Note – Twisted Kähler-Einstein equations.

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In [CDS1], the following 2-parameter family of twisted Kähler-Einstein equations is considered (equation (3.5)):

$$\begin{cases} \omega^n_{\phi_\epsilon(t,\cdot)} = e^{-t\phi_\epsilon(t,\cdot)-(\beta-t)\varphi_\epsilon+h_{\omega_0}} \dfrac{\omega_0^n}{(|S|_h^2+\epsilon)^{1-\beta}}, \\ \phi_\epsilon(0,\cdot) = \psi_\epsilon, \end{cases} \tag{1}$$

with $t \in [0,\beta]$ and $\varepsilon \in (0,1]$, for given smooth families $(\varphi_\epsilon)_{\varepsilon\in(0,1]}$ and $(\psi_\epsilon)_{\varepsilon\in(0,1]}$ admitting uniform C^0 bounds, and with, of course, the initial relation

$$\omega^n_{\psi_\epsilon} = e^{-\beta\varphi_\epsilon+h_{\omega_0}} \frac{\omega_0^n}{(|S|_h^2+\epsilon)^{1-\beta}}.$$

In the particular setting of [CDS1], one also has the relation $\varrho(\omega_{\psi_\epsilon}) \geq \beta\omega_{\varphi_\epsilon}$ for all $\epsilon \in (0,1]$. Theorem 1.2 in [CDS1] being trivial when $\beta = 1$, we assume in what follows that $\beta \in (0,1)$.

Since the authors quote here a not-too-classical argument for the resolution of these equations assuming some bound on the J-functional of the possible solutions – in particular, Yau's technique for the C^0 estimate for the Calabi problem [Yau] has to be modified –, and since some of the techniques involved might have their own interest, we shall give here a quick overview of a possible proof.

The method of resolution for (1), inspired from the classical method of resolution of the Calabi problem suggested by Calabi himself, consists in varying the parameter t for any fixed ϵ. More formally, once ϵ is fixed, one considers the set of those t up to which there exists a smooth family of solutions:

$$\mathcal{S}_\epsilon := \big\{ t \in [0,\beta] \,\big|\, \text{there exists a smooth family of solutions of (1) indexed by } s \in [0,t] \big\}.$$

By definition, $\mathcal{S}_\epsilon$ non-empty, as it contains 0. We shall see that:

- $\mathcal{S}_\epsilon$ is open in $[0,\beta]$, which amounts (up to the regularity of the solutions) to the implicit functions theorem;

- $\mathcal{S}_\epsilon$ is closed in $[0,\beta]$, which involves less standard tools, such as Tian's α-invariant, defined in [Tian].

As an obvious consequence, $\mathcal{S}_\epsilon = [0,\beta]$, and one can then consider $\phi_\epsilon(\beta,\cdot)$.

Before starting, observe that any C^2 solution $\phi_\epsilon(t,\cdot)$ of one of our equations is automatically a Kähler potential, since $\omega^n_{\phi_\epsilon(t,\cdot)}$ is always a volume form, and at some minimum of $\phi_\epsilon(t,\cdot)$, $i\partial\overline{\partial}\phi_\epsilon(t,\cdot)\geq 0$, and therefore $\omega_{\phi_\epsilon(t,\cdot)} = \omega_0 + i\partial\overline{\partial}\phi_\epsilon(t,\cdot) > 0$.

