi|^{p-1}\rho^{p\delta-(\beta+2)} \operatorname{vol}^{g_Y} = \int_Y (|\varphi|\rho^\delta)^{p-1}\rho^{\delta-(\beta+2)} \operatorname{vol}^{g_Y}$ and by Hölder inequality this is bounded above by

$$\Big(\int_Y \rho^{p(\delta-\beta)-p-1} \operatorname{vol}^{g_Y}\Big)^{1/p} \Big(\int_Y (|\varphi|\rho^\delta)^p \rho^{-1} \operatorname{vol}^{g_Y}\Big)^{(p-1)/p}.$$

The first integral converges for $p > \frac{2}{1+\beta-\delta}$, which is automatic if $p > \frac{1}{\gamma-\delta}$, and its $\frac{1}{p}$-th power tends to $\sup_Y \rho^{\delta-\beta-1}$ as p goes to infinity, so we get the constant C_1 of the statement (after multiplying by c_1), if we restrict to $p \in [\frac{2}{\gamma-\delta}, \infty)$.

Let us deal next with the summand $\int_Y |\varphi|^p \rho^{p\delta-2} \operatorname{vol}^{g_Y}$; we can take $C_2 = c_2 \sup_Y \rho^{-1}$ without more efforts, and the lemma is proved. $\square$

Under the assumptions of Proposition 3.10 and the condition $\delta \in (0, \gamma)$, it is again an easy exercise, in view of Lemmas 3.13 and 3.14, to get two constants Q_δ and C_3 depending only on the same parameters as those announced in Lemma 3.14 such that for every $p \geq p_0'$, we have $\|\varphi\rho^\delta\|_{L^p_{d\lambda}} \leq Q_\delta (C_3 p)^{-2/p}$. Letting p go to ∞, we can conclude:

Proposition 3.15 *Under the assumptions of Proposition 3.10 and assuming $\delta \in (0,\gamma)$, there exists a constant Q_δ depending only on β, γ, δ, $\|f\|_{C^0_{\beta+2}}$, $\|f\|_{C^2}$, g_Y, and $\|\operatorname{Rm}^{\omega_Y}\|_{C^0}$ such that $\|\varphi\|_{C^0_\delta} \leq Q_\delta$.*

3.2.4 $C^{5,\alpha}$ and $C^{5,\alpha}_\delta$ estimates.

We shall now state (unweighted) higher order a priori estimate:

Proposition 3.16 *Let $f \in C^0_{\beta+2} \cap C^{3,\alpha}$, $\beta \in (0,1)$, $\alpha \in (0,1)$ and $\varphi \in C^3_\gamma \cap C^5_{\mathrm{loc}}$, $0 < \gamma \leq \beta$, such that $\big(\omega_Y + i\partial\bar\partial\varphi\big)^2 = e^f \omega_Y^2$. Then $\varphi \in C^{5,\alpha}$ and there exists a constant Q_α depending only on β, γ, α, $\|f\|_{C^0_{\beta+2}}$, $\|f\|_{C^{3,\alpha}}$ and $\|g_Y\|_{C^{3,\alpha}}$ such that $\|\varphi\|_{C^{5,\alpha}} \leq Q_\alpha$.*

Proof. The proposition follows from an inductive use of the crucial Aubin-Yau formula (46), which we recall here:

$$\begin{aligned}\Delta'(\Delta\varphi) = &-2\Delta f + 4 g_Y^{\alpha\bar\lambda} g'^{\mu\bar\beta} g'^{\gamma\bar\nu} \nabla_{\alpha\bar\beta\gamma}\varphi \nabla_{\bar\lambda\mu\bar\nu}\varphi \\ &+ 4 g'^{\alpha\bar\beta} g^{\gamma\bar\delta}\big((\operatorname{Rm}^{\omega_Y})^{\bar\varepsilon}{}_{\bar\delta\gamma\bar\beta}\nabla_{\alpha\bar\varepsilon}\varphi - (\operatorname{Rm}^{\omega_Y})^{\bar\varepsilon}{}_{\bar\beta\alpha\bar\delta}\nabla_{\gamma\bar\varepsilon}\varphi\big)\end{aligned}$$

(Δ' stands for the scalar Laplacian with respect to ω_φ).

According to the assumptions on f and the conclusions of Propositions 3.10 and 3.11, we have for the moment a C^0 estimate in terms of the parameters announced on the right-hand side.

Now the geometry of Y and the asymptotics of g_Y allow us to take an atlas of Y of holomorphic balls with uniform radius, such that the pull-backs of g_Y to these balls are uniformly bounded above and below, and so are the pull-backs of g'. As a consequence the Laplacian operators associated to these latter Kähler metrics are uniformly elliptic in C^0 sense. We can ask furthermore that the family of those balls with the half radius still gives an atlas.

To say this more precisely, we have a family of holomorphic charts $\pi_i : B(0,1) \to Y$, $i \in \mathcal{I}$, such that $Y = \bigcup_{i\in\mathcal{I}} \pi_i\big(B(0,1)\big) = \bigcup_{i\in\mathcal{I}} \pi_i\big(B(0,\frac{1}{2})\big)$ and such that there exist constants $c > 0$ and C depending on the parameters announced such that for all $i \in \mathcal{I}$, $c\mathbf{e} \leq \pi_i^* g' \leq C\mathbf{e}$. Then the $\pi_i^*\Delta'$ are uniformly elliptic, in the C^0 sense.

Pulling back the formula above by any π_i, the right-hand side is bounded independently of i, and so is $\pi_i^*(\Delta\varphi)$. The standard Morrey-Schauder's for $\pi_i^*\Delta'$ theorem tells us that the $\pi_i^*(\Delta\varphi)$ is bounded in $C^{1,\alpha}$ on $B(0,\frac{3}{4})$, again independently of i; a careful reading shows quickly that this uniform bound depends only on the parameters of the statement. We can reformulate all this saying that $\Delta\varphi$ is in $C^{1,\alpha}$, and the corresponding norm is controlled in terms of the parameters.

On the other hand, $\pi_i^*(\Delta\varphi) = (\pi_i^*\Delta)(\pi_i^*\varphi)$. We can also suppose our covering is taken so that there exist constants $c > 0$ and C depending only on the parameters announced such that for all $i \in \mathcal{I}$, $c\mathbf{e} \leq \pi_i^* g_Y$ and $\|g_Y\|_{C^{1,\alpha}} \leq C$. This allows us to apply again a Schauder estimate, and conclude that we have a uniform bound on the $\pi_i^*\varphi$ in $C^{3,\alpha}\big(B(0,\frac{1}{2})\big)$, that is: $\varphi \in C^{3,\alpha}$ and the corresponding norm is again controlled in terms of the parameters announced.

