NOTE ON POINCARÉ TYPE KÄHLER METRICS AND FUTAKI CHARACTERS

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Abstract

A Poincaré type Kähler metric on the complement $X \backslash D$ of a simple normal crossing divisor D, in a compact Kähler manifold X, is a Kähler metric on $X \backslash D$ with cusp singularity along D. We relate the *Futaki character for holomorphic vector fields parallel to the divisor*, defined for any fixed Poincaré type Kähler class, to the classical Futaki character for the relative smooth class. As an application we express a numerical obstruction to the existence of extremal Poincaré type Kähler metrics, in terms of mean scalar curvatures and Futaki characters.

Introduction

A basic fact in Kähler geometry is the independence of the de Rham class of the Ricci form from the background metric on a compact Kähler manifold: it is always $-2\pi c_1(K)$, with $c_1(K)$ the first Chern class of the canonical line bundle. This topological invariance constitutes the first obstacle for a compact Kähler manifold to admit a Kähler-Einstein metric: the Chern class in question must then have a sign, which, if definite, forces Kähler-Einstein metrics to lie in a consequently fixed Kähler class.

When $c_1(K) > 0$, a (unique) Kähler-Einstein metric was obtained by Aubin and Yau, and Bochner's technique then rules out the existence of non-trivial holomorphic vector fields . Conversely, in the opposite case $c_1(K) < 0$, the so-called *"Fano case"*, non-trivial holomorphic vector fields may exist, and the existence of a Kähler-Einstein metric, which does not always hold, is noticeably more involved. More precisely, in this case – and, respectively, on any compact Kähler manifold – non-trivial holomorphic vector fields bring a constraint to the existence of a Kähler-Einstein metric – respectively, of a constant scalar curvature metric Kähler metric in a fixed Kähler class. If indeed such a canonical metric exists, a numerical function, the *Futaki character* [Fut88], defined on the Lie algebra of holomorphic vector fields – and depending only on the Kähler class under study –, has to vanish identically .

The Futaki character was later generalised by Donaldson to polarised manifolds, into an numerical function defined on *test-configurations*, which generalise the concept of (the action of) holomorphic vector fields [Don02]. In the lines of suggestions by Yau [Yau93], and after Tian's *special degenerations* [Tia97], test-configurations and their Donaldson-Futaki invariants are meant to reveal the link between algebro-geometric stability of the manifold, and existence of a Kähler-Einstein/constant scalar curvature Kähler metric:

Conjecture 1 (Yau-Tian-Donaldson) *A polarised manifold (X, L) admits a constant scalar Kähler curvature metric in $2\pi c_1(L)$ if, and only if, (X, L) is "K-stable", that is: the Donaldson-Futaki invariant is nonpositive (negative) for any (non-trivial) test-configuration.*

The "only if" direction is now established [Mab04, Sto09]; the "if" direction is still a very active area of research, and has recently been solved for Kähler-Einstein metrics in the Fano case, i.e. when $L = -K_X$ is ample, see [CDS12b, CDS12c, CDS13] and [Tia12].

In a related scope, the aim of this note is, after restriction to the relevant set of holomorphic vector fields, to generalise the Futaki character to a certain class of singular metrics on a compact manifold. Namely, fixing a simple normal crossing divisor D in a compact Kähler manifold (X, J, ω_X), we recall the definition *Poincaré type Kähler metrics on $X\backslash D$*, following [TY87, Wu08, Auv11]:

Definition 2 *A smooth positive $(1,1)$-form ω on $X\backslash D$ is called a* ***Poincaré type Kähler metric on*** *$X\backslash D$ if: on every open subset U of coordinates $(z^1, \dots, z^m)$ in X, in which D is given by $\{z^1 \cdots z^j = 0\}$, ω is mutually bounded with*

$$\omega_U^{\mathrm{mdl}} := \frac{idz^1 \wedge d\overline{z^1}}{|z^1|^2 \log^2(|z^1|^2)} + \cdots + \frac{idz^j \wedge d\overline{z^j}}{|z^j|^2 \log^2(|z^j|^2)} + idz^{j+1} \wedge d\overline{z^{j+1}} + \cdots + idz^m \wedge d\overline{z^m},$$

and has bounded derivatives at any order for this model metric.

We say moreover that ω ***is of class*** *$[\omega_X]$ if $\omega = \omega_X + dd^c\varphi$ for some φ smooth on $X\backslash D$, with $\varphi = \mathcal{O}\big(\sum_{\ell=1}^j \log[-\log(|z^\ell|^2)]\big)$ in the above coordinates and $d\varphi$ bounded at any order for ω_U^{mdl}. We then set: $\omega \in \mathscr{M}^D_{[\omega_X]}$.*

Metrics of $\mathscr{M}^D_{[\omega_X]}$ are complete, with finite volume (equal to that of X for smooth Kähler metrics of class $[\omega_X]$); they also share a common mean scalar curvature, which differs from that attached to smooth Kähler metrics of class $[\omega_X]$.

Restricting our attention to the set $\mathfrak{h}^D_{/\!/}$ of holomorphic vector fields with their normal component vanishing along D – these are the holomorphic vector fields bounded (at any order) for *any* Poincaré type Kähler metric on $X\backslash D$ –, we define in a way similar to the compact case a *Poincaré type Futaki character* $\mathscr{F}^D_{[\omega_X]}$, computed with metrics of $\mathscr{M}^D_{[\omega_X]}$, and depending only on this Poincaré class.

Results. — This Poincaré type Futaki character differs generally from the restriction of the usual smooth Futaki character $\mathscr{F}_{[\omega_X]}$ to $\mathfrak{h}^D_{/\!/}$. More precisely, our first main result is a formula giving the precise relation between these two invariants:

Theorem 3 *Let* $\mathsf{Z} \in \mathfrak{h}^D_{/\!/}$*, with Riemannian gradient potential* f *for* $\omega_X(\cdot, J\cdot)$*. Then* $\mathscr{F}^D_{[\omega_X]}(\mathsf{Z}) = \mathscr{F}_{[\omega_X]}(\mathsf{Z}) + \sum_{j=1}^N \int_{D_j} f \, \frac{(\omega_X|_{D_j})^{m-1}}{(m-1)!}$.

The gradient potential refers to that of the Hodge decomposition of (the dual 1-form) of Z.

As an application of Poincaré type Futaki character, in the framework of finding necessary conditions on canonical Kähler metrics, we provide the following *numerical constraint on the existence of extremal metrics of Poincaré type on* $X\backslash D$:

Theorem 4 *Assume that there exists an extremal metric in* $\mathscr{M}^D_{[\omega_X]}$*, and denote by* K *the Riemannian gradient of its scalar curvature. Then for all* $j \in \{1, \dots, N\}$ *indexing an irreducible component of* D*, one has:*

$$\bar{\mathbf{s}}^D < \bar{\mathbf{s}}^{D^j}_{D_j} + \frac{1}{4\pi \operatorname{Vol}(D_j)} \big(\mathscr{F}^{D-D_j}_{[\omega_X]}(\mathsf{K}) - \mathscr{F}^D_{[\omega_X]}(\mathsf{K})\big), \tag{1}$$

where $\mathscr{F}^{D-D_j}_{[\omega_X]}(K)$ *is the Futaki character for Poincaré type metrics on* $X\backslash(D - D_j) = X\backslash \sum_{\ell\neq j} D_\ell$ *of class* $[\omega_X]$*,* $\bar{\mathbf{s}}^D$ *is the mean scalar curvature attached to* $\mathscr{F}^D_{[\omega_X]}(K)$*, and* $\bar{\mathbf{s}}^{D^j}_{D_j}$ *that attached to* $\mathscr{F}^{D^j}_{[\omega_X]|_{D_j}}(K)$*, the class of Poincaré type metrics on* $D_j\backslash(D - D_j)|_{D_j}$ *of class* $[\omega_X]|_{D_j}$*.*

An extremal metrics is a Kähler metrics such that the Riemannian gradient of its scalar curvature is holomorphic. Constraint (1) is a reformulation of that of [Auv14, Prop.4.5], and as such, extends that on the existence of constant scalar curvature metrics in $\mathscr{M}^D_{[\omega_X]}$ of [Auv13], which states as: $\bar{\mathbf{s}}^D < \bar{\mathbf{s}}^{D^j}_{D_j}$ for all $j = 1, \dots, N$. As in the compact case moreover, by construction and invariance on $\mathscr{M}^D_{[\omega_X]}$, $\mathscr{F}^D_{[\omega_X]}$ vanishes identically if there exists a constant scalar curvature metric in $\mathscr{M}^D_{[\omega_X]}$, and conversely, its vanishing forces possible extremal metrics of $\mathscr{M}^D_{[\omega_X]}$ to have constant scalar curvature. By contrast, the interest of Theorem 4 is to provide constraints on extremal metrics independently of such a vanishing.