The set $\mathcal{S}_\epsilon$ is open in $[0,\beta]$. Consider the operator

$$\mathcal{F}_\epsilon : (t,\phi,a) \longmapsto \Big(\omega^n_\phi - e^{a-t\phi-(\beta-t)\varphi_\epsilon + h_{\omega_0}} \frac{\omega_0^n}{(|S|^2_h+\epsilon)^{1-\beta}}, I_\epsilon(\phi,t)\Big),$$

where (t,ϕ,a) lives in $[0,\beta]\times\{\text{Kähler potentials for } \omega_0\}\times\mathbb{R}$, and

$$I_\epsilon(\phi,t) = \begin{cases} -\displaystyle\int_X (\phi-\varphi_\epsilon)\omega^n_{\psi_\epsilon} & \text{if } t=0,\\ \displaystyle\frac{1}{t}\Big(\int_X e^{-t(\phi-\varphi_\epsilon)}\omega^n_{\psi_\epsilon} - \int_X \omega_0^n\Big) & \text{if } t\in(0,\beta] \end{cases}$$

(we use this complicated expression to unify the upcoming reasoning, and apply it as well to case $t_0>0$; indeed, at least informally, one sees in equations (1) that for $t>0$, the $\phi_\epsilon(t,\cdot)$ are automatically normalized, whereas for $t=0$, one can add any constant to ψ_ϵ and still get a solution, which might be a source of trouble when applying the implicit function theorem; somehow, the shape of I_ϵ corresponds to forcing the normalization as in the case of a smooth family of solutions – and from now on we assume, without loss of generality, that $\int_X(\psi_\varepsilon-\varphi_\epsilon)\omega^n_{\psi_\epsilon}=0$).

One easily checks that $\mathcal{F}_\epsilon$ is smooth. Differentiate it at some solution of $\mathcal{F}_\epsilon=0$ denoted by $(t_0,\phi_\epsilon(t_0,\cdot),a_{t_0})$, with respect to its second variables (ϕ,a). From our conventions, one actually always has $a_{t_0}=0$; one thus gets (up to multiplication by $-\frac{1}{2}\omega^n_{\phi_\epsilon(t_0,\cdot)}$ for the first half) the operator

$$(v,\alpha)\mapsto\Big(\Delta_{\phi_\epsilon(t_0,\cdot)}v - 2t_0 v + 2\alpha, -\int_X v e^{-t_0(\phi(t_0,\cdot)-\varphi_\epsilon)}\omega^n_{\psi_\epsilon}\Big) \tag{2}$$

(notice here the slight gap of notation with [CDS1], due to our use of the Riemannian non-negative Laplacian). We need to work in Banach spaces to apply the implicit function theorem, hence we restrict to $(v,\alpha)\in C^{4,\gamma}\times\mathbb{R}$ for any $\gamma\in(0,1)$.

Now proving that the operator (2), with image in $C^{2,\gamma}\times\mathbb{R}$, is an isomorphism, amounts to:

- seeing that $\Delta_{\psi_\epsilon} : C^{4,\gamma} \mapsto C^{2,\gamma}$ is an isomorphism when restricting on both sides to functions with zero mean for ω_{ψ_ϵ}, which is perfectly standard, for the case $t_0 = 0$;

- seeing that $\Delta_{\phi_\epsilon(t_0,\cdot)} - 2t_0 : C^{4,\gamma} \mapsto C^{2,\gamma}$ is an isomorphism, which is true thanks to equation (1) and the setting of [CDS1], telling us that $\varrho(\omega_{\phi_\epsilon(t_0,\cdot)}) = t_0\omega_{\phi_\epsilon(t_0,\cdot)} + (\beta - t_0)\omega_{\psi_\epsilon} + (1-\beta)\chi_\epsilon$ for some positive χ_ϵ, hence $\varrho(\omega_{\phi_\epsilon(t_0,\cdot)}) > t_0\omega_{\phi_\epsilon(t_0,\cdot)}$ (and the $C^{4,\gamma}$ regularity on $\phi_\epsilon(t_0,\cdot)$ legitimates these computations), for the case $t_0 \in (0,\beta]$.

We can now conclude that $\mathcal{S}_\epsilon$ contains a neighbourhood of t_0, since we can extend our family $\{\phi_\epsilon(s,\cdot)\}_{s\in[0,t_0]}$ thanks to the implicit function theorem. More precisely, we can extend it by $C^{4,\gamma}$ solutions of (1) for s slightly bigger than t_0, but then the right-hand-side of the first line of this equation is (at least) $C^{4,\gamma}$, making $\phi_\epsilon(t,\cdot)$ a $C^{6,\gamma}$ solution by Yau's solution of the Calabi problem [Yau] and so on, so that finally the solutions are smooth.