Now observe that a $C^{3,\alpha}$ bound on φ together with the assumptions on f (at least, a $C^{2,\alpha}$ bound) give a $C^{0,\alpha}$ bound on the right-hand side of (46). Applying again twice the Schauder estimates (once with Δ' which is uniformly elliptic in the $C^{0,\alpha}$ sense from the previous case, and once to Δ which is uniformly elliptic in the $C^{2,\alpha}$ sense) after refining the covering if needed so, one gets the announced $C^{4,\alpha}$ estimate on φ. A final extra application of this technique – using now the $C^{3,\alpha}$ bound on f and the $C^{4,\alpha}$ estimate on φ we just proved – provides a bound $\|\varphi\|_{C^{5,\alpha}} \leq Q$, with Q as announced. ☐

We state a similar result for the *weighted* higher order estimate:

Proposition 3.17 *Let $f \in C^{3,\alpha}_{\beta+2} \cap C$, $\beta \in (0,1)$, $\alpha \in (0,1)$ and $\varphi \in C^3_\gamma \cap C^5_{\mathrm{loc}}$, $0 < \gamma \leq \beta$, such that $\big(\omega_Y + i\partial\overline{\partial}\varphi\big)^2 = e^f\omega_Y^2$. Then $\varphi \in C^{5,\alpha}_\gamma$. Moreover, there exists a constant $Q_{\alpha,\gamma}$ depending on only β, γ, α, $\|\varphi\|_{C^3_\gamma}$ and $\|f\|_{C^{3,\alpha}_\beta}$ such that $\|\varphi\|_{C^{5,\alpha}_\gamma} \leq Q_{\alpha,\gamma}$.*

Proof. We use a technical lemma from [Joy], (Proposition 8.6.12, case $k = 3$), which remains true in our ALF setting; its utility lies in the rescaling process used in establishing elliptic weighted estimates. The lemma states, with our notations:

Lemma 3.18 *Let K_1, $K_2 > 0$, and $\lambda \in [0,1]$. Then there exists K_3 depending only on α, β, γ, $\|\varphi\|_{C^0_\gamma}$ and K_1, K_2, λ such that the following holds:*

Under the assumptions of Proposition 3.17 and if $\|f\|_{C^{3,\alpha}_{\beta+2}} \leq K_1$, $\big\|\nabla^\ell dd^c\varphi\big\|_{C^0_{\lambda\ell+2}} \leq K_2$, $\ell = 0,\ldots,3$ and $\big[\nabla^3 dd^c\varphi\big]^\alpha_{3\lambda+(\lambda-1)\alpha+2} \leq K_2$, then: $\big\|\nabla^\ell dd^c\varphi\big\|_{C^0_{\lambda\ell+\gamma+2}} \leq K_3$, $\ell = 0,\ldots,3$ and $\big[\nabla^3 dd^c\varphi\big]^\alpha_{\gamma+3\lambda+(\lambda-1)\alpha+2} \leq K_3$.

The Hölder moduli $[\cdot]^\alpha_\cdot$ are analogous to those defined by (20), with Y, g_Y and ρ instead of X, g_m and R.

Proof of Lemma 3.18. The major change here is that the injectivity radius does not grow as fast as ρ, but instead remains bounded, essentially by half the length of the fibres. But this is not an issue. Indeed, the Riemannian exponential map is still well defined, and authorises the following manipulations. Given $x \in \{\rho \geq 2\rho_0\}$ (ρ_0 determined later), identify (T_xY, J_Y, g_Y) with and $(\mathbb{C}^2, I_1, \mathbf{e})$. Take $R > 0$, and denote by $\pi_{R,x}$ the map $B_{\mathbf{e}}(0,1) \to B_{g_Y}(x,R)$, $y \mapsto \exp_x^{g_Y}(Ry)$; it is not a diffeomorphism in general, since large balls wrap following asymptotically the fibers of ϖ. We can nonetheless define the operator $P_{x,R} : C^{k+2,\alpha}\big(B_{\mathbf{e}}(0,1)\big) \to C^{k,\alpha}\big(B_{\mathbf{e}}(0,1)\big)$ by

$$P_{x,R}(v) = R^2 \frac{\big({\pi_{R,x}}^*(i\partial\overline{\partial})\big)(v) \wedge {\pi_{R,x}}^*(\omega_Y + \omega_\varphi)}{{\pi_{R,x}}^*(\omega_Y^2)}.$$

Then one can take $R = L\rho(x)^\lambda$ with $L = L(\rho_0, \lambda, g_Y)$ small enough so that $B_{g_Y}(x,R) \subset \{\rho \geq \rho_0\}$; this way one has

$$\big\|R^{-2}{\pi_{x,R}}^* g_Y - \mathbf{e}\big\|_{C^{3,\alpha}} \leq \frac{1}{2} \quad \text{for all } x \in \{\rho \geq 2\rho_0\}, \quad \text{and}$$
$$\big\|R^{-2}{\pi_{x,R}}^* \omega_Y - \omega_{\mathbf{e}}\big\|_{C^{3,\alpha}} \leq \frac{1}{2} \quad \text{for all } x \in \{\rho \geq 2\rho_0\},$$

if ρ_0 is chosen big enough, thanks to the asymptotic geometry of g_Y. Now the rest of Joyce's proof applies unchanged (in particular, one is brought to using Schauder estimates between the *fixed* balls $B_{\mathbf{e}}(0,2)$ and $B_{\mathbf{e}}(0,1)$, with a $C^{3,\alpha}$ uniformly elliptic family of operators), since the identity

$$P_{x,R}\big((\pi_{x,R})^*\varphi\big) = R^2\big(e^{(\pi_{x,R})^* f} - 1\big)$$

is again just a rewriting of the pulled-back Monge-Ampère equation verified by φ.

