Bergman kernels on punctured Riemann surfaces

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Abstract

In this paper we consider a punctured Riemann surface endowed with a Hermitian metric which equals the Poincaré metric near the punctures and a holomorphic line bundle which polarizes the metric. We show that the Bergman kernels can be localized around the singularities and its local model is the Bergman kernel of the punctured unit disc endowed with the standard Poincaré metric. As a consequence, we obtain an optimal uniform estimate of the supremum norm of the Bergman kernel, involving a fractional growth order of the tensor power.

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1 Introduction

In this paper we study the Bergman kernels of a singular Hermitian line bundle over a Riemann surface under the assumption that the curvature has singularities of Poincaré type at a finite set. Our first result shows that the Bergman kernel can be localized around the singularities and its local model is the Bergman kernel of the punctured disc endowed with the standard Poincaré metric. The proof follows the principle that the spectral gap

of the Kodaira Laplacian implies the localization of the Bergman metric [MM1]. By a detailed analysis of the local model we deduce a sharp uniform estimate of the supremum norm of the Bergman kernels.

Let us describe our setting. Let $\overline{\Sigma}$ be a compact Riemann surface and let $D = \{a_1, \ldots, a_N\} \subset \overline{\Sigma}$ be a finite set. We consider the punctured Riemann surface $\Sigma = \overline{\Sigma} \backslash D$ and a Hermitian form ω_Σ on Σ. Let L be a holomorphic line bundle on $\overline{\Sigma}$, and let h be a singular Hermitian metric on L such that:

(α) h is smooth over Σ, and for all $j = 1, \ldots, N$, there is a trivialization of L in the complex neighborhood $\overline{V_j}$ of a_j in $\overline{\Sigma}$, with associated coordinate z_j such that $|1|_h^2(z_j) = \big|\log(|z_j|^2)\big|$.

(β) There exists $\varepsilon > 0$ such that the (smooth) curvature R^L of h satisfies $iR^L \geq \varepsilon\omega_\Sigma$ over Σ and $iR^L = \omega_\Sigma$ on $V_j := \overline{V_j} \backslash \{a_j\}$; in particular, $\omega_\Sigma = \omega_{\mathbb{D}^*}$ in the local coordinate z_j on V_j and (Σ, ω_Σ) is complete.

Here $\omega_{\mathbb{D}^*}$ denotes the Poincaré metric on the punctured unit disc, normalized as follows:

$$\omega_{\mathbb{D}^*} := \frac{idz \wedge d\overline{z}}{|z|^2 \log^2(|z|^2)} \,. \tag{1.1}$$

For $p \geq 1$, let $h^p := h^{\otimes p}$ be the metric induced by h on $L^p|_\Sigma$, where $L^p := L^{\otimes p}$. We denote by $H^0_{(2)}(\Sigma, L^p)$ the space of $\boldsymbol{L}^2$-holomorphic sections of L^p relative to the metrics h^p and ω_Σ,

$$H^0_{(2)}(\Sigma, L^p) = \left\{ S \in H^0(\Sigma, L^p) : \|S\|^2_{\boldsymbol{L}^2} := \int_\Sigma |S|^2_{h^p}\, \omega_\Sigma < \infty \right\}, \tag{1.2}$$

endowed with the obvious inner product. The sections from $H^0_{(2)}(\Sigma, L^p)$ extend to holomorphic sections of L^p over $\overline{\Sigma}$, i. e., (see [MM1, (6.2.17)])

$$H^0_{(2)}(\Sigma, L^p) \subset H^0\big(\overline{\Sigma}, L^p\big). \tag{1.3}$$

In particular, the dimension d_p of $H^0_{(2)}(\Sigma, L^p)$ is finite.

We denote by $B_p \in C^\infty(\Sigma, \mathbb{R})$ the Bergman kernel function of the space $H^0_{(2)}(\Sigma, L^p)$, defined as follows: if $\{S^p_\ell\}_{\ell \geq 1}$ is an orthonormal basis of $H^0_{(2)}(\Sigma, L^p)$, then

$$B_p(x) = \sum_{\ell=1}^{d_p} |S^p_\ell(x)|^2_{h^p} \,. \tag{1.4}$$

Note that B_p is independent of the choice of basis (see [MM1, (6.1.10)] or [CM, Lemma 3.1]). Let $B_p^{\mathbb{D}^*}$ be the Bergman kernel function of $\big(\mathbb{D}^*, \omega_{\mathbb{D}^*}, \mathbb{C}, \big|\log(|z|^2)\big|^p |\cdot|\big)$.

The main result of this paper is a weighted estimate in the C^m-norm near the punctures for the global Bergman kernel B_p compared to the Bergman kernel $B_p^{\mathbb{D}^*}$ of the punctured disc, uniformly in the tensor powers of the given bundle.

Theorem 1.1 *Assume that $(\Sigma, \omega_\Sigma, L, h)$ fulfill conditions (α) and (β). Then the following estimate holds: for every integer $\ell, m \geq 0$, and every $\delta > 0$, there exists a constant $C = C(\ell, m, \delta)$ such that for all $p \in \mathbb{N}^*$, and $z \in V_1 \cup \ldots \cup V_N$ with the local coordinate z_j, in the sense of (2.13),*

$$\left| B_p - B_p^{\mathbb{D}^*} \right|_{C^m}(z_j) \leq C p^{-\ell} \left|\log(|z_j|^2)\right|^{-\delta}. \tag{1.5}$$

Remark 1.2 *Theorem 1.1 admits a generalization to orbifold Riemann surfaces. Assume that $\overline{\Sigma}$ is a compact orbifold Riemann surface, and the finite set $D = \{a_1, \dots, a_N\} \subset \overline{\Sigma}$ does not meet the (orbifold) singular set of $\overline{\Sigma}$. Assume moreover that L is a holomorphic orbifold line bundle on $\overline{\Sigma}$. Let ω_Σ be an orbifold Hermitian form on Σ and h an orbifold Hermitian metric on L in the sense of [MM1, §5.4]. The proof of Theorem 1.1 can be modified to show: If conditions $(\alpha), (\beta)$ hold in this context, then (1.5) holds. In fact, the elliptic estimate [DLM1, (4.14)] and the finite propagation speed of wave operators hold on orbifolds as observed by [M, §6], so the arguments used in this paper go through for orbifolds to get the conclusion.*

By [MM1, Theorems 6.1.1, 6.2.3], for any compact set $K \subset \Sigma$ we have the following expansion on K in any C^m-topology (see Theorem 2.1),

$$\frac{1}{p} B_p(x) = \frac{1}{2\pi} + \sum_{j=1}^{\infty} \boldsymbol{b}_j(x) p^{-j} \qquad \text{as } p \to \infty. \tag{1.6}$$

Theorem 1.1 gives a precise description of B_p near the punctures, in terms of the Bergman kernel function of the Poincaré metric on the local model of the punctured unit disc in $\mathbb{C}$. Note that in the case of smooth metrics with positive curvature the Bergman kernel can be localised and its local model is the Euclidean space endowed with a trivial bundle of positive curvature, see [MM1, Sections 4.1.2–3]. This kind of localization is inspired from the analytic localization technique of Bismut-Lebeau [BL] in local index theory. For the problem at hand here we have to overcome difficulties linked to the presence of singularities.

From a study of the model Bergman kernel functions $B_p^{\mathbb{D}^*}$ on the punctured unit disc, we get the following ratio estimate as a corollary of Theorem 1.1 and Corollary 3.6:

Corollary 1.3 *Let $(\Sigma, \omega_\Sigma, L, h)$ be as in Theorem 1.1. Then*

$$\sup_{x \in \Sigma} B_p(x) = \sup_{x \in \Sigma, \sigma \in H^0_{(2)}(\Sigma, L^p)} \frac{|\sigma(x)|^2_{h^p}}{\|\sigma\|^2_{\boldsymbol{L}^2}} = \left(\frac{p}{2\pi}\right)^{3/2} + \mathcal{O}(p) \qquad \text{as } p \to \infty. \tag{1.7}$$

It is, to our knowledge, the first example of a *uniform* $\boldsymbol{L}^\infty$ asymptotic description of the Bergman kernel function of a singular polarization. This is of particular interest in arithmetic situations. Note that the work of Burgos et al. [BrBK, BuKK] developed the arithmetic intersection theory for log-singular Hermitian metrics, showing in particular that Arakelov heights can be defined, and applied successfully the theory for the Hilbert modular surfaces. Our results provide some possible applications in this direction. For example, the classical arithmetic Hilbert-Samuel theorem [GiS] for positive Hermitian line bundles is usually used to produce global integral sections with small sup-norm; a combination of the recent work [BF] with the distortion estimate of our Corollary 1.3 should give some interesting arithmetic consequence for cusp forms on arithmetic surfaces and Hilbert modular surfaces.

Corollary 1.3 is also quite striking from a Kähler geometry point of view, as the supremum of the Bergman kernel is equivalent to $\left(\frac{p}{2\pi}\right)^n$ on compact polarized manifolds of complex dimension n (cf. Corollary 2.3).

Note also that the behavior of the Bergman kernel on singular Riemann surfaces is relevant for the theory of quantum Hall effect [LCCW] and attracted attention recently.

We give an important example where Theorem 1.1 applies. Let $\overline{\Sigma}$ be a compact Riemann surface of genus g and consider a finite set $D = \{a_1, \dots, a_N\} \subset \overline{\Sigma}$. We also denote by D the divisor $\sum_{j=1}^N a_j$ and let $\mathscr{O}_{\overline{\Sigma}}(D)$ be the associated line bundle. The following

conditions are equivalent:

(i) $\Sigma = \overline{\Sigma} \smallsetminus D$ admits a complete Kähler-Einstein metric ω_Σ with $\mathrm{Ric}_{\omega_\Sigma} = -\omega_\Sigma$,

(ii) $2g - 2 + N > 0$,

(iii) the universal cover of Σ is the upper-half plane $\mathbb{H}$,

(iv) $L = K_{\overline{\Sigma}} \otimes \mathscr{O}_{\overline{\Sigma}}(D)$ is ample.

This follows from the Uniformization Theorem [FK, Chapter IV] and the fact that the Euler characteristic of Σ equals $\chi(\Sigma) = 2-2g-N$ and the degree of L is $2g-2+N = -\chi(\Sigma)$. If one of these equivalent conditions is satisfied, the Kähler-Einstein metric ω_Σ is induced by the Poincaré metric on $\mathbb{H}$; (Σ, ω_Σ) and the formal square root of (L, h) satisfy conditions (α) and (β), see Lemma 6.2. Theorem 1.1 hence applies to this context. Let Γ be the Fuchsian group associated with the above Riemann surface Σ, that is, $\Sigma \cong \Gamma\backslash\mathbb{H}$. Then Γ is a geometrically finite Fuchsian group of the first kind, without elliptic elements. Conversely, if Γ is such a group, then $\Sigma := \Gamma\backslash\mathbb{H}$ can by compactified by finitely many points $D = \{a_1, \dots, a_N\}$ into a compact Riemann surface $\overline{\Sigma}$ such that the equivalent conditions (i)-(iv) above are fulfilled. Let $\mathcal{S}_{2p}^\Gamma$ be the space of cusp forms (Spitzenformen) of weight $2p$ of Γ endowed with the Petersson inner product. We can form the Bergman kernel function of $\mathcal{S}_p^\Gamma$ as in (1.4), denoted by B_p^Γ. We deduce from Corollary 1.3:

Corollary 1.4 *Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ be a geometrically finite Fuchsian group of the first kind without elliptic elements. Let B_p^Γ be the Bergman kernel function of cusp forms of weight $2p$. If Γ is cocompact then uniformly on $\Gamma\backslash\mathbb{H}$,*

$$B_p^\Gamma(x) = \frac{p}{\pi} + \mathcal{O}(1), \qquad \text{as } p \to \infty. \tag{1.8}$$

If Γ is not cocompact then

$$\sup_{x\in\Gamma\backslash\mathbb{H}} B_p^\Gamma(x) = \left(\frac{p}{\pi}\right)^{3/2} + \mathcal{O}(p), \qquad \text{as } p \to \infty. \tag{1.9}$$

Uniform estimates for $\sup_{x\in\Gamma\backslash\mathbb{H}} B_p^\Gamma(x)$ are relevant in arithmetic geometry and were proved in various degrees of generality and sharpness in [AU, MU, JK, FJK]. In [FJK] it is proved that in the cofinite but non-cocompact case $\sup_{x\in\Gamma\backslash\mathbb{H}} B_p^\Gamma(x) = \mathcal{O}(p^{3/2})$ and the result is optimal, at least up to an additive term in the exponent of the form $-\varepsilon$ for any $\varepsilon > 0$. Estimate (1.9) gives the precise coefficient of the leading term $p^{3/2}$ and is sharp (by killing the "ε from below" from [FJK]). Estimate (1.8) is the consequence of the general expansion of the Bergman kernel on compact manifolds [T, Bou, Ca, Z] (cf. also [DLM1, MM1] and Theorem 2.1).

It turns out that Corollary 1.4 can be formulated so as to underline a certain uniformity in Γ, in the same fashion as in [FJK]:

Theorem 1.5 *Let $\Gamma_0 \subset \mathrm{PSL}(2,\mathbb{R})$ be a fixed Fuchsian subgroup of the first kind without elliptic elements and let $\Gamma \subset \Gamma_0$ be any subgroup of finite index. If Γ_0 is cocompact, then*

$$B_p^\Gamma(x) = \frac{p}{\pi} + \mathcal{O}_{\Gamma_0}(1), \qquad \text{as } p \to \infty. \tag{1.10}$$

If Γ_0 is not cocompact then

$$\sup_{x\in\Gamma\backslash\mathbb{H}} B_p^\Gamma(x) = \left(\frac{p}{\pi}\right)^{3/2} + \mathcal{O}_{\Gamma_0}(p), \qquad \text{as } p \to \infty. \tag{1.11}$$

Here the implied constants in $\mathcal{O}_{\Gamma_0}(1)$, $\mathcal{O}_{\Gamma_0}(p)$ depend solely on Γ_0.

Note that (1.10) is a special case of a more general result which is implied in [MM1, §6.1.2] and which we state as Theorem 2.5 in Section 2.

We consider further extension of Theorem 1.5 to the case when the group Γ_0 has elliptic elements. Then the quotients $\Gamma\backslash\mathbb{H}$ are in general orbifolds. By using the result of Dai-Liu-Ma [DLM1, (5.25)] on the Bergman kernel asymptotics on orbifolds and the orbifold version of Theorem 1.1 we obtain the following.

Theorem 1.6 *Let $\Gamma_0 \subset \mathrm{PSL}(2,\mathbb{R})$ be a fixed Fuchsian subgroup of the first kind. Let $\{x_j\}_{j=1}^q$ be the orbifold points of $\Gamma_0\backslash\mathbb{H}$ and U_{x_j} be a small neighborhood of x_j in $\Gamma_0\backslash\mathbb{H}$. Let $\Gamma \subset \Gamma_0$ be any subgroup of finite index and $\pi_\Gamma : \Gamma\backslash\mathbb{H} \to \Gamma_0\backslash\mathbb{H}$ be the natural projection. If Γ_0 is cocompact, then as $p \to \infty$*

$$B_p^\Gamma(x) = \frac{p}{\pi} + \mathcal{O}_{\Gamma_0}(1), \quad \textit{uniformly on } (\Gamma\backslash\mathbb{H}) \smallsetminus \bigcup_{j=1}^q \pi_\Gamma^{-1}(U_{x_j}). \tag{1.12}$$

On each $\pi_\Gamma^{-1}(U_{x_j})$ we have as $p \to \infty$,

$$B_p^\Gamma(x) = \left(1 + \sum_{\gamma \in \Gamma_{x_j^\Gamma} \smallsetminus \{1\}} \exp\left(ip\theta_\gamma - p(1 - e^{i\theta_\gamma})|z|^2\right)\right)\frac{p}{\pi} + \mathcal{O}_{\Gamma_0}(1), \tag{1.13}$$

where $x_j^\Gamma \in \pi_\Gamma^{-1}(x_j)$ is in the same component of $\pi_\Gamma^{-1}(U_{x_j})$ as x, $e^{i\theta_\gamma}$ is the action of γ on the fiber of $K_{\Gamma\backslash\mathbb{H}}$ at x_j^Γ, and $z = z(x)$ is the coordinate of x in normal coordinates z centered at x_j^Γ in $\mathbb{H}$, and $\Gamma_y = \{\gamma \in \Gamma : \gamma y = y\}$ the stabilizer of y.
In particular, if $q_0 = \mathrm{lcm}\{|\Gamma_{0,x_j}| : j = 1,\dots,q\}$, $n_\Gamma = \max\{|\Gamma_y| : y \in \pi_\Gamma^{-1}(x_j), j = 1,\dots,q\}$, then

$$\sup_{x\in\Gamma\backslash\mathbb{H}} B_{q_0 p}^\Gamma(x) = n_\Gamma \frac{q_0 p}{\pi} + \mathcal{O}_{\Gamma_0}(1). \tag{1.14}$$

If Γ_0 is not cocompact then as $p \to \infty$

$$\sup_{x\in\Gamma\backslash\mathbb{H}} B_p^\Gamma(x) = \left(\frac{p}{\pi}\right)^{3/2} + \mathcal{O}_{\Gamma_0}(p). \tag{1.15}$$

Here again the implied constants in $\mathcal{O}_{\Gamma_0}(1)$, $\mathcal{O}_{\Gamma_0}(p)$ depend solely on Γ_0.

Theorems 1.5, 1.6 sharpen (in an optimal way) the main result of [FJK] which states that

$$\sup_{x\in\Gamma\backslash\mathbb{H}} B_p^\Gamma(x) = \begin{cases} \mathcal{O}_{\Gamma_0}(p) & \text{if } \Gamma_0 \text{ is cocompact,} \\ \mathcal{O}_{\Gamma_0}(p^{3/2}) & \text{if } \Gamma_0 \text{ is not cocompact.} \end{cases} \tag{1.16}$$

We obtain in this way the the precise leading terms in (1.16).

This paper is organized as follows. In Section 2 we recall the Bergman kernel expansion of complete Kähler manifolds and introduce the functional space we need further. In Section 3, we study our model situation: the Bergman kernel on the punctured unit disc with Poincaré metric. In Section 4, we establish the basic weighted elliptic estimate on the punctured unit disc with Poincaré metric uniformly with respect to the p-th power of the trivial line bundle with Poincaré metric. In Section 5, we develop the spectral gap properties of the Kodaira Laplacian and give a rough uniform estimate of an approximation of the Bergman kernel. In Section 6, by combining the finite propagation speed of the wave

operator and Section 5, we establish finally the main results stated in the Introduction. In the Appendix A, we prove a technical result, Lemma 3.4.

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2 Preliminaries

In Section 2.1 we recall by following [MM1] the asymptotics of the Bergman kernels on complete manifolds and prove some results of independent interest about this expansion on Riemann surfaces with locally constant curvature and also about its behavior with respect to coverings. In Section 2.2 we introduce some functional and section spaces that will be used throughout the paper.

2.1 Expansion of Bergman kernels on complete manifolds

For a Hermitian holomorphic line bundle (L, h) on a complex manifold we denote by R^L its Chern curvature and by $c_1(L, h) = \frac{i}{2\pi} R^L$ its Chern form.

Let (M, ω_M) be a complete Kähler manifold of dimension n and (L, h) be a Hermitian holomorphic line bundle on M and K_M be the canonical line bundle on M. Then the $\boldsymbol{L}^2$-norm on $C_0^\infty(M, L^p)$, the space of smooth sections of L^p with compact support, is defined for any $s \in C_0^\infty(M, L^p)$ by

$$\|s\|^2_{\boldsymbol{L}^2} = \int_M |s(x)|^2_{h^p} \frac{\omega_M^n}{n!} \cdot \tag{2.1}$$

Let $\boldsymbol{L}^2(M, L^p)$ be the $\boldsymbol{L}^2$-completion of $(C_0^\infty(M, L^p), \|\cdot\|_{\boldsymbol{L}^2})$. We denote by $\langle \cdot, \cdot \rangle$ the inner product on $\boldsymbol{L}^2(M, L^p)$ induced by this $\boldsymbol{L}^2$-norm. Then the Bergman kernel function $B_p(x) \in C^\infty(M, \mathbb{R})$ is still defined by (1.4) with $\{S_\ell^p\}_{\ell \geq 1}$ an orthonormal basis of $H^0_{(2)}(M, L^p)$, the space of $\boldsymbol{L}^2$-holomorphic sections of L^p on M with respect to (2.1). The Bergman kernel $B_p(x, y)$ is the smooth kernel of the orthonormal projection from $(\boldsymbol{L}^2(M, L^p), \|\cdot\|_{\boldsymbol{L}^2})$ onto $H^0_{(2)}(M, L^p)$. We have

$$B_p(x, y) = \sum_{\ell \geq 1} S_\ell^p(x) \otimes (S_\ell^p(y))^* \in L_x^p \otimes (L_y^p)^*, \quad \text{and } B_p(x, x) = B_p(x). \tag{2.2}$$

Here $(S_\ell^p(y))^* \in (L_y^p)^*$ is the metric dual of $S_\ell^p(y)$ with respect to h^p.

The Bergman kernel function (1.4) has the following variational characterization (see [CM, Lemma 3.1]):

$$B_p(x) = \max\Big\{ |S(x)|^2_{h^p} : S \in H^0_{(2)}(M, L^p),\ \|S\|_{\boldsymbol{L}^2} = 1 \Big\}. \tag{2.3}$$

We recall the expansion theorem for the Bergman kernel on a complete manifold [MM1, Theorem 6.1.1].

Theorem 2.1 *Let (M, ω_M) be a complete Kähler manifold of dimension n and (L, h) be a Hermitian holomorphic line bundle on M. We assume there exist $\varepsilon > 0$, $C > 0$ such that*

$iR^L \geq \varepsilon\omega_M$ and $\mathrm{Ric}_{\omega_M} \geq -C\omega_M$, where $\mathrm{Ric}_{\omega_M} = iR^{K_M^*}$ is the Ricci curvature of ω_M. Then there exist coefficients $\boldsymbol{b}_j \in C^\infty(M)$, $j \in \mathbb{N}$, such that for any compact set $K \subset M$, any $k, m \in \mathbb{N}$, there exists $C_{k,m,K} > 0$ such that for $p \in \mathbb{N}^*$,

$$\Big\| \frac{1}{p^n} B_p(x) - \sum_{j=0}^{k} \boldsymbol{b}_j(x) p^{-j} \Big\|_{C^m(K)} \leqslant C_{k,m,K}\, p^{-k-1}, \tag{2.4}$$

where

$$\boldsymbol{b}_0 = \frac{c_1(L,h)^n}{\omega_M^n}, \;\; \boldsymbol{b}_1 = \frac{\boldsymbol{b}_0}{8\pi}\,(r_\omega - 2\Delta_\omega \log \boldsymbol{b}_0), \tag{2.5}$$

and r_ω, Δ_ω, are the scalar curvature, respectively the (positive) Laplacian, of the Riemannian metric associated to $\omega := c_1(L,h)$.