Notice furthermore that the previous argument showed the uniqueness of a germ of solutions near any $\phi_\epsilon(t_0,\cdot)$, $t_0 \in [0,\beta]$. Actually, this proves a bit more, namely: given two smooth families of solutions $\{\phi_\epsilon(s,\cdot)\}_{s\in[0,t_0]}$ and $\{\phi'_\epsilon(s,\cdot)\}_{s\in[0,t]}$ (with common initial value ψ_ϵ and) with $t_0 \le t$, then $\phi_\epsilon(s,\cdot) = \phi'_\epsilon(s,\cdot)$ for all $s \le t_0$ [1].

Let us come now to the slightly more difficult part of the argument:

The set $\mathcal{S}_\epsilon$ *is closed in* $[0,\beta]$. By our definition of $\mathcal{S}_\epsilon$, the closedness of $\mathcal{S}_\epsilon$ amounts to proving that for any t which is the limit of a strictly increasing sequence (t_j) of elements of $\mathcal{S}_\epsilon$, then $t \in \mathcal{S}_\epsilon$. In the classical treatment of the Calabi problem, this part is done by deducing *a priori estimates* on the possible solutions of equation (1) from the equation itself, in order to have some compactness statements on sequences of solutions. The first and most crucial step of those estimates consists in a C^0-estimate, which comes from a non-linear version of Moser iteration scheme. Unfortunately, the $e^{-t\phi_\epsilon(t,\cdot)}$ in equation (1) impedes the transposition of this technique as such, which is why, in order to establish our C^0-estimate, we need here two extra ingredients, both due to Tian [Tian]: the α-invariant, for an upper bound on the $\phi_\epsilon(t,\cdot)$, and a Moser iteration scheme *with respect to the varying metric but with fixed Sobolev constant* for a lower bound. As we shall see, these work thanks to the J-functional estimate. Before seeing how to proceed, let us specify that the

[1] Suppose indeed $\{s \in [0,t_0] |\, \phi_\epsilon(s,\cdot) \neq \phi'_\epsilon(s,\cdot)\}$ is non-empty, and call t its infimum. From the uniqueness argument above, $t > 0$, and by continuity, $t = t_0$ is absurd. By definition, $\phi_\epsilon(s,\cdot) = \phi'_\epsilon(s,\cdot)$ for all $s \in [0,t)$, hence $\phi_\epsilon(t,\cdot) = \phi'_\epsilon(t,\cdot)$ by continuity. But from the uniqueness argument above again, $\phi_\epsilon(s,\cdot) = \phi'_\epsilon(s,\cdot)$ for all s in a neighbourhood of t, contradicting its definition.

higher order estimates follow without much trouble from the C^0-estimate, as in the classical case, which is why we are not dealing with them.

To start with, consider t as above, and thus families $\{\phi^j_\epsilon(s,\cdot)\}_{s\in[0,t_j]}$; from our concluding remark in the previous part, those families match with one another on common s-domains, so we can sum these data up into a single family $\{\phi_\epsilon(s,\cdot)\}_{s\in[0,t)}$. Define, for ϕ any Kähler potential with respect to ω_0 (of class C^2, say), the quantity $J(\phi)=\int_X\phi(\omega_0^n-\omega_\phi^n)$. It is easily seen to be nonnegative, and vanishing if and only if ϕ is constant, since $J(\phi)=\frac{1}{2}\int_X d\phi\wedge d^c\phi\wedge(\omega_0^{n-1}+\cdots+\omega_\phi^{n-1})\geq\frac{1}{2n}\int_X|d\phi|^2_{\omega_0}\omega_0^n$.