Finally, Donaldson-Futaki invariants are already considered in [CDS12a] which take into account the contribution of a divisor. These are used in the context of *Kähler metrics with conical singularities on polarised manifolds*, and the divisor term of the invariant comes with coefficient $(1-\beta)$, with $2\pi\beta$ the angle of the cone singularity. In view of Theorem 3, the Poincaré type Futaki invariant might thus be viewed – at the level of holomorphic vector fields rather than at that of test-configurations – as the limit when the conical singularity angle goes to 0, that is, roughly speaking, when *cones* become *cusps*.

Organisation of the article. — This note is divided into three parts. In the first part, we analyse holomorphic vector fields parallel to a divisor, with respect to Poincaré type Kähler metrics on the complement of this divisor. We see in particular that a Hodge decomposition analogous to that of the compact case still holds for such metrics and such vector fields. This allows us in Section 1.2 to define the Poincaré type Futaki character, as an *invariant* of a given Poincaré type Kähler class.

Theorem 3, under a slightly more general version, is stated in Section 2.1 (Proposition 2.1); it is proven in Section 2.2, and the final section 2.3 of Part 2 is devoted to a key technical lemma (Lemma 2.2) used in Section 2.2.

In Part 3 we state and prove Theorem 4: a useful extension (Proposition 3.1) of 3 (to asymptotically product Poincaré type metrics when the divisor is smooth) is given in Section 3.1; Theorem 4 is then proven in Section 3.2 (Theorem 3.2), first in the smooth divisor case using Proposition 3.1, then in the simple normal crossing case. Notice that both steps use the asymptotic properties of extremal Poincaré type metrics obtained in [Auv14].

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In all this note, X is a compact Kähler manifold, and $D \subset X$ a simple normal crossing divisor, the decomposition into irreducible smooth components of which we write as $\sum_{j=1}^{N} D_j$.

1 The Futaki character of a Poincaré class

1.1 Hodge decomposition of vector fields parallel to the divisor

Reminder: the compact case. — Fix a smooth Kähler form ω_X on X, of associated Riemannian metric g_X. Given any real holomorphic vector field Z – "$\mathsf{Z} \in \mathfrak{h}$" –, it is well-known that its g_X-dual 1-form ξ^{Z}, that is, $\mathsf{Z}^{\sharp g_X}$, enjoys the following decomposition:

$$\xi^{\mathsf{Z}} = \xi^{\mathsf{Z}}_{\mathrm{harm}} + df^{\mathsf{Z}}_{\omega_X} + d^c h^{\mathsf{Z}}_{\omega_X}, \tag{2}$$

into harmonic, d- and d^c-parts; these are uniquely determined, provided that $f^{\mathsf{Z}}_{\omega_X}$ and $h^{\mathsf{Z}}_{\omega_X}$ are taken with null mean against ω_X^m.

Decomposition (2) is called the *(dual) Hodge decomposition* of Z. Given moreover any other smooth metric $\tilde{\omega} = \omega_X + dd^c\varphi$ of $\mathscr{M}_{[\omega_X]}$, and setting $\tilde{\xi}^{\mathsf{Z}}$ for the dual 1-form of Z with respect to $\tilde{\omega}(\cdot, J\cdot)$, its Hodge decomposition is:

$$\tilde{\xi}^{\mathsf{Z}} = \xi^{\mathsf{Z}}_{\mathrm{harm}} + d\big(f^{\mathsf{Z}}_{\omega_X} + \mathsf{Z}\cdot\varphi\big) + d^c\big(h^{\mathsf{Z}}_{\omega_X} - (J\mathsf{Z})\cdot\varphi\big)$$

see [Gau, Lemma 4.5.1]– notice in particular that the harmonic part remains unchanged at the level of 1-forms; recall that on compact Kähler manifolds, the space of harmonic 1-forms is *independent of the Kähler metric.*

Extension to Poincaré type Kähler metrics for vector fields parallel to a divisor. — Consider now the simple normal crossing divisor $D = \sum_{j=1}^N D_j$ in X. The normal crossing assumption can be expressed as follows: given any $p \in (D_{j_1} \cap \cdots \cap D_{j_k}) \backslash (D_{\ell_1} \cup \cdots \cup D_{\ell_{N-k}})$, with $\{j_1, \dots, j_k\} \sqcup \{\ell_1 \dots, \ell_{N-k}\} = \{1, \dots, N\}$, one can find in X an open neighbourhood U of p, of holomorphic coordinates $(z^1, \dots, z^m)$ such that $U \cap D_{j_s} = \{z^s = 0\}$ for $s = 1, \dots, k$ (in particular, $k \leq m$).

We define a restricted class of holomorphic vector fields on X, the use of which is natural when working with Poincaré type Kähler metrics on $X\backslash D$:

Definition 1.1 *Let $\mathsf{Z} \in \mathfrak{h}$. We say that Z **is parallel to** D, denoted $\mathsf{Z} \in \mathfrak{h}^D_{/\!/}$, if: writing Z as $\mathfrak{Re}\big(f_1\frac{\partial}{\partial z^1} + \cdots + f_m\frac{\partial}{\partial z^m}\big)$ in local holomorphic coordinates as above, one has $f_1 \equiv \cdots \equiv f_s \equiv 0$ on D.*

*Given $j \in \{1, \dots, N\}$, we then define **the restriction** $\mathsf{Z}|_{D_j}$ **of** Z **to** D_j by setting locally $\mathsf{Z}_{D_j} = \mathfrak{Re}\big(f_2|_{D_j}\frac{\partial}{\partial z^2} + \cdots + f_m|_{D_j}\frac{\partial}{\partial z^m}\big)$, whenever $j_1 = j$ in the above coordinates.*

One checks in particular that the definition of Z_{D_j} is independent of the choice of holomorphic coordinates, as long as the first coordinate is a local equation of D_j; one also checks easily that $\mathfrak{h}^D_{/\!/}$ is a Lie subalgebra of $\mathfrak{h}$.

Holomorphic vector fields parallel to D are relevant when working with Poincaré type Kähler metrics on D for the following reason (see the proof of Lemma 5.2 in [Auv11]): *any holomorphic vector field on $X\backslash D$ which is bounded – or actually, L^2 – with respect to a Poincaré type Kähler metric on $X\backslash D$ extends to a holomorphic vector field on X, parallel to D.* Conversely, any holomorphic vector field on X parallel to D gives on $X\backslash D$ a vector field bounded at any order for any Poincaré type metric on $X\backslash D$.

We now provide *a Hodge decomposition of holomorphic vector fields parallel to D with respect to Poincaré type Kähler metrics on $X\backslash D$*, analogous to the decomposition of the compact setting:

Proposition 1.2 *Let $\mathsf{Z} \in \mathfrak{h}_{/\!/}^D$, and let $\omega = \omega_X + dd^c\varphi \in \mathscr{M}^D_{[\omega_X]}$. Let ξ_φ^{Z} be the dual 1-form of Z with respect to $\omega(\cdot, J\cdot)$. Then*

$$\xi_\varphi^{\mathsf{Z}} = \xi_{\mathrm{harm}}^{\mathsf{Z}} + d\big(f_{\omega_X}^{\mathsf{Z}} + \mathsf{Z}\cdot\varphi\big) + d^c\big(h_{\omega_X}^{\mathsf{Z}} - (J\mathsf{Z})\cdot\varphi\big) \tag{3}$$

on $X\backslash D$, with the same harmonic part $\xi_{\mathrm{harm}}^{\mathsf{Z}}$ as in the compact case, and this decomposition is unique. Moreover,

$$\int_{X\backslash D} \big(f_{\omega_X}^{\mathsf{Z}} + \mathsf{Z}\cdot\varphi\big)\omega^m = \int_{X\backslash D} \big(h_{\omega_X}^{\mathsf{Z}} - (J\mathsf{Z})\cdot\varphi\big)\omega^m = 0.$$

The uniqueness we state here is understood as follows: if $\xi_\varphi^{\mathsf{Z}} = \alpha + d\beta + d^c\gamma$ with α harmonic on $X\backslash D$, and α, β, γ bounded for ω of Poincaré type, then $\alpha = \xi_{\mathrm{harm}}^{\mathsf{Z}}$, and $\beta = f_{\omega_X}^{\mathsf{Z}} + \mathsf{Z}\cdot\varphi$ and $\gamma = h_{\omega_X}^{\mathsf{Z}} - (J\mathsf{Z})\cdot\varphi$ up to a constant. This justifies:

Notation 1.3 *With the notations of Proposition 1.2, we set*

$$f_\omega^{\mathsf{Z}} = f_{\omega_X}^{\mathsf{Z}} + \mathsf{Z}\cdot\varphi \quad \textit{and} \quad h_\omega^{\mathsf{Z}} = h_{\omega_X}^{\mathsf{Z}} - (J\mathsf{Z})\cdot\varphi.$$

Proof of Proposition 1.2. — With the notations of the statement, we first prove that equality (3) holds on $X\backslash D$. This identity is purely local; it is thus sufficient to establish it for any Kähler metric equal to ω in the neighbourhood of any given point of $X\backslash D$. More concretely, as $\omega = \omega_X + dd^c\varphi$ is of Poincaré type, local analysis provides that $\varphi \to -\infty$ near D. Consider a convex function $\chi : \mathbb{R} \to \mathbb{R}$, with $\chi(t) = 0$ if $t \leq -1$, $\chi(t) = t$ if $t \geq 1$ – and thus $0 \leq \chi'(t) \leq 1$ for all t. Given $K \in \mathbb{R}$, one now easily checks that

$$\omega_K := \omega_X + dd^c\big(\chi\circ(\varphi + K)\big)$$

is a smooth metric on X, equal to ω on $\{\varphi \geq 1 - K\}$ (compact in $X\backslash D$), and to ω_X on $\{\varphi \leq -(K+1)\}$. Now (3) follows on $\{\varphi > 1 - K\}$ by the smooth case of Hodge decomposition applied to ω_K, thus on all $X\backslash D$ by letting $K \to \infty$.