To complete the present proof though, one has to deal with the compact subset $\{\rho \leq 2\rho_0\}$; in that case, the estimates of the statement are an immediate consequence of the unweighted estimates of Proposition 3.16. ■

Now Proposition 3.17 follows from a repeated use of Lemma 3.18, as in [Joy, p. 195]. At the end of the argument, we pass to $\lambda = 1$ in this lemma, which precisely gives estimates on the $\|\nabla^\ell dd^c\varphi\|_{C^0_{\ell+\gamma}}$, $\ell = 0, \dots, 3$, and on $\big[\nabla^3 dd^c\varphi\big]^\alpha_{\gamma+5}$, i.e. on $\|dd^c\varphi\|_{C^{3,\alpha}_{\gamma+2}}$, hence on $\|\Delta\varphi\|_{C^{3,\alpha}_{\gamma+2}}$. Conclusion follows from the assumption $\varphi \in C^0_\gamma$ and the isomorphism $\Delta : C^{5,\alpha}_\gamma \to C^{3,\alpha}_{\gamma+2}$ (see [BM, App. A]). □

3.2.5 Closedness of S

We conclude this section by the proof of the closedness of S. Take a sequence (t_j) of elements of S, converging to some $t_\infty \in [0,1]$; in particular, each $\varphi_{t_j} \in C^{5,\alpha}_{\beta/2}$. Pick $\alpha \in (0,1)$. From Proposition 3.17, since the $t_j f$ now play the role of the f of that Proposition, we have a bound on $\|\varphi_{t_j}\|_{C^{5,\alpha}_{\beta/2}}$ which depends only on $\|f\|_{C^{3,\alpha}_\beta}$ and the $\|\varphi_{t_j}\|_{C^0_{\beta/2}}$. But from Proposition 3.15, we also have a bound on the $\|\varphi_{t_j}\|_{C^0_{\beta/2}}$, which depends only on $\|f\|_{C^{3,\alpha}_{\beta+2}}$. In other words, (φ_{t_j}) is bounded in $C^{5,\alpha}_{\beta/2}$, and thus converges weakly to some $\varphi_\infty \in C^{5,\alpha}_{\beta/2}$. Now the inclusion $C^{5,\alpha}_{\beta/2} \hookrightarrow C^{5,\alpha/2}_{\beta/4}$ is compact; we can thus assume that (φ_{t_j}) converges strongly to φ_∞ in $C^{5,\alpha/2}_{\beta/4}$. In particular the (E_{t_j}) pass to the limit to give:

$$\big(\omega_Y + i\partial\overline{\partial}\varphi_\infty\big)^2 = e^{t_\infty f}\omega_Y^2.$$

Developing this latter equation yields: $\omega_Y^2 + 2\omega_Y \wedge i\partial\overline{\partial}\varphi_\infty + (i\partial\overline{\partial}\varphi_\infty)^2 = e^{f t_\infty}\omega_Y^2$, that is:

$$-\frac{1}{2}\Delta\varphi_\infty = \big(e^{t_\infty f} - 1\big) + \frac{(i\partial\overline{\partial}\varphi_\infty)^2}{\omega_Y^2} \in C^{3,\alpha/2}_{\beta+2}.$$

On the other hand, $\Delta : C^{5,\alpha/2}_\gamma \to C^{3,\alpha/2}_{\gamma+2}$ is an isomorphism for any $\gamma \in (0,1)$; this follows from [BM, App. A]. In view of the latter equation, we hence get $\varphi_\infty \in C^{5,\alpha/2}_\beta$. Apply Proposition 3.17 to conclude that $\varphi_\infty \in C^{5,\alpha}_\beta$, and thus that $t_\infty \in S$. $\square$

3.3 Openness of S

The only point we are left with is the openness of S; we settle it now. Let $t \in S$; write ω_t for $\omega_Y + i\partial\overline{\partial}\varphi_t$. We already know that ω_t is Kähler. Moreover, the linearisation of the Monge-Ampère operator

$$\begin{aligned} C^{5,\alpha}_\beta &\longrightarrow C^{3,\alpha}_{\beta+2} \\ \psi &\longmapsto \frac{\big(\omega_Y + i\partial\overline{\partial}\psi\big)^2}{(\omega_Y)^2}, \end{aligned}$$

with $\alpha \in (0,1)$, at point φ_t, is $-\frac{e^{tf}}{2}\Delta_t$, where Δ_t is the Laplacian of ω_t. Now ω_t is $C^\infty_{\beta+2}$ close to ω_Y; it follows that Δ_t, and hence $-\frac{e^{tf}}{2}\Delta_t$, are isomorphisms $C^{5,\alpha}_\beta \to C^{3,\alpha}_{\beta+2}$. Consequently, (E_s) has a (necessarily unique) solution $\varphi_s \in C^{5,\alpha}_\beta$ for all s close to t. At last, we apply Proposition 3.17 to get that those φ_s are in C^∞_β, which ends the proof of the openness of S, and that of Theorem 0.4. $\square$

A The Taub-NUT metric on $\mathbb{C}^2$

A.1 A potential for the Taub-NUT metric on $\mathbb{C}^2$

In [LeB] C. LeBrun leaves the following exercise to his reader: let m be a positive parameter, and u and v implicitly defined on $\mathbb{C}^2$ by formulas:

$$\begin{aligned} |z_1| &= e^{m(u^2-v^2)}u, \\ |z_2| &= e^{m(v^2-u^2)}v \end{aligned} \tag{48}$$

(we do not make the dependence on m appearent here, since for now we see this parameter as fixed; we shall only add m as an index by places to emphasise this dependence). One has:

Proposition A.1 (LeBrun) *The metric* $\mathbf{f}$ *associated to the form*

$$\omega_{\mathbf{f}} := \frac{1}{4} dd^c\big(u^2 + v^2 + m(u^4 + v^4)\big)$$

for the standard complex structure I_1 *on* $\mathbb{C}^2$ *is the Taub-NUT metric.*

We shall give our own, direct proof here. Before this, we shall mention that LeBrun's potential may be obtained by hyperkähler quotient considerations; we chose to give a less conceptual proof though since it exhibits several objects we use back in this article.

Lemma A.2 *The metric* $\mathbf{f}$ *is Ricci-flat; more precisely,* $\omega_{\mathbf{f}}^2 = 2\Omega_{\mathbf{e}}$*, where we recall that* $\Omega_{\mathbf{e}}$ *is the standard volume form* $\frac{idz_1\wedge d\overline{z_1}\wedge idz_2\wedge d\overline{z_2}}{4}$.