Now from the special setting of [CDS1] and the computation p.12, one has that $s\mapsto J\big(\phi_\epsilon(s,\cdot)\big)$ is bounded independently of ϵ and t. We shall see first: there exists C independent of ϵ and s such that: $\sup_{s\in[0,t)}\sup_X\phi_\epsilon(s,\cdot)\leq C$.

Here we introduce Tian's α-invariant $\alpha(X,[\omega_0])$ defined in [Tian] as

$$\alpha(X,[\omega_0])=\sup\Big\{\alpha>0\Big|\,\exists C\geq 0,\,\forall\phi\in\mathcal{H}^0_{\omega_0},\,\int_X e^{-\alpha\phi}\omega_0^n\leq C\Big\},\tag{3}$$

where $\mathcal{H}^0_{\omega_0}$ is the space of ω_0-Kähler potentials with supremum 0. The point in that definition is that the set on which a supremum is taken is *non-empty*, and thus $\alpha(X,[\omega_0])>0$. Moreover, and although we are not using it, as the notation suggests, this invariant only depends on the cohomology class of the metric with respect to which it is computed.

Now fix $\alpha\in\big(0,\alpha(X,[\omega_0])\big)$. Take $s\in[0,t)$, and apply definition (3) to $\phi=\phi_\epsilon(s,\cdot)-\sup_X\phi_\epsilon(s,\cdot)$: there exists C independent of ϵ, s, such that:

$$\int_X e^{-\alpha(\phi_\epsilon(s,\cdot)-\sup_X\phi_\epsilon(s,\cdot))}\omega_0^n\leq C,$$

By convexity of the exponential and after rearranging the terms of the inequality, one has:

$$(n!\,\mathrm{Vol})\sup_X\phi_\epsilon(s,\cdot)=\int_X\sup_X\phi_\epsilon(s,\cdot)\omega_0^n\leq\int_X\phi_\epsilon(s,\cdot)\omega_0^n+\frac{\log C}{\alpha};$$

we are thus left with finding an upper bound on $\int_X\phi_\epsilon(s,\cdot)\omega_0^n$. Since $\int_X\phi_\epsilon(s,\cdot)\omega_0^n=J_0\big(\phi_\epsilon(s,\cdot)\big)+\int_X\phi_\epsilon(s,\cdot)\omega^n_{\phi_\epsilon(s,\cdot)}$, an upper bound on $\int_X\phi_\epsilon(s,\cdot)\omega^n_{\phi_\epsilon(s,\cdot)}$ will do as well. But from equation (1) with parameters 0 and s, we have:

$$\omega^n_{\phi_\epsilon(s,\cdot)}=e^{-s(\phi_\epsilon(s,\cdot)-\varphi_\epsilon)}\omega^n_{\psi_\epsilon},\qquad\text{i.e.}\qquad\omega^n_{\psi_\epsilon}=e^{s(\phi_\epsilon(s,\cdot)-\varphi_\epsilon)}\omega^n_{\phi_\epsilon(s,\cdot)}.$$

Both members have integral $(n!\,\mathrm{Vol})$ over X, so that:

$$\begin{aligned}n!\,\mathrm{Vol}=\int_X\omega^n_{\psi_\epsilon}=\int_X e^{s(\phi_\epsilon(s,\cdot)-\varphi_\epsilon)}\omega^n_{\phi_\epsilon(s,\cdot)}\geq\int_X\big(1+s(\phi_\epsilon(s,\cdot)-\varphi_\epsilon)\big)\omega^n_{\phi_\epsilon(s,\cdot)}\\=n!\,\mathrm{Vol}+s\int_X(\phi_\epsilon(s,\cdot)-\varphi_\epsilon)\omega^n_{\phi_\epsilon(s,\cdot)},\end{aligned}$$

and thus $\int_X \phi_\epsilon(s,\cdot)\omega^n_{\phi_\epsilon(s,\cdot)} \leq \int_X \varphi_\epsilon \omega^n_{\phi_\epsilon(s,\cdot)}$. This latter integral is bounded above independently of $(t,)$ s and ϵ, since φ_ϵ is, and $\int_X \omega^n_{\phi_\epsilon(s,\cdot)} = n!\,\mathrm{Vol}$, which finally gives us the desired upper bound on $\sup_X \phi_\epsilon(s,\cdot)$.