Observe that $\xi_{\mathrm{harm}}^{\mathsf{Z}}$ is still harmonic with respect to ω; again, this condition is local, implied, thanks to the Kähler identities, by the closedness and the d^c-closedness of $\xi_{\mathrm{harm}}^{\mathsf{Z}}$. These latter conditions are independent of the Kähler metric, and indeed implied by the harmonicity of $\xi_{\mathrm{harm}}^{\mathsf{Z}}$ for the smooth ω_X, as X is compact.

As $\xi_{\mathrm{harm}}^{\mathsf{Z}}$ is bounded for ω_X, it is so for ω, which dominates ω_X. Similarly, $f_{\omega_X}^{\mathsf{Z}}$ and $h_{\omega_X}^{\mathsf{Z}}$ are bounded at any order for ω_X hence for ω, and as Z is parallel to D, it is bounded at any order for ω, as well as $d\varphi$ by definition; consequently, f_ω^{Z} and h_ω^{Z} are bounded at any order for ω. From this the uniqueness of Hodge decomposition easily follows. Write $\xi_\varphi^{\mathsf{Z}} = \alpha + d\beta + d^c\gamma$ with α, β, γ as above. As $d\alpha = d^c\alpha = 0$ (α is bounded and harmonic for ω hence bounded at any order by uniform ellipticity in quasi-coordinates, and one can thus integrate by parts without boundary terms), one gets $dd^c(f_\omega^{\mathsf{Z}} - \beta) = dd^c(h_\omega^{\mathsf{Z}} - \gamma) = 0$ on $X\backslash D$. Therefore $f_\omega^{\mathsf{Z}} - \beta$ and $h_\omega^{\mathsf{Z}} - \gamma$ are constant (use e.g. Yau's maximum principle [Wu08, p.406]) as wanted, and thus $\alpha = \xi_{\mathrm{harm}}^{\mathsf{Z}}$.

We are left with the mean assertion on f_ω^{Z} and h_ω^{Z}. For $t \in [0,1]$, set $\omega_t = \omega_X + t dd^c\varphi$, $f_t = f_{\omega_X}^{\mathsf{Z}} + t(\mathsf{Z}\cdot\varphi)$, and consider the function $t \mapsto \int_{X\backslash D} f_t\, \omega_t^m$. Thanks to the growths near D, this function is smooth, with derivative $\int_{X\backslash D} (\mathsf{Z}\cdot\varphi)\, \omega_t^m + m\int_{X\backslash D} f_t dd^c\varphi \wedge \omega_t^{m-1}$. Now,

$$m\int_{X\backslash D} f_t dd^c\varphi \wedge \omega_t^{m-1} = -m\int_{X\backslash D} df_t \wedge d^c\varphi \wedge \omega_t^{m-1} = -\int_{X\backslash D} \langle df_t, d\varphi\rangle_{\omega_t}\, \omega_t^m$$

(no boundary terms). On the other hand, we now know that for all t, $\xi_{t\varphi}^{\mathsf{Z}} = \xi_{\mathrm{harm}}^{\mathsf{Z}} + d\big(f_{\omega_X}^{\mathsf{Z}} + t(\mathsf{Z}\cdot\varphi)\big) + d^c\big(h_{\omega_X}^{\mathsf{Z}} - t(J\mathsf{Z})\cdot\varphi\big)$. Notice that $\int_{X\backslash D} \langle \xi_{\mathrm{harm}}^{\mathsf{Z}}, d\varphi\rangle_{\omega_t}\, \omega_t^m = \int_{X\backslash D} \varphi(\delta_{\omega_t}\xi_{\mathrm{harm}}^{\mathsf{Z}})\, \omega_t^m = 0$ ($\xi_{\mathrm{harm}}^{\mathsf{Z}}$ is co-closed for ω_t, as $\delta_{\omega_t} = \Lambda_{\omega_t} d^c$ on 1-forms), and $\int_{X\backslash D} \langle d^c\big(h_{\omega_X}^{\mathsf{Z}} - t(J\mathsf{Z})\cdot\varphi\big), d\varphi\rangle_{\omega_t}\, \omega_t^m = -m\int_{X\backslash D} d\big(h_{\omega_X}^{\mathsf{Z}} - t(J\mathsf{Z})\cdot\varphi\big) \wedge d\varphi \wedge \omega_t^{m-1} = 0$ (the integrand is closed, and there are no boundary terms). This way $\int_{X\backslash D} \langle df_t, d\varphi\rangle_{\omega_t}\, \omega_t^m = \int_{X\backslash D} \langle \xi_{t\varphi}^{\mathsf{Z}}, d\varphi\rangle_{\omega_t}\, \omega_t^m = \int_{X\backslash D} (\mathsf{Z}\cdot\varphi)\, \omega_t^m$, hence: $\int_{X\backslash D} f_t\, \omega_t^m$ is constant, which gives (take $t = 0, 1$): $\int_{X\backslash D} f_\omega^{\mathsf{Z}}\, \omega^m = \int_{X\backslash D} f_{\omega_X}^{\mathsf{Z}}\, \omega_X^m = 0$. The mean of h_ω^{Z} against ω^m is seen to vanish likewise. □

1.2 The Poincaré type Futaki character

Definition. — We can now generalise to Poincaré type Kähler metrics/classes, and holomorphic vector fields parallel to the divisor, a well-known invariant [Fut88] of compact Kähler manifolds:

Definition 1.4 *For* $\mathsf{Z} \in \mathfrak{h}_{/\!/}^D$ *and* $\omega \in \mathscr{M}_{[\omega_X]}^D$*, we call* ***Poincaré type Futaki character of*** Z ***with respect to*** D *the quantity*

$$\mathscr{F}_{[\omega_X]}^D(\mathsf{Z}) = \int_{X\backslash D} \mathbf{s}(\omega) f_\omega^{\mathsf{Z}} \frac{\omega^m}{m!}. \tag{4}$$

Here, $\mathbf{s}(\omega)$ denotes the *(Riemannian) scalar curvature* of ω, that one can compute for instance via: $\mathbf{s}(\omega)\frac{\omega^m}{m!} = 2\varrho(\omega) \wedge \frac{\omega^{m-1}}{(m-1)!}$, with $\varrho(\omega)$ the Ricci-form of ω.

Independence from the reference metric. — As terminology and notation suggest, this Poincaré type Futaki character does not depend on ω of class $[\omega_X]$, *provided it is of Poincaré type*:

Proposition 1.5 *Let $\tilde{\omega}$ be any Poincaré type metric in $\mathscr{M}^D_{[\omega_X]}$, and $\mathsf{Z} \in \mathfrak{h}^D_{/\!\!/}$. Then $\mathscr{F}^D_{[\omega_X]}(\mathsf{Z}) = \int_{X\backslash D} \mathbf{s}(\tilde{\omega}) f^{\mathsf{Z}}_{\tilde{\omega}} \frac{\tilde{\omega}^m}{m!}$.*

Observe nonetheless that we take $\tilde{\omega}$ of Poincaré type in this proposition; the relation between the usual smooth Futaki character, and our Poincaré type Futaki character, is the purpose of next part. For now, let us address the proof.

