Partial degeneration of Hodge to de Rham spectral sequences and Kodaira type vanishing theorems for locally complete intersections in positive characteristic

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ABSTRACT. We generalize to the lci case decomposition, degeneration, and vanishing theorems of [18]. The main tool is a comparison theorem between certain derived de Rham and de Rham-Witt complexes modulo p^{th}-steps of Hodge and Nygaard filtrations.

CONTENTS

0. Introduction

Let k be a perfect field of characteristic $p > 0$. Let X be a k-scheme. Let X' be the pull-back of X by the Frobenius automorphism of k, and let $F : X \to X'$ denote the relative Frobenius morphism. One of the main results of [18] is that, if X is smooth and liftable to $W_2(k)$, a lifting of X to $W_2(k)$ defines a decomposition

(0.1) $$\oplus_{0 \leqslant i < p} \Omega^i_{X'/k}[-i] \xrightarrow{\sim} \tau_{<p} F_* \Omega^\bullet_{X/k}$$

in $D(X', \mathcal{O}_{X'})$. We extend this to the lci case. We prove (see 4.8 for a slightly more general statement):

Theorem 0.2. Assume X is lci and admits a (flat) lifting to $W_2(k)$. Then any such lifting defines a decomposition

(0.2.1) $$\oplus_{0 \leqslant i < p} L\Omega^i_{X'/k}[-i] \xrightarrow{\sim} \mathrm{Fil}^{\mathrm{conj}}_{p-1} F_* L\Omega^\bullet_{X/k}$$

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in $D(X', \mathcal{O}_{X'})$.

Here $L\Omega^\bullet_{X/k}$ denotes the derived de Rham complex, and $L\Omega^i_{X'/k} := L\Lambda^i L_{X'/k}$ is the i-th piece of the associated graded for the Hodge filtration, where $L_{X'/k} = L\Omega^1_{X'/k}$ is the cotangent complex of X'/k.

Combining 0.2 with cohomological amplitude estimates due to Bhatt (5.3) we deduce the following degeneration and vanishing results:

Theorem 0.3 (6.1). Assume X/k is proper, lci, of pure dimension $d < p$, and liftable to $W_2(k)$. Let s be the dimension of the singular locus of X. Then, for $n < d - s - 1$ we have

$$\dim_k H^n(X, L\widehat{\Omega}^\bullet_{X/k}) = \sum_{0 \leqslant i \leqslant d} \dim_k H^n(X, L\Omega^i_{X/k}[-i]),$$

and

$$H^n(X, L\Omega^i_{X/k}[-i]) = 0$$

for $i > d$.

Here $L\widehat{\Omega}^\bullet_{X/k}$ is the derived Hodge completion of $L\Omega^\bullet_{X/k}$, i.e., $R\varprojlim_n L\Omega^\bullet_{X/k}/\mathrm{Fil}^n$, where Fil^n denotes the Hodge filtration.

Theorem 0.4 (7.1). Let X/k satisfy the assumptions of (0.3). Let L be an ample invertible sheaf on X. Then, for $n < \min(d, d - s - 1)$ and all i,

$$H^n(X, L\Omega^i_{X/k}[-i] \otimes L^{-1}) = 0.$$

By the usual spreading out arguments, 0.3 and 0.4 imply similar results in characteristic zero, with no restriction on the dimension (6.4, 7.5). In particular, we recover a slightly weaker form of the vanishing theorem of Bhatt-Blickle-Lyubeznik-Singh-Zhang ([14], Th. 3.2) (i.e., with $d - s - 1$ instead of $d - s$).