We see now how to get a uniform lower bound on our $\phi_\epsilon(s,\cdot)$, i.e. a constant C depending only on t such that: $\sup_{s\in[t/2,t)} \big(-\inf_X \phi_\epsilon(s,\cdot)\big) \leq C$. This we do using a Moser's iteration scheme, which is also used in Yau's C^0-estimate for the Calabi problem. The difference here nonetheless lies in that we are applying our scheme to some $\phi_- = \max\{0, -\phi_\epsilon(s,\cdot)\}$, and that the metric we use to apply Sobolev embedding is *the varying metric* $\omega_{\phi_\epsilon(s,\cdot)}$.

Let us settle this last point. We are looking for a uniform estimate (at least, independent of s and ϵ); we hence want a constant S such that for all v in $L^{2,1}$ (L^2 functions with L^2 differential),

$$\Big(\int_X |v|^{\frac{2n}{n-1}}\, \omega^n_{\phi_\epsilon(s,\cdot)}\Big)^{\frac{n-1}{n}} \leq S\Big(\int_X |dv|^2_{\omega_{\phi_\epsilon(s,\cdot)}} \omega^n_{\phi_\epsilon(s,\cdot)} + \int_X v^2\, \omega^n_{\phi_\epsilon(s,\cdot)}\Big) \tag{4}$$

We know however that the volume of X is the same for all the $\omega_{\phi_\epsilon(s,\cdot)}$, and for all $s \in [t/2,t)$, as underlined before, $\varrho\big(\omega_{\phi_\epsilon(s,\cdot)}\big) \geq s\omega_{\phi_\epsilon(s,\cdot)} \geq \frac{t}{2}\omega_{\phi_\epsilon(s,\cdot)}$. This way, using results from Croke [Cro] and Li [Li] as Tian does [Tian, p.234], we can thus indeed assert that inequality (4) holds for all $s \in [t/2,t)$ with a constant S depending only on t, and in particular independent of ϵ.

Now fix ϵ and $s \in [t/2,t)$, set $\phi_- = \max\{0, -\phi_\epsilon(s,\cdot)\}$, and assume that the following holds: for all $p \geq 2$,

$$\int_X \big|d\phi_-^{p/2}\big|^2_{\omega_{\phi_\epsilon(s,\cdot)}}\, \omega^n_{\phi_\epsilon(s,\cdot)} \leq \frac{np^2}{2(p-1)} \int_X \phi_-^{p-1}(\omega^n_{\phi_\epsilon(s,\cdot)} - \omega_0^n). \tag{5}$$

Then for all $p > 2$, by (4) applied to $\phi_-^{p/2}$,

$$\begin{aligned}
\|\phi_-\|^p_{L^{pn/(n-1)}_{\omega_{\phi_\epsilon(s,\cdot)}}} = \Big(\int_X \phi_-^{\frac{pn}{n-1}} \omega^n_{\phi_\epsilon(s,\cdot)}\Big)^{\frac{n-1}{n}} &\leq S\Big(\int_X \big|d\phi_-^{p/2}\big|^2_{\omega_{\phi_\epsilon(s,\cdot)}} \omega^n_{\phi_\epsilon(s,\cdot)} + \int_X \phi_-^p\, \omega^n_{\phi_\epsilon(s,\cdot)}\Big)\\
&\leq S\Big(\frac{np^2}{2(p-1)} \int_X \phi_-^{p-1}\, \omega^n_{\phi_\epsilon(s,\cdot)} + \int_X \phi_-^p\, \omega^n_{\phi_\epsilon(s,\cdot)}\Big) \qquad \text{by (5)}\\
&= S\Big(\frac{np^2}{2(p-1)} \|\phi_-\|^{p-1}_{L^{p-1}_{\omega_{\phi_\epsilon(s,\cdot)}}} + \|\phi_-\|^p_{L^p_{\omega_{\phi_\epsilon(s,\cdot)}}}\Big),
\end{aligned}$$