Proof. We start by the computation of $\omega_{\mathbf{f}}$, which goes through that of $\frac{\partial u}{\partial z_j}$, $\frac{\partial v}{\partial z_j}$, $j = 1, 2$. One has:

$$\begin{aligned} \frac{\partial u}{\partial z_1} &= \frac{1+2mv^2}{(2z_1)\big(1+2m(u^2+v^2)\big)}u, & \frac{\partial u}{\partial z_2} &= \frac{muv^2}{z_2\big(1+2m(u^2+v^2)\big)}, \\ \frac{\partial v}{\partial z_1} &= \frac{mu^2v}{z_1\big(1+2m(u^2+v^2)\big)}, & \frac{\partial v}{\partial z_2} &= \frac{1+2mu^2}{(2z_2)\big(1+2m(u^2+v^2)\big)}v. \end{aligned} \tag{49}$$

Indeed, differentiating the relation $|z_1| = e^{m(u^2-v^2)}u$ with $\frac{\partial}{\partial z_1}$ yields:

$$\frac{1}{2}\frac{|z_1|}{z_1} = \Big[m\Big(2u^2\frac{\partial u}{\partial z_1} - 2uv\frac{\partial v}{\partial z_1}\Big) + \frac{\partial u}{\partial z_1}\Big]e^{m(u^2-v^2)},$$

hence, writing $e^{m(u^2-v^2)} = \frac{|z_1|}{u}$, $u = 2z_1\big[(1+2mu^2)\frac{\partial u}{\partial z_1} - 2muv\frac{\partial v}{\partial z_1}\big]$. Similarly, applying $\frac{\partial}{\partial z_1}$ to the relation $|z_2| = e^{m(v^2-u^2)}v$, one gets $0 = (1+2mv^2)\frac{\partial v}{\partial z_1} - 2muv\frac{\partial u}{\partial z_1}$, that is $\frac{\partial v}{\partial z_1} = \frac{2muv}{1+2mv^2}\frac{\partial u}{\partial z_1}$. Substituting in the previous equality, one gets formulas (49) for $\frac{\partial u}{\partial z_1}, \dots, \frac{\partial v}{\partial z_2}$.

Set now $\varphi = \frac{1}{4}\big(u^2 + v^2 + m(u^4 + v^4)\big)$. According to formulas (49),

$$2\frac{\partial \varphi}{\partial z_1} = u(1+2mu^2)\frac{\partial u}{\partial z_1} + v(1+2mv^2)\frac{\partial v}{\partial z_1} = \frac{(1+2mv^2)u^2}{2z_1}$$

and $2\frac{\partial\varphi}{\partial z_2}=\frac{(1+2mu^2)v^2}{2z_2}$, i.e. $\frac{\partial\varphi}{\partial\overline{z_1}}=\frac{(1+2mv^2)u^2}{4\overline{z_1}}$ and $\frac{\partial\varphi}{\partial\overline{z_2}}=\frac{(1+2mu^2)v^2}{4\overline{z_2}}$ by conjugation. Apply again $\frac{\partial}{\partial z_1}$ and $\frac{\partial}{\partial z_2}$ to those equalities, as well as the relation $uv=|z_1z_2|$ and formulas (49); then, setting $R=\frac{1}{2}(u^2+v^2)$:

$$\begin{aligned}\omega_{\mathbf{f}}=&dd^c\varphi\\=&\Big(\frac{u^2(1+2mv^2)}{2|z_1|^2(1+4mR)}+m|z_2|^2\Big)idz_1\wedge d\overline{z_1}+m\overline{z_2}z_1\Big(1+\frac{1}{1+4mR}\Big)idz_2\wedge d\overline{z_1}\\&+m\overline{z_1}z_2\Big(1+\frac{1}{1+4mR}\Big)idz_1\wedge d\overline{z_2}+\Big(\frac{v^2(1+2mu^2)}{2|z_2|^2(1+4mR)}+m|z_1|^2\Big)idz_2\wedge d\overline{z_2}.\end{aligned}$$

A direct computation of $\omega_{\mathbf{f}}^2$, using again $uv=|z_1z_2|$, brings the conclusion. □

Remark A.3 *With the above definition of R and formulas* (48)*, one gets:*

$$2R\le r^2\le 2Re^{4mR}, \tag{50}$$

(with equality along $\{|z_1|=|z_2|\}$ and $\{z_1z_2=0\}$, respectively); this implies that R is proper on $\mathbb{C}^2$.

Recall $\mathbb{S}^1$ acts on $\mathbb{C}^2$ by $\alpha\cdot(z_1,z_2)=(e^{i\alpha}z_1,e^{-i\alpha}z_2)$; the associated infinitesimal action is generated by the vector field $\xi=i\big(z_1\frac{\partial}{\partial z_1}+\overline{z_2}\frac{\partial}{\partial\overline{z_2}}-z_2\frac{\partial}{\partial z_2}-\overline{z_1}\frac{\partial}{\partial\overline{z_1}}\big)$. By invariance of u and v under this circle action, clearly, $\xi\cdot u=\xi\cdot v=\xi\cdot\varphi=0$, and similarly $\mathcal{L}_\xi\omega_{\mathbf{f}}=0$. This holds as well fo the holomorphic symplectic $(2,0)$-form $\Theta:=dz_1\wedge dz_2$, – notice that $\Theta\wedge\overline{\Theta}=4\Omega_{\mathbf{e}}=2\omega_{\mathbf{f}}^2$ –, thus: $\mathcal{L}_\xi\Theta=0$. More precisely, $\iota_\xi\Theta=(z_1dz_2+z_2dz_1)=d(iz_1z_2)$; a complex hamiltonian $H=y_2+iy_3$ for the S^1-action on $(\mathbb{C}^2,\Theta)$ is thus given by: $y_2:=\mathfrak{Im}(z_1z_2)$ and $y_3:=-\mathfrak{Re}(z_1z_2)$.

In the same way, $\mathcal{L}_\xi d^c\varphi=0$; as $\mathcal{L}_\xi d^c\varphi=\iota_\xi dd^c\varphi+d(\iota_\xi d^c\varphi)$ (Cartan's formula), i.e. $\iota_\xi\omega_{\mathbf{f}}=-d\big(d^c\varphi(\xi)\big)$, we are led to setting $y_1=d^c\varphi(\xi)$. All computations done:

Lemma A.4 *One has $y_1=\frac{1}{2}(u^2-v^2)$, and thus R indeed equals $(y_1^2+y_2^2+y_3^2)^{1/2}$.*

Proof. To see that, $y_1=\frac{1}{2}(u^2-v^2)$, write, according to the proof of Lemma (A.2),

$$d^c\varphi=i(1+2mv^2)u^2\Big(\frac{d\overline{z_1}}{2\overline{z_1}}-\frac{dz_1}{2z_1}\Big)+i(1+2mu^2)v^2\Big(\frac{d\overline{z_2}}{2\overline{z_2}}-\frac{dz_2}{2z_2}\Big),$$

hence the result, from the identity $\xi=i\big(z_1\frac{\partial}{\partial z_1}+\overline{z_2}\frac{\partial}{\partial\overline{z_2}}-z_2\frac{\partial}{\partial z_2}-\overline{z_1}\frac{\partial}{\partial\overline{z_1}}\big)$. Noticing that $y_2^2+y_3^2=|z_1z_2|^2=u^2v^2$ suffices to get $y_1^2+y_2^2+y_3^2=\frac{1}{4}(u^2+v^2)^2$. □