Liftings of X to $W_2(k)$ are controlled by $\tau_{\geqslant -1} L_{X/W_2(k)}$. In this lci case, $\tau_{\geqslant -1} L_{X/W_2(k)} \xrightarrow{\sim} L_{X/W}$, and we deduce 0.2 from a general comparison theorem between suitably truncated derived de Rham complexes of X/W and derived de Rham-Witt complexes of X. More precisely:

Theorem 0.5 (2.9). One can construct, functorially in the k-scheme X, a filtered isomorphism in the derived ∞-category $\mathcal{D}(X, W)$

$$L\Omega^\bullet_{X/W}/\mathrm{Fil}^p \xrightarrow{\sim} LW\Omega^\bullet_X/\mathcal{N}^p, \tag{0.5.1}$$

where the left hand side is filtered by the quotient of the Hodge filtration Fil^r and the right hand side by the quotient of the Nygaard filtration $\mathcal{N}^r$.

On the associated graded pieces this isomorphism induces, for $r < p$, an isomorphism in $\mathcal{D}(X', \mathcal{O}_{X'})$

$$L\Omega^r_{X'/W}[-r] \xrightarrow{\sim} \mathrm{Fil}^{\mathrm{conj}}_r F_* L\Omega^\bullet_{X/k}. \tag{0.5.2}$$

The existence of such an isomorphism was suggested to me by the standard proof of the comparison theorem between crystalline and de Rham-Witt cohomology ([25], II 1.4), and a conversation with Mathieu Florence about ([15], §5) which pointed to a connection between liftings mod p^2 and the Nygaard filtration.

Let me now briefly describe the contents of the paper.

The morphism (0.5.1) is constructed by left Kan extension from finitely generated polynomial algebras over k. These are lci over $W(k)$. In section 1, we examine more generally the case of a *weakly lci* A-algebra B (1.1), for a $\mathbf{Z}_{(p)}$-algebra A, and construct a model for the truncated derived de Rham complex $L\Omega^\bullet_{B/A}/\mathrm{Fil}^p$ in terms of usual de Rham complexes $\Omega^\bullet_{P/A}$, for P a polynomial algebra mapping surjectively to B (1.8.1). The truncation by Fil^p originates in Quillen's décalage formula for $L\Lambda^i$ and the fact that, if I is the ideal of $P \to B$, the canonical morphism $S^n_B(I/I^2) \to \Gamma^n_B(I/I^2)$ is an isomorphism only for $n < p$ in general. In section 2, we specialize to $A = W(k)$, and construct (0.5.1) in two steps: (i) for $X = \mathrm{Spec}(B)$, B a finitely generated polynomial algebra over k, by using the model (1.8.1) and a lifting of Frobenius on P; (ii) in the general case, by left Kan extension from (i) and sheafification. A crucial ingredient in the proof of 0.5 is the concrete description of the Nygaard filtration of $W\Omega^\bullet_B$ for B a finitely generated polynomial algebra over k, in terms of the de Rham complex $\Omega^\bullet_{P/W}$ of a finitely generated polynomial algebra P over W *lifting* B, equipped with a lifting of Frobenius, via the interpretation of the de Rham-Witt complex $W\Omega^\bullet_B$ as the strict saturation of $\Omega^\bullet_{P/W}$ ([10], 8.3.5).

The rest of the paper builds on 0.5. In section 4, using 0.5 and standard deformation theory, we show that, for X lci over k, liftings of X to $W_2(k)$ correspond bijectively, up to isomorphisms, to splitting of the 1-step of the conjugate filtration $\mathrm{Fil}^{\mathrm{conj}}_1 F_* L\Omega^\bullet_{X/k}$ (4.4). For this we need a slight refinement (3.3) of (0.5.2) for $r = 1$. The decomposition theorem 0.2 follows. If X is an lci k-scheme, its cotangent complex $L_{X/k} = L\Omega^1_{X/k}$ is of perfect amplitude in $[-1, 0]$, hence $L\Omega^i_{X/k}[-i]$ is of perfect amplitude in $[0, i]$, but already for isolated singularities, does not vanish for i large ([8], 2.1). However, Bhatt proved that, if X is of pure dimension d and the singular locus of X is of dimension s, then, for all $i > d$, $L\Omega^i_{X/k}[-i]$ belongs to $D^{\geqslant d-s}(X, \mathcal{O}_X)$ (and similar results hold for derived boundaries and cycles) (5.3 (b)). In section 6, we derive 0.3 from these estimates and (0.2.1). Th. 0.4 generalizes the vanishing theorem of ([18], (2.8.2)). Its proof is similar, based again on (0.2.1) and the estimates of section 5. The key tool is a variant (7.3) of Raynaud's lemma ([18], 2.9).