We write (2.4) shortly as

$$B_p(x) = \sum_{j=0}^{k} \boldsymbol{b}_j(x) p^{n-j} + \mathcal{O}(p^{n-k-1}). \tag{2.6}$$

For compact or certain complete Kähler-Einstein manifolds the expansion was obtain by Tian [T] for $k = 0$ and $m = 2$. For general k, m and compact manifolds the existence of the expansion was first obtained in [Ca, Z].

The proof of [MM1, Theorem 6.1.1] yields immediately the following localization principle for Bergman kernels used in the proof of Corollary 2.4. Namely, the asymptotics of $B_p(x)$ depend only on the geometric data in any neighborhood of $x \in M$. Hence, the Bergman kernel function asymptotics are the same on two open sets (in two possibly different manifolds) over which the geometric data are isometric.

Theorem 2.2 *Let (M_1, ω_{M_1}), (M_2, ω_{M_2}) be complete Kähler manifolds of dimension n and $(L_1, h_1) \to M_1$, $(L_2, h_2) \to M_2$ be Hermitian holomorphic line bundles. We assume there exist $\varepsilon > 0$, $C > 0$ such that for $j = 1, 2$ we have $iR^{L_j} \geq \varepsilon\omega_{M_j}$ and $\mathrm{Ric}_{\omega_{M_j}} \geq -C\omega_{M_j}$. Assume moreover that there are open sets $U_j \subset M_j$, $j = 1, 2$, and biholomorphic isometries $\Phi : U_1 \to U_2$, $\Psi : (L_1|_{U_1}, h_1) \to \Phi^*((L_2|_{U_2}, h_2))$, where Ψ is also a bundle isomorphism. Let us denote by $B_{j,p}$ the Bergman kernel functions of $H^0_{(2)}(M_j, L_j^p)$, $j = 1, 2$. Then for any $k, m \in \mathbb{N}$ and any compact set $K \subset U_1$, we have*

$$B_{1,p} - B_{2,p} \circ \Phi = \mathcal{O}(p^{-k}) \;\; \text{in } C^m(K) \;\; \text{as } p \to \infty.$$

In particular, if $\boldsymbol{b}_{1,j}$ and $\boldsymbol{b}_{2,j}$ denote the coefficients of the expansion (2.6) of $B_{1,p}$ and $B_{2,p}$, then $\boldsymbol{b}_{1,j} = \boldsymbol{b}_{2,j} \circ \Phi$ on U_1 for all $j \in \mathbb{N}$.

We immediately obtain from Theorem 2.1 uniform sup-norm bounds for the Bergman kernel on compact subsets.

Corollary 2.3 *Under the hypotheses of Theorem 2.1, let $K \subset M$ be a compact subset such that $iR^L = \omega_M$ on K. Then uniformly on K,*

$$B_p(x) = \Big(\frac{p}{2\pi}\Big)^n + \mathcal{O}(p^{n-1}), \qquad \text{as } p \to \infty. \tag{2.7}$$

In the case of dimension one and constant curvature we can state the following.

Corollary 2.4 *Assume that M in Theorem 2.1 is a Riemann surface and there exists an open set V such that ω_M has scalar curvature -4 and $iR^L = \omega_M$ on V. Then for any $k, m \in \mathbb{N}$ and any compact set $K \subset V$,*

$$B_p(x) = \frac{1}{2\pi}p - \frac{1}{2\pi} + \mathcal{O}(p^{-k}) \ \text{ in } C^m(K) \text{ as } p \to \infty. \tag{2.8}$$

Proof. — From (2.5) follows that $\boldsymbol{b}_0 = \frac{1}{2\pi}$ and $\boldsymbol{b}_1 = -\frac{1}{2\pi}$ (note that $r_\omega = -8\pi$), thus the task is to prove that the coefficients $\boldsymbol{b}_j$ of the expansions (1.6), (2.6) vanish on V for $j \geq 2$. We divide the proof in three steps.

Firstly, it is easy to observe that $\boldsymbol{b}_j$ are constant functions on V for all $j \in \mathbb{N}$. Indeed, by [MM1, Theorem 4.1.1] we know that $\boldsymbol{b}_j$, $j \in \mathbb{N}$, are polynomials in the curvatures R^L and $R^{T^{(1,0)}M}$ and their derivatives. On V we have $iR^L = \omega_M$ and $iR^{T^{(1,0)}M} = -2\omega_M$. Thus all the derivatives alluded to above vanish on V, hence $\boldsymbol{b}_j$ are polynomials just in R^L and $R^{T^{(1,0)}M}$, hence constant functions on V, for all $j \in \mathbb{N}$.

Secondly, we prove the assertion of the Corollary for a compact Riemann surface Σ_1 with genus $g \geq 2$, such that $\Sigma_1 \sim \Gamma_1 \backslash \mathbb{H}$, with Γ_1 a cocompact Fuchsian group. We endow Σ_1 with the metric ω_{Σ_1} induced from the Poincaré metric of $\mathbb{H}$ with scalar curvature -4. We consider the line bundle $L_1 = T^{*(1,0)}\Sigma_1 = K_{\Sigma_1}$ endowed with the metric h_1 induced by ω_{Σ_1}. Thus $iR^{L_1} = 2\omega_{\Sigma_1}$. Let $B_{1,p}(x)$ be the Bergman kernel function of $H^0(\Sigma_1, L_1^p)$. By our observation above, the coefficients $\boldsymbol{b}_{1,j}$ of the expansion (2.6) are constant functions on Σ_1 for all $j \in \mathbb{N}$. Thus

$$B_{1,p}(x) = \sum_{j=0}^{k} \boldsymbol{b}_{1,j} p^{1-j} + \mathcal{O}(p^{-k-1}). \tag{2.9}$$

By the Riemann-Roch theorem, for $p > 1$,

$$\int_{\Sigma_1} B_{1,p}(x)\omega_{\Sigma_1} = \dim H^0(\Sigma_1, L_1^p) = \int_{\Sigma_1} \Big(p - \frac{1}{2}\Big) c_1(K_{\Sigma_1}, h_1), \tag{2.10}$$

and $c_1(K_{\Sigma_1}, h_1) = \frac{1}{\pi}\omega_{\Sigma_1}$. By plugging the expansion (2.9) into (2.10), identifying the coefficients of the powers of p and taking into account that $\boldsymbol{b}_{1,j}$ are constants we get

$$\boldsymbol{b}_{1,0} = \frac{1}{\pi}, \quad \boldsymbol{b}_{1,1} = -\frac{1}{2\pi}, \quad \boldsymbol{b}_{1,j} = 0 \quad \text{for } j \geq 2. \tag{2.11}$$

Thirdly, we use the localization principle for Bergman kernels formulated in Theorem 2.2. We now identify holomorphically and isometrically L^2 on a neighborhood of $x \in V$ to L_1 on an open set of Σ_1. Indeed, by [W, Theorem 2.5.17 and Corollary 2.5.18], near x, the surface is locally isometric to the Poincaré upper half-plane $\mathbb{H}$, and the holomorphic structure of the surface is determinated by the conformal structure fixed by the metric, thus we obtain a holomorphic and isometric identification Ψ of a convex neighborhood U of $x \in V$ to an open set of Σ_1. Then the curvature of the Chern connection on the line bundle $L^2 \otimes \Psi^* K_{\Sigma_1}^{-1}$ with the induced metric h is zero on U. If σ is a holomorphic frame of $L^2 \otimes \Psi^* K_{\Sigma_1}^{-1}$ on U, this means that $\partial\overline{\partial} \log |\sigma|_h^2 = 0$, so there is a holomorphic function f on U such that $\log |\sigma|_h^2 = 2\mathrm{Im} f$ (which holds in any dimension). Now $e^{-f}\sigma$ is a holomorphic frame of $L^2 \otimes \Psi^* K_{\Sigma_1}^{-1}$ such that $|e^{-f}\sigma|_h^2 = 1$ on U, and this yields a holomorphic and isometric identification of L^2 to $\Psi^* K_{\Sigma_1}$.

By Theorem 2.2, we know the asymptotics of $B_p(x)$ is as same as of $B_{1,p}$, thus from (2.9) and (2.11), we get that (2.8) holds uniformly on $K \subset V$. □

Observe that if $(\Sigma, \omega_\Sigma, L, h)$ fulfill conditions (α) and (β), the hypotheses of Corollary 2.4 are satisfied for $V = V_1 \cup \ldots \cup V_N$ (note that the scalar curvature of the Poincaré metric (1.1) equals -4), thus (2.8) holds on any compact set $K \subset V_1 \cup \ldots \cup V_N$.

The following result is a direct consequence of the proof of [MM1, Theorem 6.1.4], and for completeness, we include the proof in Section 6.

Theorem 2.5 *Let (M, ω_M, L, h) be in Theorem 2.1 and assume moreover that M is compact. Let $\pi_1(M)$ be the fundamental group of $M$and $\widetilde{M}$ be the universal covering of M. For any subgroup $\Gamma \subset \pi_1(M)$ with finite index, we define the Bergman kernel $B_p^\Gamma(x,y)$ on $\Gamma\backslash\widetilde{M}$ with the pull-back objects from $\pi_\Gamma : \Gamma\backslash\widetilde{M} \to M$. Then for any $k, m \in \mathbb{N}$, there exists $C_{k,m} > 0$ such that for any Γ as above we have*

$$\Big\| \frac{1}{p^n} B_p^\Gamma(x) - \sum_{j=0}^{k} (\pi_\Gamma^* \boldsymbol{b}_j)(x) p^{-j} \Big\|_{C^m(\Gamma\backslash\widetilde{M})} \leqslant C_{k,m}\, p^{-k-1}, \tag{2.12}$$

where $\boldsymbol{b}_j$ are the coefficients of the expansion (2.4) *on M.*

2.2 Functional spaces, section spaces

We define a few functional spaces, that will be much helpful in what follows.

(i) $C^0(\mathbb{D}^*, \omega_{\mathbb{D}^*})$ is merely the space of bounded continuous functions on $\mathbb{D}^*$, endowed with the sup norm; notice that the reference to the metric, needed when considering bounds on derivatives, is superfluous here.

(ii) Let $U \subset \Sigma$ be an open set. The space $C^k(U, \omega_\Sigma)$ is defined as the set of C^k functions on U bounded up to order k on U with respect to the metric ω_Σ, and endowed with the natural norm:

$$C^k(U, \omega_\Sigma) = \big\{ f \in C^k(U) : \|f\|_{C^k(U,\omega_\Sigma)} < \infty \big\},$$

with

$$\|f\|_{C^k(U,\omega_\Sigma)} = \sup_{x\in U} |f|_{C^k}(x), \quad |f|_{C^k}(x) = \big(|f| + |\nabla^\Sigma f|_{\omega_\Sigma} + \ldots + |(\nabla^\Sigma)^k f|_{\omega_\Sigma} \big)(x), \tag{2.13}$$

∇^Σ being the Levi-Civita connection attached to ω_Σ.

In the same vein, $C^k(U, \omega_\Sigma, L^p, h^p)$ is the space of C^k sections of L^p on U such that the following norm is bounded for $\sigma \in C^k(U, \omega_\Sigma, L^p, h^p)$:

$$\begin{aligned} &|\sigma|_{C^k(h^p)}(x) = \big(|\sigma|_{h^p} + \big|\nabla^{p,\Sigma}\sigma\big|_{h^p,\omega_\Sigma} + \ldots + \big|(\nabla^{p,\Sigma})^k \sigma\big|_{h^p,\omega_\Sigma} \big)(x), \\ &\|\sigma\|_{C^k(U,\omega_\Sigma)} := \sup_{x\in U} |\sigma|_{C^k(h^p)}(x) < \infty, \end{aligned} \tag{2.14}$$

with $\nabla^{p,\Sigma}$ is the connection on $(T\Sigma)^{\otimes \ell} \otimes L^p$ induced by the Levi-Civita connection associated with ω_Σ and the Chern connection relative to h^p.

(iii) For $k \geq 1$, the space $\boldsymbol{L}^{2,k}\big(\Sigma, \omega_\Sigma, L^p, h^p\big)$ is the Sobolev space of sections of the line bundle L^p endowed with the Hermitian norm h^p over Σ, which are $\boldsymbol{L}^2$ up to order k, with respect to ω_Σ and h^p. This way, elements of $\boldsymbol{L}^{2,k}\big(\Sigma, \omega_\Sigma, L^p, h^p\big)$ are sections σ of L^p with $\boldsymbol{L}^{2,k}_{\mathrm{loc}}$ regularity on Σ, such that:

$$\|\sigma\|^2_{\boldsymbol{L}^{2,k}_p(h)} := \int_\Sigma \big(|\sigma|^2_{h^p} + \big|\nabla^{p,\Sigma}\sigma\big|^2_{h^p,\omega_\Sigma} + \ldots + \big|(\nabla^{p,\Sigma})^k \sigma\big|^2_{h^p,\omega_\Sigma} \big) \omega_\Sigma < \infty. \tag{2.15}$$

Alternatively, $\boldsymbol{L}^{2,k}\big(\Sigma,\omega_\Sigma,L^p,h^p\big)$ is the $\|\cdot\|_{\boldsymbol{L}^{2,k}_p(h)}$ closure of the space of smooth and compactly supported sections of L^p over Σ, with $\|\cdot\|^2_{\boldsymbol{L}^{2,k}_p(h)}$ defined in (2.15). For $k=0$ we simply denote $\|\cdot\|_{\boldsymbol{L}^{2,0}_p(h)}$ by $\|\cdot\|_{\boldsymbol{L}^2_p(h)}$ and the corresponding inner product by $\langle\cdot,\cdot\rangle_p$.

When we apply this definition for the trivial line bundle $\mathbb{C}$ endowed with the non-trivial Hermitian norm $\big|\log(|z|^2)\big|^p h_0$ (the trivial Hermitian norm being h_0), we get the space $\boldsymbol{L}^{2,k}\big(\mathbb{D}^*,\omega_{\mathbb{D}^*},\mathbb{C},\big|\log(|z|^2)\big|^p h_0\big)$ and the norm $\|\cdot\|_{\boldsymbol{L}^{2,k}_p(\mathbb{D}^*)}$.

(iv) We also need in the localization procedure below some *weighted Sobolev spaces on* (Σ,ω_Σ) (*resp. on a double copy of* (Σ,ω_Σ)). We first define the weight function ρ on Σ as a smooth function, equal to 1 far from the punctures, to $\big|\log(|z_j|^2)\big|$ near the puncture a_j, and everywhere ≥ 1. Let now $k\in\mathbb{N}$, and $q\geq 1$; the weighted Sobolev space $\boldsymbol{L}^{q,k}_{\mathrm{wtd}}(\Sigma,\omega_\Sigma)$ is defined as the space of $\boldsymbol{L}^{q,k}_{\mathrm{loc}}$ functions f on Σ, such that:

$$\|f\|^q_{\boldsymbol{L}^{q,k}_{\mathrm{wtd}}}:=\int_\Sigma \rho\big(|f|^q+\ldots+|(\nabla^\Sigma)^k f|^q_{\omega_\Sigma}\big)\omega_\Sigma<\infty. \tag{2.16}$$

Notice moreover that $\Sigma\times\Sigma$ is the complement of a simple normal crossing divisor in a compact Kähler manifold, namely $\Sigma\times\Sigma=\big(\overline{\Sigma}^2\big)\smallsetminus\mathbf{D}$ with $\mathbf{D}=(D\times\Sigma)+(\Sigma\times D)$. This way, the natural product metric $\omega_{\Sigma\times\Sigma}$ is a Kähler metric of Poincaré type on $\Sigma\times\Sigma$ (see e.g. [Auv, Def. 0.1]). Analogously, $\boldsymbol{L}^{q,k}_{\mathrm{wtd}}(\Sigma\times\Sigma,\omega_{\Sigma\times\Sigma})$ is the space of $\boldsymbol{L}^{q,k}_{\mathrm{loc}}$ functions f on $\Sigma\times\Sigma$, endowed with the product metric $\omega_{\Sigma\times\Sigma}(x,y)=\omega_\Sigma(x)+\omega_\Sigma(y)$ such that

$$\|f\|^q_{\boldsymbol{L}^{q,k}_{\mathrm{wtd}}}:=\int\limits_{(x,y)\in\Sigma\times\Sigma}\rho(x)\rho(y)\big(|f(x,y)|^q+\ldots+|(\nabla^{\Sigma\times\Sigma})^k f(x,y)|^q_{\omega_{\Sigma\times\Sigma}}\big)\omega_\Sigma(x)\omega_\Sigma(y) \tag{2.17}$$

is finite.

Lemma 2.6 *a) We have* $\boldsymbol{L}^{1,3}_{\mathrm{wtd}}(\Sigma,\omega_\Sigma)\hookrightarrow C^0(\Sigma)$*, i. e., there exists* $c_0>0$ *such that for all* $f\in\boldsymbol{L}^{1,3}_{\mathrm{wtd}}(\Sigma,\omega_\Sigma)$ *we have*

$$\|f\|_{C^0(\Sigma,\omega_\Sigma)}\leq c_0\|f\|_{\boldsymbol{L}^{1,3}_{\mathrm{wtd}}}. \tag{2.18}$$

b) There are continuous embeddings

$$\boldsymbol{L}^{2,k}_{\mathrm{wtd}}(\Sigma\times\Sigma,\omega_{\Sigma\times\Sigma})\hookrightarrow C^m(\Sigma\times\Sigma,\omega_{\Sigma\times\Sigma}) \tag{2.19}$$

for all k*,* m *such that* $k>m+2$*.*

Proof. — a) is from [Biq, §4.A and Lemme 4.5]. For b), after noticing that the proof of [Biq, Lemme 4.5] remains valid close to the divisor $\mathbf{D}\subset\overline{\Sigma}\times\overline{\Sigma}$ but far from the crossings $(a_{j_1},a_{j_2})\in\overline{\Sigma}\times\overline{\Sigma}$, we work around one of these, just as in the proof of [Auv, Lemma 4.2]. More precisely, we choose two small punctured discs $\mathbb{D}^*_{r_1}$ and $\mathbb{D}^*_{r_2}$ around a_{j_1} and a_{j_2} in each Σ respectively, and cover the product $\mathbb{D}^*_{r_1}\times\mathbb{D}^*_{r_2}$ in $\Sigma\times\Sigma$ with help of (self-overlapping) holomorphic polydiscs:

$$\begin{aligned}\Phi_{\ell_1,\ell_2}:\mathbb{D}_\epsilon\times\mathbb{D}_\epsilon&\longrightarrow\mathbb{D}^*_{r_1}\times\mathbb{D}^*_{r_2}\\(u,v)&\longmapsto\Big(e^{-2^{\ell_1}\frac{1+u}{1-u}},e^{-2^{\ell_2}\frac{1+v}{1-v}}\Big),\end{aligned}$$

with ℓ_1, $\ell_2 \geq 0$, and $0 < \epsilon < 1$ fixed independently of ℓ_1 and ℓ_2. This way, $\mathbb{D}^*_{r_1} \times \mathbb{D}^*_{r_2} \subset \bigcup_{\ell_1,\ell_2=0}^{\infty} \Phi_{\ell_1,\ell_2}(\mathbb{D}_\epsilon \times \mathbb{D}_\epsilon)$, and we can even assume that $\mathbb{D}^*_{r_1} \times \mathbb{D}^*_{r_2} \subset \bigcup_{\ell_1,\ell_2=0}^{\infty} \Phi_{\ell_1,\ell_2}(\mathbb{D}_{\epsilon/2} \times \mathbb{D}_{\epsilon/2})$. Moreover, for *any* (ℓ_1, ℓ_2),

$$(\Phi_{\ell_1,\ell_2})^* \omega_{\Sigma\times\Sigma} = \frac{i du \wedge d\overline{u}}{(1-|u|^2)^2} + \frac{i dv \wedge d\overline{v}}{(1-|v|^2)^2} := \varpi, \tag{2.20}$$

which does not depend on (ℓ_1, ℓ_2). On the other hand, $(\Phi_{\ell_1,\ell_2})^*\rho(x) = 2^{\ell_1+1}\big|\frac{1+u}{1-u}\big|$, which is of size 2^{ℓ_1+1} for $|u| \leq \epsilon$, with derivatives (at every order) of the same size for ϖ, and similarly for $(\Phi_{\ell_1,\ell_2})^*\rho(y)$ with 2^{ℓ_2+1}.

Set $U = \mathbb{D}_{r_1} \times \mathbb{D}_{r_2} \subset \overline{\Sigma} \times \overline{\Sigma}$, so that $\mathbb{D}^*_{r_1} \times \mathbb{D}^*_{r_2} = U \smallsetminus \mathbf{D}$. Take $w \in \boldsymbol{L}^{q,k}_{\mathrm{wtd}}(\Sigma \times \Sigma, \omega_{\Sigma\times\Sigma})$, $q \geq 1$, $k \geq 0$, and pick $m \geq 0$, $m < k - \frac{2}{q}$, so that $w \in C^m(U \smallsetminus \mathbf{D})$; what precedes thus yields:

$$\begin{aligned}
\|w\|^q_{C^m(U\smallsetminus\mathbf{D})} &\leq \sup_{\ell_1,\ell_2\geq 0} \|(\Phi_{\ell_1,\ell_2})^* w\|^q_{C^m(\mathbb{D}_{\epsilon/2}\times\mathbb{D}_{\epsilon/2},\varpi)} \\
&\leq \sum_{\ell_1,\ell_2=0}^{\infty} \|(\Phi_{\ell_1,\ell_2})^* w\|^q_{C^m(\mathbb{D}_{\epsilon/2}\times\mathbb{D}_{\epsilon/2},\varpi)} \\
&= \sum_{\ell_1,\ell_2=0}^{\infty} \frac{1}{2^{\ell_1+\ell_2+2}} 2^{\ell_1+\ell_2+2} \|(\Phi_{\ell_1,\ell_2})^* w\|^q_{C^m(\mathbb{D}_{\epsilon/2}\times\mathbb{D}_{\epsilon/2},\varpi)} \\
&\simeq \sum_{\ell_1,\ell_2=0}^{\infty} \frac{1}{2^{\ell_1+\ell_2+2}} \big\|(\Phi_{\ell_1,\ell_2})^*\big(\rho(x)^{\frac{1}{q}}\rho(y)^{\frac{1}{q}} w\big)\big\|^q_{C^m(\mathbb{D}_{\epsilon/2}\times\mathbb{D}_{\epsilon/2},\varpi)} \\
&\leq c \sum_{\ell_1,\ell_2=0}^{\infty} \frac{1}{2^{\ell_1+\ell_2}} \big\|(\Phi_{\ell_1,\ell_2})^*\big(\rho(x)^{\frac{1}{q}}\rho(y)^{\frac{1}{q}} w\big)\big\|^q_{\boldsymbol{L}^{q,k}(\mathbb{D}_\epsilon\times\mathbb{D}_\epsilon,\varpi)}
\end{aligned} \tag{2.21}$$

by the *fixed* usual Sobolev embedding (or, more exactly, continuous restriction)

$$\boldsymbol{L}^{q,k}(\mathbb{D}_\epsilon \times \mathbb{D}_\epsilon, \varpi) \longrightarrow C^m(\mathbb{D}_{\epsilon/2} \times \mathbb{D}_{\epsilon/2}, \varpi)$$

applied to *all* the $(\Phi_{\ell_1,\ell_2})^*\big(\rho(x)^{\frac{1}{q}}\rho(y)^{\frac{1}{q}} w\big)$. Now, observe that our choices provide

$$\begin{aligned}
\sum_{\ell_1,\ell_2=0}^{\infty} \frac{1}{2^{\ell_1+\ell_2}} \big\|(\Phi_{\ell_1,\ell_2})^*\big(\rho(x)^{\frac{1}{q}}\rho(y)^{\frac{1}{q}} w\big)\big\|^q_{\boldsymbol{L}^{q,k}(\mathbb{D}_\epsilon\times\mathbb{D}_\epsilon,\varpi)} \\
\leq C(q)\big\|\big(\rho(x)^{\frac{1}{q}}\rho(y)^{\frac{1}{q}} w\big)\big\|^q_{\boldsymbol{L}^{q,k}(\Sigma\times\Sigma,\omega_{\Sigma\times\Sigma})},
\end{aligned} \tag{2.22}$$

as the $\Phi_{\ell_1,\ell_2}(\mathbb{D}_\epsilon \times \mathbb{D}_\epsilon)$ self-overlaps are of order $2^{\ell_1+\ell_2}$, and as each $\Phi_{\ell_1,\ell_2}(\mathbb{D}_\epsilon \times \mathbb{D}_\epsilon)$ overlaps only a finite number of other $\Phi_{\ell'_1,\ell'_2}(\mathbb{D}_\epsilon \times \mathbb{D}_\epsilon)$, this number being bounded independently of ℓ_1 and ℓ_2. Hence

$$\|w\|^q_{C^m(U\smallsetminus\mathbf{D})} \leq C\big\|\big(\rho(x)^{\frac{1}{q}}\rho(y)^{\frac{1}{q}} w\big)\big\|^q_{\boldsymbol{L}^{q,k}(\Sigma\times\Sigma,\omega_{\Sigma\times\Sigma})} \simeq C\|w\|^q_{\boldsymbol{L}^{q,k}_{\mathrm{wtd}}},$$

and one concludes by specializing to $q = 2$, and gathering such estimates around the crossings (a_{j_1}, a_{j_2}) with analogous estimates along the divisor $\mathbf{D} \subset \overline{\Sigma} \times \overline{\Sigma}$ and far from the crossings, and estimates far from the divisor. □

3 Bergman kernels on the punctured unit disc

In this section we give a detailed description of the Bergman kernel on the punctured unit disc. We first obtain an explicit formula in §3.1 and then in §3.2 we get precise asymptotics near the puncture by using a natural rescaling.