Lemma A.5 *Set $V=|\xi|_{\mathbf{f}}^{-2}$. Then $|\xi|_{\mathbf{f}}^2=\frac{2R}{1+4mR}$, and hence $V=2m\big(1+\frac{1}{4mR}\big)$.*

Proof. One has $I_1\xi = -z_1\frac{\partial}{\partial z_1} - \overline{z_1}\frac{\partial}{\partial \overline{z_1}} + \overline{z_2}\frac{\partial}{\partial \overline{z_2}} + z_2\frac{\partial}{\partial z_2}$; the easy but tedious calculation of $|\xi|^2_{\mathbf{f}} = \omega_{\mathbf{f}}(\xi, I_1\xi)$ then follows, which can be made easier by noticing that $idz_1 \wedge d\overline{z_1}(\xi, I_1\xi) = 2|z_1|^2$, $idz_1 \wedge d\overline{z_2}(\xi, I_1\xi) = -2z_1\overline{z_2}$, and so on. □

To get the Taub-NUT metric back under its classical form, we need finally a 1-form η, which is also a connection 1-form for the circle fibration $\varpi : \mathbb{C}^2\backslash\{0\} \mapsto \mathbb{R}^3\backslash\{0\}$, $(z_1, z_2) \mapsto (y_1, y_2, y_3)$. The natural candidate is given by: $\eta := VI_1dy_1$.

Lemma A.6 *On $\mathbb{C}^2\backslash\{z_1z_2 = 0\}$, one has*

$$\eta = \frac{i}{4R}\Big[u^2\Big(\frac{d\overline{z_1}}{\overline{z_1}} - \frac{dz_1}{z_1}\Big) - v^2\Big(\frac{d\overline{z_2}}{\overline{z_2}} - \frac{dz_2}{z_2}\Big)\Big], \tag{51}$$

and $\eta(\xi) = 1$ outside of 0.

Proof. By definition, $\eta = Vd^cy_1 = \frac{1}{2}iV\big(2u(\overline{\partial}u - \partial u) - 2v(\overline{\partial}v - \partial v)\big)$. We then apply formula (49), which we rewrite as:

$$V\frac{\partial u}{\partial z_1} = \frac{1 + 2mv^2}{4z_1R}u, \quad V\frac{\partial u}{\partial z_2} = \frac{muv^2}{2z_2R}, \quad V\frac{\partial v}{\partial z_1} = \frac{mu^2v}{2z_1R}, \quad V\frac{\partial v}{\partial z_2} = \frac{1 + 2mu^2}{4z_2R}v,$$

hence the component of η in the dz_1 direction is $-iu\frac{1+2mv^2}{4z_1R}u + iv\frac{mu^2v}{2z_1R} = -\frac{iu^2}{4z_1R}$, and so on. A straightforward computation suffices to see that $\eta(\xi) = 1$. □

We shall now recover the Taub-NUT metric under a more familiar shape:

Lemma A.7 *On $\mathbb{C}^2\backslash\{0\}$, $\omega_{\mathbf{f}} = dy_1 \wedge \eta + Vdy_2 \wedge dy_3$, hence $\mathbf{f} = V(dy_1^2 + dy_2^2 + dy_3^2) + V^{-1}\eta^2$.*

Proof. Clearly, $\{dy_2, dy_3\}$ is linearly independent at all points of $\mathbb{C}^2\backslash\{0\}$, and those forms vanish against ξ by $\mathbb{S}^1$-invariance. They vanish as well against $I_1\xi$, as $I_1dy_2 = dy_3$. Since $dy_1(\xi) = 0$ (y_1 is S^1-invariant) and $dy_1(I_1\xi = -V^{-1} \neq 0$ as $I_1dy_1 = V^{-1}\eta$, and since $\eta(I_1\xi) = 0$ and $\eta(\xi) = 1$, we deduce that $\{dy_1, dy_2, dy_3, \eta\}$ is linearly independent outside $\{0\}$. Consequently, on $\mathbb{C}^2\backslash\{0\}$, one has can write:

$$\omega_{\mathbf{f}} = \alpha dy_1 \wedge \eta + \beta dy_2 \wedge \eta + \gamma dy_3 \wedge \eta + \delta dy_1 \wedge dy_2 + \varepsilon dy_1 \wedge dy_3 + \zeta dy_2 \wedge dy_3$$

for some functions $\alpha, \dots, \zeta$. Now $\alpha dy_1 + \beta dy_2 + \gamma dy_3 = -\iota_\xi\omega_{\mathbf{f}} = dy_1$, thus $\alpha = 1$ and $\beta = \gamma = 0$; as $\omega_{\mathbf{f}}$ is of type $I_1 - (1,1)$, one also has $\delta = \varepsilon = 0$.

To determine ζ, one evaluates $\zeta dy_1 \wedge \eta \wedge dy_2 \wedge dy_3 = \frac{1}{2}\omega_{\mathbf{f}}^2 = \Omega_{\mathbf{e}}$ on $(-I_1\zeta, \zeta)$; this gives: $V^{-1}\zeta dy_2 \wedge dy_3 = \Omega_{\mathbf{e}}(-I_1\xi, \xi, \cdot, \cdot) = \frac{1}{2}(|z_2|^2 idz_1 \wedge d\overline{z_1} + |z_1|^2 idz_2 \wedge d\overline{z_2} + \overline{z_1}z_2 idz_1 \wedge d\overline{z_2} + z_1\overline{z_2} idz_2 \wedge d\overline{z_1}) = dy_2 \wedge dy_3$, hence $\zeta = V$. □

One easily checks that η is a connection 1-form away from 0 for the fibration $\varpi = (y_1, y_2, y_3)$: it is $\mathbb{S}^1$-invariant, and at any point p but $0 \in \mathbb{C}^2$, as $\{\eta, dy_1, dy_2, dy_3\}$ is a basis of $T_p^*\mathbb{C}^2$, necessarily, $T_p\mathbb{C}^2 = \ker\eta + \ker T\varpi$. Finally, $d\eta$ has the expected shape:

Lemma A.8 *The differential of η is given on $\mathbb{C}^2\backslash\{0\}$ by:*

$$d\eta = *_{\mathbb{R}^3} dV.$$

Proof. The 1-form η is $\mathbb{S}^1$-invariant and $\eta(\xi)$ is constant; by Cartan's formula, $0 = \mathcal{L}_\xi\eta = \iota_\xi d\eta + d(\iota_\xi\eta) = \iota_\xi d\eta$, i.e.: the components of $d\eta$ in the $dy_j\wedge\eta$-directions vanish. Moreover $d\omega_{\mathbf{f}} = 0$ thus according to Lemma A.7, $d\eta = \frac{\partial V}{\partial y_1} dy_2\wedge dy_3 + \alpha_2 dy_3\wedge dy_1 + \alpha_3 dy_1\wedge dy_2$. For the computation of α_2 and α_3, observe that:

$$4\eta = \Big(1+\frac{y_1}{R}\Big) d^c\log\big(|z_1|^2\big) - \Big(1-\frac{y_1}{R}\Big) d^c\log\big(|z_2|^2\big),$$

as $u^2 = R + y_1$ and $v^2 = R - y_1$. Since $dd^c\log(|z_1|^2) = dd^c\log(|z_2|^2) = 0$ outside of $\{z_1z_2 = 0\}$, we thus have $d\eta = \frac{1}{4} d\big(\frac{y_1}{R}\big)\wedge d^c\log(|z_1z_2|^2) = \frac{1}{4} d\big(\frac{y_1}{R}\big)\wedge d^c\log(y_2^2+y_3^2)$. Now

$$d\Big(\frac{y_1}{R}\Big) = \frac{1}{R^3}\Big((y_2^2+y_3^2)dy_1 - y_1y_2dy_2 - y_1y_3dy_3\Big)$$

and

$$d^c\log(y_2^2+y_3^2) = I_1 d\log(y_2^2+y_3^2) = 2\frac{y_2dy_3 - y_3dy_2}{y_2^2+y_3^2};$$

this clearly provides $\alpha_j = -\frac{y_j}{2R^3} = \frac{\partial V}{\partial y_j}$, $j = 2,3$. The lemma is proved, outside of $\{z_1z_2 = 0\}$, and the formula extends at once to $\mathbb{C}^2\backslash\{0\}$ by continuity. $\square$

A.2 Comparison of the Euclidean and the Taub-NUT metrics

A.2.1 Mutual control

The metrics $\mathbf{e}$ and $\mathbf{f}$ are far from being globally mutually bounded; an example of this geometric gap can be read in the scale of the ball volume growth: r^4 in the euclidean regime, but R^3 for Taub-NUT – notice that R plays the role of the distance to 0 on $(\mathbb{C}^2, \mathbf{f})$. Another example of the geometric gap is given by the length of the orbit of the $\mathbb{S}^1$-action on $\mathbb{C}^2$ used above: the orbit of $x\in\mathbb{C}^2\backslash\{0\}$ has length $2\pi|x|_{\mathbf{e}}$ under $\mathbf{e}$, and length $2\pi V(x)^{-1/2}$ when measured by $\mathbf{f}$; this latter length tends to $\pi\sqrt{2/m}$ when x goes ∞ – which gives us a geometric interpretation of the parameter m. We can nonetheless still compare $\mathbf{e}$ and $\mathbf{f}$ as follows:

Proposition A.9 *There exists some constant $C > 0$ such that on $\mathbb{C}^2$ minus its unit ball,*

$$C^{-1}r^{-2}\mathbf{e} \le \mathbf{f} \le Cr^2\mathbf{e}.$$

Proof. As $\mathbf{f} = V(dy_1^2 + dy_2^2 + dy_3^2) + V^{-1}\eta^2$, with $\eta = I_1 V dy_1$ and $dy_3 = I_1 dy_2$, we evaluate $|dy_1|_{\mathbf{e}}$ and $|dy_2|_{\mathbf{e}}$ first; since $dy_2 = \frac{i}{2}(z_1 dz_2 + z_2 dz_1 - \overline{z_1}d\overline{z_2} - \overline{z_2}d\overline{z_1})$, we readily get $|dy_2|_{\mathbf{e}} = cr$. Now, we rearrange formulas (48) to write

$$dy_1 = \frac{1}{2(1+4mR)}\big(e^{-4my_1}(\overline{z_1}dz_1 + z_1 d\overline{z_1}) - e^{4my_1}(\overline{z_2}dz_2 + z_2 d\overline{z_2})\big). \tag{52}$$

This provides $|dy_1|^2_{\mathbf{e}} = \frac{c}{(1+4mR)^2}(|z_1|^2 e^{-8my_1} + |z_2|^2 e^{8my_1})$. But $|z_1|^2 e^{-4my_1} = u^2$ and $|z_2|^2 e^{4my_1} = v^2$, so $|dy_1|^2_{\mathbf{e}} = \frac{c}{(1+4mR)^2}(e^{-4my_1}u^2 + e^{4my_1}v^2) = \frac{c'}{(1+4mR)^2}(R\cosh(4my_1) - y_1\sinh(4my_1))$. Now $R\cosh(4my_1) - y_1\sinh(4my_1) \leq R\cosh(4my_1) + y_1\sinh(4my_1)$ is obvious, and rearranging formulas (48) gives also

$$2\big(R\cosh(4my_1) + y_1\sinh(4my_1)\big) = r^2, \tag{53}$$

so finally $|dy_1|^2_{\mathbf{e}} \leq c\frac{r^2}{R^2}$. Those estimates give us the bound $\mathbf{f} \leq Cr^2\mathbf{e}$.

The reverse bound $\mathbf{e} \leq Cr^2\mathbf{f}$ follows at once, as $\mathbf{e}$ and $\mathbf{f}$ are hermitian, have the same volume form, and as we are in complex dimension 2. □

A.2.2 Expressing euclidean objects in Taub-NUT vocabulary

We give here some further material useful in the comparison between $\mathbf{e}$ and $\mathbf{f}$ on $\mathbb{C}^2$. In Lemma A.10 we introduce a vector field ζ helping to complete the dual frame of $(V^{-1/2}\eta, V^{1/2}dy_1, V^{1/2}dy_2, V^{1/2}dy_3)$ for $\mathbf{f}$. Then in Lemma A.11, we express the canonical frames of 1-forms and vector fields of $\mathbf{e}$, i.e. the dx_j and the $\frac{\partial}{\partial x_j}$, in terms of those of $\mathbf{f}$. The essential point in those expressions lies in their computational consequences; indeed, they allow to compute objects like $\nabla^{\mathbf{f}} dx_j$, and estimate quantities like $\big|\nabla^{\mathbf{f}} dx_j\big|_{\mathbf{f}}$ – which is required when manipulating euclidean objects in the Taub-NUT setting; see e.g. the proof of Proposition 1.7.