This paper was completed in March, 2020. Since then, Bhargav Bhatt and Akhil Mathew have proposed far reaching generalizations and refinements of its main results, especially 2.9, 4.4, 4.8. I hope these new developments will be worked out soon.

1. Weakly lci morphisms and truncated derived de Rham complexes

1.1. Let $A \to B$ be a homomorphism of rings. We say that B is *weakly lci* over A if

$$\mathrm{toramp}(L_{B/A}) \subset [-1, 0].$$

This definition is similar to that of a *quasisyntomic map* in ([9], Def. 4.9), except that we don't impose any condition of p-flatness or p-completion. It was introduced in ([23], III 3.3.4). If A is noetherian and B is of finite type over A, then if B is lci over A, B is weakly lci over A by a theorem of Quillen ([30], Th. 5.4), and the converse is true by a theorem of Avramov [4].

1.2. Let B be a weakly lci A-algebra, and let $P \to B$ a surjective homomorphism of A-algebras, of ideal I, with P a free A-algebra i.e., of the form $A[(X_i)_{i\in S}]$ for a set S). Then we have a natural isomorphism in $D(B)$

$$L_{B/A} \xrightarrow{\sim} (I/I^2 \xrightarrow{d} B \otimes_P \Omega^1_{P/A}), \tag{1.2.1}$$

where I/I^2 is placed in degre -1, d is induced by $d_{P/A}$, and I/I^2 (resp. $\Omega^1_{P/A}$ is flat over B(resp. P) ([23], III 3.3.6). If $Q \to B$ is a second surjective homomorphism from a free A-algebra Q, with ideal J, and $P \to Q$ is a homomorphism of A-algebras compatible with the surjections to B, then we have a commutative diagram

$$\begin{tikzcd} L_{B/A} \arrow[r] \arrow[dr] & (I/I^2 \xrightarrow{d} B \otimes_P \Omega^1_{P/A}) \arrow[d] \\ & (J/J^2 \xrightarrow{d} B \otimes_Q \Omega^1_{Q/A}) \end{tikzcd} , \tag{1.2.2}$$

where the vertical map is the natural map, and, hence, is a quasi-isomorphism.

1.3. With $P \to B$ as in 1.2, consider the filtration on the de Rham complex $\Omega^\bullet_{P/A}$ given by

(1.3.1)

$$I^n\Omega^\bullet_{P/A} := (I^n \to I^{n-1}\Omega^1_{P/A} \to \cdots \to I\Omega^{n-1}_{P/A} \to \Omega^n_{P/A} \to \Omega^{n+1} \to \cdots)$$

for $n \in \mathbf{N}$. This is a decreasing, multiplicative filtration, with $I^0\Omega^\bullet_{P/A} = \Omega^\bullet_{P/A}$. The associated graded object is a B-dga (differential graded algebra), with

$$\mathrm{gr}^1_I \Omega^\bullet_{P/A} = (I/I^2 \xrightarrow{d} B \otimes_P \Omega^1_{P/A}), \tag{1.3.2}$$

i.e., the complex on the right hand side of (1.2.1), with I/I^2 placed in degree 0. As I is weakly regular ([23], III, 3.3.1, 3.3.6), the canonical map

$$S_B(I/I^2) \to \mathrm{gr}_I P$$

is an isomorphism, and it induces an isomorphism of B-dga

$$(S_B(I/I^2) \otimes_B \Lambda(B \otimes_P \Omega^1_{P/A}), d) \xrightarrow{\sim} \mathrm{gr}_I \Omega^\bullet_{P/A}, \tag{1.3.3}$$

where the left hand side is the Koszul algebra constructed on gr^1 (1.3.2), defined similarly to $\Gamma_B(I/I^2) \otimes_B \Lambda(B \otimes_P \Omega^1_{P/A})$ ([23], I 4.3.1.2), where d is the unique B-derivation of bidegree (-1,1) such that $d(x \otimes 1) = 1 \otimes dx$ for $x \in I/I^2$ and $d(1 \otimes x) = 0$ for $x \in B \otimes_P \Omega^1_{P/A}$.