Proof of Proposition 1.5. — Take $\mathsf{Z} \in \mathfrak{h}^D_{/\!\!/}$. Fix $\omega = \omega_X + dd^c\varphi$ and $\tilde{\omega} = \omega_X + dd^c\tilde{\varphi}$ in $\mathscr{M}^D_{[\omega_X]}$, and for $t \in [0,1]$, set $\omega_t = (1-t)\omega + t\tilde{\omega} = \omega_X + dd^c\varphi_t$, $\varphi_t = (1-t)\varphi + t\tilde{\varphi}$; the ω_t are metrics of Poincaré type, uniformly bounded below by $c\omega$, say. As a consequence, the $\mathbf{s}(\omega_t)$ are uniformly bounded, at any order for ω, and for all $t_0 \in [0,1]$, $\mathbf{s}(\omega_t) = \mathbf{s}(\omega_{t_0}) + (t-t_0)\dot{\mathbf{s}}_{t_0} + (t-t_0)^2 w_{t_0,t}$, with $\dot{\mathbf{s}}_{t_0} = -\frac{1}{2}\Delta^2_{\omega_{t_0}}(\tilde{\varphi}-\varphi) - \langle \varrho(\omega_t), dd^c(\tilde{\varphi}-\varphi)\rangle_t$, and $w_{t_0,t}$ (uniformly) bounded at any order. Uniform bounds at any order hold as well for the $f^{\mathsf{Z}}_{\omega_t} = f^{\mathsf{Z}}_{\omega_X} + \mathsf{Z}\cdot\varphi_t = f^{\mathsf{Z}}_{\omega} + t\mathsf{Z}\cdot(\tilde{\varphi}-\varphi)$; these growth conditions near D thus ensure us that

$$t \longmapsto \mathscr{F}_t := \int_{X\backslash D} \mathbf{s}(\omega_t) f^{\mathsf{Z}}_{\omega_t} \frac{\omega_t^m}{m!}$$

is a smooth function of t, with derivative

$$t \longmapsto \dot{\mathscr{F}}_t = \int_{X\backslash D} \big(\dot{\mathbf{s}}_t f^{\mathsf{Z}}_{\omega_t} + \mathbf{s}(\omega_t)\big[\mathsf{Z}\cdot(\tilde{\varphi}-\varphi)\big]\big)\frac{\omega_t^m}{m!} + \int_{X\backslash D} \mathbf{s}(\omega_t) f^{\mathsf{Z}}_{\omega_t}\, dd^c(\tilde{\varphi}-\varphi)\wedge\frac{\omega_t^{m-1}}{(m-1)!},$$

just as in the compact case. And as in the compact case, integrations by parts can be performed without boundary terms, again thanks to the bounds mentioned above; one thus ends with $\dot{\mathscr{F}}_t = 0$ for all $t \in [0,1]$ (see e.g. [Gau, Prop. 4.12.1]), hence the result. ☐

Remark 1.6 *The word "character" for the function $\mathscr{F}^D_{[\omega_X]} : \mathfrak{h}^D_{/\!\!/} \to \mathbb{R}$ might appear slightly abusive, as long as we have not checked that $\mathscr{F}^D_{[\omega_X]}([\mathsf{Z}_1,\mathsf{Z}_2]) = 0$ for all $\mathsf{Z}_1,\mathsf{Z}_2 \in \mathfrak{h}^D_{/\!\!/}$. As in the compact case, this identity however follows at once from the invariance of $\mathscr{F}^D_{[\omega_X]}$ along $\mathscr{M}^D_{[\omega_X]}$, and the stability of this class under automorphisms of X parallel to D and homotopic to id_X.*

2 Link between smooth and Poincaré type Futaki characters

2.1 Statement

We keep the notations of the previous part; in particular, ω_X is a smooth Kähler metric on X compact, and $\mathscr{F}^D_{[\omega_X]}:\mathfrak{h}^D_{/\!/}\to\mathbb{R}$ denotes the Futaki character associated to the space $\mathscr{M}^D_{[\omega_X]}$ of Poincaré type Kähler metrics on $X\backslash D$ of class $[\omega_X]$.

Recall moreover that if $\mathsf{Z}\in\mathfrak{h}$, we set $f^{\mathsf{Z}}_{\omega_X}$ for the normalised potential of its (Riemannian) gradient part, relatively to ω_X. The purpose of this part is to state and prove an explicit relation between smooth and Poincaré type Futaki characters on $\mathfrak{h}^D_{/\!/}$; this is the main result of this note. We use for this, as intermediates, Futaki characters with respect to sub-divisors of D, e.g. $D-D_j=\sum_{\ell=1,\ell\neq j}^N D_\ell$ if $D=\sum_{\ell=1}^N D_\ell$; the Futaki character is denoted by $\mathscr{F}^{D-D_j}_{[\omega_X]}$ in this case, and is still defined on $\mathfrak{h}^D_{/\!/}$. We denote by $\mathscr{F}_{[\omega_X]}$ the usual Futaki character on X:

Proposition 2.1 *For all $\mathsf{Z}\in\mathfrak{h}^D_{/\!/}$ and for all $j=1,\dots,N$, one has:*

$$\mathscr{F}^D_{[\omega_X]}(\mathsf{Z})=\mathscr{F}^{D-D_j}_{[\omega_X]}(\mathsf{Z})+4\pi\int_{D_j} f^{\mathsf{Z}}_{\omega_X}\frac{(\omega_X|_{D_j})^{m-1}}{(m-1)!}. \tag{5}$$

Consequently, for all $\mathsf{Z}\in\mathfrak{h}^D_{/\!/}$,

$$\mathscr{F}^D_{[\omega_X]}(\mathsf{Z})=\mathscr{F}_{[\omega_X]}(\mathsf{Z})+4\pi\sum_{j=1}^N\int_{D_j} f^{\mathsf{Z}}_{\omega_X}\frac{(\omega_X|_{D_j})^{m-1}}{(m-1)!}. \tag{6}$$

2.2 Proof of Proposition 2.1

Identity (6) clearly follows from an inductive use of identity (5), the proof of which we focus on for the rest of this part.

Fix $Z\in\mathfrak{h}^D_{/\!/}$. To compute $\mathscr{F}^D_{[\omega_X]}(Z)$, we first fix a Poincaré type Kähler metric $\omega\in\mathscr{M}^D_{[\omega_X]}$ as follows. We take $\omega=\omega_X-dd^c\sum_{j=1}^N\log\big(-\log(|\sigma_j|_j^2)\big)$, with $\sigma_j\in\mathscr{O}([D_j])$ such that $D_j=\{\sigma_j=0\}$, and the $|\cdot|_j$ smooth hermitian metrics on the $[D_j]$, chosen so that ω is indeed a (Poincaré type) metric on $X\backslash D$ – see [Auv11, §1.1.1] for details.

Fix now $j\in\{1,\dots,N\}$, set $\varphi_j=-\log\big(-\log(|\sigma_j|_j^2)\big)$, $\psi_j=-\sum_{\ell\neq j}\log\big(-\log(|\sigma_j|_j^2)\big)$, and define $\omega_t=\omega_X+dd^c(\psi_j+t\varphi_j)$ for $t\in[0,1]$. Notice that *these are metrics of Poincaré type on $X\backslash D$ for $t\in(0,1]$ only*, as $\omega_{t=0}=\omega_X-dd^c\sum_{\ell\neq j}\log\big(-\log(|\sigma_j|_j^2)\big)$ *is of Poincaré type on $X\backslash(D-D_j)$* – assuming a good choice of the $|\cdot|_\ell$

for the positivity assertion. Now by Proposition 1.5,

$$\mathscr{F}^{D}_{[\omega_X]}(Z) = \int_{X\backslash D} \mathbf{s}(\omega_t) f^{\mathsf{Z}}_{\omega_t} \frac{\omega_t^m}{m!} \tag{7}$$

for all $t \in (0,1]$. Observe however that the integrand tends uniformly to $\mathbf{s}(\omega_0) f^{\mathsf{Z}}_{\omega_0} \frac{\omega_0^m}{m!}$ away from D_j, as t goes to 0. Our strategy is hence to show that, for the price of the correction $4\pi \int_{D_j} f^{\mathsf{Z}}_{\omega_X} \frac{(\omega_X|_{D_j})^{m-1}}{(m-1)!}$, the formal limit $\int_{X\backslash(D-D_j)} \mathbf{s}(\omega_0) f^{\mathsf{Z}}_{\omega_0} \frac{\omega_0^m}{m!}$ is the limit of (7) as t goes to 0; in other words, we want to show that:

$$\lim_{t\searrow 0} \int_{X\backslash D} \mathbf{s}(\omega_t) f^{\mathsf{Z}}_{\omega_t} \frac{\omega_t^m}{m!} = \int_{X\backslash(D-D_j)} \mathbf{s}(\omega_0) f^{\mathsf{Z}}_{\omega_0} \frac{\omega_0^m}{m!} + 4\pi \int_{D_j} f^{\mathsf{Z}}_{\omega_X} \frac{(\omega_X|_{D_j})^{m-1}}{(m-1)!}, \tag{8}$$

which provides (5), by definition of $\mathscr{F}^{D}_{[\omega_X]}$ – and its independence from $t \in (0,1]$ in (7) –, and of $\mathscr{F}^{D-D_j}_{[\omega_X]}$.