and similarly, $\|\phi_-\|^2_{L^{2n/(n-1)}_{\omega_{\phi_\epsilon(s,\cdot)}}} \leq S\big(2n\int_X \phi_-(\omega^n_{\phi_\epsilon(s,\cdot)} - \omega_0^n)) + \int_X \phi_-^2\, \omega^n_{\phi_\epsilon(s,\cdot)}\big)$. It is now an easy exercise to prove that there exists $C = C(S,n)$ such that

$$-\inf_X \phi_\epsilon(s,\cdot) = \|\phi_-\|_{L^\infty} = \lim_{p\to\infty} \|\phi_-\|_{L^p_{\omega_{\phi_\epsilon(s,\cdot)}}} \leq C\Big(\int_X \phi_-(\omega^n_{\phi_\epsilon(s,\cdot)} - \omega_0^n) + \int_X \phi_-^2\, \omega^n_{\phi_\epsilon(s,\cdot)}\Big)$$

(control inductively $L^{pn/(n-1)}_{\omega_{\phi_\epsilon(s,\cdot)}}$-norms with help of $L^p_{\omega_{\phi_\epsilon(s,\cdot)}}$-norms thanks to the recursive formula above – notice that taking p-th roots largely counterbalance the coefficient of order p in this formula).

Hence we are done if we can control $\int_X \phi_-(\omega^n_{\phi_\epsilon(s,\cdot)} - \omega_0^n)$ and $\int_X \phi_-^2\, \omega^n_{\phi_\epsilon(s,\cdot)}$ in terms of $J\big(\phi_\epsilon(s,\cdot)\big)$ and constants independent of ϵ, say. For the latter, apply Poincaré inequality with respect to $\omega^n_{\phi_\epsilon(s,\cdot)}$ (and thus with a uniform constant C, coming from the uniform positive Ricci lower bound):

$$\begin{aligned}\int_X \phi_-^2\, \omega^n_{\phi_\epsilon(s,\cdot)} &\leq C\Big(\int_X |d\phi_-|^2_{\omega_{\phi_\epsilon(s,\cdot)}}\, \omega^n_{\phi_\epsilon(s,\cdot)} + \big(\int_X \phi_-\, \omega^n_{\phi_\epsilon(s,\cdot)}\big)^2\Big) \\ &\leq C\Big(n\int_X \phi_-(\omega^n_{\phi_\epsilon(s,\cdot)} - \omega_0^n) + \big(\int_X \phi_-\, \omega^n_{\phi_\epsilon(s,\cdot)}\big)^2\Big).\end{aligned}$$

Let us control $\int_X \phi_-\, \omega^n_{\phi_\epsilon(s,\cdot)}$ as announced; it is nonnegative, so we seek an upper bound. We have $\int_X \phi_-\, \omega^n_{\phi_\epsilon(s,\cdot)} = -\int_X \phi_\epsilon(s,\cdot)\, \omega^n_{\phi_\epsilon(s,\cdot)} + \int_{\{\phi_\epsilon(s,\cdot)>0\}} \phi_\epsilon(s,\cdot)\, \omega^n_{\phi_\epsilon(s,\cdot)}$, and on the one hand, $-\int_X \phi_\epsilon(s,\cdot)\, \omega^n_{\phi_\epsilon(s,\cdot)} = J\big(\phi_\epsilon(s,\cdot)\big) + \int_X \phi_\epsilon(s,\cdot)\omega_0^n \leq J\big(\phi_\epsilon(s,\cdot)\big) + (n!\,\mathrm{Vol})\sup_X \phi_\epsilon(s,\cdot)$, whereas on the other hand, $\int_{\{\phi_\epsilon(s,\cdot)>0\}} \phi_\epsilon(s,\cdot)\, \omega^n_{\phi_\epsilon(s,\cdot)} \leq (n!\,\mathrm{Vol})$ $\sup_X \phi_\epsilon(s,\cdot)$, and we control $\sup_X \phi_\epsilon(s,\cdot)$ in terms of the announced parameters.