1.4. Let p be a prime number, and assume that A is a $\mathbf{Z}_{(p)}$-algebra. Recall that if M is a flat B-module, then, for each $n \geqslant 1$, the canonical map $\Gamma^n_B(M) \to \mathrm{TS}^n_B(M)$ is an isomorphism ([2], XVII 5.5.2.5), and that, by ([31], Prop. III.3), the canonical map $S^n_B(M) \to \Gamma^n_B(M)$ is an isomorphism for $n < p$. With the notation of 1.3, consider the complex

$$\Omega^\bullet_{P/A}/I^p := \Omega^\bullet_{P/A}/I^p\Omega^\bullet_{P/A}.$$

It is filtered by the quotient filtration of the filtration (1.3.1), and

$$\mathrm{gr}_I(\Omega^\bullet_{P/A}/I^p) = (\mathrm{gr}_I \Omega^\bullet_{P/A})_{<p},$$

where $< p$ denotes the part of degree $< p$. Let $(\Gamma_B(I/I^2) \otimes_B \Lambda(B \otimes_P \Omega^1_{P/A}), d)$ be the Koszul algebra constructed on gr^1 (1.3.2) ([23], I 4.3.1.2). By what we have recalled, as I/I^2 is flat over B, the canonical morphism (of B-dga)

$$(S_B(I/I^2) \otimes_B \Lambda(B \otimes_P \Omega^1_{P/A}), d) \to (\Gamma_B(I/I^2) \otimes_B \Lambda(B \otimes_P \Omega^1_{P/A}), d)$$

induces an isomorphism in degree $< p$. From (1.3.3) we thus get isomorphisms

$$(\Gamma_B(I/I^2) \otimes_B \Lambda(B \otimes_P \Omega^1_{P/A}), d)_{<p} \xleftarrow{\sim} (S_B(I/I^2) \otimes_B \Lambda(B \otimes_P \Omega^1_{P/A}), d)_{<p}$$

$$\xrightarrow{\sim} \mathrm{gr}_I(\Omega^\bullet_{P/A}/I^p). \tag{1.4.1}$$

By ([27], Lemma 1.2.6), for all $n \geq 1$, (1.2.1) induces an isomorphism of $D(B)$

$$L\Lambda^n_B L_{B/A}[-n] \xrightarrow{\sim} (\Gamma_B(I/I^2) \otimes_B \Lambda(B \otimes_P \Omega^1_{P/A}), d)_n$$

(where the index n means the homogeneous component of degree n), and therefore, in view of (1.4.1), an isomorphism (of $D(B)$)

(1.4.2).
$$\oplus_{n<p} L\Lambda^n_B L_{B/A}[-n] \xrightarrow{\sim} \mathrm{gr}_I(\Omega^\bullet_{P/A}/I^p).$$

1.5. Let $A \to B$ be as in 1.4, and, as in 1.2, let $u : P \to Q$ be a homomorphism of free A-algebras compatible with surjective homomorphisms $P \to B$, $Q \to B$ of ideals I and J. The homomorphism u induces a homomorphism of A-dga

$$u : \Omega^\bullet_{P/A} \to \Omega^\bullet_{Q/A},$$

compatible with the filtrations I and J, hence a homomorphism of B-dga

$$\mathrm{gr}(u) : \mathrm{gr}_I \Omega^\bullet_{P/A} \to \mathrm{gr}_J \Omega^\bullet_{Q/A},$$

compatible in an obvious way with the isomorphisms (1.4.1). Therefore we get a commutative diagram