Set $D^j := (D - D_j)|_{D_j}$; admitting momentarily that

$$\int_{D_j} f^{\mathsf{Z}}_{\omega_X} \frac{(\omega_X|_{D_j})^{m-1}}{(m-1)!} = \int_{D_j\backslash D^j} f^{\mathsf{Z}}_{\omega_0} \frac{(\omega_0|_{D_j\backslash D^j})^{m-1}}{(m-1)!}, \tag{9}$$

our aim is to prove (8) with $\int_{D_j\backslash D^j} f^{\mathsf{Z}}_{\omega_0} \frac{(\omega_0|_{D_j\backslash D^j})^{m-1}}{(m-1)!}$ instead of $\int_{D_j} f^{\mathsf{Z}}_{\omega_X} \frac{(\omega_X|_{D_j})^{m-1}}{(m-1)!}$. The key point is the following technical lemma:

Lemma 2.2 *Let $f \in C^\infty\big(X\backslash(D-D_j)\big)$, and $w \in C^\infty_1(X\backslash D_j)$. Then*

$$\lim_{t\searrow 0} \int_{X\backslash D} \mathbf{s}(\omega_t)(f+w) \frac{\omega_t^m}{m!} = \int_{X\backslash(D-D_j)} \mathbf{s}(\omega_0)(f+w) \frac{\omega_0^m}{m!} + 4\pi \int_{D_j\backslash D^j} f \frac{(\omega_0|_{D_j\backslash D^j})^{m-1}}{(m-1)!}. \tag{10}$$

By “$f \in C^\infty\big(X\backslash(D-D_j)\big)$”, we mean: f is smooth on $X\backslash(D-D_j)$, with derivatives bounded at any order with respect to any Poincaré type metric on $X\backslash(D-D_j)$, e.g. ω_0; by “$w \in C^\infty_1(X\backslash D_j)$”, we mean: w smooth on $X\backslash D_j$, with derivatives at any order $\mathcal{O}\big(\big|\log|\sigma_j|_j\big|^{-1}\big)$ with respect to any Poincaré type metric on $X\backslash D_j$, e.g. $\omega_X + dd^c\varphi_j$.

Lemma 2.2 is proven in next section. Let us see for now how it applies to our situation. One has: $f^{\mathsf{Z}}_{\omega_t} = f^{\mathsf{Z}}_{\omega_0} + t(\mathsf{Z}\cdot\varphi_j)$; we already know that $f^{\mathsf{Z}}_{\omega_0} \in C^\infty\big(X\backslash(D-D_j)\big)$, and we check easily that $(\mathsf{Z}\cdot\varphi_j) \in C^\infty_{-1}(X\backslash D_j)$ thanks to the assumption that Z is parallel to D_j. This way, by Lemma 2.2, $\int_{X\backslash D} \mathbf{s}(\omega_t) f^{\mathsf{Z}}_{\omega_0}\, \omega_t^m/m!$ tends to $\int_{X\backslash D} \mathbf{s}(\omega_0) f^{\mathsf{Z}}_{\omega_0}\, \omega_0^m/m! + 4\pi \int_{D_j\backslash D^j} f^{\mathsf{Z}}_{\omega_0}\, (\omega_0|_{D_j\backslash D^j})^{m-1}/(m-1)!$, and $\int_{X\backslash D} \mathbf{s}(\omega_t)(\mathsf{Z}\cdot\varphi_j)\, \omega_t^m/m!$ tends to $\int_{X\backslash D} \mathbf{s}(\omega_0)(\mathsf{Z}\cdot\varphi_j)\, \omega_t^m/m!$ as t goes to 0 – all that matters here is actually this limit existing and being finite. As a result,

$$\int_{X\backslash D} \mathbf{s}(\omega_t) f^{\mathsf{Z}}_{\omega_t} \frac{\omega_t^m}{m!} = \int_{X\backslash D} \mathbf{s}(\omega_t) f^{\mathsf{Z}}_{\omega_0} \frac{\omega_t^m}{m!} + t \int_{X\backslash D} \mathbf{s}(\omega_t)(\mathsf{Z}\cdot\varphi_j) \frac{\omega_t^m}{m!}$$
$$\xrightarrow{t\searrow 0} \int_{X\backslash(D-D_j)} \mathbf{s}(\omega_0) f^{\mathsf{Z}}_{\omega_0} \frac{\omega_0^m}{m!} + 4\pi \int_{D_j\backslash D^j} f^{\mathsf{Z}}_{\omega_0} \frac{(\omega_0|_{D_j\backslash D^j})^{m-1}}{(m-1)!},$$

as wanted.

Apart from the proof of Lemma 2.2, we are left with that of equality (9). We work on $D_j\backslash D^j$ – recall the notation $D^j = (D - D_j)|_{D_j}$ –, where we set $\varpi_s = (1-s)(\omega_X|_{D_j}) + s(\omega_0|_{D_j\backslash D^j})$; these are Poincaré type metrics for $s>0$. In the same fashion as in the proof of Proposition 1.2, growths near D^j allow us to say that

$$s \longmapsto \int_{D_j\backslash D^j} \big(f^{\mathsf{Z}}_{\omega_X} + s(\mathsf{Z}\cdot\psi_j)\big)\varpi_s^{m-1}$$

is smooth, with derivative

$$\int_{D_j\backslash D^j} (\mathsf{Z}\cdot\psi_j)\,\varpi_s^{m-1} + (m-1)\int_{D_j\backslash D^j} \big(f^{\mathsf{Z}}_{\omega_X} + s(\mathsf{Z}\cdot\psi_j)\big) dd^c\psi_j \wedge \varpi_s^{m-2}.$$

In order to conclude as in the proof of Proposition 1.2, since $(\mathsf{Z}\cdot\psi_j)|_{D_j\backslash D^j} = (\mathsf{Z}|_{D_j\backslash D^j})\cdot(\psi_j|_{D_j\backslash D^j})$ as Z is parallel to D_j, we check that the Hodge decomposition out of D_j induces a Hodge decomposition on D_j, up to the mean of the Riemannian/symplectic gradient potentials. Namely, we check that

$$\xi^{\mathsf{Z}|_{D_j}}_{\omega_X|_{D_j}} := (\mathsf{Z}|_{D_j})^{\sharp(g_X|_{D_j})} = \xi_{\mathrm{harm}}|_{D_j} + d(f^{\mathsf{Z}}_{\omega_X}|_{D_j}) + d^c(h^{\mathsf{Z}}_{\omega_X}|_{D_j}), \tag{11}$$

the extension to couples (Poincaré type metric ϖ on $X\backslash(D-D_j)$, restriction of ϖ on $D_j\backslash D^j$) being dealt with as in Proposition 1.2. Now, as harmonic 1-forms are exactly $d-$ and d^c-closed 1-forms on compact Kähler manifolds, (11) is immediate from $\xi^{\mathsf{Z}|_{D_j}}_{\omega_X|_{D_j}} = \xi^{\mathsf{Z}}_{\omega_X}|_{D_j}$, and this latter identity follows at once from the definition of $\mathsf{Z}|_{D_j}$. Indeed, in local holomorphic coordinates $(z^1,\dots,z^m)$ such that D_j is given by $z^1=0$, write $\mathsf{Z} = \mathsf{Z}^k\frac{\partial}{\partial z^k} + \overline{\mathsf{Z}^k}\frac{\partial}{\partial \overline{z^k}}$, and thus $\mathsf{Z}|_{D_j} = \mathsf{Z}^\alpha|_{D_j}\frac{\partial}{\partial z^\alpha} + \overline{\mathsf{Z}}^\alpha|_{D_j}\frac{\partial}{\partial \overline{z^\alpha}}$ – we implicitly sum on repeated Latin indices over $\{1,\dots,m\}$, and on Greek indices over $\{2,\dots,m\}$. The dual 1-forms are given by:

$$\xi^{\mathsf{Z}}_{\omega_X} = \overline{\mathsf{Z}^\ell}(g_X)_{k\bar\ell}dz^k + \mathsf{Z}^\ell(g_X)_{\ell\bar k}d\overline{z^k},\ \ \xi^{\mathsf{Z}|_{D_j}}_{\omega_X|_{D_j}} = \overline{\mathsf{Z}^\beta}|_{D_j}(g_X|_{D_j})_{\alpha\bar\beta}dz^\alpha + \mathsf{Z}^\beta|_{D_j}(g_X|_{D_j})_{\beta\bar\alpha}d\overline{z^\alpha},$$

hence the result after restriction to D_j of $\xi^{\mathsf{Z}}_{\omega_X}$, as $\mathsf{Z}^1|_{D_j} \equiv \overline{\mathsf{Z}^1}|_{D_j} \equiv 0$. □

2.3 Main technical argument: proof of Lemma 2.2

Localisation of the problem. Recall that $\omega_0 = \omega_X + dd^c\psi_j$ is of Poincaré type on $X\backslash(D - D_j)$, and that the $\omega_t = \omega_X + dd^c(t\varphi_j + \psi_j)$, $t \in (0,1]$, are of Poincaré type on $X\backslash D$. Now for all $t \in [0,1]$, $\mathbf{s}(\omega_t)\omega_t^m = 2m\varrho(\omega_0) \wedge \omega_t^m - m dd^c \log\big(\frac{\omega_t^m}{\omega_0^m}\big) \wedge \omega_t^{m-1}$. On the one hand, for f and w as in the statement, as $(f+w)\varrho(\omega_0)\wedge\omega_t^m$ is uniformly dominated by ω^m,

$$2m\int_{X\backslash D}(f+w)\varrho(\omega_0)\wedge\omega_t^m \to 2m\int_{X\backslash D}(f+w)\varrho(\omega_0)\wedge\omega_0^m = \int_{X\backslash D}\mathbf{s}(\omega_0)(f+w)\,\omega_0^m$$

as t tends to 0; one recognises the first term in the right-hand side of (10).