To conclude, we hence need an upper bound on $\int_X \phi_-(\omega^n_{\phi_\epsilon(s,\cdot)} - \omega_0^n)$. Now as we shall see below, $\int_X \phi_-(\omega^n_{\phi_\epsilon(s,\cdot)} - \omega_0^n) = \frac{1}{2}\int_{\{\phi_\epsilon(s,\cdot)\leq 0\}} d\phi_\epsilon(s,\cdot)\wedge d^c\phi_\epsilon(s,\cdot)\wedge(\omega^{n-1}_{\phi_\epsilon(s,\cdot)} + \cdots + \omega_0^n)$, and since $d\phi_\epsilon(s,\cdot)\wedge d^c\phi_\epsilon(s,\cdot)\wedge(\omega^{n-1}_{\phi_\epsilon(s,\cdot)} + \cdots + \omega_0^n)$ is always nonnegative on X, this is $\leq \frac{1}{2}\int_X d\phi_\epsilon(s,\cdot)\wedge d^c\phi_\epsilon(s,\cdot)\wedge(\omega^{n-1}_{\phi_\epsilon(s,\cdot)} + \cdots + \omega_0^n) = J\big(\phi_\epsilon(s,\cdot)\big)$.

Our last task is proving formula (5); we prove actually for all $p \geq 2$:

$$\int_X \phi_-^{p-2} d\phi_-\wedge d^c\phi_-\wedge(\omega^{n-1}_{\phi_\epsilon(s,\cdot)} + \cdots + \omega_0^n) = \frac{2}{p-1}\int_X \phi_-^{p-1}(\omega^n_{\phi_\epsilon(s,\cdot)} - \omega_0^n).$$

This is done by taking a smooth cut-off function χ, equal to 1 on $(-\infty, 0]$ and vanishing on $[1,+\infty)$. For $k>0$ and $p>2$ (this is similar for $p=2$), if one sets $\chi_k = \chi\big(k\phi_\epsilon(s,\cdot)\big)$,

$$\int_X d\big(\chi_k|\phi_\epsilon(s,\cdot)|^{p-1} d^c\phi_\epsilon(s,\cdot)\wedge(\omega^{n-1}_{\phi_\epsilon(s,\cdot)} + \cdots + \omega_0^n)\big) = 0.$$

Now the integrand can be rewritten as

$$\begin{aligned}&(p-1)\chi_k|\phi_\epsilon(s,\cdot)|^{p-3}\phi_\epsilon(s,\cdot)d\phi_\epsilon(s,\cdot)\wedge d^c\phi_\epsilon(s,\cdot)\wedge(\omega^{n-1}_{\phi_\epsilon(s,\cdot)} + \cdots + \omega_0^n) \\ &+ \chi_k|\phi_\epsilon(s,\cdot)|^{p-1} dd^c\phi_\epsilon(s,\cdot)\wedge(\omega^{n-1}_{\phi_\epsilon(s,\cdot)} + \cdots + \omega_0^n) \\ &+ \chi'\big(k\phi_\epsilon(s,\cdot)\big)k|\phi_\epsilon(s,\cdot)|^{p-1} d\phi_\epsilon(s,\cdot)\wedge d^c\phi_\epsilon(s,\cdot)\wedge(\omega^{n-1}_{\phi_\epsilon(s,\cdot)} + \cdots + \omega_0^n)\big).\end{aligned}$$

