**Workshop: Extremal Kähler metrics, reductive groups compactifications and stationnary Lagrangians**

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Abstracts and references.

**Boris Pasquier I: Introduction to reductive algebraic groups**

*Abstract :* In this talk, we introduce the theory of reductive algebraic groups and their representations. We fix all notations used in the next three talks. We illustrate all this by the example of $\mathrm{SL_n}$ (or $\mathrm{PSL_n}$).

*Main reference :* T.A. Springer, Linear algebraic groups. Second edition. Progress in Mathematics, 9. Birkhäuser Boston, Inc., Boston, MA, 1998.

**Boris Pasquier II: Introduction to spherical homogeneous spaces and spherical varieties**

*Abstract :* Let $X$ be a normal (or smooth) algebraic variety. If the torus $T = (\mathbb{C}^*)^n$ acts on $X$ with an open orbit isomorphic to $T$, then $X$ is called a toric variety. In that case, the geometry of $X$ is very well-known and is described combinatorially by fans (some sets of strictly convex polyhedral cones). Now, consider the case where a reductive algebraic group acts on $X$ with an open orbit isomorphic to an homogeneous space $G/H$. Then, if $G/H$ is "nice enough" (ie spherical) we can generalize the theory of toric varieties and describe $X$ in terms of colored fans. The aim of this talk is to define spherical homogeneous spaces, and to give the description spherical varieties in terms of colored fans.

*Main reference :* M. Brion, Variétés sphériques, Notes de la session de la S. M. F. "Opérations hamiltoniennes et opérations de groupes algébriques" (Grenoble, 1997), available at http://www-fourier.ujf-grenoble.fr/~mbrion/notes.html

**Boris Pasquier III & IV: Fano horospherical varieties**

*Abstract:* Spherical varieties that are Fano (ie such that their anticanonical divisor is Cartier and ample) are classified. This classification permits to get several results and to test conjectures on Fano varieties. To simplify the talk, I will focus on the case of horospherical varieties and I will not speak about the generalizations (due to G. Gagliardi and J. Hofscheier) to the case of spherical varieties. If we have enough time, I will say few words about birational geometry of these varieties (in particular, how to run easily the Minimal Model Program for horospherical varieties).

*References:*

B. Pasquier, Variétés horosphériques de Fano, Ph.D. thesis, Université Joseph Fourier, Grenoble 1, available at http://tel.archives-ouvertes.fr/tel-00111912, 2006.

B. Pasquier, Variétés horosphériques de Fano, Bull. Soc. Math. France 136 (2008), no. 2, 195-225.

B. Pasquier, On some smooth projective two-orbit varieties with Picard number 1, Math. Ann. 344 (2009), no. 4, 963-987.

B. Pasquier, The pseudo-index of horospherical Fano varieties, Internat. J. Math. 21 (2010), no. 9, 1147-1156.

B. Pasquier and N. Perrin, Local rigidity of quasi-regular varieties, Mathematische Zeitschrift 265 (2010), no. 3, 589-600.

*More references:*

G. Gagliardi and J. Hofscheier, Gorenstein spherical Fano varieties, Geom. Dedicata 178 (2015), 111-133.

G. Gagliardi and J. Hofscheier, The generalized mukai conjecture for symmetric varieties, preprint available at https://arxiv.org/abs/1412.6084 (2014).

B. Pasquier, An approach of the Minimal Model Program for horospherical varieties via moment polytopes, J. Reine Angew. Math. 708 (2015), 173-212.

B. Pasquier, Birational geometry of spherical varieties (Habilitation à Diriger des Recherches), avalaible at http://www.math.univ-montp2.fr/~pasquier/Publi2.html
B. Pasquier, Lectures notes, Lectures given in KIAS in 2009, available at \http://www.math.univ-montp2.fr/~pasquier/Talks2.html.

**Thibaut Delcroix I: Hermitian metrics on polarized group compactifications**
*Abstract:* More precisely, I will introduce the convex potential of such a metric, and explain the relation between its asymptotics and the moment polytope. I will then show how to compute the curvature form of the metric in terms of the convex potential, and give examples of computations of integrals (for example the volume of the line bundle, or the Futaki invariant on smooth Fano group compactifications).