3.1 Expression of the Bergman kernels on the punctured unit disc

Let $p \in \mathbb{N}^*$, and

$$H^p_{(2)}(\mathbb{D}^*) := H^0_{(2)}\big(\mathbb{D}^*, \omega_{\mathbb{D}^*}, \mathbb{C}, \big|\log(|z|^2)\big|^p h_0\big), \tag{3.1}$$

be the space of holomorphic functions S on $\mathbb{D}^*$ with finite $\boldsymbol{L}^2$-norm defined in Section 2.2 (iii) for $k = 0$. The purpose here is to study of the Bergman kernel of $H^p_{(2)}(\mathbb{D}^*)$, as $p \to \infty$.

Lemma 3.1 *For $p \geq 2$, the set*

$$\Big\{\Big(\frac{\ell^p}{2\pi(p-1)!}\Big)^{1/2} z^\ell : \ell \in \mathbb{N},\, \ell \geq 1\Big\} \tag{3.2}$$

forms an orthonormal basis of $H^p_{(2)}(\mathbb{D}^*)$.

Proof. — Let $H^0(\mathbb{D}, \mathbb{C})$ be the space of holomorphic function on $\mathbb{D}$. By [MM1, (6.2.17)], we know

$$H^p_{(2)}(\mathbb{D}^*) \subset H^0(\mathbb{D}, \mathbb{C}). \tag{3.3}$$

Note that for $p \geq 2$, $\ell \geq 1$,

$$\begin{aligned} \int_{\mathbb{D}^*} \big|\log(|z|^2)\big|^p \omega_{\mathbb{D}^*} &= \int_{\mathbb{D}^*} \big|\log(|z|^2)\big|^{p-2} \frac{i dz \wedge d\overline{z}}{|z|^2} \\ &= \int_{\mathbb{S}^1} 2^{p-1}\, d\theta \int_0^1 |\log r|^{p-2} \frac{dr}{r} = \infty, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \int_{\mathbb{D}^*} |z^\ell|^2 \big|\log(|z|^2)\big|^p \omega_{\mathbb{D}^*} &= \int_{\mathbb{D}^*} \big|\log(|z|^2)\big|^{p-2} |z|^{2\ell} \frac{i dz \wedge d\overline{z}}{|z|^2} \\ &= \int_{\mathbb{S}^1} 2^{p-1}\, d\theta \int_0^1 r^{2\ell-1} |\log r|^{p-2}\, dr \\ &= 2^p \pi \cdot (2\ell)^{1-p} \cdot \Gamma(p-1) = \frac{2\pi(p-2)!}{\ell^{p-1}} < \infty. \end{aligned} \tag{3.5}$$

By (3.4), (3.5) and the circle invariance of $\omega_{\mathbb{D}^*}$ and $\big|\log(|z|^2)\big|^p h_0$, the set (3.2) forms an orthonormal basis of $H^p_{(2)}(\mathbb{D}^*)$. □

Remark 3.2 *Notice that a similar computation shows that the elements of $H^0_{(2)}(\Sigma, L^p)$ are, for $p \geq 2$, exactly the sections of L^p over the whole $\overline{\Sigma}$ vanishing on the puncture divisor $D = \{a_1, \ldots, a_N\}$.*

Back to $\mathbb{D}^*$ and according to Lemma 3.1, the Bergman kernel of $H^p_{(2)}(\mathbb{D}^*)$, for any $p \geq 2$, is thus

$$B^{\mathbb{D}^*}_p(x, y) = \frac{\big|\log(|y|^2)\big|^p}{2\pi(p-2)!} \sum_{\ell=1}^{\infty} \ell^{p-1} x^\ell \overline{y}^\ell. \tag{3.6}$$

Here the metric dual of the canonical section 1 with respect to h_0 is identified to 1, hence the metric dual of 1 with respect to $\big|\log(|z|^2)\big|^p h_0$ is $1^*(z) = \big|\log(|z|^2)\big|^p 1$. Specializing to the diagonal, we get in particular the Bergman kernel function of $H^p_{(2)}(\mathbb{D}^*)$ for all $p \geq 2$,

$$B^{\mathbb{D}^*}_p(z) = \frac{\big|\log(|z|^2)\big|^p}{2\pi(p-2)!} \sum_{\ell=1}^{\infty} \ell^{p-1} |z|^{2\ell}. \tag{3.7}$$

This readily provides the behavior of $B^{\mathbb{D}^*}_p$ far from $0 \in \mathbb{D}$.

Proposition 3.3 *For any $0 < a < 1$ and any $m \geq 0$, there exists $c = c(a) > 0$ such that*

$$\left\| B_p^{\mathbb{D}^*}(z) - \frac{p-1}{2\pi} \right\|_{C^m(\{a \leq |z| < 1\}, \omega_{\mathbb{D}^*})} = \mathcal{O}(e^{-cp}) \quad \text{as } p \to \infty\,. \tag{3.8}$$

More generally, for any $0 < a < 1$ and $0 < \gamma < \frac{1}{2}$, there exists $c = c(a, \gamma) > 0$ such that

$$\left\| B_p^{\mathbb{D}^*}(z) - \frac{p-1}{2\pi} \right\|_{C^m(\{ae^{-p^\gamma} \leq |z| < 1\}, \omega_{\mathbb{D}^*})} = \mathcal{O}\big(e^{-cp^{1-2\gamma}}\big) \quad \text{as } p \to \infty\,. \tag{3.9}$$

Proof. — Let us recall the celebrated formula from complex analysis:

$$\frac{1}{\sin^2 w} = \sum_{k \in \mathbb{Z}} \frac{1}{(w - k\pi)^2} \qquad \text{on } \mathbb{C} \smallsetminus \pi\mathbb{Z}.$$

Thus for $t > 0$,

$$\sum_{k \in \mathbb{Z}} \frac{1}{(2ki\pi + t)^2} = (e^{t/2} - e^{-t/2})^{-2} = \sum_{\ell=1}^{\infty} \ell e^{-\ell t}. \tag{3.10}$$

Combining (3.7), (3.10) with an easy induction on $p \geq 2$, one gets the identity

$$\begin{aligned} B_p^{\mathbb{D}^*}(z) &= \frac{(p-1)}{2\pi} \sum_{k \in \mathbb{Z}} \frac{\big|\log(|z|^2)\big|^p}{(2ik\pi + |\log(|z|^2)|)^p} \\ &= \frac{(p-1)}{2\pi} \left(1 + \sum_{k \in \mathbb{Z},\, k \neq 0} \frac{\big|\log(|z|^2)\big|^p}{(2ik\pi + |\log(|z|^2)|)^p} \right). \end{aligned} \tag{3.11}$$

To obtain (3.8) for $m = 0$, from (3.11), for $p \geq 2$, we use

$$\begin{aligned} &\sum_{k \in \mathbb{Z},\, k \neq 0} \frac{\big|\log(|z|^2)\big|^p}{|2ik\pi + |\log(|z|^2)||^p} \\ &\qquad < 2 \left(1 + \frac{(2\pi)^2}{\big|\log(|z|^2)\big|^2} \right)^{-\frac{p-2}{2}} \sum_{k=1}^{\infty} \frac{\big|\log(|z|^2)\big|^2}{(2k\pi)^2 + |\log(|z|^2)|^2} \quad \text{for } 0 < |z| \leq e^{-1/2}, \\ &\sum_{k \in \mathbb{Z},\, k \neq 0} \frac{\big|\log(|z|^2)\big|^p}{|2ik\pi + |\log(|z|^2)||^p} \leq 2\big|\log(|z|^2)\big|^p \sum_{k=1}^{\infty} (2k\pi)^{-p} \quad \text{for } e^{-1/2} \leq |z| < 1. \end{aligned} \tag{3.12}$$

For $m \geq 1$, by considering separately $a < |z| \leq e^{-1/2}$ and $e^{-1/2} \leq |z| < 1$ as above, we get again (3.8) from (3.11).

For $ae^{-p^\gamma} \leq |z| \leq e^{-1/2}$, from

$$\begin{aligned} &\left| \log \left(1 + \frac{(2\pi)^2}{\big|\log(|z|^2)\big|^2} \right) \right| \geq \frac{C}{|\log(|z|^2)|^2} \geq Cp^{-2\gamma}, \\ &\sum_{k=1}^{\infty} \frac{\big|\log(|z|^2)\big|^2}{(2k\pi)^2 + |\log(|z|^2)|^2} \leq \sum_{k=1}^{\infty} \frac{Cp^{2\gamma}}{(2k\pi)^2 + 1}, \end{aligned} \tag{3.13}$$

and (3.12), we get also (3.9). □

Observe that the expected behavior for an Einstein metric of scalar curvature -4 such as $\omega_{\mathbb{D}^*}$, at least on compact subsets of $\mathbb{D}^*$, according to (1.6), Theorem 2.1 and Corollary 2.4, is $B_p^{\mathbb{D}^*}(z) - \frac{(p-1)}{2\pi} = \mathcal{O}(p^{-\infty})$. From our explicit description of $B_p^{\mathbb{D}^*}$, we hence benefit an improvement, namely *exponential decay* of the remainder, and extension of such an asymptotic result *up to the exterior boundary* $\partial\mathbb{D}$ of $\mathbb{D}^*$, as well as an estimate on how close to the singularity $0 \in \mathbb{D}$ such an exponential decay holds.

3.2 Asymptotics of the density functions near the puncture

We are also interested in a global description of $B_p^{\mathbb{D}^*}$ *up to the singularity* $0 \in \mathbb{D}$, especially in the geometric context of Theorem 1.1, and such a description requires another angle of attack. Let us simplify notations: for $p \in \mathbb{N}^*$, set

$$\begin{aligned} b_p(y) &= \frac{|\log y|^{p+1}}{2\pi(p-1)!} \sum_{\ell=1}^{\infty} \ell^p y^\ell \qquad \text{for } y \in (0,1), \\ \varphi(\xi) &= e\,\xi\,|\log \xi| \qquad \text{for } \xi \in (0,1), \\ \nu(p) &= (2\pi p)^{1/2} p^p e^{-p} (p!)^{-1} - 1. \end{aligned} \tag{3.14}$$

Note that by Stirling's formula and (3.14),

$$\nu(p) = \mathcal{O}(p^{-1}) \quad \text{as } p \to \infty. \tag{3.15}$$

By (3.7) and (3.14), we have

$$B_{p+1}^{\mathbb{D}^*}(z) = b_p(|z|^2) \quad \text{for } z \in \mathbb{D}^*. \tag{3.16}$$

Motivated by the observation that at fixed y, the index of the largest term of the sum $\sum_{\ell=1}^{\infty} \ell^p y^\ell$ is determined by $y^{1/p}$, we further proceed to the change of variable $x = y^{1/p}$, and focus on the function $f_p : (0,1) \to \mathbb{R}$ given by:

$$\begin{aligned} f_p(x) := b_p(x^p) &= \frac{|\log(x^p)|^{p+1}}{2\pi(p-1)!} \sum_{\ell=1}^{\infty} \ell^p x^{p\ell} \\ &= \frac{p^{p+2}e^{-p}}{2\pi p!} |\log x| \sum_{\ell=1}^{\infty} \big(e x^\ell |\log(x^\ell)|\big)^p \\ &= \Big(\frac{p}{2\pi}\Big)^{3/2} \big(1+\nu(p)\big) |\log x| \sum_{\ell=1}^{\infty} \big(\varphi(x^\ell)\big)^p. \end{aligned} \tag{3.17}$$

The smooth function φ maps $(0,1)$ to $(0,1]$, with $\varphi(\xi) = 1$ iff $\xi = e^{-1}$. Thus, for ℓ fixed, $x \mapsto \big(\varphi(x^\ell)\big)^p$ heuristically converges to a thinner and thinner Gaussian-shaped bump of height 1 centered at $e^{-1/\ell}$, and $|\log x| \sum_{\ell=1}^{\infty} \big(\varphi(x^\ell)\big)^p$ can thus be thought of as a series of these bumps centered at e^{-1}, $e^{-1/2}$, $e^{-1/3}$, of respective heights 1, $\frac{1}{2}$, $\frac{1}{3}$ (because of the factor $|\log x|$) and so on; this actually holds for x in "low regime" ($x \le e^{-p^{-\delta}}$, $\delta > 1/2$, say), the "tail" ($x \ge e^{-p^{-\delta}}$, $0 < \delta < 1/2$) consisting in an agglomeration of such bumps mixing up with one another to follow an almost constant behavior near $x = 1$: see Figure 1 below.

We develop in the following lines some elementary analysis that justifies these heuristic considerations. First, set

$$\begin{aligned} \psi_p(\zeta) &= \big(\varphi(e^{-\zeta})\big)^p = e^{p(1-\zeta+\log\zeta)} \quad \text{for } \zeta > 0, \\ \mathcal{G}_0(\eta) &= e^{-\eta^2/2}, \qquad \mathcal{G}_1(\eta) = \eta^3 e^{-\eta^2/2} \quad \text{for } \eta \in \mathbb{R}. \end{aligned} \tag{3.18}$$

We prove in the appendix A the following estimate, linking ψ_p to the Gaussian-type functions $\mathcal{G}_0$ and $\mathcal{G}_1$:

Lemma 3.4 *There exists a constant C such that for all $\zeta > 0$ and all $p \ge 1$,*

$$\Big|\psi_p(\zeta) - \mathcal{G}_0\big(\sqrt{p}(1-\zeta)\big) + \frac{1}{3\sqrt{p}} \mathcal{G}_1\big(\sqrt{p}(1-\zeta)\big)\Big| \le \frac{C}{p(1+p(1-\zeta)^2)}.$$

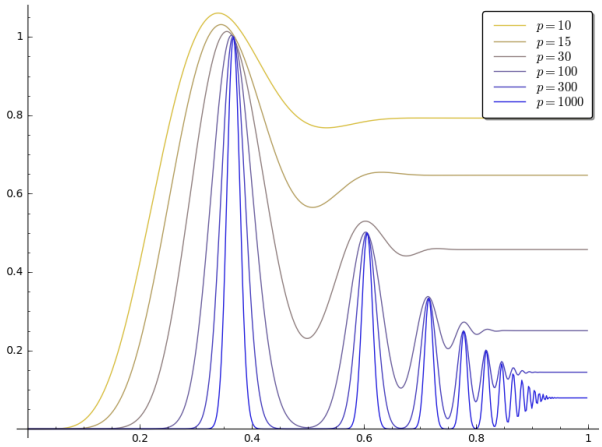

Figure 1 – The scaled functions $\left(\frac{2\pi}{p}\right)^{3/2} f_p$ on $(0,1)$

For $p \geq 1$ and $x \in (0,1)$, set

$$\mathbb{G}_p(x) = |\log x| \Big(\sum_{\ell=1}^{\infty} \mathcal{G}_0\big(\sqrt{p}[1+\log(x^\ell)]\big) - \frac{1}{3\sqrt{p}} \sum_{\ell=1}^{\infty} \mathcal{G}_1\big(\sqrt{p}[1+\log(x^\ell)]\big)\Big). \tag{3.19}$$

Remembering that we are looking for an approximation of $|\log x| \sum_{\ell=1}^{\infty} \big(\varphi(x^\ell)\big)^p$ and keeping in mind the relation (3.18) between φ, p and ψ_p, we state:

Proposition 3.5 *There exists a constant C such that for all $p \geq 1$ and $x \in (0,1)$,*

$$\Big| |\log x| \sum_{\ell=1}^{\infty} \big(\varphi(x^\ell)\big)^p - \mathbb{G}_p(x) \Big| \leq \frac{C}{p + p|\log x|}. \tag{3.20}$$

Corollary 3.6 *There exists a constant C such that for all $p \geq 1$ and $z \in \mathbb{D}^*$,*

$$\Big| \Big(\frac{2\pi}{p}\Big)^{3/2} \big(1+\nu(p)\big)^{-1} B_{p+1}^{\mathbb{D}^*}(z) - \mathbb{G}_p(|z|^{2/p}) \Big| \leq \frac{C}{p + 2|\log|z||}. \tag{3.21}$$

In particular,

$$\sup_{z \in \mathbb{D}^*} B_p^{\mathbb{D}^*}(z) = \big(\frac{p}{2\pi}\big)^{3/2} + \mathcal{O}(p). \tag{3.22}$$

Proof of Proposition 3.5. — Setting

$$\delta_p(\zeta) = \psi_p(\zeta) - \mathcal{G}_0\big(\sqrt{p}(1-\zeta)\big) + \frac{1}{3\sqrt{p}} \mathcal{G}_1\big(\sqrt{p}(1-\zeta)\big). \tag{3.23}$$

For all $p \geq 1$ and $x \in (0,1)$, by (3.18), (3.19) and (3.23),

$$\begin{aligned} \Big| |\log x| \sum_{\ell=1}^{\infty} \big(\varphi(x^\ell)\big)^p - \mathbb{G}_p(x) \Big| &= |\log x| \Big| \sum_{\ell=1}^{\infty} \delta_p\big(-\log(x^\ell)\big) \Big| \\ &\leq |\log x| \sum_{\ell=1,\, \ell \neq |\log x|^{-1}}^{\infty} \big|\delta_p\big(-\log(x^\ell)\big)\big|; \end{aligned} \tag{3.24}$$

this takes into account the vanishing of $\delta_p(\zeta)$ at $\zeta = 1$. By Lemma 3.4, the latter is bounded above by

$$\frac{C}{p}|\log x| \sum_{\ell=1, \ell\neq|\log x|^{-1}}^{\infty} \frac{1}{1+p(\ell\log x+1)^2}; \tag{3.25}$$

we can thus conclude if we bound this quantity above by an expression of type $\frac{C}{p(1+|\log x|)}$.

If $0 < |\log x| \leq 2$, by bounding the terms associated with $\ell = \lfloor -(\log x)^{-1}\rfloor, \lfloor -(\log x)^{-1}\rfloor + 1$ by 1, where we note $\lfloor u \rfloor$ the integer part of $u \in \mathbb{R}$, we get

$$\begin{aligned}\sum_{\substack{\ell=1\\ \ell\neq|\log x|^{-1}}}^{\infty} \frac{1}{1+p(\ell\log x+1)^2} &\leq 2 + \frac{1}{|\log x|}\int_0^{\infty} \frac{d\alpha}{1+p(\alpha-1)^2}\\ &\leq 2 + \frac{1}{|\log x|}\int_{-\infty}^{\infty} \frac{d\alpha}{1+p(\alpha-1)^2}\\ &= 2 + \frac{\pi}{\sqrt{p}|\log x|},\end{aligned}$$

Thus by (3.24) and (3.25), for $0 < |\log x| \leq 2$,

$$\Big||\log x|\sum_{\ell=1}^{\infty}\big(\varphi(x^\ell)\big)^p - \mathbb{G}_p(x)\Big| \leq \frac{C|\log x|}{p} + \frac{C}{p^{3/2}} \leq \frac{C}{p}, \tag{3.26}$$

and this yields the upper bound $\dfrac{C}{p(1+|\log x|)}$ as $\dfrac{1}{1+|\log x|} \geq \dfrac{1}{3}$. This way, the estimate (3.20) is proved on the region $\{0 < |\log x| \leq 2\}$.

Let us assume now that $|\log x| \geq 2$. Then for all $\ell \geq 1$, $\ell|\log x| - 1 \geq \dfrac{\ell|\log x|}{2}$, thus

$$(\ell\log x+1)^2 = (\ell|\log x|-1)^2 \geq \frac{\ell^2|\log x|^2}{4},$$

and

$$\sum_{\substack{\ell=1\\ \ell\neq|\log x|^{-1}}}^{\infty} \frac{|\log x|^2}{1+p(\ell\log x+1)^2} \leq \sum_{\ell=1}^{\infty} \frac{|\log x|^2}{p\big(\frac{\ell^2|\log x|^2}{4}\big)} = \frac{2\pi^2}{3p}. \tag{3.27}$$

In other words, by (3.24), (3.25) and (3.27), on the region $\{|\log x| \geq 2\}$,

$$\Big||\log x|\sum_{\ell=1}^{\infty}\big(\varphi(x^\ell)\big)^p - \mathbb{G}_p(x)\Big| \leq \frac{C}{p^2|\log x|}, \tag{3.28}$$

and this upper bound yields here again an upper bound $\dfrac{C}{p(1+|\log x|)}$, since $\dfrac{1}{|\log x|} \leq \dfrac{3}{2(1+|\log x|)}$ when $|\log x| \geq 2$. By (3.26) and (3.28), we get (3.20). □

Proof of Corollary 3.6. — The first part of the corollary follows at once from Proposition 3.5. The second part is an immediate consequence of the estimate

$$\sup_{x\in(0,1)} |\log x|\mathbb{G}_p(x) = 1 + \mathcal{O}(p^{-1/2}). \tag{3.29}$$

To establish this estimate, let us prove first that

$$1 \leq \sup_{x\in(0,1)} |\log x| \sum_{\ell=1}^{\infty} \mathcal{G}_0\big(\sqrt{p}[1+\log(x^\ell)]\big) = 1 + \mathcal{O}(p^{-1/2}). \tag{3.30}$$

As putting $x = e^{-1}$ in $|\log x| \sum_{\ell=1}^{\infty} \mathcal{G}_0\big(\sqrt{p}[1+\log(x^\ell)]\big)$ gives $1+\sum_{\ell=2}^{\infty} \mathcal{G}_0\big(\sqrt{p}[1+\log(x^\ell)]\big) \geq 1$, we get already that the sup in (3.30) is bounded below by 1.