In Section A.1, we used the vector field ξ on $\mathbb{C}^2$, which verified $\eta(\xi) = 1$, $dy_j(\xi) = 0$, $j = 1,2,3$, and $dy_1(I_1\xi) = -\frac{1}{V}$, $\eta(I_1\xi) = dy_2(I_1\xi) = dy_3(I_1\xi) = 0$. We shall complete our dual frame with the help of another vector field:

Lemma A.10 *Define on $\mathbb{C}^2\backslash\{0\}$ the vector field*

$$\zeta = \frac{1}{2iR}\Big(e^{4my_1}\big(z_2\frac{\partial}{\partial\overline{z_1}} - \overline{z_2}\frac{\partial}{\partial z_1}\big) + e^{-4my_1}\big(z_1\frac{\partial}{\partial\overline{z_2}} - \overline{z_1}\frac{\partial}{\partial z_2}\big)\Big).$$

Then $dy_2(\zeta) = 1$ whereas $\eta(\zeta) = dy_1(\zeta) = dy_3(\zeta) = 0$, and $dy_3(I_1\zeta) = 1$ whereas $\eta(I_1\zeta) = dy_1(I_1\zeta) = dy_2(I_1\zeta) = 0$. Moreover, $[\xi,\zeta] = 0$.

Proof. We only need to check the first list of equalities, as $dy_3 = I_1 dy_2$ and $\eta = I_1 V dy_1$. Since $dy_2 = \frac{1}{2i}(z_1 dz_2 + z_2 dz_1 - \overline{z_1}d\overline{z_2} - \overline{z_2}d\overline{z_1})$, we get

$$dy_2(\zeta) = \frac{1}{2R}\big(e^{4my_1}|z_2|^2 + e^{-4my_1}|z_1|^2\big);$$

now $e^{4my_1}|z_2|^2 = v^2$, $e^{-4my_1}|z_1|^2 = u^2$, and $R = \frac{1}{2}(u^2+v^2)$, hence $dy_2(\zeta) = 1$. Using that $dy_3 = -\frac{1}{2}(z_1 dz_2 + z_2 dz_1 + \overline{z_1}d\overline{z_2} + \overline{z_2}d\overline{z_1})$ readily gives $dy_3(\zeta) = 0$. Now for the equality $dy_1(\zeta) = 0$, we use formula (52) to write $dy_1(\zeta) = \frac{1}{4iR(1+4mR)}(z_1 z_2 - \overline{z_1 z_2} - z_1 z_2 + \overline{z_1 z_2}) = 0$; likewise, equality $\eta(\zeta) = 0$ follows from formula (51).

Finally, the $\mathbb{S}^1$-invariance of ζ provides $[\xi, \zeta] = 0$. □

Lemma A.11 *One has the following formulas for 1-forms:*

$$\begin{aligned}
dx_1 &= Vx_1dy_1 - x_2\eta + \frac{e^{4my_1}}{2R}(x_4dy_2 - x_3dy_3),\\
dx_2 &= Vx_2dy_1 + x_1\eta + \frac{e^{4my_1}}{2R}(x_3dy_2 + x_4dy_3),\\
dx_3 &= -Vx_3dy_1 + x_4\eta + \frac{e^{-4my_1}}{2R}(x_2dy_2 - x_1dy_3),\\
dx_4 &= -Vx_4dy_1 - x_3\eta + \frac{e^{-4my_1}}{2R}(x_1dy_2 + x_2dy_3);
\end{aligned}$$

and for vector fields:

$$\begin{aligned}
\frac{\partial}{\partial x_1} &= -\frac{e^{-4my_1}}{2R}(x_2\xi + x_1I_1\xi) + (x_4\zeta - x_3I_1\zeta),\\
\frac{\partial}{\partial x_2} &= \frac{e^{-4my_1}}{2R}(x_1\xi - x_2I_1\xi) + (x_3\zeta + x_4I_1\zeta),\\
\frac{\partial}{\partial x_3} &= \frac{e^{4my_1}}{2R}(x_4\xi + x_3I_1\xi) + (x_2\zeta - x_1I_1\zeta),\\
\frac{\partial}{\partial x_4} &= \frac{e^{4my_1}}{2R}(-x_3\xi + x_4I_1\xi) + (x_1\zeta + x_2I_1\zeta).
\end{aligned}$$

Proof. We shall only see how those formulas arise for dx_1 and $\frac{\partial}{\partial x_1}$; the other identities are then easily deduced with the relations $dx_2 = I_1dx_1$, $dx_3 = \tau^*dx_1$, $dx_4 = I_1dx_3$, etc., on the euclidean side, and $\tau^*y_j = -y_j$, $\tau^*\eta = -\eta$, $\tau^*\xi = -\xi$, $\tau^*\zeta = -d\zeta$, etc., on the Taub-NUT side.

Write $dx_1 = \alpha dy_1 + \beta\eta + \gamma dy_2 + \delta dy_3$. By duality between $(\xi, -VI_1\xi, \zeta, I_1\zeta)$ and (η, dy_1, dy_2, dy_3), $dx_1(\xi) = \beta$, $dx_1(I_1\xi) = -\frac{\alpha}{V}$, $dx_1(\zeta) = \gamma$ and $dx_1(I_1\zeta) = \delta$. On the other hand, $dx_1(\xi) = \frac{1}{2}i(z_1 - \overline{z_1}) = -x_2$, $dx_1(I_1\xi) = -\frac{1}{2}(z_1 + \overline{z_1}) = -x_1$, $dx_1(\zeta) = \frac{1}{2iR}(e^{4my_1}\frac{1}{2}(z_2 - \overline{z_2}+) + 0) = \frac{e^{4my_1}}{2R}x_4$ and $dx_1(I_1\zeta) = \frac{i}{2R}(e^{4my_1}\frac{i}{2}(z_2 + \overline{z_2}) + 0) = -\frac{e^{4my_1}}{2R}x_3$, hence the result.