So we can conclude after noticing that $dd^c\phi_\epsilon(s,\cdot)\wedge(\omega^{n-1}_{\phi_\epsilon(s,\cdot)}+\cdots+\omega^n_0) = 2(\omega^n_{\phi_\epsilon(s,\cdot)} - \omega^n_0)$, that χ_k is uniformly bounded by $\sup_{\mathbb{R}}\chi$ and $\chi'(k\phi_\epsilon(s,\cdot))k|\phi_\epsilon(s,\cdot)|$ by $\sup_{[0,1]}|\chi'|$ (since whenever $|k\phi_\epsilon(s,\cdot)|\geq 1$, $\chi'(k\phi_\epsilon(s,\cdot)) = 0$), that χ_k converges pointwise to 1 on $\{\phi_\epsilon(s,\cdot)\leq 0\}$ and to 0 on $\{\phi_\epsilon(s,\cdot)>0\}$, and $\chi'(k\phi_\epsilon(s,\cdot))t|\phi_\epsilon(s,\cdot)|$ to 0 pointwise on X, which legitimates an easy use of dominated convergence.

We now conclude with the closedness of $\mathcal{S}_\epsilon$. Take $\gamma\in(0,1)$; our C^0-bound on the $\phi_\epsilon(s,\cdot)$ turns into a $C^{4,\gamma}$-bound by Yau's techniques. One can thus consider a subsequence converging in $C^{4,\gamma/2}$, with $C^{4,\gamma}$ limit we call $\phi_\epsilon(t,\cdot)$. As before, this limit is in fact smooth. We can once more apply implicit functions theorem to $\mathcal{F}_\epsilon$ at $(t,\phi_\epsilon(t,\cdot),0)$; this gives us more specifically (after the bootstrap argument) a smooth family of solutions $\{\phi'_\epsilon(s,\cdot)\}_{s\in(t_1,t]}$ with $\phi'_\epsilon(t\cdot)=\phi_\epsilon(t,\cdot)$, and such that moreover $\phi'_\epsilon(s,\cdot)=\phi_\epsilon(s,\cdot)$ for those s indexing our converging subsequence and close enough to t. Again by unique continuation, this equality holds for all $s\in(t',t)$, so that at last, $\{\phi_\epsilon(s,\cdot)\}_{s\in[0,t]}$ is a smooth family of solutions, or in other words, $t\in\mathcal{S}_\epsilon$.

Let us conclude by the following remark. We insisted that the upper bound on the $\phi_\epsilon(s,\cdot)$ is independent of s and ϵ, and that the lower bound might depend on the range of s, and more precisely on a positive lower bound for s, and is also independent of ϵ. We can thus draw from what precedes a constant C such that for all $\epsilon\in(0,1]$, $\sup_X|\phi_\epsilon(\beta,\cdot)|\leq C$. With other techniques form [CDS1], this gives enough control to show that $\{\omega_{\phi_\epsilon(\beta,\cdot)}\}_{\epsilon\in(0,1]}$, the Ricci tensor of which is at least β, converge in the Gromov-Hausdorff sense to the singular Kähler-Einstein metric ω_{φ_β}.

References

[CDS1] X.X. Chen, S.K. Donaldson, S. Sun, *Kähler-Einstein metrics on Fano manifolds, I: approximation of metrics with cone singularities*, arXiv:1211.4566v1 [math.DG].

[Cro] C. Croke, *Some isoperimetric inequalities and eigenvalue estimates*, Ann. Sci. École Norm. Sup. (4) 13 (1980), no. 4, 419-435.

[Li] P. Li, *On the Sobolev constant and the p-spectrum of a compact Riemannian manifold*, Ann. Sci. École Norm. Sup. (4) 13 (1980), no. 4, 451-468.

[Tian] G. Tian, *On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M)>0$*, Invent. Math. 89 (1987), no. 2, 225-246.

[Yau] S.T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I*, Comm. Pure Appl. Math. 31 (1978), no. 3, 339-411.