Now we have

$$\sum_{\ell=\lfloor -(\log x)^{-1}\rfloor}^{\lfloor -(\log x)^{-1}\rfloor+1} e^{-p(1+\ell\log x)^2} \leq 1 + e^{-p(\log x)^2/4} \quad \text{if } |\log x| \leq 1, \tag{3.31}$$

and as a function of $s > 0$, $e^{-p(1+s\log x)^2}$ increases when $s < \lfloor -(\log x)^{-1}\rfloor$ and decreases when $s > \lfloor -(\log x)^{-1}\rfloor + 1$, thus

$$\begin{aligned} &\Bigg(\sum_{\ell=1}^{\lfloor -(\log x)^{-1}\rfloor-1} + \sum_{\ell=\lfloor -(\log x)^{-1}\rfloor+2}^{\infty}\Bigg)\mathcal{G}_0\big(\sqrt{p}\big[1+\log(x^\ell)\big]\big) \\ &\qquad \leq \int_{\mathbb{R}} \mathcal{G}_0\big(\sqrt{p}\big[1 - s|\log(x)|\big]\big)\, ds \\ &\qquad = \int_{\mathbb{R}} \mathcal{G}_0\big(-\sqrt{p}s|\log(x)|\big)\, ds = \frac{C}{\sqrt{p}\,|\log x|}, \end{aligned} \tag{3.32}$$

where we just omit the sum $\sum_{\ell=1}^{\lfloor -(\log x)^{-1}\rfloor-1}$, if $|\log x| \geq 1$; the transition from the second to the third line simply comes from the translation $s \leftarrow s + |\log x|^{-1}$.

Now $|\log x| e^{-p(1+\log x)^2} = (|\log x| - 1)e^{-p(|\log x|-1)^2} + e^{-p(|\log x|-1)^2}$. By using that the function $\eta \mapsto \eta e^{-\eta^2/2}$ is bounded on $\mathbb{R}$, we get from (3.31), (3.32) that for $x \in (0,1)$,

$$|\log x| \sum_{\ell=1}^{\infty} \mathcal{G}_0\big(\sqrt{p}[1+\log(x^\ell)]\big) = \inf\{1, |\log x|\} + \mathcal{O}(p^{-1/2}),\ \ p \to \infty. \tag{3.33}$$

With similar methods, one proves that

$$\sup_{x\in(0,1)} |\log x| \sum_{\ell=1}^{\infty} \mathcal{G}_1\big(\sqrt{p}[1+\log(x^\ell)]\big) = \mathcal{O}(1),\ \ p \to \infty. \tag{3.34}$$

From (3.30) and (3.34), we get (3.29). □

4 Elliptic Estimates for Kodaira Laplacians on $\mathbb{D}^*$ and Σ

In this section, we establish a weighted elliptic estimate for Kodaira Laplacians on $(\mathbb{D}^*, \omega_{\mathbb{D}^*})$ with weight $\big|\log(|z|^2)\big|^p$ such that the estimate is uniform on p, and on $\mathbb{D}^*$. This is the essential analyze input in comparison with the compact situation.

Let $\overline{\partial}^{L^p*}$ be the adjoint of the Dolbeault operator $\overline{\partial}^{L^p}$ on (L^p, h^p) over (Σ, ω_Σ). Then the Kodaira Laplacian is defined as

$$\square_p := (\overline{\partial}^{L^p} + \overline{\partial}^{L^p*})^2 = \overline{\partial}^{L^p}\overline{\partial}^{L^p*} + \overline{\partial}^{L^p*}\overline{\partial}^{L^p} : \Omega^{(0,\bullet)}(\Sigma, L^p) \to \Omega^{(0,\bullet)}(\Sigma, L^p). \tag{4.1}$$

We denote by $\square_p^{\mathbb{D}^*}$ the above operator when $\Sigma = \mathbb{D}^*$.

4.1 Estimate on the punctured disc $\mathbb{D}^*$: degree 0

Note that the Poincaré metric (1.1) on the punctured disc can be written as

$$\omega_{\mathbb{D}^*} = -i\partial\overline{\partial}\log\big(-\log(|z|^2)\big). \tag{4.2}$$

Recall that the norm $\|\cdot\|_{\boldsymbol{L}^{2,2}_p(\mathbb{D}^*)}$ was defined in Section 2.2 (iii). In what follows, we adopt the notation L for the trivial line bundle $\mathbb{C}$ over the open unit disc $\mathbb{D}$, thought of as endowed with the singular Hermitian metric $h_{\mathbb{D}^*} := \big|\log(|z|^2)\big|h_0$; similarly, for $p \geq 1$, L^p will implicitly refer to $\big(\mathbb{C}, \big|\log(|z|^2)\big|^p h_0\big) = \big(\mathbb{C}, h^p_{\mathbb{D}^*}\big)$. Notice that with these conventions, (4.2) can be interpreted as:

$$-i\omega_{\mathbb{D}^*} \text{ is the curvature of } (L, h_{\mathbb{D}^*}) \text{ (and thus } -i\,p\omega_{\mathbb{D}^*} \text{ is that of } (L^p, h^p_{\mathbb{D}^*})). \tag{4.3}$$

We prove in this section the following basic elliptic estimate on the Kodaira Laplacians $\square^{\mathbb{D}^*}_p$, associated to the data $\big(\mathbb{D}^*, \omega_{\mathbb{D}^*}, L^p, h^p_{\mathbb{D}^*}\big)$.

Proposition 4.1 *Let $s \geq 1$. Then there exists $C = C(s, h_{\mathbb{D}^*})$ such that for all $p \geq 1$, and all $\sigma \in \boldsymbol{L}^{2,2s}_p(\mathbb{D}^*)$,*

$$\|\sigma\|^2_{\boldsymbol{L}^{2,2s}_p(\mathbb{D}^*)} \leq C\sum_{j=0}^{s} p^{4(s-j)}\big\|(\square^{\mathbb{D}^*}_p)^j\sigma\big\|^2_{\boldsymbol{L}^2_p(\mathbb{D}^*)}. \tag{4.4}$$

Our strategy is as follows. We will write the detail proof for $s = 1$, then by induction, we get it for $s \geq 2$.

Fixing $f \in C^\infty_0(\mathbb{D}^*)$, for $s = 1$, we first establish an estimate analogous to (4.4) with Δ_0, the Laplace-Beltrami operator of $\omega_{\mathbb{D}^*}$, instead of the Kodaira Laplacian $\square^{\mathbb{D}^*}_p$ associated to $(\mathbb{D}^*, \omega_{\mathbb{D}^*}, L^p, h^p_{\mathbb{D}^*})$. Then we deduce (4.4) by Kähler identities.

To facilitate the computation, we introduce first a new coordinate for $\mathbb{D}^*$ and explain some basic geometric facts.

For $z \in \mathbb{D}^*$, we will use the coordinates $(t, \theta) \in \mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z})$ with

$$t := \log\big(-\log(|z|^2)\big)), \quad z = |z|e^{i\theta}. \tag{4.5}$$

We denote also $\frac{\partial}{\partial t}$ by ∂_t, and $\frac{\partial}{\partial\theta}$ by ∂_θ. Then we compute

$$\overline{z}\log(|z|^2)\frac{\partial}{\partial\overline{z}} = \partial_t - \frac{i}{2}e^t\partial_\theta, \quad (\overline{z}\log(|z|^2))^{-1}d\overline{z} = \frac{1}{2}dt + i\,e^{-t}d\theta. \tag{4.6}$$

Thus we have

$$\overline{\partial} = d\overline{z}\wedge\frac{\partial}{\partial\overline{z}} = (\frac{1}{2}dt + i\,e^{-t}d\theta)(\partial_t - \frac{i}{2}e^t\partial_\theta), \quad \partial t = \frac{1}{2}dt - i\,e^{-t}d\theta. \tag{4.7}$$

From (4.7), we obtain the following useful relation

$$\omega_{\mathbb{D}^*} = -e^{-t}dt\wedge d\theta, \quad \big|\log(|z|^2)\big|^p\omega_{\mathbb{D}^*} = -e^{(p-1)t}dt\wedge d\theta, \tag{4.8}$$

and the metric associated with $\omega_{\mathbb{D}^*}$ in the coordinates (t, θ) is

$$\frac{1}{2}(dt)^2 + 2e^{-2t}(d\theta)^2, \tag{4.9}$$

thus $(\sqrt{2}\partial_t, \frac{1}{\sqrt{2}}e^t\partial_\theta)$ is an orthonormal frame of $\omega_{\mathbb{D}^*}$.

Let $\nabla^{\mathbb{D}^*}$ be the Levi-Civita connection on $(\mathbb{D}^*, \omega_{\mathbb{D}^*})$. Using (4.9) and the equality

$$\nabla^{\mathbb{D}^*}_{\partial_t}\partial_\theta - \nabla^{\mathbb{D}^*}_{\partial_\theta}\partial_t = [\partial_t, \partial_\theta] = 0,$$

we compute that

$$\left\langle \nabla^{\mathbb{D}^*}_{\partial_\theta}\partial_\theta, \partial_t \right\rangle = -\left\langle \nabla^{\mathbb{D}^*}_{\partial_t}\partial_\theta, \partial_\theta \right\rangle = -\frac{1}{2}\partial_t \left\langle \partial_\theta, \partial_\theta \right\rangle = 2e^{-2t}. \tag{4.10}$$

>From (4.10), we get

$$\begin{aligned} &\nabla^{\mathbb{D}^*}_{\partial_t}\partial_t = 0, \quad \nabla^{\mathbb{D}^*}_{\partial_\theta}\partial_\theta = 4e^{-2t}\partial_t, \\ &\nabla^{\mathbb{D}^*}_{\partial_t}\partial_\theta = \nabla^{\mathbb{D}^*}_{\partial_\theta}\partial_t = -\partial_\theta. \end{aligned} \tag{4.11}$$

From (4.11), we get

$$\begin{aligned} &\nabla^{\mathbb{D}^*} d\theta = d\theta \otimes dt + dt \otimes d\theta, \quad \nabla^{\mathbb{D}^*} dt = -4e^{-2t} d\theta \otimes d\theta, \\ &\Delta_0 = -2(\partial_t\partial_t - \partial_t) - \frac{1}{2}e^{2t}\partial_\theta\partial_\theta. \end{aligned} \tag{4.12}$$

Let $\overline{\partial}^*$ (resp. $\overline{\partial}^{L^p *}$) be the adjoint of $\overline{\partial}$ on the trivial line bundle $(\mathbb{C}, h_0)$ (resp. on $(\mathbb{C}, \left|\log(|z|^2)\right|^p h_0)$) over $(\mathbb{D}^*, \omega_{\mathbb{D}^*})$. By (4.6) and (4.8), we have the following expressions in the coordinates (t, θ),

$$\overline{\partial}^{L^p *} = \overline{\partial}^* - p(\overline{\partial}t\wedge)^* \quad \text{and} \quad (\overline{\partial}t\wedge)^* d\overline{z} = \left\langle d\overline{z}, \overline{\partial}t \right\rangle = \overline{z}\log(|z|^2). \tag{4.13}$$

By (4.6) and (4.8), we get for $f \in C^\infty(\mathbb{D}^*)$,

$$\overline{\partial}^{L^p *}(f\overline{\partial}t) = -z\log(|z|^2)\frac{\partial}{\partial z}f + (1-p)f. \tag{4.14}$$

Thus the Kodaira Laplacian associated with $(\mathbb{C}, \left|\log(|z|^2)\right|^p h_0)$ has the form

$$\begin{aligned} \square^{\mathbb{D}^*}_p = \overline{\partial}^{L^p *}\overline{\partial} + \overline{\partial}\,\overline{\partial}^{L^p *} &= \overline{\partial}^*\overline{\partial} + \overline{\partial}\,\overline{\partial}^* - p(\overline{\partial}(\overline{\partial}t\wedge)^* + (\overline{\partial}t\wedge)^*\overline{\partial}) \\ &= \frac{1}{2}\Delta_0 - p\left(\overline{\partial}(\overline{\partial}t\wedge)^* + (\overline{\partial}t\wedge)^*\overline{\partial}\right), \end{aligned} \tag{4.15}$$

where we used the Kähler identity for the last equality.

Proof of Proposition 4.1. — Notice that since the Hermitian line bundles L^p we consider here are powers of the line bundle $\left(\mathbb{C}, \left|\log(|z|^2)\right||\cdot|\right)$, the Chern connections ∇^p acting on the sections of these bundles, which are functions, are given by

$$\nabla^p f = df + pf\,\partial t\,, \;\; f \in C^\infty(\mathbb{D}^*, L^p). \tag{4.16}$$

Therefore, for these f, and $p > 1$,

$$\begin{aligned} \|f\|^2_{L^{2,1}_p(\mathbb{D}^*)} &= \int_{\mathbb{D}^*} \left(|f|^2 + |\nabla^p_{\sqrt{2}\partial_t} f|^2 + |\nabla^p_{(1/\sqrt{2})e^t\partial_\theta} f|^2\right)\left|\log(|z|^2)\right|^p \omega_{\mathbb{D}^*} \\ &\leq 2\int_{\mathbb{D}^*} \left((p^2+1)|f|^2 + 2|\partial_t f|^2 + \tfrac{1}{2}|e^t\partial_\theta f|^2\right)\left|\log(|z|^2)\right|^p \omega_{\mathbb{D}^*}, \end{aligned} \tag{4.17}$$

and, similarly

$$\begin{aligned} \|f\|^2_{\boldsymbol{L}^{2,2}_p(\mathbb{D}^*)} \lesssim \int_{\mathbb{D}^*} &\left(p^4|f|^2 + p^2(|\partial_t f|^2 + |e^t\partial_\theta f|^2)\right. \\ &\left. + |\partial_t^2 f|^2 + |e^t\partial_t\partial_\theta f|^2 + |e^{2t}\partial_\theta^2 f|^2\right)\left|\log(|z|^2)\right|^p \omega_{\mathbb{D}^*}, \end{aligned} \tag{4.18}$$

with the constants hidden in $\lesssim$ *independent of* p.

We will compute everything by using the coordinate (t,θ), then $\int$ means $\int_{\mathbb{R}\times(\mathbb{R}/2\pi\mathbb{Z})}$ and sometimes, we identify S^1 to $\mathbb{R}/2\pi\mathbb{Z}$. Thus by (4.8) and simple integrations by parts, we get

$$\int_{\mathbb{D}^*} |e^t\partial_\theta f|^2\big|\log(|z|^2)\big|^p\omega_{\mathbb{D}^*} = \int |e^t\partial_\theta f|^2 e^{(p-1)t}dtd\theta = \int (e^{2t}\partial_\theta^2 f)fe^{(p-1)t}dtd\theta, \tag{4.19}$$

and

$$\int_{\mathbb{D}^*} |\partial_t f|^2\big|\log(|z|^2)\big|^p\omega_{\mathbb{D}^*} = -\int (\partial_t^2 f)fe^{(p-1)t}dtd\theta + \frac{(p-1)^2}{2}\int f^2e^{(p-1)t}dtd\theta. \tag{4.20}$$

This way, by the Peter-Paul inequality, $2xy \le \frac{x^2}{\varepsilon} + \varepsilon y^2$ for $x,y\ge 0$, $\varepsilon>0$, we obtain for every $\varepsilon>0$,

$$\begin{aligned}&\int_{\mathbb{D}^*} (|\partial_t f|^2 + |e^t\partial_\theta f|^2)\big|\log(|z|^2)\big|^p\omega_{\mathbb{D}^*}\\ &\le\Big(\varepsilon^{-1} + \frac{(p-1)^2}{2}\Big)\int f^2e^{(p-1)t}dtd\theta + \frac{\varepsilon}{2}\int \big(|\partial_t^2 f|^2 + |e^{2t}\partial_\theta^2 f|^2\big)e^{(p-1)t}dtd\theta.\end{aligned} \tag{4.21}$$

Taking $\varepsilon = p^{-2}$ we get

$$\begin{aligned}&\int_{\mathbb{D}^*} p^2(|\partial_t f|^2 + |e^t\partial_\theta f|^2)\big|\log(|z|^2)\big|^p\omega_{\mathbb{D}^*}\\ &\qquad\le 2\int \big(p^4|f|^2 + (|\partial_t^2 f|^2 + |e^{2t}\partial_\theta^2 f|^2)\big)e^{(p-1)t}dtd\theta.\end{aligned} \tag{4.22}$$

Thus, from (4.18), for $f\in \boldsymbol{L}_p^{2,2}(\mathbb{D}^*)$,

$$\|f\|^2_{\boldsymbol{L}_p^{2,2}(\mathbb{D}^*)} \lesssim \int_{\mathbb{D}^*} \big(p^4|f|^2 + |\partial_t^2 f|^2 + |e^t\partial_t\partial_\theta f|^2 + |e^{2t}\partial_\theta^2 f|^2\big)e^{(p-1)t}dtd\theta, \tag{4.23}$$

with the implied constant independent of p.

By (4.12),

$$\begin{aligned}&\int_{\mathbb{D}^*} (\Delta_0 f)^2\big|\log(|z|^2)\big|^p\omega_{\mathbb{D}^*}\\ &= 4\int \big((\partial_t^2 f)^2 + (\partial_t f)^2 + (\tfrac{e^{2t}}{4}\partial_\theta^2 f)^2\big)e^{(p-1)t}\,dtd\theta - 8\int (\partial_t^2 f)(\partial_t f)e^{(p-1)t}\,dtd\theta\\ &\quad + 8\int (\partial_t^2 f)(\tfrac{e^{2t}}{4}\partial_\theta^2 f)e^{(p-1)t}\,dtd\theta - 8\int (\partial_t f)(\tfrac{e^{2t}}{4}\partial_\theta^2 f)e^{(p-1)t}\,dtd\theta.\end{aligned} \tag{4.24}$$

We deal with the mixed terms as follows:

- $-8\int(\partial_t^2 f)(\partial_t f)e^{(p-1)t}\,dtd\theta$: an integration by parts yields:

$$-8\int (\partial_t^2 f)(\partial_t f)e^{(p-1)t}\,dtd\theta = 4(p-1)\int (\partial_t f)^2e^{(p-1)t}\,dtd\theta, \tag{4.25}$$

  and we do not provide more efforts, as this quantity has the favorable sign already – remember we want a bound below on the $\boldsymbol{L}_p^2(\mathbb{D}^*)$-norm of $\Delta_0 f$;

- $8\int(\partial_t^2 f)(\frac{e^{2t}}{4}\partial_\theta^2 f)e^{(p-1)t}\,dtd\theta$: exchanging ∂_t and ∂_θ via integrations by parts, we get:

$$8\int(\partial_t^2 f)(\tfrac{e^{2t}}{4}\partial_\theta^2 f)e^{(p-1)t}\,dtd\theta = 8\int(\tfrac{e^t}{2}\partial_t\partial_\theta f)^2 e^{(p-1)t}\,dtd\theta - 8(p+1)\int(\partial_t f)(\tfrac{e^{2t}}{4}\partial_\theta^2 f)e^{(p-1)t}\,dtd\theta, \tag{4.26}$$

and collect the extra term $-8(p+1)\int(\partial_t f)(\frac{e^{2t}}{4}\partial_\theta^2 f)e^{(p-1)t}\,dtd\theta$ together with the left over right-hand-side mixed term in (4.24), i.e. we deal with:

- $-8(p+2)\int(\partial_t f)(\frac{e^{2t}}{4}\partial_\theta^2 f)e^{(p-1)t}\,dtd\theta$:

$$-8(p+2)\int(\partial_t f)(\tfrac{e^{2t}}{4}\partial_\theta^2 f)e^{(p-1)t}\,dtd\theta \geq -2\int(\tfrac{e^{2t}}{4}\partial_\theta^2 f)^2 e^{(p-1)t}\,dtd\theta - 8(p+2)^2\int(\partial_t f)^2 e^{(p-1)t}\,dtd\theta, \tag{4.27}$$

By Cauchy-Schwarz inequalities and (4.20), we get

$$-8(p+2)^2\int(\partial_t f)^2 e^{(p-1)t}\,dtd\theta \geq -2\int(\partial_t^2 f)^2 e^{(p-1)t}\,dtd\theta - \big(8(p+2)^4 + 4(p-1)^2(p+2)^2\big)\int f^2 e^{(p-1)t}\,dtd\theta.$$

We sum up what precedes as:

$$\int_{\mathbb{D}^*}(\Delta_0 f)^2\big|\log(|z|^2)\big|^p\omega_{\mathbb{D}^*} \geq 2\int\big((\partial_t^2 f)^2 + (\tfrac{e^t}{2}\partial_t\partial_\theta^2 f)^2 + (\tfrac{e^{2t}}{4}\partial_\theta^2 f)^2\big)e^{(p-1)t}\,dtd\theta - \big(8(p+2)^4 + 4(p-1)^2(p+2)^2\big)\int f^2 e^{(p-1)t}\,dtd\theta, \tag{4.28}$$

By (4.23) and (4.28), we get

$$\|f\|^2_{\boldsymbol{L}_p^{2,2}(\mathbb{D}^*)} \leq C\left(\|\Delta_0 f\|^2_{\boldsymbol{L}_p^2(\mathbb{D}^*)} + p^4\|f\|^2_{\boldsymbol{L}_p^2(\mathbb{D}^*)}\right), \tag{4.29}$$

for some $C>0$ independent of both $p\geq 1$ and $f\in C_0^\infty(\mathbb{D}^*)$ with real values; by density, this readily generalizes to $f\in\boldsymbol{L}_p^{2,2}(\mathbb{D}^*)$ with complex values, as Δ_0 is a real operator.