Similarly, if $\frac{\partial}{\partial x_1} = \alpha\xi + \beta I_1\xi + \gamma\zeta + \delta I_1\zeta$, then $\alpha = \eta(\frac{\partial}{\partial x_1}) = -e^{-4my_1}\frac{x_2}{2R}$, $\beta = -Vdy_1(\frac{\partial}{\partial x_1}) = -e^{-4my_1}\frac{x_1}{2R}$, $\gamma = dy_2(\frac{\partial}{\partial x_1}) = x_4$ and $\delta = dy_3(\frac{\partial}{\partial x_1}) = -x_3$. □

A.2.3 Derivatives

Consider the $\mathbf{f}$-orthonormal frame $(e_j)_{j=0,\dots,3}$ of vector fields given by

$$(e_0, e_1, e_2, e_3) = \big(V^{1/2}\xi, -V^{1/2}I_1\xi, V^{-1/2}\zeta, V^{-1/2}I_1\zeta\big)$$

away from 0. In Part 1, we have to estimate the $\nabla^{\mathbf{f}}_{e_i} e_j$. This we do in the following:

Lemma A.12 *One has* $[e_0, e_i] = \frac{y_i}{4R^3V^{3/2}} e_0$ *for* $i = 1, 2, 3$, *and*

$$[e_i, e_j] = \frac{1}{4R^3V^{3/2}} (y_i e_j - y_j e_i + 2y_k e_0)$$

for any triple $(i, j, k) \in \mathcal{I} = \big\{(1, 2, 3), (2, 3, 1), (3, 1, 2)\big\}$. *As a consequence,*

$$\nabla^{\mathbf{f}} e_0 = \frac{1}{4R^3V^{3/2}} \sum_{(i,j,k)\in\mathcal{I}} e_i \otimes \big(y_k e_j^* - y_j e_k^* - y_i e_0^*\big)$$

with $(e_0^*, e_1^*, e_2^*, e_3^*) = \big(V^{-1/2}\eta, V^{1/2}dy_1, V^{1/2}dy_2, V^{1/2}dy_3\big)$.

Remark A.13 *Defining* J_j *by* $\mathbf{f}(J_j\cdot, \cdot) = \omega_j^{\mathbf{e}}$, $j = 2, 3$, *we get two complex structures verifying with* $J_1 := I_1$ *the quaternionic relations, just as we did for* J_2^Y *and* J_3^Y *at the end of §1.4.4. By Lemma A.11, we see moreover that* ω_2 *is exactly* $dy_2 \wedge \eta + V dy_1 \wedge dy_3$ *– and likewise for* ω_3 *– so that, for instance,* $e_0 = J_1e_1 = J_2e_2 = J_3e_3$.

Proof of Lemma A.12. Once the statement on the Lie brackets is proved, the formula for $\nabla^{\mathbf{f}} e_0$ follows from Koszul formula for the Levi-Civita connection $\nabla^{\mathbf{f}}$ and the orthonormality of the frame (e_i). Moreover, because of the symmetric roles of e_1, e_2, e_3, we shall only see how to compute $[e_0, e_1]$ and $[e_1, e_2]$.

- $[e_0, e_1]$: this bracket is rather easy to compute. Recall that $e_0 = V^{1/2}\xi$, $e_1 = -V^{1/2}I_1\xi$, and ξ is holomorphic for I_1, so that $[\xi, I_1\xi] = 0$. Moreover, as V is $\mathbb{S}^1$-invariant, $\xi \cdot V = 0$, and $(I_1\xi) \cdot V = -V^{-1}\frac{\partial V}{\partial y_1}$. Thus:

$$\begin{aligned}[e_0, e_1] =& \mathcal{L}_{e_0}(-V^{1/2}I_1\xi) = (-e_0 \cdot V^{1/2})I_1\xi - V^{1/2}\mathcal{L}_{e_0}(I_1\xi) = 0 + V^{1/2}\mathcal{L}_{I_1\xi}e_0 \\ =& V^{1/2}\big((I_1\xi)\cdot V^{1/2})\big)\xi + V\mathcal{L}_{I_1\xi}e_0 = \frac{1}{2}\big((I_1\xi)\cdot V\big)\xi + 0 = -\frac{1}{2V}\frac{\partial V}{\partial y_1}\xi,\end{aligned}$$

  hence the result, as $\frac{\partial V}{\partial y_1} = \frac{\partial R}{\partial y_1}\frac{dV}{dR} = -\frac{y_1}{R}\frac{1}{2R^2}$.

- $[e_1, e_2]$: as $e_1 = -V^{1/2}I_1\xi$ and $e_2 = V^{-1/2}\zeta$, by Leibniz rule,

$$\begin{aligned}[e_1, e_2] &= (e_2 \cdot V^{1/2})I_1\xi + V^{1/2}\mathcal{L}_{e_2}(I_1\xi) \\ &= -V^{-1}(\zeta \cdot V^{1/2})e_1 - V\big((I_1\xi)\cdot V^{-1/2}\big)e_2 - \mathcal{L}_{I_1\xi}\zeta.\end{aligned} \tag{54}$$

We already know $(I_1\xi)\cdot V^{-1/2} = \frac{1}{2V^{5/2}}\frac{\partial V}{\partial y_1} = -\frac{y_1}{4R^3V^{1/2}}$; similarly, $\zeta\cdot V^{1/2} = \frac{1}{2}V^{-1/2}\frac{\partial V}{\partial y_2} = -\frac{y_2}{4R^3V^{1/2}}$. We are thus left with $\mathcal{L}_{I_1\xi}\zeta$. Now $e^{tI_1\xi}(z_1,z_2) = (e^{-t}z_1, e^t z_2)$, hence $\omega_2 = dx_1\wedge dx_3 + dx_4\wedge dx_2$ is invariant under this flow: $\mathcal{L}_{I_1\xi}\omega_2 = 0$. Besides, $\omega_2 = dy_2\wedge\eta + dy_3\wedge dy_1$, so that $\omega_2(\zeta,\cdot) = \eta$, hence:

$$(d\eta)(I_1\xi,\cdot) = \mathcal{L}_{I_1\xi}\eta = \mathcal{L}_{I_1\xi}\big(\omega_2(\zeta,\cdot)\big) = \omega_2(\mathcal{L}_{I_1\xi}\zeta,\cdot),$$

the first equality coming from Cartan's formula and the identity $\eta(I_1\xi)=0$. Now by Lemma A.8, $(d\eta)(I_1\xi,\cdot) = V^{-1}\big(\frac{\partial V}{\partial y_2}dy_3 - \frac{\partial V}{\partial y_3}dy_2\big)$, and thus

$$\mathcal{L}_{I_1\xi}\zeta = -\frac{1}{2R^3V}(y_3\xi + y_2 I_1\xi) = -\frac{1}{2R^3V^{3/2}}(y_3e_0 - y_2e_1).$$

The conclusion follows from plugging this back into (54). □

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