On the other hand, thanks to the uniform convergence of $dd^c \log\big(\frac{\omega_t^m}{\omega_0^m}\big) \wedge \omega_t^{m-1}$ to 0 far from D_j (for ω_0, say), as t tends to 0, we can restrict to f and w with compact supports in a neighbourhood U of holomorphic coordinates $(z^1, \dots, z^m)$ centred at any point of D_j; we also assume that $|z_\ell| \leq e^{-1}$ on U for all ℓ, that $D_j \leq U = \{z^1 = 0\}$, and that the possible other components of D intersecting U are respectively given by $\{z^2 = 0\}, \dots, \{z^k = 0\}$ for the appropriate $k \in \{2, \dots, m\}$.

For fixed $t > 0$, we can write $\omega_t^m/\omega_0^m = v_t/[|z^1|^2\log^2(|z^1|^2)]$ on $U\backslash D$, with v_t positively bounded below, and bounded up to order 2, for $\omega = \omega_{t=1}$; these bounds are not uniform in t though, as $(\omega_t^m/\omega_0^m) \to 1$ far from D_j when $t \searrow 0$. We rather write $|\log(\omega_t^m/\omega_0^m)| \leq C + \log\big(1 + t/[|z^1|^2\log^2(|z^1|^2)]\big)$ for a control uniform in t, with $C > 0$ independent of t.

Both controls come from the expansion $\omega_t = \omega_0 + t\frac{idz^1\wedge d\overline{z^1}}{|z^1|^2\log^2(|z^1|^2)} + \varepsilon_t$, with $|\varepsilon_t|_\omega, |\nabla^\omega\varepsilon_t|_\omega, |(\nabla^\omega)^2\varepsilon_t|_\omega \leq Ct\big|\log|z^1|\big|^{-1}$, where $C > 0$ is independent of t.

Integration by parts. Now as $dd^c\log(|z^1|^2) = 0$ in $U\backslash D_j$, for fixed $t > 0$,

$$\begin{aligned}\int_{U\backslash D}(f+w)dd^c\log\Big(\frac{\omega_t^m}{\omega_0^m}\Big)\wedge\omega_t^{m-1} &= \int_{U\backslash D}(f+w)dd^c\log\Big(\frac{v_t}{\log^2(|z^1|^2)}\Big)\wedge\omega_t^{m-1}\\ &= \int_{U\backslash D}\log\Big(\frac{v_t}{\log^2(|z^1|^2)}\Big)dd^c(f+w)\wedge\omega_t^{m-1};\end{aligned}$$

we perform this integration by parts without boundary terms, as integrands are L^1 at every stage (including the intermediate step, where the integrand is $d(f+w)\wedge d^c\log\big(\frac{v_t}{\log^2(|z^1|^2)}\big)\wedge\omega_t^{m-1}$).

Expand now ω_t^{m-1} as $\omega^{m-1} + (m-1)t\frac{idz^1\wedge d\overline{z^1}}{|z^1|^2\log^2(|z^1|^2)}\wedge\omega_0^{m-2} + \tilde{\varepsilon}_t$, with $|\tilde{\varepsilon}_t|_\omega \leq$

$Ct\big|\log|z^1|\big|^{-1}$; this way,

$$
\begin{aligned}
&\int_{U\backslash D}\log\Big(\frac{v_t}{\log^2(|z^1|^2)}\Big)dd^c(f+w)\wedge\omega_t^{m-1}\\
&\quad=\int_{U\backslash D}\log\Big(\frac{v_t}{\log^2(|z^1|^2)}\Big)dd^c(f+w)\wedge\omega_0^{m-1}\\
&\qquad+(m-1)\int_{U\backslash D}\log\Big(\frac{v_t}{\log^2(|z^1|^2)}\Big)dd^c(f+w)\wedge\frac{t\,idz^1\wedge d\overline{z^1}}{|z^1|^2\log^2(|z^1|^2)}\wedge\omega_0^{m-2}\\
&\qquad+\int_{U\backslash D}\log\Big(\frac{v_t}{\log^2(|z^1|^2)}\Big)dd^c(f+w)\wedge\tilde{\varepsilon}_t.
\end{aligned}
\tag{12}
$$

We deal with the three summands of the right-hand side separately; the aim is to show that when t goes to 0, the first summand provides the "$\int_{D_j}$-term" of (10), whereas the other two tend to 0.

First summand. As $w|_{D_j}=0$, (an easy adaptation of) the classical Lelong formula yields: $\int_{U\backslash D}\log(|z^1|^2)dd^c(f+w)\wedge\omega_0^{m-1}=-4\pi\int_{U\cap(D_j\backslash D^j)}f(\omega_0|_{D_j\backslash D^j})^{m-1}$. Consequently, for $t>0$, as $\omega_t^m/\omega_0^m=v_t/[|z^1|^2\log^2(|z^1|^2)]$,

$$
\begin{aligned}
&\int_{U\backslash D}\log\Big(\frac{v_t}{\log^2(|z^1|^2)}\Big)dd^c(f+w)\wedge\omega_0^{m-1}\\
&\qquad=\int_{U\backslash D}\log\Big(\frac{\omega_t}{\omega_0}\Big)dd^c(f+w)\wedge\omega_0^{m-1}+4\pi\int_{U\cap(D_j\backslash D^j)}f(\omega_0|_{D_j\backslash D^j})^{m-1}.
\end{aligned}
$$

The uniform controls $|\log(\omega_t^m/\omega_0^m)|\leq C+\log\big(1+1/[|z^1|^2\log^2(|z^1|^2)]\big)$, $|(dd^cf\wedge\omega_0^{m-1})/\omega_0^m|\leq C$, $|(dd^cw\wedge\omega_0^{m-1})/\omega^m\big|\leq C\big|\log|z^1|\big|^{-1}$ now allow us[1] to argue by dominated convergence on the first summand of the right-hand side in the latter identity; since the integrand tends to 0 as $t\searrow 0$, we get:

$$
\lim_{t\searrow 0}\int_{U\backslash D}\log\Big(\frac{v_t}{\log^2(|z^1|^2)}\Big)dd^c(f+w)\wedge\omega_0^{m-1}=4\pi\int_{U\cap(D_j\backslash D^j)}f(\omega_0|_{D_j\backslash D^j})^{m-1}.
$$

Third summand of the right-hand side of (12). Use the control on $\tilde{\varepsilon}_t$ to write:

$$
\begin{aligned}
&\Big|\int_{U\backslash D}\log\Big(\frac{v_t}{\log^2(|z^1|^2)}\Big)dd^c(f+w)\wedge\tilde{\varepsilon}_t\Big|\\
&\qquad\leq Ct\|dd^c(f+w)\|_\omega\int_{U\backslash D}\Big|\log\Big(\frac{\omega_t^m}{\omega_0^m}\Big)+\log(|z^1|^2)\Big|\frac{\omega^m}{|\log(|z^1|^2)|};
\end{aligned}
$$

[1] the worst term to deal with is $\int_{U\backslash D}\log\big(1+1/[|z^1|^2\log^2(|z^1|^2)]\big)/\big|\log|z^1|\big|\,\omega^m$, which is finite, as $\log\big(1+1/[|z^1|^2\log^2(|z^1|^2)]\big)/\big|\log|z^1|\big|=1+o(1)$ for z^1 small

the integral of the right-hand side is indeed finite (same argument as in the footnote above), and the left-hand side thus tends to 0 as $t \searrow 0$.