We now carry out the replacement of Δ_0 by $\square_p^{\mathbb{D}^*}$ in (4.29), to get the desired estimate (4.4). By (4.13) and (4.15), acting on function on $\mathbb{D}^*$, we have

$$\square_p^{\mathbb{D}^*} = \frac{1}{2}\Delta_0 - p\widetilde{\partial} \quad \text{with } \widetilde{\partial} = \overline{z}\log(|z|^2)\frac{\partial}{\partial\overline{z}} = \frac{\partial}{\partial t} - \frac{i}{2}e^t\frac{\partial}{\partial\theta}\,. \tag{4.30}$$

Let $f\in\boldsymbol{L}_p^{2,2}(\mathbb{D}^*)$. Using inequality (4.21) with $\varepsilon>0$ to be adjusted, (4.23) and (4.29), we have:

$$\int\big|\widetilde{\partial}f\big|^2 e^{(p-1)t}dtd\theta \leq 2\int(|\partial_t f|^2 + |\tfrac{e^t}{2}\partial_\theta f|^2)e^{(p-1)t}dtd\theta \leq \big(2\varepsilon^{-1} + p^2 + \varepsilon C p^4\big)\|f\|^2_{\boldsymbol{L}_p^2(\mathbb{D}^*)} + \varepsilon C\|\Delta_0 f\|^2_{\boldsymbol{L}_p^2(\mathbb{D}^*)}. \tag{4.31}$$

From (4.30) and (4.31), we are led to:

$$\begin{aligned}\|\Delta_0 f\|^2_{\boldsymbol{L}_p^2(\mathbb{D}^*)} &= \big\|2\big(\square_p^{\mathbb{D}^*} + p\widetilde{\partial}\big)f\big\|^2_{\boldsymbol{L}_p^2(\mathbb{D}^*)} \leq 8\|\square_p^{\mathbb{D}^*}f\|^2_{\boldsymbol{L}_p^2(\mathbb{D}^*)} + 8p^2\big\|\widetilde{\partial}f\big\|^2_{\boldsymbol{L}_p^2(\mathbb{D}^*)}\\ &\leq 8\|\square_p^{\mathbb{D}^*}f\|^2_{\boldsymbol{L}_p^2(\mathbb{D}^*)} + 8p^2\big(2\varepsilon^{-1} + p^2 + \varepsilon C p^4\big)\|f\|^2_{\boldsymbol{L}_p^2(\mathbb{D}^*)} + 8p^2\varepsilon C\|\Delta_0 f\|^2_{\boldsymbol{L}_p^2(\mathbb{D}^*)}.\end{aligned} \tag{4.32}$$

Take $\varepsilon = \dfrac{1}{16Cp^2}$ to conclude that:

$$\|\Delta_0 f\|^2_{\boldsymbol{L}^2_p(\mathbb{D}^*)} \leq 16\|\Box_p f\|^2_{\boldsymbol{L}^2_p(\mathbb{D}^*)} + Ap^4\|f\|^2_{\boldsymbol{L}^2_p(\mathbb{D}^*)} \tag{4.33}$$

with $A = 256C + 9$. Plugged back into (4.29), this estimate gives exactly (4.4), with a (new) constant $C > 0$, uniform for $p \geq 1$ and $f \in \boldsymbol{L}^{2,2}_p(\mathbb{D}^*)$.

The proof of Proposition 4.1 for $s = 1$ is completed. Continuing by induction we get it for all $s \geq 2$. □

4.2 Estimate on the punctured Riemann surface Σ : degree 0

We now consider the geometric situation of a punctured polarized Riemann surface $(\Sigma, \omega_\Sigma, L, h)$ satisfying conditions (α) and (β). Let $a \in D$. By assumption the following holds: *there exists a trivialization of L around a such that in the associated local complex coordinate $z \in \mathbb{D}$, we have $h = \big|\log(|z|^2)\big||\cdot|$ on the coordinate disc $\mathbb{D}_r$ centered at a and of radius $r \in (0, e^{-1})$.* This way, the curvature ω_Σ of h coincides with $\omega_{\mathbb{D}^*}$ on $\mathbb{D}^*_r := \mathbb{D}_r \smallsetminus \{0\}$.

Proposition 4.2 *For every $s \in \mathbb{N}^*$ there exists $C = C(s, h)$ such that for all $p \gg 1$, and all $\sigma \in \boldsymbol{L}^{2,2s}_p(h) = \boldsymbol{L}^{2,2s}\big(\Sigma, \omega_\Sigma, L^p, h^p\big)$,*

$$\|\sigma\|^2_{\boldsymbol{L}^{2,2s}_p(h)} \leq C \sum_{j=0}^{s} p^{4(s-j)} \|(\Box_p)^j \sigma\|^2_{\boldsymbol{L}^2_p(h)}, \tag{4.34}$$

where $\Box_p$ is the Kodaira Laplacian on Σ associated to ω_Σ and h^p.

Proof. — Again, we do it for $s = 1$.

In the situation of the Proposition, we denote by $\overline{h}$ a smooth Hermitian metric on L on the whole $\overline{\Sigma}$ such that it coincides with h on $\Sigma \smallsetminus \mathbb{D}_{r/2}$. It is an easy exercise to construct $\overline{h}$ so that its curvature, ω say, is Kähler over the whole (compact) $\overline{\Sigma}$, which we take for granted until the end of this proof. Notice that ω_Σ and ω coincide on $\Sigma \smallsetminus \mathbb{D}_{r/2}$.

Now the principle of the proof is to glue estimate (4.4) to the analogous estimate for $\big(\overline{\Sigma}, L, \overline{h}\big)$, that states the existence of a constant C such that for all $p \gg 1$, and all $\sigma \in \boldsymbol{L}^{2,2}_p\big(\overline{h}\big) := \boldsymbol{L}^{2,2}\big(\overline{\Sigma}, \omega, L^p, \overline{h}^p\big)$,

$$\|\sigma\|^2_{\boldsymbol{L}^{2,2}_p(\overline{h})} \leq C\big(\big\|\Box^{\overline{\Sigma}}_p \sigma\big\|^2_{\boldsymbol{L}^2_p(\overline{h})} + p^4\|\sigma\|^2_{\boldsymbol{L}^2_p(\overline{h})}\big). \tag{4.35}$$

This estimate, as well as its generalization for $\sigma \in \boldsymbol{L}^{2,2s}_p\big(\overline{h}\big)$, $s \geq 1$, can be found for instance in [DLM1, (4.14)] or [MM1, §1.6.2].

We denote $\nabla^{p,\Sigma *}$ the formal adjoint of $\nabla^{p,\Sigma}$ action on $\Lambda(T^{*(0,1)}\Sigma) \otimes L^p$. By Lichnerowicz formula [MM1, Remark 1.4.8],

$$2\Box_p = \nabla^{p,\Sigma *}\nabla^{p,\Sigma} - pR^L(w, \overline{w}) + (2pR^L + R^{T^{(1,0)}\Sigma})(w, \overline{w})\overline{w}^* \wedge i_{\overline{w}}, \tag{4.36}$$

and w is an orthonormal frame of $T^{(1,0)}\Sigma$. By (4.3),

$$R^L(w, \overline{w}) = 1, \quad R^{T^{(1,0)}\Sigma}(w, \overline{w}) = -2 \quad \text{on } V = V_1 \cup \ldots \cup V_N. \tag{4.37}$$

From (4.36) and (4.37), for any $\sigma \in \boldsymbol{L}^{2,2}_p(h)$,

$$\Big|\|\nabla^{p,\Sigma}\sigma\|^2_{\boldsymbol{L}^2_p(h)} - 2\left\langle \Box_p \sigma, \sigma \right\rangle_p\Big| \leq Cp\|\sigma\|^2_{\boldsymbol{L}^2_p(h)}. \tag{4.38}$$

Let χ be a cut-off function supported near a; assume, more precisely, that

$$\chi \in C^\infty\big(\overline{\Sigma}\big),\ \ 0 \le \chi \le 1,\ \ \chi\,|_{\overline{\mathbb{D}}_{r/2}} \equiv 1,\ \ \chi\,|_{\Sigma\smallsetminus\mathbb{D}_{2r/3}} \equiv 0. \tag{4.39}$$

Let $p \ge 1$, and $\sigma \in \boldsymbol{L}_p^{2,2}\big(h\big) = \boldsymbol{L}^{2,2}\big(\Sigma, \omega_\Sigma, L^p, h^p\big)$. Then $(1-\chi)\sigma \in \boldsymbol{L}_p^{2,2}(\overline{h})$ and on its support, h coincides with $\overline{h}$; likewise, $\chi\sigma$ can be interpreted as an element of $\boldsymbol{L}_p^{2,2}(\mathbb{D}^*)$ and on its support, h can be regarded as $h_{\mathbb{D}^*}$. Therefore,

$$\begin{aligned}\|\sigma\|^2_{\boldsymbol{L}_p^{2,2}(h)} &= \|\chi\sigma + (1-\chi)\sigma\|^2_{\boldsymbol{L}_p^{2,2}(h)} \le 2\big(\|\chi\sigma\|^2_{\boldsymbol{L}_p^{2,2}(\mathbb{D}^*)} + \|(1-\chi)\sigma\|^2_{\boldsymbol{L}_p^{2,2}(\overline{h})}\big)\\ &\le 2C\big(\big\|\square_p^{\mathbb{D}^*}(\chi\sigma)\big\|^2_{\boldsymbol{L}_p^2(\mathbb{D}^*)} + \big\|\square_p^{\overline{\Sigma}}[(1-\chi)\sigma]\big\|^2_{\boldsymbol{L}_p^2(\overline{h})}\big)\\ &\qquad + 2Cp^4\big(\|\chi\sigma\|^2_{\boldsymbol{L}_p^2(\mathbb{D}^*)} + \|(1-\chi)\sigma\|^2_{\boldsymbol{L}_p^2(\overline{h})}\big)\end{aligned} \tag{4.40}$$

where $C = \sup\big(C(h_{\mathbb{D}^*}), C(\overline{h})\big)$, with $C(h_{\mathbb{D}^*})$, resp. $C(\overline{h})$, the constant from (4.4), resp. from its analogue for $\big(\overline{\Sigma}, L, \overline{h}\big)$. Thus defined, C is independent of σ and p.

Now $\|\chi\sigma\|^2_{\boldsymbol{L}_p^2(h_{\mathbb{D}^*})} = \|\chi\sigma\|^2_{\boldsymbol{L}_p^2(h)} \le \|\sigma\|^2_{\boldsymbol{L}_p^2(h)}$, and $\|(1-\chi)\sigma\|^2_{\boldsymbol{L}_p^2(\overline{h})} \le \|\sigma\|^2_{\boldsymbol{L}_p^2(h)}$ as well. The treatments of $\big\|\square_p^{\mathbb{D}^*}(\chi\sigma)\big\|^2_{\boldsymbol{L}_p^2(h_{\mathbb{D}^*})}$ and $\big\|\square_p^{\overline{\Sigma}}[(1-\chi)\sigma]\big\|^2_{\boldsymbol{L}_p^2(\overline{h})}$ are done in the same spirit, but require a little extra work. For instance, on $\mathbb{D}_{2r/3}$, by (4.15) and (4.16),

$$\square_p^{\mathbb{D}^*}(\chi\sigma) = \chi\square_p^{\mathbb{D}^*}\sigma - \big(\overline{\partial}\chi, \overline{(\nabla^p)^{1,0}\sigma}\big)_{T^*\mathbb{D}^*} - \big(\overline{\partial}\sigma, \overline{\partial\chi}\big)_{T^*\mathbb{D}^*} + \Big(\frac{1}{2}\Delta_0\chi\Big)\sigma, \tag{4.41}$$

hence

$$\begin{aligned}\big\|\square_p^{\mathbb{D}^*}(\chi\sigma)\big\|^2_{\boldsymbol{L}_p^2(h_{\mathbb{D}^*})} &\le 4\Big(\big\|\chi\square_p^{\mathbb{D}^*}\sigma\big\|^2_{\boldsymbol{L}_p^2(h_{\mathbb{D}^*})} + \big\||\partial\chi|\overline{\partial}\sigma\big\|^2_{\boldsymbol{L}_p^2(h_{\mathbb{D}^*})}\\ &\qquad + \big\||\partial\chi|(\nabla^p)^{1,0}\sigma\big\|^2_{\boldsymbol{L}_p^2(h_{\mathbb{D}^*})} + \big\|(\tfrac{1}{2}\Delta_0\chi)\sigma\big\|^2_{\boldsymbol{L}_p^2(h_{\mathbb{D}^*})}\Big)\\ &= 4\Big(\big\|\chi\square_p\sigma\big\|^2_{\boldsymbol{L}_p^2(h)} + \big\||\partial\chi|\overline{\partial}\sigma\big\|^2_{\boldsymbol{L}_p^2(h)}\\ &\qquad + \big\||\partial\chi|(\nabla^{p,\Sigma})^{1,0}\sigma\big\|^2_{\boldsymbol{L}_p^2(h)} + \big\|(\tfrac{1}{2}\Delta_0\chi)\sigma\big\|^2_{\boldsymbol{L}_p^2(h)}\Big),\end{aligned} \tag{4.42}$$

with the $\boldsymbol{L}_p^2(h_{\mathbb{D}^*})$-norms, resp. $\boldsymbol{L}_p^2(h)$-norms, for 1-forms, resp. 1-forms with value in L^p, computed with $\omega_{\mathbb{D}^*} \otimes h^p_{\mathbb{D}^*}$ on $\mathbb{D}^*$, resp. with $\omega_\Sigma \otimes h^p$. Consequently,

$$\big\|\square_p^{\mathbb{D}^*}(\chi\sigma)\big\|^2_{\boldsymbol{L}_p^2(h_{\mathbb{D}^*})} \le 4C\Big(\big\|\square_p\sigma\big\|^2_{\boldsymbol{L}_p^2(h)} + \big\|\nabla^{p,\Sigma}\sigma\big\|^2_{\boldsymbol{L}_p^2(h)} + \big\|\sigma\big\|^2_{\boldsymbol{L}_p^2(h)}\Big) \tag{4.43}$$

with $C = 1 + \max\big\{\big\|\frac{1}{2}\Delta_0\chi\big\|^2_{C^0(\mathbb{D}^*,\omega_{\mathbb{D}^*})}, \big\|\overline{\partial}\chi\big\|^2_{C^0(\mathbb{D}^*,\omega_{\mathbb{D}^*})}\big\}$, that does not depend on p.

From (4.38) and (4.42), we get

$$\big\|\square_p^{\mathbb{D}^*}(\chi\sigma)\big\|^2_{\boldsymbol{L}_p^2(h_{\mathbb{D}^*})} \le C\big(\big\|\square_p\sigma\big\|^2_{\boldsymbol{L}_p^2(h)} + p\|\sigma\big\|^2_{\boldsymbol{L}_p^2(h)}\big) \tag{4.44}$$

for some C independent of p and σ. Similarly,

$$\big\|\square_p^{\overline{\Sigma}}\big((1-\chi)\sigma\big)\big\|^2_{\boldsymbol{L}_p^2(\overline{h})} \le C\big(\big\|\square_p\sigma\big\|^2_{\boldsymbol{L}_p^2(h)} + p\|\sigma\big\|^2_{\boldsymbol{L}_p^2(h)}\big) \tag{4.45}$$

with C again independent of p and σ.

In conclusion, it follows from (4.40), (4.44) and (4.45), there exists $C > 0$ such that for any $p \gg 1$ and $\sigma \in \boldsymbol{L}_p^{2,2}(h)$, we have

$$\|\sigma\|^2_{\boldsymbol{L}_p^{2,2}(h)} \le C\big(\big\|\square_p\sigma\big\|^2_{\boldsymbol{L}_p^2(h)} + p^4\|\sigma\|^2_{\boldsymbol{L}_p^2(h)}\big). \tag{4.46}$$

The proof of Proposition 4.2 for $s = 1$ is completed. The proof for general $s \in \mathbb{N}^*$ follows by induction with the help of Proposition 4.1. $\square$

4.3 Bidegree $(0,1)$

This subsection will not be used in the rest of this paper, we include it here only for completeness and its independent interest.

To prove that Propositions 4.1 and 4.2 still hold in bidegree $(0,1)$, or, namely, for σ a section of $T^{*(0,1)}\mathbb{D}^*\otimes L^p$ or $T^{*(0,1)}\Sigma\otimes L^p$, an easy procedure is to observe that the following diagram:

$$\begin{tikzcd} C_0^\infty(\mathbb{D}^*) \arrow[r, "\cdot\otimes\frac{d\overline{z}}{\overline{z}\log(|z|^2)}"] \arrow[d, "\square_p - ie^t\partial_\theta"'] & C_0^\infty(\mathbb{D}^*, T^{*(0,1)}\mathbb{D}^*) \arrow[d, "\square_p"] \\ C_0^\infty(\mathbb{D}^*) \arrow[r, "\cdot\otimes\frac{d\overline{z}}{\overline{z}\log(|z|^2)}"'] & C_0^\infty(\mathbb{D}^*, T^{*(0,1)}\mathbb{D}^*) \end{tikzcd} \tag{4.47}$$

commutes, where the horizontal arrows are isometries under $h^p_{\mathbb{D}^*}$ and $(h_{\mathbb{D}^*})^p\otimes\omega_{\mathbb{D}^*}$. Indeed, by (4.6) and (4.7), $\frac{d\overline{z}}{\overline{z}\log(|z|^2)}=\overline{\partial}t$, and by (4.7) and (4.14), for $g\in C^\infty(\mathbb{D}^*)$, we have

$$\begin{aligned}\overline{\partial}\,\overline{\partial}^{L^p*}(g\overline{\partial}t) &= \Big[\overline{\partial}^{L^p*}\overline{\partial}g+\overline{z}\log(|z|^2)\frac{\partial}{\partial\overline{z}}g - z\log(|z|^2)\frac{\partial}{\partial z}g\Big]\overline{\partial}t \\ &= (\square_p g - ie^t\partial_\theta g)\overline{\partial}t.\end{aligned} \tag{4.48}$$

Proposition 4.3 *Let $s\geq 1$. Then there exists $C=C(s,h_{\mathbb{D}^*})$ such that for all $p\geq 1$, and all $\sigma\in \boldsymbol{L}^{2,2s}_p(\mathbb{D}^*)=\boldsymbol{L}^{2,2s}\big(\mathbb{D}^*,\omega_{\mathbb{D}^*},T^{*(0,1)}\mathbb{D}^*\otimes L^p,\omega_{\mathbb{D}^*}\otimes h^p_{\mathbb{D}^*}\big)$,*

$$\|\sigma\|^2_{\boldsymbol{L}^{2,2s}_p(\mathbb{D}^*)}\leq C\sum_{j=0}^{s}p^{4(s-j)}\big\|(\square_p^{\mathbb{D}^*})^j\sigma\big\|^2_{\boldsymbol{L}^2_p(\mathbb{D}^*)}. \tag{4.49}$$

Proof. — Indeed, take $\sigma=f\frac{d\overline{z}}{\overline{z}\log(|z|^2)}=f\overline{\partial}t\in C^\infty(\mathbb{D}^*,T^{*(0,1)\mathbb{D}^*})$. Then for instance,

$$(\nabla^p)^2\sigma=(\nabla^p)^2f\otimes\overline{\partial}t+2\nabla^pf\otimes\nabla^{\mathbb{D}^*}\overline{\partial}t+f\otimes(\nabla^{\mathbb{D}^*})^2\overline{\partial}t$$

where $\nabla^{\mathbb{D}^*}$ is the Levi-Civita connection of $\omega_{\mathbb{D}^*}$. By (4.7), (4.9) and (4.11), $\overline{\partial}t$ is uniformly bounded at any order with respect to $\omega_{\mathbb{D}^*}$ on $\mathbb{D}^*$, we get that

$$\int_{\mathbb{D}^*}\big|(\nabla^p)^2\sigma\big|^2_p\omega_{\mathbb{D}^*}\lesssim\int_{\mathbb{D}^*}\big(\big|(\nabla^p)^2f\big|^2_p+|\nabla^pf|^2_p+|f|^2\big)\,\omega_{\mathbb{D}^*}=\|f\|^2_{\boldsymbol{L}^{2,2}_p(\mathbb{D}^*)},$$

independently of p. By Proposition 4.1, we thus have

$$\int_{\mathbb{D}^*}\big|(\nabla^p)^2\sigma\big|^2_p\omega_{\mathbb{D}^*}\leq C\int_{\mathbb{D}^*}\big(\big|\square_pf\big|^2_p+p^4|f|^2\big)\,\omega_{\mathbb{D}^*} \tag{4.50}$$

for $p\gg 1$, with C independent of p. By (4.47), we have

$$\begin{aligned}\int_{\mathbb{D}^*}\big|\square_p\sigma\big|^2_p\omega_{\mathbb{D}^*} &= \int_{\mathbb{D}^*}\big|\square_pf-ie^t\partial_\theta f\big|^2_p\omega_{\mathbb{D}^*} \\ &\geq\frac{1}{2}\int_{\mathbb{D}^*}|\square_pf|^2_p\omega_{\mathbb{D}^*}-\int\big|e^t\partial_\theta f\big|^2e^{(p-1)t}dtd\theta.\end{aligned} \tag{4.51}$$

By (4.21) with $\varepsilon=\dfrac{1}{2^9p^2}$, (4.28) and (4.33), as in (4.31), we get

$$\int\big|e^t\partial_\theta f\big|^2e^{(p-1)t}dtd\theta\leq Cp^2\|f\|^2_{\boldsymbol{L}^2_p(\mathbb{D}^*)}+\frac{1}{4p^2}\|\square_pf\|^2_{\boldsymbol{L}^2_p(\mathbb{D}^*)}. \tag{4.52}$$

As $\int_{\mathbb{D}^*} |\sigma|_p^2 \omega_{\mathbb{D}^*} = \int_{\mathbb{D}^*} |f|_p^2 \omega_{\mathbb{D}^*}$, from (4.51) and (4.52), we get

$$\int_{\mathbb{D}^*} \left|\Box_p f\right|_p^2 \omega_{\mathbb{D}^*} \leq 4 \int_{\mathbb{D}^*} |\Box_p \sigma|_p^2 \omega_{\mathbb{D}^*} + Cp^2 \int_{\mathbb{D}^*} |\sigma|_p^2 \omega_{\mathbb{D}^*}.$$

This yields, coming back to (4.50),

$$\int_{\mathbb{D}^*} \left|(\nabla^p)^2 \sigma\right|_p^2 \omega_{\mathbb{D}^*} \leq C \int_{\mathbb{D}^*} \left(\left|\Box_p \sigma\right|_p^2 + p^4 |\sigma|^2\right) \omega_{\mathbb{D}^*}. \tag{4.53}$$

Consequently, for $p \gg 1$, we get

$$\|\sigma\|^2_{\boldsymbol{L}^{2,2}_p(\mathbb{D}^*)} \leq C\big(\|\Box_p \sigma\|^2_{\boldsymbol{L}^2_p(\mathbb{D}^*)} + p^4 \|\sigma\|^2_{\boldsymbol{L}^2_p(\mathbb{D}^*)}\big). \tag{4.54}$$

Now by induction on s, we get Proposition 4.3 for $s \geq 1$. $\Box$

Using moreover the same gluing procedure as in proving Proposition 4.2, we get the analogue of (4.49) on Σ:

Proposition 4.4 *Let $s \in \mathbb{N}^*$. Then there exists $C = C(s,h)$ such that for all $p \geq 1$, and all $\sigma \in \boldsymbol{L}^{2,2s}_p(\Sigma) = \boldsymbol{L}^{2,2s}\big(\Sigma, \omega_\Sigma, T^{*(0,1)}\Sigma \otimes L^p, \omega_\Sigma \otimes h^p\big)$,*

$$\|\sigma\|^2_{\boldsymbol{L}^{2,2s}_p(\Sigma)} \leq C \sum_{j=0}^{s} p^{4(s-j)} \left\|(\Box_p)^j \sigma\right\|^2_{\boldsymbol{L}^2_p(\Sigma)}. \tag{4.55}$$

5 Spectral Gap and Localization

We follow in this Section the localization scheme based on the spectral gap and finite propagation speed [MM1] and show that the Bergman kernel localizes near the singularities. As a consequence we obtain a first rough estimate, which will be improved in the next Section.