**Thibaut Delcroix II: Existence of Kähler-Einstein metrics**
*Abstract:* I will present a proof of an existence criterion for Kähler-Einstein metrics on smooth Fano group compatifications, in terms of the moment polytope. The necessary condition follows easily from the results of Part 1 and provides an interesting example of a group compactification with no Kähler-Ricci solitons, and not K-semistable. The sufficient condition is obtained by the continuity method and provides new examples of Kähler-Einstein Fano manifolds.
*References:*
T. Delcroix (thesis) https://tel.archives-ouvertes.fr/tel-01286292
T. Delcroix, Log canonical thresholds on group compactifications. http://arxiv.org/abs/1510.05079
T. Delcroix, Kähler-Einstein metrics on group compactifications. http://arxiv.org/abs/1510.07384
X.-J. Wang and X. Zhu. Kähler-Ricci solitons on toric manifolds with positive first Chern class. Adv. Math., 188(1):87–103, 2004.

**Ludmil Katzarkov: Multiplier Ideal Sheaves, Kaehler metrics and Stability conditions.**
*Abstract:* In this talk we will build a parallel between multiplier ideal sheaves and stability of categories.

*References :*
L. Ein and M. Mustata, Invariants of singularities of pairs https://arxiv.org/abs/math/0604601
G. Dimitrov and L. Katzarkov, Bridgeland stability conditions on wild Kronecker quivers https://arxiv.org/abs/1602.09117

**Yann Rollin: Smoothing cscK surfaces (and minimal Lagrangians)**
*Abstract:* We consider smoothings of a complex surface with singularities of class T and no nontrivial holomorphic vector field. Under an hypothesis of non degeneracy of the smoothing at each singular point, we prove that if the singular surface admits an extremal metric, then the smoothings also admit extremal metrics in nearby Kähler classes. In addition, we construct small Lagrangian stationary spheres which represent Lagrangian vanishing cycles for surfaces close to the singular one.
*References :*
Biquard, O. Rollin, Y., Smoothing singular constant scalar curvature Kähler surfaces and minimal Lagrangians. Adv. Math. 285 (2015), 980–1024.

**Tommaso Pacini: What's the volume of a totally real submanifold?**
*Abstract:* You probably think you already know the answer, and everyone from Joseph Plateau to Nicos Kapouleas would agree with you. I will try to convince you that you're all wrong, and Jason Lotay will (I hope) be backing me up. I will also try to explain why, if you're interested in Kahler-Einstein geometry, this question might be relevant to you.

**Jason Lotay: From Lagrangian to totally real geometry: coupled flows and calibrations**

*Abstract:* Lagrangian mean curvature flow gives an attractive potential means for detecting and analysing minimal Lagrangians in Ricci-flat Kaehler manifolds. We show that the properties of Lagrangian mean curvature flow are a special case of a more general phenomenon, concerning couplings between geometric flows of the ambient space and of totally real submanifolds. To this end we explore the geometry of totally real submanifolds, defining (i) a new geometric flow in terms of the ambient canonical bundle, (ii) a modified volume functional which takes into account the totally real condition. We also discuss possible applications to Lagrangian submanifolds and calibrated geometry.

*References:*

J. Lotay, T. Pacini, Complexified diffeomorphism groups, totally real submanifolds and Kähler-Einstein geometry. https://arxiv.org/abs/1506.04630

J. Lotay, T. Pacini, From Lagrangian to totally real geometry: coupled flows and calibrations, https://arxiv.org/abs/1404.4227

**David Petrecca: Lagrangian submanifolds and related structures; homogeneous examples and Hamiltonian stability**

*Abstract:* We will start with a result of Bedulli and Gori that classifies the holomorphic Hamiltonian actions of compact Lie groups on compact Kähler manifolds that admit a Lagrangian orbit.

For the case of $\mathbb{CP}^n$, they relate the existence of such an orbit with the notion of prehomogeneous vector space (PVS) and, using a classification of the latter, they provide the classification of simple Lie groups that admit a Lagrangian orbit.

PVS's come with a transformation called castling transform. We will describe how this can be use to construct new examples of Lagrangian orbits out of lower dimensional ones and describe an example of such an orbit and prove its Hamiltonian stability.

Finally, if time allows, the odd-dimensional analogue of these notions will be discussed.

Based on joint work with F. Podestà (2012), S. Calamai (2014) and L. Schäfer (2016).