Second summand of the right-hand side of (12). This is probably the most delicate. We rewrite the integral in play as

$$\int_{0<|z^1|\leq 1/e} \frac{t\, idz_1 \wedge d\overline{z^1}}{|z^1|^2 \log^2(|z^1|^2)} \int_{V_{z^1}} \log\Big(\frac{v_t}{\log^2(|z^1|^2)}\Big)\big(dd^c(f+w)\big)|_{V_{z^1}} \wedge (\omega_0|_{V_{z^1}})^{m-2}$$

where the V_{z^1} are the slices $\{z^1 = \text{constant}\}$ of $U\backslash D$. On each such slice, (the restriction of) $f+w$, $d(f+w)$ and $dd^c(f+w)$ are bounded, with respect to (the restriction of) ω_0, hence $\int_{V_{z^1}} \big(dd^c(f+w)\big)|_{V_{z^1}} \wedge (\omega_0|_{V_{z^1}})^{m-2} = 0$ for all $z^1 \neq 0$. Our integral can thus be rewritten as

$$\int_{0<|z^1|\leq 1/e} \frac{t\, idz_1 \wedge d\overline{z^1}}{|z^1|^2 \log^2(|z^1|^2)} \int_{V_{z^1}} \log(v_t)\big(dd^c(f+w)\big)|_{V_{z^1}} \wedge (\omega_0|_{V_{z^1}})^{m-2},$$

that is:

$$\int_{0<|z^1|\leq 1/e} idz_1 \wedge d\overline{z^1} \int_{V_{z^1}} \frac{t \log[|z^1|^2 \log^2(|z^1|^2)\cdot \omega_t^m/\omega_0^m]}{|z^1|^2 \log^2(|z^1|^2)} \big(dd^c(f+w)\big)|_{V_{z^1}} \wedge (\omega_0|_{V_{z^1}})^{m-2}.$$

Now for all $z^1 \neq 0$, $t \in (0,1]$,

$$\begin{aligned}
&\Big|\int_{V_{z^1}} \frac{t \log[|z^1|^2 \log^2(|z^1|^2)\cdot \omega_t^m/\omega_0^m]}{|z^1|^2 \log^2(|z^1|^2)} \big(dd^c(f+w)\big)|_{V_{z^1}} \wedge (\omega_0|_{V_{z^1}})^{m-2}\Big| \\
&\leq C \big\|\big(dd^c(f+w)\big)|_{V_{z^1}}\big\|_{\omega_0|_{V_{z^1}}} \operatorname{Vol}(V_{z^1}) \frac{t}{|z^1|^2 \log^2(|z^1|^2)} \Big[1 + \big|\log[t + |z^1|^2 \log^2(|z^1|^2)]\big|\Big],
\end{aligned}$$

where $\operatorname{Vol}(V_{z^1}) = \int_{V_{z^1}} \omega_0^{m-1}$. This volume, as well as the supremums $\big\|\big(dd^c(f+w)\big)|_{V_{z^1}}\big\|_{\omega_0|_{V_{z^1}}}$ are bounded below independently of z^1 (and of t!) – notice that we restrict to directions parallel to D_j, along which ω_0 and ω are comparable. Now,

$$\begin{aligned}
&\int_{\{0<|z^1|\leq 1/e\}} \frac{t\, idz^1 \wedge d\overline{z^1}}{|z^1|^2 \log^2(|z^1|^2)} \Big[1 + \big|\log[t + |z^1|^2 \log^2(|z^1|^2)]\big|\Big] \\
&\qquad = t\int_{\{0<|z^1|\leq 1/e\}} \frac{idz^1 \wedge d\overline{z^1}}{|z^1|^2 \log^2(|z^1|^2)} + t|\log t| \int_{\{0<|z^1|\leq 1/e\}} \frac{idz^1 \wedge d\overline{z^1}}{|z^1|^2 \log^2(|z^1|^2)} \\
&\qquad\quad + \int_{\{0<|z^1|\leq 1/e\}} \frac{t}{|z^1|^2 \log^2(|z^1|^2)} \log\Big(1 + \frac{|z^1|^2 \log^2(|z^1|^2)}{t}\Big) idz^1 \wedge d\overline{z^1}.
\end{aligned}$$

As $t \searrow 0$, the first two summands of the right-hand side clearly tend to 0; as for the integrand of the third summand, an elementary study of the function $x \mapsto x\log(1+1/x)$ on $(0,\infty)$ shows that it is bounded above by 1, and tends to 0 as $t \searrow 0$. A last use of dominated convergence thus gives that this third summand, hence the whole second summand of (12), tend to 0 as $t \searrow 0$.

Summing up the above analysis of the three summands of the right-hand side of (12), we get:

$$\int_{U\backslash D} (f+w)dd^c \log\left(\frac{\omega_t^m}{\omega_0^m}\right) \wedge \omega_t^{m-1} \xrightarrow{t\searrow 0} 4\pi \int_{U\cap(D_j\backslash D^j)} f\left(\omega_0|_{(D_j\backslash D^j)}\right)^{m-1},$$

and we saw this is equivalent to Lemma 2.2 for our (localised) f and w. □

3 Application to extremal metrics of Poincaré type

3.1 Extension of Proposition 2.1 (smooth divisor)

Noticed that the integral term in (5) does not depend on the smooth metric $\omega_X \in \mathscr{M}_{[\omega_X]}$, as neither $\mathscr{F}^D_{[\omega_X]}(\mathsf{Z})$ nor $\mathscr{F}^{D-D_j}_{[\omega_X]}(\mathsf{Z})$ do. Considerations similar to those invoked when proving (9) tell us moreover that for the price of replacing D_j by $D_j\backslash D^j$, one can replace ω_X by *any* $\omega \in \mathscr{M}^{D-D_j}_{[\omega_X]}$, $\omega|_{D_j\backslash D^j}$ being in that case an element of $\mathscr{M}^{D^j}_{[\omega_X]|_{D_j}}$.

One can go a step further, at least when the divisor is smooth, and take an $\omega \in \mathscr{M}^D_{[\omega_X]}$ which is asymptotically a product near D_j, i.e. for which there exist $a > 0$, $\omega_j \in \mathscr{M}_{[\omega_X]|_{D_j}}$ and $\delta > 0$ such that as soon as $D_j = \{z^1 = 0\}$ in local holomorphic coordinates $(z^1,\dots,z^m)$, then

$$\omega = \frac{a\, idz^1\wedge d\overline{z^1}}{|z^1|^2\log^2(|z^1|^2)} + p^*\omega_j + \mathcal{O}\big(\big|\log|z^1|\big|^{-\delta}\big),$$

where $p(z^1,\dots,z^m) = (z^2,\dots z^m)$, and with the $\mathcal{O}$ understood at any order for ω. This way $\omega|_{D_j}$ still makes sense as an element of $\mathscr{M}_{[\omega_X]|_{D_j}}$, as well as $f^{\mathsf{Z}}_\omega|_{D_j}$, and:

Proposition 3.1 (**D smooth**) *Let $\omega \in \mathscr{M}^D_{[\omega_X]}$, and assume that ω is asymptotically a product near D_j, for $j \in \{1,\dots,N\}$. Then for all $\mathsf{Z} \in \mathfrak{h}^D_{/\!/}$, one has:*

$$\mathscr{F}^D_{[\omega_X]}(\mathsf{Z}) = \mathscr{F}^{D-D_j}_{[\omega_X]}(\mathsf{Z}) + 4\pi\int_{D_j} f^{\mathsf{Z}}_\omega \frac{(\omega|_{D_j})^{m-1}}{(m-1)!}. \tag{13}$$

Proof. — Assume that ω is asymptotically a product as above; then $\omega = \omega_X + dd^c\big(\varphi + \tilde{\psi}\big)$, with $\varphi = -a\log\big(-\log(|z^1|^2)\big)$, and in local holomorphic coordinates $(z^1,\dots,z^m)$ such that $D_j = \{z^1 = 0\}$, $\tilde{\psi} = p^*\psi + \mathcal{O}\big(\big|\log|z^1|\big|^{-\delta}\big)$, where the $\mathcal{O}$ is understood at any order for ω, and where $\psi \in C^\infty(D_j)$ is such that $\omega_{D_j}^{\psi} := \omega_X|_{D_j} + dd^c\psi \in \mathscr{M}_{[\omega_X]}$, and $\omega|_{D_j} = \omega_{D_j}^{\psi}$.