Let (M, ω_M) be a complete Kähler manifold. We will denote by $R^{\det}$ the curvature of the anticanonical line bundle $(K_M^*, h^{K_M^*})$, where $h^{K_M^*}$ is induced by ω_M.

Let (E, h^E) be a Hermitian holomorphic line bundle on M. Let $\overline{\partial}^*$ be the formal adjoint of $\overline{\partial}$ with respect to $\langle \cdot, \cdot \rangle$ (cf. (2.1)). Let $\Box^E = \overline{\partial}^* \overline{\partial}$ be the Kodaira Laplace operator. By [MM1, Corollary 3.3.4] the operator $\Box^E : C_0^\infty(M,E) \to C_0^\infty(M,E)$ is essentially self-adjoint and we will denote its unique self-adjoint extension with the same symbol $\Box^E$. Note that the domain of this extension is $\mathrm{Dom}(\Box^E) = \{\sigma \in \boldsymbol{L}^2(M,E) : \Box^E \sigma \in \boldsymbol{L}^2(M,E)\}$.

Consider now a Hermitian holomorphic line bundle (L,h) and denote by $\Box_p := \Box^{L^p}$ the Kodaira Laplace operator corresponding to (L^p, h^p). By [MM1, Theorem 6.1.1] and its proof we have the following.

Proposition 5.1 (Spectral gap) *Let (M, ω_M) be a complete Kähler manifold and (L,h) be a Hermitian holomorphic line bundle on M. We assume there exist $\varepsilon > 0$, $C > 0$ such that $iR^L \geq \varepsilon \omega_M$ and $iR^{\det} \geq -C\omega_M$. Then there exists $c = c(C, \varepsilon) > 0$ such that for all $p \gg 1$ we have*

$$\mathrm{Spec}(\Box_p) \subset \{0\} \cup [cp, +\infty)\,. \tag{5.1}$$

Corollary 5.2 *The spectral gap (5.1) holds for the Laplacian $\Box_p$ in the following situations:*

(1) $(M, \omega_M) = (\mathbb{D}^*, \omega_{\mathbb{D}^*})$, $(L,h) = (\mathbb{C}, \left|\log(|z|^2)\right| h_0)$,

(2) $(M, \omega_M) = (\Sigma, \omega_\Sigma)$, (L,h) as in Theorem 1.1.

Indeed, by (4.3) and $iR^{\det} = -\omega_{\mathbb{D}^*}$ holds on $\mathbb{D}^*$, combining the condition (β), we know $iR^L \geq \varepsilon\omega_\Sigma$, $iR^{\det} \geq -C\omega_\Sigma$ on Σ, for some $C > 0$. Thus we can apply Proposition 5.1 to Corollary 5.2.

We assume here, without loss of generality, that the puncture divisor D in Σ is reduced to one point a. Let $\mathfrak{e}$ be the holomorphic frame of L near a corresponding to the trivialization in the condition (α).

By the assumption (α), (β), under our trivialization $\mathfrak{e}$ of L on the coordinate z on $\mathbb{D}^*_r$ for some $0 < r < e^{-1}$, we have the identification of the geometric data

$$(\Sigma, \omega_\Sigma, L, h)|_{\mathbb{D}^*_r} = (\mathbb{D}^*, \omega_{\mathbb{D}^*}, \mathbb{C}, h^p_{\mathbb{D}^*})|_{\mathbb{D}^*_r}. \tag{5.2}$$

We set:

- F is the normalized Fourier transform of a smooth cut-off function as in [MM1, §4.1], namely

$$F(u) = \Big(\int_{\mathbb{R}} f(v)\,dv\Big)^{-1} \int_{\mathbb{R}} e^{ivu} f(v)\,dv \tag{5.3}$$

  with $f : \mathbb{R} \to [0,1]$ a smooth even function such that $f(v) = 1$ if $|v| \leq \epsilon/2$ and $f(v) = 0$ if $|v| \geq \epsilon$ for $\epsilon > 0$. Thus F is an even function in the Schwartz space $\mathscr{S}(\mathbb{R})$ with $F(0) = 1$. Let $\widetilde{F}$ be the function satisfying $\widetilde{F}(u^2) = F(u)$ for all $u \in \mathbb{R}$. We consider the function

$$\phi_p : \mathbb{R} \longrightarrow \mathbb{R},\ \ u \longmapsto \mathbb{1}_{[cp,+\infty)}(|u|)\widetilde{F}(u) \tag{5.4}$$

  where $c > 0$ is defined in (5.1); let $K_p := \phi_p(\Box_p)$ and let $K_p(\cdot,\cdot)$ be the associated kernel; we denote by $f_p(\cdot,\cdot)$ the function associated to $K_p(\cdot,\cdot)$ via the doubled trivialization around a used above; for $x \in \mathbb{D}^*_r$, we set $f_{p,x}$ for the one-variable function $y \mapsto f_p(x,y)$; then

$$K_p(x,y) = f_p(x,y)\mathfrak{e}^p(x) \otimes (\mathfrak{e}^p(y))^*, \tag{5.5}$$

  and $(\mathfrak{e}^p(y))^*$ is the metric dual of $\mathfrak{e}^p(y)$ with respect to h^p, that is, $\mathfrak{e}^p(y)^* \cdot \mathfrak{e}^p(y) = |\mathfrak{e}^p(y)|^2_{h^p}$.
- χ a cut-off function as in (4.39);
- $\rho : \Sigma \to [1, +\infty)$ is a smooth function such that $\rho(z) = \big|\log(|z|^2)\big|$ on $\mathbb{D}^*_r$.

Proposition 5.3 *For any $\ell, m \geq 0$, $\gamma > \frac{1}{2}$, there exists $C_{\ell,m,\gamma} > 0$ such that for any $p > 1$, we have*

$$\big\|\rho(x)^{-\gamma}\rho(y)^{-\gamma}K_p(x,y)\big\|_{C^m(h^p)} \leq C_{\ell,m,\gamma}p^{-\ell}, \tag{5.6}$$

in the sense of (2.14).

Proof. — We proceed as follows. Take $p \gg 1$, and pick $\mathbf{a}$ and b two real parameters to

be determined later; then by definition of f_p and K_p :

(5.7)
$$\begin{aligned}
&\big\|\rho(x)^{\mathbf{a}}\rho(y)^b\chi(x)\chi(y)f_p(x,y)\big\|^2_{\boldsymbol{L}^2(\Sigma\times\Sigma)}\\
&=\int\limits_{x\in\mathbb{D}^*}\rho(x)^{2\mathbf{a}}\chi(x)^2\omega_\Sigma(x)\int\limits_{y\in\mathbb{D}^*}\rho(y)^{2b}\chi(y)^2\big|f_p(x,y)\big|^2\,\omega_\Sigma(y)\\
&=\int\limits_{x\in\mathbb{D}^*}\rho(x)^{2\mathbf{a}-p}\chi(x)^2\Big\langle\int\limits_{y\in\mathbb{D}^*}\big\langle K_p(x,y),\rho(y)^{2b-p}\chi(y)^2f_p(x,y)\mathfrak{e}^p(y)\big\rangle_{h^p_{\mathbb{D}^*}}\omega_{\mathbb{D}^*}(y),\mathfrak{e}^p(x)\Big\rangle_{h^p_{\mathbb{D}^*}}\omega_{\mathbb{D}^*}(x)\\
&=\int\limits_{x\in\mathbb{D}^*}\rho(x)^{2\mathbf{a}-p}\chi(x)^2\Big\langle K_p(\rho^{2b-p}\chi^2f_{p,x}\mathfrak{e}^p),\mathfrak{e}^p\Big\rangle_{h^p_{\mathbb{D}^*}}(x)\,\omega_{\mathbb{D}^*}(x).
\end{aligned}$$

As $|\mathfrak{e}^p|_{h^p_{\mathbb{D}^*}}=\rho^{p/2}$, from (5.7), we have:

$$\begin{aligned}
&\big\|\rho(x)^{\mathbf{a}}\rho(y)^b\chi(x)\chi(y)f_p(x,y)\big\|^2_{\boldsymbol{L}^2(\Sigma\times\Sigma)}\\
&\qquad=\int_{x\in\mathbb{D}^*}\rho(x)^{2\mathbf{a}-\frac{p}{2}}\chi(x)^2\big|K_p(\rho^{2b-p}\chi^2f_{p,x}\mathfrak{e}^p)(x)\big|_{h^p_{\mathbb{D}^*}}\omega_{\mathbb{D}^*}(x).
\end{aligned}\tag{5.8}$$

Now for all $x\in\mathbb{D}^*_{2r/3}$,

$$\begin{aligned}
\big|K_p(\rho^{2b-p}\chi^2f_{p,x}\mathfrak{e}^p)(x)\big|_{h^p_{\mathbb{D}^*}}&\le\rho(x)^{1/2}\big\|\rho^{-1/2}K_p(\rho^{2b-p}\chi^2f_{p,x}\mathfrak{e}^p)\big\|_{C^0(\mathbb{D}^*_{2r/3})}\\
&=\rho(x)^{1/2}\big\|\rho^{-1}|K_p(\rho^{2b-p}\chi^2f_{p,x}\mathfrak{e}^p)|^2_{h^p_{\mathbb{D}^*}}\big\|^{1/2}_{C^0(\mathbb{D}^*_{2r/3})},
\end{aligned}\tag{5.9}$$

and thus:

$$\begin{aligned}
&\big\|\rho(x)^{\mathbf{a}}\rho(y)^b\chi(x)\chi(y)f_p(x,y)\big\|^2_{\boldsymbol{L}^2(\Sigma\times\Sigma)}\\
&\qquad\le\int_{x\in\mathbb{D}^*}\rho(x)^{2\mathbf{a}+\frac{1-p}{2}}\chi(x)^2\big\|\rho^{-1}|K_p(\rho^{2b-p}\chi^2f_{p,x}\mathfrak{e}^p)|^2_{h^p_{\mathbb{D}^*}}\big\|^{1/2}_{C^0(\mathbb{D}^*_{2r/3})}\omega_{\mathbb{D}^*}(x).
\end{aligned}\tag{5.10}$$

We momentarily set $g(z)=|K_p(\rho^{2b-p}\chi^2f_{p,x}\mathfrak{e}^p)|^2_{h^p_{\mathbb{D}^*}}(z)$, and use the embedding $\boldsymbol{L}^{1,3}_{\mathrm{wtd}}(\Sigma,\omega_\Sigma)\hookrightarrow C^0(\Sigma,\omega_\Sigma)$ (cf. Lemma 2.6 a)), that gives, taking supports into account:

$$\begin{aligned}
\big\|\rho^{-1}g\big\|_{C^0(\mathbb{D}^*_{2r/3})}&\le c_0\int_{\mathbb{D}^*_r}\rho\big(|\rho^{-1}g|+\ldots+\big|(\nabla^\Sigma)^3(\rho^{-1}g)\big|\big)\omega_\Sigma\\
&\le A^3c_0\int_{\mathbb{D}^*_r}\big(|g|+\ldots+\big|(\nabla^\Sigma)^3g\big|\big)\omega_\Sigma
\end{aligned}\tag{5.11}$$

with $A=1+\|d\log\rho\|_{C^2(\mathbb{D}^*_r,\omega_{\mathbb{D}^*})}$, which is finite by $\log\rho=\log\big(-\log(|z|^2)\big)=t$ and (4.12). Moreover,

$$\begin{aligned}
&|g|+\ldots+\big|(\nabla^\Sigma)^3g\big|\\
&\qquad\le C\big(|K_p(\rho^{2b-p}\chi^2f_x\mathfrak{e}^p)|^2_{h^p_{\mathbb{D}^*}}+\ldots+|(\nabla^{p,\Sigma})^3K_p(\rho^{2b-p}\chi^2f_{p,x}\mathfrak{e}^p)|^2_{h^p_{\mathbb{D}^*}}\big).
\end{aligned}\tag{5.12}$$

From (5.11), (5.12), we obtain

$$\big\|\rho^{-1}g\big\|_{C^0(\mathbb{D}^*_{2r/3})}\le C\big\|K_p(\rho^{2b-p}\chi^2f_{p,x}\mathfrak{e}^p)\big\|^2_{\boldsymbol{L}^{2,3}_p(h)}.\tag{5.13}$$

By Propositions 4.2 and 5.1, (5.4) and $\widetilde{F} \in \mathscr{S}(\mathbb{R})$, we have for any fixed ℓ, there exists $C_\ell > 0$ such that for any $x \in \mathbb{D}^*_r, p \in \mathbb{N}^*$,

$$\big\|K_p(\rho^{2b-p}\chi^2 f_{p,x}\mathfrak{e}^p)\big\|_{\boldsymbol{L}^{2,3}_p(h)} \leq C_\ell p^{-\ell}\big\|\rho^{2b-p}\chi^2 f_{p,x}\mathfrak{e}^p\big\|_{\boldsymbol{L}^2_p(h)}, \tag{5.14}$$

and thus (5.10), (5.13), (5.14) yield

(5.15)

$$\begin{aligned}
&\big\|\rho(x)^{\mathbf{a}}\rho(y)^b\chi(x)\chi(y)f_p(x,y)\big\|^2_{\boldsymbol{L}^2(\Sigma\times\Sigma)}\\
&\leq C_\ell\, p^{-\ell}\int_{x\in\mathbb{D}^*}\chi(x)^2\rho(x)^{2\mathbf{a}+\frac{1-p}{2}}\big\|\rho^{2b-p}\chi^2 f_{p,x}\mathfrak{e}^p\big\|_{\boldsymbol{L}^2_p(h)}\omega_{\mathbb{D}^*}(x)\\
&\leq C_\ell\, p^{-\ell}\bigg(\int_{\mathbb{D}^*}\chi^2\rho^{2\mathbf{a}-p+1}\omega_{\mathbb{D}^*}\bigg)^{\frac12}\bigg(\int_{x\in\mathbb{D}^*}\chi(x)^2\rho(x)^{2\mathbf{a}}\big\|\rho^{2b-p}\chi^2 f_{p,x}\mathfrak{e}^p\big\|^2_{\boldsymbol{L}^2_p(h)}\omega_{\mathbb{D}^*}(x)\bigg)^{\frac12}\\
&\leq C_\ell\, p^{-\ell}\bigg(\int_{\mathbb{D}^*}\chi^2\rho^{2\mathbf{a}-p+1}\omega_{\mathbb{D}^*}\bigg)^{\frac12}\big\|\rho(x)^{\mathbf{a}}\rho(y)^{2b-\frac p2}\chi(x)\chi(y)f_p(x,y)\big\|_{\boldsymbol{L}^2(\Sigma\times\Sigma)},
\end{aligned}$$

since

$$\begin{aligned}
&\int_{x\in\mathbb{D}^*}\chi(x)^2\rho(x)^{2\mathbf{a}}\big\|\rho^{2b-p}\chi^2 f_{p,x}\mathfrak{e}^p\big\|^2_{\boldsymbol{L}^2_p(h)}\omega_{\mathbb{D}^*}(x)\\
&\qquad=\int_{x\in\mathbb{D}^*}\chi(x)^2\rho(x)^{2\mathbf{a}}\omega_{\mathbb{D}^*}(x)\int_{y\in\mathbb{D}^*}\rho(y)^{4b-p}\chi(y)^4|f_p(x,y)|^2\omega_{\mathbb{D}^*}(y)\\
&\qquad\leq\int_{x\in\mathbb{D}^*}\chi(x)^2\rho(x)^{2\mathbf{a}}\omega_{\mathbb{D}^*}(x)\int_{y\in\mathbb{D}^*}\rho(y)^{4b-p}\chi(y)^2|f_p(x,y)|^2\omega_{\mathbb{D}^*}(y)\\
&\qquad=\big\|\rho(x)^{\mathbf{a}}\rho(y)^{2b-\frac p2}\chi(x)\chi(y)f_p(x,y)\big\|^2_{\boldsymbol{L}^2(\Sigma\times\Sigma)}.
\end{aligned}$$

Moreover, $\int_{\mathbb{D}^*}\chi^2\rho^{2\mathbf{a}-p+1}\omega_{\mathbb{D}^*}$ is finite as soon as $\mathbf{a} < \frac p2$. Fixing $\mathbf{a} = \frac p2 - \delta$, $\delta > 0$, $\int_{\mathbb{D}^*}\chi^2\rho^{2\mathbf{a}-p+1}\omega_{\mathbb{D}^*} = \int_{\mathbb{D}^*}\chi^2\rho^{1-2\delta}\omega_{\mathbb{D}^*} = \int_{\mathbb{D}^*}\chi^2 e^{-2\delta t}dt\,d\theta < \infty$ is independent on p, and consequently, the previous inequality reads:

$$\big\|\rho(x)^{\frac p2-\delta}\rho(y)^b\chi(x)\chi(y)f_p(x,y)\big\|_{\boldsymbol{L}^2(\Sigma\times\Sigma)} \leq C_{\ell,\delta}\, p^{-\ell} \tag{5.16}$$

for all $b \leq \frac p2$, and with $C_{\ell,\delta}$ independent of b, hence in particular for $p \geq 1$,

$$\big\|\rho(x)^{\frac p2-\delta}\rho(y)^{\frac p2-\delta}\chi(x)\chi(y)f_p(x,y)\big\|_{\boldsymbol{L}^2(\Sigma\times\Sigma)} \leq C_{\ell,\delta}p^{-\ell}. \tag{5.17}$$

With the same techniques, we extend this estimate to higher orders, namely for any $k \geq 0$,

$$\big\|\big(\rho(x)\rho(y)\big)^{\frac p2-\delta}\chi(x)\chi(y)f_p(x,y)\big\|_{\boldsymbol{L}^{2,k}(\Sigma\times\Sigma)} \leq C_{\ell,k,\delta}p^{-\ell}, \tag{5.18}$$

Observe that by (2.13) and (4.12), $d\log\rho = dt$ is $C^m(\Sigma,\omega_\Sigma)$ bounded, and the factor ρ in the definition of $L^{2,k}_{\mathrm{wtd}}$, thus (5.18) implies

$$\big\|\big(\rho(x)\rho(y)\big)^{\frac p2-\gamma}\chi(x)\chi(y)f_p(x,y)\big\|_{\boldsymbol{L}^{2,k}_{\mathrm{wtd}}(\Sigma\times\Sigma)} \leq C_{\ell,k,\gamma}p^{-\ell}, \tag{5.19}$$

with $\gamma = \delta + \frac12 > \frac12$. By Lemma 2.6 b) and (5.19), for every $\ell, m \geq 0$, $\gamma > \frac12$ there exists $C_{\ell,m,\gamma} > 0$ such that for all $p \geq 1$ we have

$$\big\|\big(\rho(x)\rho(y)\big)^{\frac p2-\gamma}\chi(x)\chi(y)f_p(x,y)\big\|_{C^m(\Sigma\times\Sigma)} \leq C_{\ell,m,\gamma}p^{-\ell}, \tag{5.20}$$

which can be rewritten in the sense of (2.14) as

$$\left\|\rho(x)^{-\gamma}\rho(y)^{-\gamma}\chi(x)\chi(y)K_p(x,y)\right\|_{C^m(h^p)} \leq C_{\ell,m,\gamma}p^{-\ell}. \tag{5.21}$$

Such an estimate is already well-known far from $D_a = (\{a\}\times\Sigma)+(\Sigma\times\{a\})$ in $\Sigma\times\Sigma$, where the weights ρ can be omitted. We moreover prove the analogous estimates on

$$\big(\mathbb{D}^*_{r/2}\times(\Sigma\smallsetminus\mathbb{D}_{r/3})\big)\cup\big((\Sigma\smallsetminus\mathbb{D}_{r/3})\times\mathbb{D}^*_{r/2}\big)$$

along the same lines. We thus come to the conclusion that (5.6) holds. □

Estimates (5.6) might look a bit disappointing, as these are made with *negative* weights; so far, this does not even tell us that $K_p(x,y)$ is bounded *near* the divisor $(\{a\}\times\Sigma)+(\Sigma\times\{a\})$ in $\overline{\Sigma}\times\overline{\Sigma}$ for the product Hermitian norm, which contrasts with our knowledge that the Bergman kernels of L^p do vanish *along* this divisor.

We see in next part that this rough estimate, together with the vanishing property, suffice however to estimate sharply Bergman kernels on the whole $\Sigma\times\Sigma$.

Remark 5.4 *The embedding $\boldsymbol{L}^{1,3}_{\mathrm{wtd}}\hookrightarrow C^0$ does not hold on the whole $\mathbb{D}^*$. Now Proposition 5.3 still holds on $\mathbb{D}^*\times\mathbb{D}^*$ near $\{0\}+\{0\}$ and, more generally, far from $\partial\mathbb{D}\times\partial\mathbb{D}$, as functions with supports in $\mathbb{D}^*_r$ or $\mathbb{D}^*_{2r/3}$ in $\mathbb{D}^*$ can be thought of as functions around the punctures in Σ, to which Lemma 2.6 applies. More precisely, let $K^{\mathbb{D}^*}_p(x,y)$ be the kernel of $\phi_p(\square^{\mathbb{D}^*}_p)$ on $\mathbb{D}^*$ with respect to $\omega_{\mathbb{D}^*}$; then for all $0<r<1$ and all $\ell,m\geq 0$, and $\gamma>\frac{1}{2}$, there exists $C_{\ell,m,\gamma}=C_{\ell,m,\gamma}(r)>0$ such that for all $p\geq 1$,*

$$\left\|\rho(x)^{-\gamma}\rho(y)^{-\gamma}\chi(x)\chi(y)K^{\mathbb{D}^*}_p(x,y)\right\|_{C^m(\mathbb{D}^*_r\times\mathbb{D}^*_r)} \leq C_{\ell,m,\gamma}p^{-\ell}. \tag{5.22}$$

6 Proofs of the main results

In this section, we will establish the results stated in the Introduction. We first complete the proofs of Theorem 1.1 and Corollary 1.3, then that of Corollary 1.4, and finally give the details needed to establish Theorems 2.5 and 1.6.

Theorem 1.1 and Corollary 1.3 will be a consequence of Proposition 5.3. The principal idea is to combine Proposition 5.3 and the *holomorphicity* of sections associated to Bergman kernels, together with the fact that for $p\geq 2$, $\boldsymbol{L}^2$ holomorphic sections of (L^p,h^p) over (Σ,ω_Σ) *vanish at* D, and similarly for the holomorphic sections in $\boldsymbol{L}^2\big(\mathbb{D}^*,\omega_{\mathbb{D}^*},\mathbb{C},h^p_{\mathbb{D}^*}\big)$, which vanish at 0, as already noticed.

Let us fix a point $a\in D$ and work around this point, in coordinates centered at a. For $x,y\in\mathbb{D}^*_r$, under our identification (5.2) and convention after (3.6), we write

$$\begin{aligned} B^{\mathbb{D}^*}_p(x,y) &= \big|\log(|y|^2)\big|^p\beta^{\mathbb{D}^*}_p(x,y);\\ B_p(x,y) &= \big|\log(|y|^2)\big|^p\beta^{\Sigma}_p(x,y), \end{aligned} \tag{6.1}$$

where, by (3.6), $\beta^{\mathbb{D}^*}_p$ is a holomorphic function of x and $\overline{y}$, namely

$$\beta^{\mathbb{D}^*}_p(x,y) = \frac{1}{2\pi(p-2)!}\sum_{\ell=1}^{\infty}\ell^{p-1}x^\ell\overline{y}^\ell, \tag{6.2}$$

which vanishes along $\{x=0\}+\{y=0\}=\{xy=0\}\subset\mathbb{D}\times\mathbb{D}$. By Remark 3.2, (2.2) and the convention after (3.6), β^{Σ}_p is a holomorphic function of x and $\overline{y}$ vanishing along $\{x=0\}+\{y=0\}\subset\mathbb{D}_r\times\mathbb{D}_r$, and

$$\Big|(B_p-B^{\mathbb{D}^*}_p)(x,y)\Big|_{h^p} = \big|\log(|x|^2)\big|^{p/2}\big|\log(|y|^2)\big|^{p/2}\Big|(\beta^{\mathbb{D}^*}_p-\beta^{\Sigma}_p)(x,y)\Big|. \tag{6.3}$$

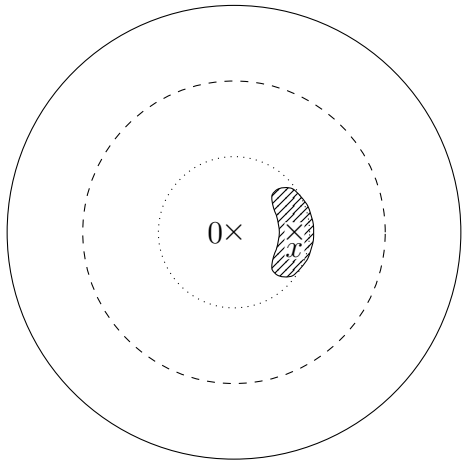

Figure 2 – In (6.7), $\tilde{F}(\Box_p)(x,\cdot)$ depends only on the restriction of $\Box_p$ to a *geodesic* ball of radius $\frac{\epsilon}{\sqrt{2}}$ around x, and vanishes outside this ball (the circles $\{|z|=r/2\}$ and $\{|z|=r\}$ are at Poincaré distance $\frac{1}{2}\log\big(1-\frac{\log 2}{\log r}\big)$ from each other).

Theorem 6.1 *For any $k\in\mathbb{N},\ell>0$, $\alpha\geq 0$, there exists a constant $C_{k,\ell,\alpha}$ such that for $p\gg 1$, on $\mathbb{D}^*_{r/2}\times\mathbb{D}^*_{r/2}$, we have in the sense of (2.14),*

$$\Big|B_p^{\mathbb{D}^*}(x,y)-B_p(x,y)\Big|_{C^k(h^p)}\leq C_{k,\ell,\alpha}p^{-\ell}\big|\log(|x|^2)\big|^{-\alpha}\big|\log(|y|^2)\big|^{-\alpha}. \tag{6.4}$$

Proof — By (5.4), we have

$$\begin{aligned} F\big(\Box_p\big)(x,y)-B_p(x,y)&=\phi_p\big(\Box_p\big)(x,y)=K_p(x,y),\\ F\big(\Box_p^{\mathbb{D}^*}\big)(x,y)-B_p^{\mathbb{D}^*}(x,y)&=\phi_p\big(\Box_p^{\mathbb{D}^*}\big)(x,y)=K_p^{\mathbb{D}^*}(x,y). \end{aligned} \tag{6.5}$$

By the finite propagation speed for the wave operators [MM1, Theorem D.2.1], we have

$$\begin{aligned} &\operatorname{supp}\widetilde{F}(\Box_p)(x,\cdot)\subset B\Big(x,\frac{\epsilon}{\sqrt{2}}\Big)\text{ and }\widetilde{F}(\Box_p)(x,\cdot)\\ &\text{depends only on the restriction of }\Box_p\text{ to }B\Big(x,\frac{\epsilon}{\sqrt{2}}\Big). \end{aligned} \tag{6.6}$$

Here $B(x,\frac{\epsilon}{\sqrt{2}})$ is the geodesic ball with center at x and radius of $\frac{\epsilon}{\sqrt{2}}$ for $\omega_{\mathbb{D}^*}$.

Thus from (5.2) and (6.6) (here we fix $\epsilon>0$ such that $\epsilon\leq d(\partial D_r,\partial D_{r/2})$), we have

$$\widetilde{F}\big(\Box_p\big)(x,y)=\widetilde{F}\big(\Box_p^{\mathbb{D}^*}\big)(x,y)\quad\text{for all }x,y\in\mathbb{D}^*_{r/2}\,. \tag{6.7}$$

From (6.5) and (6.7), we have

$$B_p(x,y)-B_p^{\mathbb{D}^*}(x,y)=K_p^{\mathbb{D}^*}(x,y)-K_p(x,y)\quad\text{for all }x,y\in\mathbb{D}^*_{r/2}\,. \tag{6.8}$$

Note that by (4.7) and (4.12),

$$\Big|\,|\log|z|^2|^p\Big|_{C^k}=\Big|e^{tp}\Big|_{C^k}\leq C_kp^ke^{tp}. \tag{6.9}$$

By (2.13), (2.14), (4.7), (4.12), (4.16), (5.2) and (6.9), for any $k \in \mathbb{N}$, there exists $C > 0$ such that for any $p \in \mathbb{N}^*$, $x, y \in \mathbb{D}^*_{r/2}$, we have

$$\begin{aligned} &\big|B_p^{\mathbb{D}^*} - B_p\big|_{C^k(h^p)}(x,y) \leq Cp^k\big|\log(|x|^2)\big|^{p/2}\big|\log(|y|^2)\big|^{p/2}\big|\beta_p^{\mathbb{D}^*} - \beta_p^{\Sigma}\big|_{C^k}(x,y), \\ &\big|\beta_p^{\mathbb{D}^*} - \beta_p^{\Sigma}\big|_{C^k}(x,y) \leq Cp^k\big|\log(|x|^2)\big|^{-p/2}\big|\log(|y|^2)\big|^{-p/2}\big|B_p^{\mathbb{D}^*} - B_p\big|_{C^k(h^p)}(x,y). \end{aligned} \tag{6.10}$$

By Proposition 5.3, (6.8) and (6.10), we get for $x, y \in \mathbb{D}^*_{r/2}$,

$$\big|\beta_p^{\mathbb{D}^*} - \beta_p^{\Sigma}\big|_{C^k}(x,y) \leq C_{\ell,\gamma}p^{-\ell}\big|\log(|x|^2)\big|^{\gamma-p/2}\big|\log(|y|^2)\big|^{\gamma-p/2}. \tag{6.11}$$

Our task is thus to refine estimate (6.11) by working directly on $\beta_p^{\mathbb{D}^*} - \beta_p^{\Sigma}$. We set:

$$\beta_p^{\mathbb{D}^*,\Sigma} = \beta_p^{\mathbb{D}^*} - \beta_p^{\Sigma}. \tag{6.12}$$

As $\beta_p^{\mathbb{D}^*,\Sigma}$ is a holomorphic function of x and $\overline{y}$ vanishing along $\{xy = 0\}$, one can write

$$\beta_p^{\mathbb{D}^*,\Sigma}(x,y) = x\overline{y}g_p(x,y) \tag{6.13}$$

for some smooth g_p, holomorphic in x and $\overline{y}$. Now for $\varepsilon > 0$ small,

$$\sup_{|x|,|y|\leq\varepsilon^{1/2}} |g_p(x,y)| = \sup_{|x|=|y|=\varepsilon^{1/2}} |g_p(x,y)| = \varepsilon^{-1} \sup_{|x|=|y|=\varepsilon^{1/2}} \big|\beta_p^{\mathbb{D}^*,\Sigma}(x,y)\big| \tag{6.14}$$

by holomorphicity of g_p. From (6.11) and (6.14), we get:

$$\sup_{|x|,|y|\leq\varepsilon^{1/2}} |g_p(x,y)| \leq C_{\ell,\gamma}p^{-\ell}\varepsilon^{-1}|\log\varepsilon|^{-p+2\gamma}. \tag{6.15}$$

From (6.13), (6.15), for $|x|, |y| \leq e^{-p}$, we have

$$\begin{aligned} \big|\beta_p^{\mathbb{D}^*,\Sigma}(x,y)\big| &\leq |x||y| \sup_{|x|,|y|\leq e^{-p}} |g_p(x,y)| \\ &\leq C_{\ell,\gamma}p^{-\ell}e^{2p}(2p)^{-p+2\gamma}|x||y|. \end{aligned} \tag{6.16}$$

Pick $\alpha \geq 0$. As the function $u \mapsto u\big(-\log(u^2)\big)^{\frac{p}{2}+\alpha}$ is increasing on $(0, e^{-\frac{p}{2}-\alpha}]$, hence on $(0, e^{-p}]$, thus

$$|x| \leq e^{-p}(2p)^{\frac{p}{2}+\alpha}\big|\log(|x|^2)\big|^{-\frac{p}{2}-\alpha} \quad \text{if } |x| \leq e^{-p}. \tag{6.17}$$

We thus convert (6.16) into:

$$\big|\beta_p^{\mathbb{D}^*,\Sigma}(x,y)\big| \leq 2^{2(\gamma+\alpha)}C_{\ell,\gamma}p^{-\ell+2(\gamma+\alpha)}\big|\log(|x|^2)\big|^{-\frac{p}{2}-\alpha}\big|\log(|y|^2)\big|^{-\frac{p}{2}-\alpha} \tag{6.18}$$

on $|x|, |y| \leq e^{-p}$; up to increasing ℓ and adjusting the constant, we thus have

$$\big|\beta_p^{\mathbb{D}^*,\Sigma}(x,y)\big| \leq C_{\ell,\alpha}p^{-\ell}\big|\log(|x|^2)\big|^{-\frac{p}{2}-\alpha}\big|\log(|y|^2)\big|^{-\frac{p}{2}-\alpha} \tag{6.19}$$

on $\{|x|, |y| \leq e^{-p}\}$. Notice that such an estimate holds on $\{e^{-p} \leq |x|, |y| \leq r \leq e^{-1}\}$ as well, since then $\big|\log(|x|^2)\big|, \big|\log(|y|^2)\big| \leq 2p$, and thus (6.11) gives for any $\beta \geq 0$,

$$\big|\beta_p^{\mathbb{D}^*,\Sigma}(x,y)\big| \leq C_{\ell,\gamma}p^{-\ell}(2p)^{2(\gamma+\beta)}\big|\log(|x|^2)\big|^{-\beta-p/2}\big|\log(|y|^2)\big|^{-\beta-p/2}. \tag{6.20}$$

We conclude by the case $|x| \le e^{-p}$, $e^{-p} \le |y| \le r \le e^{-1}$; fixing y, by the holomorphicity of x and (6.11),

$$\begin{aligned}\sup_{|x|\le e^{-p}} \Big|\frac{1}{x}\beta_p^{\mathbb{D}^*,\Sigma}(x,y)\Big| &= \sup_{|x|=e^{-p}} \Big|\frac{1}{x}\beta_p^{\mathbb{D}^*,\Sigma}(x,y)\Big| \\ &= e^p \sup_{|x|=e^{-p}} \Big|\beta_p^{\mathbb{D}^*,\Sigma}(x,y)\Big| \\ &\le e^p C_{\ell,\gamma} p^{-\ell}(2p)^{-\frac{p}{2}+\gamma}\big|\log(|y|^2)\big|^{-\frac{p}{2}+\gamma}\end{aligned} \tag{6.21}$$

so that

$$\big|\beta_p^{\mathbb{D}^*,\Sigma}(x,y)\big| \le e^p C_{\ell,\gamma} p^{-\ell}(2p)^{-\frac{p}{2}+\gamma}|x|\big|\log(|y|^2)\big|^{-\frac{p}{2}+\gamma} \tag{6.22}$$

on $\{|x| \le e^{-p},\ e^{-p} \le |y| \le r \le e^{-1}\}$. Now on this set, $\big|\log(|y|^2)\big|^{\frac{p}{2}+\gamma} \le (2p)^{\alpha+\gamma}\big|\log(|y|^2)\big|^{-\frac{p}{2}-\alpha}$ by (6.17) and (6.22), we get

$$\big|\beta_p^{\mathbb{D}^*,\Sigma}(x,y)\big| \le 2^{2(\alpha+\gamma)} C_{\ell,\gamma} p^{-\ell+2(\alpha+\gamma)}\big|\log(|x|^2)\big|^{-\frac{p}{2}-\alpha}\big|\log(|y|^2)\big|^{-\frac{p}{2}-\alpha} \tag{6.23}$$

on $\{|x| \le e^{-p},\ e^{-p} \le |y| \le r \le e^{-1}\}$. This holds $\{|y| \le e^{-p},\ e^{-p} \le |x| \le r \le e^{-1}\}$ as well by symmetry.

From (6.19)—(6.23), we get: For all $\ell > 0$, $\alpha \ge 0$, there exists a constant $C_{\ell,\alpha}$ such that for $p > 1$, on $\mathbb{D}^*_{r/2} \times \mathbb{D}^*_{r/2}$,

$$\big|(\beta_p^{\mathbb{D}^*} - \beta_p^{\Sigma})(x,y)\big| \le C_{\ell,\alpha} p^{-\ell}\big|\log(|x|^2)\big|^{-\frac{p}{2}-\alpha}\big|\log(|y|^2)\big|^{-\frac{p}{2}-\alpha}. \tag{6.24}$$

From (6.3) and (6.24), we get (6.4) for $k = 0$.

Observe that by (4.5), (4.14), for any $k \in \mathbb{N}$, there exists $C_k > 0$ such that on $\mathbb{D}^*_r$,

$$\Big|\log|z|^2\Big|_{C^k} \le C_k\Big|\log|z|^2\Big|, \quad |z|_{C^k} \le C_k\Big|\log|z|^2\Big|^k|z|. \tag{6.25}$$

From (6.13) and (6.25), we have

$$\big|\beta_p^{\mathbb{D}^*,\Sigma}(x,y)\big|_{C^k} \le C_k|x||y| \sum_{k_1+k_2+k_3\le k} \big|\log(|x|^2)\big|^{k_1}\big|\log(|y|^2)\big|^{k_2}\Big|g_p(x,y)\Big|_{C^{k_3}}. \tag{6.26}$$

By (4.6) and (4.12), we get

$$\begin{aligned}&\Big|g_p(x,y)\Big|_{C^j} \\ &\quad\le C \sum_{j_1+j_2\le j} \big|\log(|x|^2)\big|^{j_1}\big|\log(|y|^2)\big|^{j_2}|x|^{\min(1,j_1)}|y|^{\min(1,j_2)}\Big|\frac{\partial^{j_1+j_2}}{\partial x^{j_1}\partial \overline{y}^{j_2}} g_p(x,y)\Big|.\end{aligned} \tag{6.27}$$

Thus from (6.26) and (6.27), for any $k \in \mathbb{N}$, there exists $C_k > 0$ such that for $x, y \in \mathbb{D}^*_r$,

$$\big|\beta_p^{\mathbb{D}^*,\Sigma}(x,y)\big|_{C^k} \le C_k|x||y|\big|\log(|x|^2)\big|^k\big|\log(|y|^2)\big|^k \sum_{j_1+j_2\le k} \Big|\frac{\partial^{j_1+j_2}}{\partial x^{j_1}\partial \overline{y}^{j_2}} g_p(x,y)\Big|. \tag{6.28}$$

Now we can combine the argument for (6.24) and (6.28) to get (6.4) for $k \ge 1$. The proof of Theorem 6.1 is completed. □

Proof of Theorem 1.1 — This follows immediately by taking $x = y$ in Theorem 6.1. □

Proof of Corollary 1.3 — By Theorem 1.1, Proposition 3.3 and (3.22), we get

$$\sup_{|x|\leq r\leq e^{-1}} \big|B_p(x,x)\big|_{h^p} = \frac{p^{3/2}}{(2\pi)^{3/2}} + \mathcal{O}(p) \qquad \text{as } p\to\infty. \tag{6.29}$$

Combining (1.6), (2.3) and (6.29), we obtain the desired conclusion. □

We turn now to the proof of Corollary 1.4. Let $\overline{\Sigma}$ be a compact Riemann surface of genus g, let $D=\{a_1,\dots,a_N\}\subset\overline{\Sigma}$ be a finite set and $\Sigma=\overline{\Sigma}\smallsetminus D$.

The Uniformization Theorem, see [FK, Theorems IV.5.6, IV.6.3, IV.6.4, IV.8.6] readily implies that conditions (i)-(iv) from the introduction are equivalent, taking into account that $\chi(\Sigma)=2-2g-N$ and the degree of L equals $-\chi(\Sigma)$: From [FK, Theorem IV.8.6], we see first that (i) and (iii) are equivalent, and combining [FK, Theorems IV.6.3, IV.6.4], we know that (iii) implies (ii), and the Riemann surfaces with universal covering the sphere or $\mathbb{C}$, are the sphere, $\mathbb{C}$, $\mathbb{C}^*$ or torus. This means that (ii) implies (iii). Finally by [GrH, p. 214], (ii) and (iv) are equivalent.

Lemma 6.2 *Let $\overline{\Sigma}$ be a compact Riemann surface of genus g and $D=\{a_1,\dots,a_N\}\subset\overline{\Sigma}$ a finite set such that $2g-2+N>0$. Denote $\Sigma=\overline{\Sigma}\smallsetminus D$ and $L=K_{\overline{\Sigma}}\otimes\mathscr{O}_{\overline{\Sigma}}(D)$. There exists a metric ω_Σ on Σ and a singular Hermitian metric h on L, such that (Σ,ω_Σ) and the formal square root of (L,h) satisfy the conditions (α) and (β).*

Proof. — Since $\chi(\Sigma)=2-2g-N<0$, the universal covering of Σ is $\mathbb{H}$ and Σ admits a Kähler-Einstein metric ω_Σ of constant negative curvature -4, induced by the Poincaré metric $\omega_{\mathbb{H}}=\frac{idz\wedge d\bar z}{4|\mathrm{Im}z|^2}$ on $\mathbb{H}$. It is a classical fact that every $a\in D$ has a coordinate neighborhood $\big(\overline{U}_a,z\big)$ in $\overline{\Sigma}$ such that in this coordinate ω_Σ is exactly given by $\omega_{\mathbb{D}^*}(z)$ on $U_a=\overline{U}_a\smallsetminus\{a\}$, see e.g. [Bor, p. 79, (6.7)].

Note that ω_Σ extends to a closed strictly positive $(1,1)$-current $\omega_{\overline{\Sigma}}$ on $\overline{\Sigma}$. Let h^{K_Σ} be the metric on K_Σ induced by ω_Σ. Then we have

$$|dz|^2_{h^{K_\Sigma}} = |z|^2\log^2(|z|^2) \text{ in } (\overline{U}_a,z). \tag{6.30}$$

Let σ be the canonical section of $\mathscr{O}_{\overline{\Sigma}}(D)$. The singular metric $h^{\mathscr{O}_{\overline{\Sigma}}(D)}$ on $\mathscr{O}_{\overline{\Sigma}}(D)$ is defined by $|\sigma|^2_{h^{\mathscr{O}_{\overline{\Sigma}}(D)}}=1$. The isomorphism

$$K_\Sigma\to K_\Sigma\otimes\mathscr{O}_{\overline{\Sigma}}(D)|_\Sigma = L|_\Sigma,\ \ s\to s\otimes\sigma$$

over Σ and the metrics h^{K_Σ} and $h^{\mathscr{O}_{\overline{\Sigma}}(D)}$ induce the metric h on $L|_\Sigma$. The curvature of the line bundle $(L|_\Sigma,h)$ is given by $-2i\omega_\Sigma$. Since $\frac{\sigma}{z}$ is a holomorphic frame of $\mathscr{O}_{\overline{\Sigma}}(D)$ on $\overline{U}_a$, $dz\otimes\frac{\sigma}{z}$ is a holomorphic frame of L on $\overline{U}_a$. Then $|dz\otimes\frac{\sigma}{z}|^2_h=(\log(|z|^2))^2$, and thus (Σ,ω_Σ) and the (formal) square root of (L,h) satisfy conditions (α) and (β). □

Let $\Gamma\cong\pi_1(\Sigma)$ be the group of deck transformations of the covering $\mathbb{H}\to\Sigma$. Then $\Gamma\backslash\mathbb{H}\cong\Sigma$ has finite hyperbolic volume and Γ is a Fuchsian group of the first kind without elliptic elements. We denote by $\pi:\mathbb{H}\to\Gamma\backslash\mathbb{H}$ the canonical projection.

The space $\mathcal{M}^\Gamma_{2p}$ of Γ-modular forms of weight $2p$ is by definition the space of holomorphic functions $f\in\mathscr{O}(\mathbb{H})$ satisfying the functional equation

$$f(\gamma z)=(cz+d)^{2p}f(z),\quad z\in\mathbb{H},\ \gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma, \tag{6.31}$$

and which extend holomorphically to the cusps of Γ (fixed points of the parabolic elements). If $f\in\mathscr{O}(\mathbb{H})$ satisfies (6.31), then $fdz^{\otimes p}\in H^0(\mathbb{H},K^p_{\mathbb{H}})$ descends to a holomorphic

section $\Phi(f)$ of $H^0(\Sigma, K_\Sigma^p) \cong H^0(\Sigma, L^p)$. By [Mu, Proposition 3.3, 3.4(b)], Φ induces an isomorphism $\Phi : \mathcal{M}_{2p}^\Gamma \to H^0\big(\overline{\Sigma}, L^p\big)$.

The subspace of $\mathcal{M}_{2p}^\Gamma$ consisting of modular forms vanishing at the cusps is called the space of *cusp forms* (Spitzenformen) of weight $2p$ of Γ, denoted by $\mathcal{S}_{2p}^\Gamma$. The space of cusps forms is endowed with the Petersson scalar product

$$\langle f, g\rangle := \int_U f(z)\overline{g(z)}(2y)^{2p}\, dv_{\mathbb{H}}(z),$$

where U is a fundamental domain for Γ and $dv_{\mathbb{H}} = \frac{1}{2}y^{-2}dx \wedge dy$ is the hyperbolic volume form. The Bergman density function of $\big(\mathcal{S}_{2p}^\Gamma, \langle\cdot,\cdot\rangle\big)$ is defined by taking any orthonormal basis (f_j) and setting

$$S_p^\Gamma(z) = \sum_j |f_j(z)|^2 (2y)^{2p}\,, \quad z \in U.$$

Under the above isomorphism, $\mathcal{S}_{2p}^\Gamma$ is identified to the space $H^0\big(\overline{\Sigma}, L^p \otimes \mathscr{O}_{\overline{\Sigma}}(D)^{-1}\big) = H^0\big(\overline{\Sigma}, K_{\overline{\Sigma}}^p \otimes \mathscr{O}_{\overline{\Sigma}}(D)^{p-1}\big)$ of holomorphic sections of L^p over $\overline{\Sigma}$ vanishing on D.