Taking $\mathsf{Z} \in \mathfrak{h}_{/\!/}^D$, $\mathsf{Z}\cdot\varphi = \mathcal{O}\big(\big|\log|z^1|\big|^{-1}\big)$ in coordinates as above, so that $f_\omega^{\mathsf{Z}} = f_{\omega_X}^{\mathsf{Z}} + \mathsf{Z}\cdot\big(\varphi + \tilde{\psi}\big)$, restricts to $f_\omega^{\mathsf{Z}}|_{D_j} + (\mathsf{Z}|_{D_j})\cdot\psi$ on D_j. Now we know from the treatment of equality (9) in the proof of Proposition 2.1 that $d\big(f_\omega^{\mathsf{Z}}|_{D_j} + (\mathsf{Z}|_{D_j})\cdot\psi\big)$ is the gradient part in the Hodge decomposition of the dual 1-form of $(\mathsf{Z}|_{D_j})$ for $\omega_{D_j}^{\psi}$. The analogue moreover holds when replacing ψ by $t\psi$ for $t \in [0,1]$; setting $\omega_{D_j}^t = \omega_X|_{D_j} + t dd^c\psi$ and $f_t = f_\omega^{\mathsf{Z}}|_{D_j} + t(\mathsf{Z}|_{D_j})\cdot\psi$, we thus see that the derivative of $\int_{D_j} f_t\,(\omega_{D_j}^t)^{m-1}$ vanishes thanks to the usual integration by parts, hence the result, in view of (5). ☐

3.2 A numerical constraint on extremal metrics of Poincaré type

We apply what precedes to reformulate the numerical obstruction of [Auv14, §4.2.2], which is a constraint on *extremal Poincaré type metrics* of class $[\omega_X]$:

Theorem 3.2 *Assume that there exists an extremal metric of Poincaré type of class $[\omega_X]$ on $X\backslash D$, and let $\mathsf{K} \in \mathfrak{h}_{/\!/}^D$ be the Riemannian gradient of its scalar curvature. Then for all $j = 1,\dots,N$, setting $D^j = (D - D_j)|_{D_j}$,*

$$\overline{\mathbf{s}}^D < \overline{\mathbf{s}}_{D_j}^{D^j} + \frac{1}{4\pi\,\mathrm{Vol}(D_j)}\big(\mathscr{F}_{[\omega_X]}^{D-D_j}(\mathsf{K}) - \mathscr{F}_{[\omega_X]}^{D}(\mathsf{K})\big), \tag{14}$$

where $\overline{\mathbf{s}}^D$ (resp. $\overline{\mathbf{s}}_{D_j}^{D^j}$) denotes the mean scalar curvature attached to $\mathscr{M}_{[\omega_X]}^D$ (resp. to $\mathscr{M}_{[\omega_X]|_{D_j}}^{D^j}$).

Proof. — Assume for a start that D is smooth. Let $\omega \in \mathscr{M}_{[\omega_X]}^D$ be extremal, and let $\mathsf{K} = \nabla\mathbf{s}(\omega) \in \mathfrak{h}_{/\!/}^D$, where the (Riemannian) gradient ∇ is computed with respect to (the Riemannian metric associated to) ω. According to [Auv14, Thm. 3], ω is asymptotically a product near the divisor, and induces an extremal metric $\omega_j \in \mathscr{M}_{[\omega_X]|_{D_j}}$ for all $j = 1,\dots,N$. We fix one of these j; as $f_\omega^{\mathsf{K}} = \mathbf{s}(\omega) - \overline{\mathbf{s}}^D$, Proposition 3.1 implies:

$$\begin{aligned}\mathscr{F}^{D}_{[\omega_X]}(\mathsf{K}) =&\mathscr{F}^{D-D_j}_{[\omega_X]}(\mathsf{K}) + 4\pi\int_{D_j}(\mathbf{s}(\omega)-\overline{\mathbf{s}}^D)\frac{\omega_j^{m-1}}{(m-1)!}\\ =&\mathscr{F}^{D-D_j}_{[\omega_X]}(\mathsf{K}) - 4\pi\,\mathrm{Vol}(D_j)\overline{\mathbf{s}}^D + 4\pi\int_{D_j}\Big(\mathbf{s}(\omega_j)-\frac{2}{a_j}\Big)\frac{\omega_j^{m-1}}{(m-1)!}\\ =&\mathscr{F}^{D-D_j}_{[\omega_X]}(\mathsf{K}) - 4\pi\,\mathrm{Vol}(D_j)\Big(\overline{\mathbf{s}}^D-\overline{\mathbf{s}}_{D_j}+\frac{2}{a_j}\Big),\end{aligned}$$

where $a_j\in(0,\infty)$ is such that: given a neighbourhood of holomorphic coordinates $(z^1,\dots,z^m)$ in X of any point of D_j such that D_j locally corresponds to $z^1=0$, then $\omega = a_j\frac{idz^1\wedge d\overline{z^1}}{|z^1|^2\log^2(|z^1|^2)}+p^*\omega_j+\mathcal{O}\big(\frac{1}{|\log(|z^1|)|^\delta}\big)$ for some $\delta>0$, and with $p(z^1,\dots,z^m)=(z^2,\dots,z^m)$. As a_j is positive, one gets:

$$\mathscr{F}^{D-D_j}_{[\omega_X]}(\mathsf{K}) > \mathscr{F}^{D}_{[\omega_X]}(\mathsf{K}) + 4\pi\,\mathrm{Vol}(D_j)(\overline{\mathbf{s}}^D-\overline{\mathbf{s}}_{D_j}),$$

of which (14) is simply a rewriting – as D is smooth, $D^j=0$ on D_j.

The simple normal crossing case. The asymptotically product behaviour of the extremal metric ω is not clear anymore when the divisor admits (simple normal) crossings; we thus content ourselves with applying Proposition 2.1, with $\mathsf{Z}=\mathsf{K}$, and ω_X smooth, and adapt our argument as follows. Let φ so that $\omega=\omega_X+dd^c\varphi$; then $f^{\mathsf{K}}_{\omega}=f^{\mathsf{K}}_{\omega_X}+\mathsf{K}\cdot\varphi$, that is, $f^{\mathsf{K}}_{\omega_X}=f^{\mathsf{K}}_{\omega}-\mathsf{K}\cdot\varphi=\mathbf{s}(\omega)-\overline{\mathbf{s}}^D-\mathsf{K}\cdot\varphi$. Remember that $f^{\mathsf{K}}_{\omega_X}$ is smooth on X, and set for the following lines $\omega_{D_j}=\omega_X|_{D_j}$; To compute $f^{\mathsf{K}}_{\omega_X}|_{D_j}$, notice that by Remarks 4.4 and 4.7 in [Auv14], one can find "tubes" around D_j in $X\backslash D$ such that: $\mathbf{s}(\omega)$ and $\mathsf{K}\cdot\varphi$ tend uniformly on compact subsets of these tubes, respectively to $\mathbf{s}(\omega_{D_j}+dd^c\psi)-2/a_j$ and $\mathsf{K}_{D_j}\cdot\psi$, and where: ψ is smooth on $D_j\backslash D^j$, such that $\omega^{\psi}_{D_j}:=\omega_{D_j}+dd^c\psi\in\mathscr{M}^{D^j}_{[\omega_{D_j}]}$, and $a_j>0$ is the inverse of the left-hand side of inequality (35) in [Auv14, Prop. 4.5].

As a consequence, $f^{\mathsf{K}}_{\omega_X}|_{D_j\backslash D^j}=\mathbf{s}(\omega_{D_j}+dd^c\psi)-\overline{\mathbf{s}}^D-2/a_j-\mathsf{K}_{D_j}\cdot\psi$, and Proposition 2.1 yields

$$\mathscr{F}^{D}_{[\omega_X]}(\mathsf{K}) = \mathscr{F}^{D-D_j}_{[\omega_X]}(\mathsf{K}) - 4\pi\int_{D_j\backslash D^j}\Big(\mathbf{s}(\omega^{\psi}_{D_j})-\overline{\mathbf{s}}^D-\frac{2}{a_j}-\mathsf{K}_{D_j}\cdot\psi\Big)\frac{(\omega_{D_j})^{m-1}}{(m-1)!}. \tag{15}$$

We can be more specific when analysing ω near D_j, and see that $\omega^{\psi}_{D_j}$ is extremal, with $\mathsf{K}_{D_j}=\nabla\mathbf{s}(\omega^{\psi}_{D_j})$ and ∇ the Riemannian gradient with respect to $\omega^{\psi}_{D_j}$. In other words, $f^{\mathsf{K}_{D_j}}_{\omega^{\psi}_{D_j}}=\mathbf{s}(\omega^{\psi}_{D_j})-\overline{\mathbf{s}}^{D^j}_{D_j}$, hence

$$f^{\mathsf{K}_{D_j}}_{\omega_{D_j}} = \mathbf{s}(\omega^{\psi}_{D_j})-\overline{\mathbf{s}}^{D^j}_{D_j}-\mathsf{K}_{D_j}\cdot\psi. \tag{16}$$

As $\int_{D_j} f_{\omega_{D_j}}^{\mathsf{K}_{D_j}} \omega_{D_j}^{m-1} = 0$ by definition of the normalised holomorphic potential, using (16), we can rewrite equation (15) as:

$$\mathscr{F}^{D}_{[\omega_X]}(\mathsf{K}) = \mathscr{F}^{D-D_j}_{[\omega_X]}(\mathsf{K}) - 4\pi \operatorname{Vol}(D_j)\big(\overline{\mathbf{s}}^{D^j}_{D_j} - \overline{\mathbf{s}}^{D} - \frac{2}{a_j}\big).$$

We now conclude as in the smooth divisor case, using the positivity of a_j. □

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