If we endow $K_{\mathbb{H}}$ with the Hermitian metric induced by the Poincaré metric on $\mathbb{H}$, the scalar product of two elements $udz^{\otimes p}, vdz^{\otimes p} \in K_{\mathbb{H},z}^p$ is $\langle udz^{\otimes p}, vdz^{\otimes p}\rangle = u\overline{v}(2y)^{2p}$. Hence, the Petersson scalar product corresponds to the $\boldsymbol{L}^2$ scalar product of pluricanonical forms on Σ,

$$\langle f, g\rangle = \int_\Sigma \langle \Phi(f), \Phi(g)\rangle\, \omega_\Sigma\,, \quad f, g \in \mathcal{S}_{2p}^\Gamma\,.$$

The isomorphism Φ gives thus an isometry (see also [CM, Section 6.4])

$$\mathcal{S}_{2p}^\Gamma \cong H^0\big(\overline{\Sigma}, L^p \otimes \mathscr{O}_{\overline{\Sigma}}(D)^{-1}\big) \cong H^0_{(2)}(\Sigma, K_\Sigma^p) \cong H^0_{(2)}(\Sigma, L^p), \tag{6.32}$$

where $H^0_{(2)}(\Sigma, L^p)$ is the space of holomorphic sections of L^p which are square-integrable with respect to the volume form ω_Σ and the metric h^p on L^p, with h introduced in Lemma 6.2. Moreover, $H^0_{(2)}(\Sigma, K_\Sigma^p)$ is the space of $\boldsymbol{L}^2$-pluricanonical sections with respect to the metric $h^{K_\Sigma^p}$ and the volume form ω_Σ, where we denote by h^{K_Σ} the Hermitian metric induced by ω_Σ on K_Σ. We let now B_p^Γ be the Bergman density function of $H^0_{(2)}(\Sigma, L^p)$, defined as in (1.4). We have

$$S_p^\Gamma(z) = B_p^\Gamma(\pi(z)), \quad z \in U. \tag{6.33}$$

We thus identify the space of cusp forms $\mathcal{S}_{2p}^\Gamma$ to a subspace of holomorphic sections of L^p by (6.32) and its Bergman density function S_p^Γ to B_p^Γ by (6.33).

Proof of Corollary 1.4 — In view of Lemma 6.2, this follows immediately from Corollary 1.3 applied for the even powers of the square root of $L = K_{\overline{\Sigma}} \otimes \mathscr{O}_{\overline{\Sigma}}(D)$, and from (6.33). Even if the square root of L is a formal line bundle we can apply Corollary 1.3 to its even powers, i. e., to L^p, which has the effect of scaling p to $2p$. This explains the occurence of p/π in the leading term of (1.8) and (1.9), as well as in Theorems 1.5 and 1.6. □

Proof of Theorem 2.5 — Since we have the same Sobolev constants for $\Gamma\backslash\widetilde{M}$ and M (cf. [MM1, Theorem A.1.6, (A.1.15)]), the proof of [DLM1, Proposition 4.1] or of [MM1, Proposition 4.1.5] shows that for any $l, m \in \mathbb{N}$, there exists $C_{l,m} > 0$ (independent of $\Gamma \subset \pi_1(M)$) such that for any $p \in \mathbb{N}^*, x, y \in \Gamma\backslash\widetilde{M}$

$$\Big|K_p^\Gamma(x,y) - B_p^\Gamma(x,y)\Big|_{C^m(\Gamma\backslash\widetilde{M})} \le C_{l,m}p^{-l}. \tag{6.34}$$

By the finite propagation speed of the wave operator, for $d(x,y) \leq \epsilon$, we have

$$K_p^\Gamma(x,y) = \pi_\Gamma^* K_p(\pi_\Gamma(x), \pi_\Gamma(y)). \tag{6.35}$$

From (6.34) and (6.35) we conclude Theorem 2.5. □

Proof of Theorem 1.5 — For a subgroup $\Gamma \subset \Gamma_0$ of finite index let $\pi_\Gamma : \Gamma\backslash\mathbb{H} \to \Gamma_0\backslash\mathbb{H}$ be the canonical finite covering. We will add a superscript Γ for the various objects living on $\Gamma\backslash\mathbb{H}$. We fix $0 < r < e^{-1}$ such that the ends $V_1, \ldots, V_N$ of $\Gamma_0\backslash\mathbb{H}$ are of the form $(\mathbb{D}_r^*, \omega_{\mathbb{D}^*}, \mathbb{C}, h_{\mathbb{D}^*})$. Now for any finite index subgroup $\Gamma \subset \Gamma_0$, the ends of $\Gamma\backslash\mathbb{H}$ are the connected components $\{V_i^\Gamma\}_i$ of $\pi_\Gamma^{-1}(V_j)$, $j = 1, \ldots, N$. Moreover, if V_i^Γ is a connected component of $\pi_\Gamma^{-1}(V_j)$, there exists $n \in \mathbb{N}^*$ such that the map $\pi_\Gamma : V_i^\Gamma \to V_j$ is given by $\mathbb{D}_{r^{1/n}}^* \to \mathbb{D}_r^* : z \mapsto z^n$ (cf. [Fo, Theorem 5.10]). Let

$$V_{i,r}^\Gamma = V_i^\Gamma \cap \pi_\Gamma^{-1}(\mathbb{D}_{r^n}^*)(= \mathbb{D}_r^*). \tag{6.36}$$

On $(\Gamma\backslash\mathbb{H}) \smallsetminus \bigcup_i V_{i,r/4}^\Gamma$, the Sobolev constants are the same as the ones on $(\Gamma_0\backslash\mathbb{H}) \smallsetminus \bigcup_j V_{j,r/4}$, where $V_{j,r/4} = \mathbb{D}_{r/4}^*$ under the identification of V_j and $\mathbb{D}_r^*$.

Arguing as in the proof of Theorem 2.5, we see that for any $k \in \mathbb{N}^*$, there exists $C_{\Gamma_0,k} > 0$ (depending only on Γ_0 and k) such that for any $p \geq 1$, $x \in (\Gamma\backslash\mathbb{H}) \smallsetminus \bigcup_i V_{i,r/4}^\Gamma$,

$$\Big| B_p^\Gamma(x) - \frac{1}{\pi}p + \frac{1}{2\pi} \Big|_{C^m} \leq C_{\Gamma_0,k} p^{-k}. \tag{6.37}$$

Now on $V_{i,r}^\Gamma$ we have

$$\big(V_{i,r}^\Gamma, \omega_{\Gamma\backslash\mathbb{H}}, L, h\big) \simeq (\mathbb{D}_r^*, \omega_{\mathbb{D}^*}, \mathbb{C}, h_{\mathbb{D}^*}). \tag{6.38}$$

As the curvatures on $\Gamma\backslash\mathbb{H}$ are pull-back of the corresponding curvatures on $\Gamma_0\backslash\mathbb{H}$, we see from the proof of [MM1, Theorem 6.1.1] that the spectral gap property, Proposition 5.1, holds uniformly on the set of subgroups $\Gamma \subset \Gamma_0$, i. e., there exists $c_{\Gamma_0} > 0$, $p_0 > 0$ such that for all $p \geq p_0$, and all subgroups $\Gamma \subset \Gamma_0$, we have

$$\mathrm{Spec}(\square_p^\Gamma) \subset \{0\} \cup [c_{\Gamma_0} p, +\infty)\,. \tag{6.39}$$

From the proof of Proposition 5.3 and (6.38), by using (6.39) in (5.14), we get for any $\ell, m \geq 0$, $\gamma > \frac{1}{2}$, there exists $C_{\ell,m,\gamma} > 0$ such that for any $p \geq p_0$ and any $\Gamma \subset \Gamma_0$ with finite index, we have

$$\big\| \rho(x)^{-\gamma} \rho(y)^{-\gamma} K_p^\Gamma(x,y) \big\|_{C^m(h^p)} \leq C_{\ell,m,\gamma} p^{-\ell}, \quad \text{for any } x, y \in V_{i,r}^\Gamma. \tag{6.40}$$

Finally from (6.40) and the proof of Theorem 6.1, for any $\ell, m \geq 0$, and every $\delta > 0$, there exists a constant $C_{\Gamma_0} > 0$ such that for all $p \in \mathbb{N}^*$ and any $\Gamma \subset \Gamma_0$ with finite index,

$$\Big\| B_p^\Gamma - B_p^{\mathbb{D}^*} \Big\|_{C^m}(z) \leq C_{\Gamma_0} p^{-\ell} \big| \log(|z|^2) \big|^{-\delta} \quad \text{for } \ z \in \cup_i V_{i,r}^\Gamma. \tag{6.41}$$

Now it is clear from Proposition 3.3, (3.22), (6.37) and (6.41), that (1.11) holds. □

Proof of Theorem 1.6. — As the proof of (6.34), there exists $C_{l,m} > 0$ (independent of $\Gamma \subset \Gamma_0$) such that for any $p \in \mathbb{N}^*$, $x, y \in (\Gamma\backslash\mathbb{H}) \smallsetminus \cup_i V_{i,r/4}^\Gamma$,

$$\big| K_p^\Gamma(x,y) - B_p^\Gamma(x,y) \big|_{C^m} \leq C_{l,m} p^{-l}. \tag{6.42}$$

Note that the property of the finite propagation speed of solutions of hyperbolic equations still holds on orbifolds, as shown in [M, §6.6]. Hence (6.42) implies that (6.37)

holds for $x \in (\Gamma\backslash\mathbb{H}) \smallsetminus \big(\bigcup_i V^{\Gamma}_{i,r/4} \cup \bigcup_j \pi_\Gamma^{-1}(U_{x_j})\big)$. Moreover, on each end, the argument (6.39)–(6.41) goes through, thus we get the uniform estimate (6.41) with the constant C_{Γ_0} depending only on Γ_0 and independent of $\Gamma \subset \Gamma_0$.

Now on each component of $\pi_\Gamma^{-1}(U_{x_j})$, observe that the stabilizer group $\Gamma_{x_j^\Gamma}$ of x_j^Γ acts in the normal coordinate of x_j^Γ in $\mathbb{H}$ by rotation, with x_j^Γ being the unique fixed point. We denote by Z the (real) normal coordinates around a point $x \in \mathbb{H}$. By [DLM1, (4.114), (5.21), (5.23)] (cf. [MM1, Theorem 5.4.11], [DLM2, Theorem 0.2]) for the $2p$-th tensor power of $(L|_{\Gamma\backslash\mathbb{H}})^{1/2} = K^{1/2}_{\Gamma\backslash\mathbb{H}}$, there exist $C_0 > 0$ such that for any $k, l > 0$, there exist $N > 0$, $C_{k,l} > 0$, depending only on Γ_0, k, l, such that in the normal coordinates around x_j^Γ in $\mathbb{H}$, we have when $p \to \infty$,

$$\begin{aligned}\Bigg| \frac{1}{2p} B_p^\Gamma(Z,Z) &- \sum_{\nu=0}^{k} \boldsymbol{b}_\nu(Z)(2p)^{-\nu} \\ &- \sum_{\nu=0}^{2k} (2p)^{-\frac{\nu}{2}} \sum_{1\neq\gamma\in\Gamma_{x_j^\Gamma}} e^{i\theta_\gamma p} \mathcal{K}_\nu(\sqrt{2p}Z) e^{-p(1-e^{i\theta_\gamma})|z|^2} \Bigg|_{C^l} \\ &\leqslant C_{k,l}\left(p^{-k-1} + p^{-k+\frac{l-1}{2}}\,(1+\sqrt{p}|Z|)^N\, e^{-\sqrt{C_0\,p}\,|Z|}\right),\end{aligned} \tag{6.43}$$

where $\mathcal{K}_\nu(Z)$ are polynomials in the real coordinates Z. Note also that $\boldsymbol{b}_\nu$ are given by (2.11). By [DLM1, (4.107), (4.105), (4.117), (5.4)] (or [MM1, Remark 4.1.26, (4.1.84), (4.1.92)]), we have $\mathcal{K}_0 = \frac{1}{2\pi}, \mathcal{K}_1 = 0$. This implies in particular that (1.13) holds on $\pi_\Gamma^{-1}(U_{x_j})$.

Combining the above arguments, we establish (1.12), (1.13) and (1.15). To obtain (1.14), notice that by our choice of q_0, all factors $e^{i\theta_\gamma q_0 p}$ in (6.43) are 1. Thus (1.13) implies (1.14). □

A Proof of Lemma 3.4

We prove in this appendix the existence of a constant C such that for all $\zeta > 0$ and all $p \geq 1$,

$$p\big(1 + p(1-\zeta)^2\big)|\delta_p(\zeta)| \leq C, \tag{A.1}$$

where we recall the notation from (3.23):

$$\delta_p(\zeta) = e^{p(1-\zeta+\log\zeta)} - e^{-\frac{p}{2}(1-\zeta)^2}\Big(1 - \frac{p}{3}(1-\zeta)^3\Big).$$

Clearly, (A.1) holds for any fixed p, that is: for any $p \geq 1$, there exists C_p such that for all $\zeta > 0$, $p\big(1 + p(1-\zeta)^2\big)|\delta_p(\zeta)| \leq C_p$. We thus want to show that the C_p can be chosen independent of p; this we do arguing by contradiction: we hence assume that there exists a positive sequence $(\zeta_p)_{p\geq 1}$ such that, up to passing to a subsequence,

$$p\big(1 + p(1-\zeta_p)^2\big)|\delta_p(\zeta_p)| \xrightarrow{p\to\infty} \infty. \tag{A.2}$$

Up to passing to a subsequence, (ζ_p) converges in $[0,\infty]$; we first distinguish the three cases $(\zeta_p) \to 0$, $(\zeta_p) \to \infty$, and $(\zeta_p) \to \ell \in \mathbb{R}_{>0}$.

1. $(\zeta_p) \to 0$: here, $p\big(1+p(1-\zeta_p)^2\big) \sim p^2$, whereas $e^{p(1-\zeta_p+\log\zeta_p)} \leq e^{-p}$ for p large, hence

$$p\big(1 + p(1-\zeta_p)^2\big)e^{p(1-\zeta_p+\log\zeta_p)} \longrightarrow 0;$$

likewise, $e^{-\frac{p}{2}(1-\zeta)^2} \le e^{-\frac{p}{4}}$ for p large, and $\big(1-\frac{p}{3}(1-\zeta)^3\big) \sim -\frac{p}{3}$, hence

$$p\big(1+p(1-\zeta_p)^2\big)e^{-\frac{p}{2}(1-\zeta_p)^2}\big(1-\frac{p}{3}(1-\zeta_p)^3\big) \longrightarrow 0. \tag{A.3}$$

This way, $p\big(1+p(1-\zeta_p)^2\big)|\delta_p(\zeta_p)| \to 0$, which contradicts (A.2).

2. $(\zeta_p) \to \infty$: one has $p\big(1+p(1-\zeta_p)^2\big) \sim p^2\zeta_p^2$ and $e^{p(1-\zeta_p+\log\zeta_p)} \le e^{-\frac{p\zeta_p}{2}}$ for p large, hence

$$p\big(1+p(1-\zeta_p)^2\big)e^{p(1-\zeta_p+\log\zeta_p)} \lesssim p^2\zeta_p^2 e^{-\frac{p\zeta_p}{2}} \longrightarrow 0; \tag{A.4}$$

moreover $e^{-\frac{p}{2}(1-\zeta_p)^2} \le e^{-\frac{p\zeta_p}{4}}$ for p large, and $\big(1-\frac{p}{3}(1-\zeta_p)^3\big) \sim \frac{p}{3}\zeta_p^3$, hence

$$p\big(1+p(1-\zeta_p)^2\big)e^{-\frac{p}{2}(1-\zeta_p)^2}\big(1-\frac{p}{3}(1-\zeta_p)^3\big) \lesssim \frac{p}{3}\zeta_p^3 e^{-\frac{p\zeta_p}{4}} = \frac{1}{3p^2}(p\zeta_p)^3 e^{-\frac{p\zeta_p}{4}} \longrightarrow 0.$$

Here again, $p\big(1+p(1-\zeta_p)^2\big)|\delta_p(\zeta_p)| \to 0$, and (A.2) is contradicted.

3. $(\zeta_p) \to \ell$: one must deal here with the dichotomy $\ell \ne 1/\ell = 1$.
   (a) $\ell \ne 1$: $p\big(1+p(1-\zeta_p)^2\big) \sim (1-\ell)^2p^2$, and as $\zeta \mapsto 1-\zeta+\log\zeta$ is strictly convex and attains 0 at $\zeta = 1$, $e^{p(1-\zeta_p+\log\zeta_p)} \le e^{-\varepsilon p}$ for p large, with some $\varepsilon > 0$, hence

   $$p\big(1+p(1-\zeta_p)^2\big)e^{p(1-\zeta_p+\log\zeta_p)} \lesssim (1-\ell)^2p^2e^{-\varepsilon p} \longrightarrow 0;$$

   furthermore, $e^{-\frac{p}{2}(1-\zeta_p)^2} \le e^{-\frac{p(1-\ell)^2p}{4}}$ for p large, and $\big(1-\frac{p}{3}(1-\zeta_p)^3\big) \sim \frac{p}{3}(1-\ell)^3$, hence

   $$p\big(1+p(1-\zeta_p)^2\big)e^{-\frac{p}{2}(1-\zeta_p)^2}\big|1-\frac{p}{3}(1-\zeta_p)^3\big| \lesssim \frac{p}{3}|1-\ell|^3e^{-\frac{p(1-\ell)^2}{4}} \longrightarrow 0. \tag{A.5}$$

   Once more, this yields $p\big(1+p(1-\zeta_p)^2\big)|\delta_p(\zeta_p)| \to 0$, and a contradiction to (A.2).

   (b) $\ell = 1$: setting $z_p = \zeta_p - 1 \to 0$, we must one last time distinguish between three different cases: $|pz_p^3| \to \infty$, $pz_p^3 \to \mu = \lambda^3 \in \mathbb{R}^*$ and $pz_p^3 \to 0$ (up to passing to a subsequence).
      i. $|pz_p^3| \to \infty$: in particular, $pz_p^2 = |pz_p^3|^{2/3}p^{1/3} \to \infty$. Here $p\big(1+p(1-\zeta_p)^2\big) \sim p^2z_p^2$ and $1-\zeta_p+\log\zeta_p = -z_p+\log(1+z_p) \le -\frac{z_p^2}{3}$ for p large, thus

      $$\begin{aligned} p\big(1+p(1-\zeta_p)^2\big)e^{p(1-\zeta_p+\log\zeta_p)} &\lesssim p^2z_p^2e^{-\frac{pz_p^2}{3}} = p^2z_p^2e^{-\frac{|pz_p^3|^{2/3}p^{1/3}}{3}} \\ &\le p^2z_p^2e^{-(|pz_p^3|^{2/3}+p^{1/3})} = p^{4/3}e^{-p^{1/3}}|pz_p^3|^{2/3}e^{-|pz_p^3|^{2/3}} \longrightarrow 0, \end{aligned} \tag{A.6}$$

      where the last inequality holds as soon as $p^{1/3}$ and $|pz_p^3|^{2/3} \ge 6$. As for the other summand,

      $$\begin{aligned} &p\big(1+p(1-\zeta_p)^2\big)e^{-\frac{pz_p^2}{2}}\Big|1-\frac{p}{3}z_p^3\Big| \\ &\sim |p^3z_p^5|e^{-\frac{pz_p^2}{2}} = p^{4/3}|pz_p^3|^{5/3}e^{-\frac{|pz_p^3|^{2/3}p^{1/3}}{2}} \\ &\le p^{4/3}|pz_p^3|^{5/3}e^{-(|pz_p^3|^{2/3}+p^{1/3})} = p^{4/3}e^{-p^{1/3}}|pz_p^3|^{5/3}e^{-|pz_p^3|^{2/3}} \longrightarrow 0, \end{aligned} \tag{A.7}$$

      (the last inequality holds as soon as $p^{1/3}$ and $|pz_p^3|^{2/3} \ge 6$). In conclusion, $p\big(1+p(1-\zeta_p)^2\big)|\delta_p(\zeta_p)| \to 0$, in contradiction with (A.2).

ii. $pz_p^3 \to \mu = \lambda^3 \neq 0$, that is: $z_p \sim \lambda p^{-1/3}$. First, $p\big(1+p(1-\zeta_p)^2\big) \sim p^2 z_p^2 \sim \lambda^2 p^{4/3}$; moreover $1-\zeta_p+\log\zeta_p = -z_p+\log(1+z_p) = -\frac{z_p^2}{2}+\frac{z_p^3}{3}+\mathcal{O}(p^{-4/3})$, hence

$$\begin{aligned} p\big(1+p(1-\zeta_p)^2\big)|\delta_p(\zeta_p)| &\sim \lambda^2 p^{4/3} e^{-\frac{pz_p^2}{2}}\Big|e^{\frac{pz_p^3}{3}+\mathcal{O}(p^{-1/3})}-1+\frac{pz_p^3}{3}\Big| \\ &\lesssim \lambda^2 p^{4/3} e^{-\frac{\lambda^2 p^{1/3}}{3}}\Big|e^{\mu/3}-1-\frac{\mu}{3}\Big| \longrightarrow 0, \end{aligned} \tag{A.8}$$

a contradiction with (A.2).

iii. $pz_p^3 \to 0$: In this very last case, where again $pz_p^4 = pz_p^3 \cdot z_p \to 0$,

$$\begin{aligned} \delta_p(\zeta_p) &= e^{-\frac{pz_p^2}{2}}\Big(e^{p(\frac{z_p^3}{3}+\mathcal{O}(z_p^4))}-1-\frac{p}{3}z_p^3\Big) \\ &= e^{-\frac{pz_p^2}{2}}\Big(\big(1+\frac{pz_p^3}{3}+\mathcal{O}(p^2z_p^6)\big)\big(1+\mathcal{O}(pz_p^4)\big)-1-\frac{p}{3}z_p^3\Big) \\ &= e^{-\frac{pz_p^2}{2}}\big(\mathcal{O}(p^2z_p^6)+\mathcal{O}(pz_p^4)\big), \end{aligned} \tag{A.9}$$

hence

$$\begin{aligned} (p+p^2z_p^2)\delta_p(\zeta_p) =& \mathcal{O}\big(p^3z_p^6e^{-\frac{pz_p^2}{2}}\big)+\mathcal{O}\big(p^2z_p^4e^{-\frac{pz_p^2}{2}}\big)+\mathcal{O}\big(p^4z_p^8e^{-\frac{pz_p^2}{2}}\big) \\ =& \mathcal{O}(1) \end{aligned} \tag{A.10}$$

independently of the behavior of $p^2z_p^2$, as $z \mapsto z^k e^{-\frac{z^2}{2}}$, $k=2,3,4$, are bounded on $\mathbb{R}$. In other words, $p\big(1+p(1-\zeta_p)^2\big)\delta_p(\zeta_p) = \mathcal{O}(1)$, and this final contradiction of (A.2) ends the proof of Lemma 3.4. □

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