(1.5.1),
$$\begin{tikzcd}
\oplus_{i<p} L\Lambda^i_B L_{B/A}[-i] \arrow[r] \arrow[dr] & \mathrm{gr}_I(\Omega^\bullet_{P/A}/I^p) \arrow[d, "\mathrm{gr}(u)"'] \\
 & \mathrm{gr}_J(\Omega^\bullet_{Q/A}/J^p)
\end{tikzcd}$$

in which the horizontal and the slanted arrow are isomorphisms in $D(B)$, and hence the vertical arrow is a quasi-isomorphism. In particular, the map

(1.5.2)
$$u : \Omega^\bullet_{P/A}/I^p \to \Omega^\bullet_{Q/A}/J^p$$

is a filtered quasi-isomorphism.

For $i = 1, 2$, let $v_i : P_i \to B$ be a surjective homomorphism, with P_i a free A-algebra, and ideal I_i. We have a commutative diagram

(1.5.3)
$$\begin{tikzcd}
P_1 \arrow[r, "j_1"] \arrow[dr, "v_1"'] & P_1 \otimes_A P_2 \arrow[d, "v_1 v_2"'] & P_2 \arrow[l, "j_2"'] \arrow[dl, "v_2"] \\
 & B &
\end{tikzcd},$$

where $j_1(x) = x \otimes 1$, $j_2(y) = 1 \otimes y$, and $v_1v_2(x \otimes y) = v_1(x)v_2(x)$. Let I be the ideal of v. By (1.5.2) we get filtered quasi-isomorphisms

$$\Omega^\bullet_{P_1/A}/I_1^p \to \Omega^\bullet_{(P_1 \otimes_A P_2)/A}/I^p \leftarrow \Omega^\bullet_{P_2/A}/I_2^p,$$

hence a filtered isomorphism

$$\varepsilon(P_2, P_1) : \Omega^{\bullet}_{P_1/A}/I_1^p \xrightarrow{\sim} \Omega^{\bullet}_{P_2/A}/I_2^p \tag{1.5.3}$$

in the derived ∞-category $\mathcal{D}(A)$, inducing a graded isomorphism

$$\operatorname{gr} \varepsilon(P_2, P_1) : \operatorname{gr}(\Omega^{\bullet}_{P_1/A}/I_1^p) \xrightarrow{\sim} \operatorname{gr}(\Omega^{\bullet}_{P_2/A}/I_2^p) \tag{1.5.4}$$

in the derived ∞-category $\mathcal{D}(B)$. For $P \to B$ running through the (small) set of surjective A-homomorphisms with P free over A, these isomorphisms ε form a transitive system, i.e., satisfy $\varepsilon(P_3, P_2)\varepsilon(P_2, P_1) = \varepsilon(P_3, P_1)$, as one sees by considering $v_1 v_2 v_3 : P_1 \otimes P_2 \otimes P_3 \to B$. In particular, for $u : P_1 \to P_2$ such that $v_2 u = v_1$, the maps u of (1.5.2) coincides with $\varepsilon(P_2, P_1)$ (as is seen by considering $P_1 \otimes P_2 \to P_2, x \otimes y \mapsto u(x)y$). Imitating the convention of ([17], 1.1), we denote by

$$\widetilde{\Omega}^{\bullet}_{B/A}/\mathcal{I}^p \tag{1.5.5}$$

the projective limit (in $\mathcal{D}(A)$) of the $\Omega^{\bullet}_{B/A}/I^p$ along the isomorphisms ε. It is a filtered object, with associated graded object in $\mathcal{D}(B)$ the projective limit of the $\operatorname{gr}(\Omega^{\bullet}_{P/A}/I^p)$ along the isomorphism $\operatorname{gr} \varepsilon$. By definition, $\widetilde{\Omega}^{\bullet}_{B/A}/\mathcal{I}^p$ comes equipped with (filtered) isomorphisms

$$\sigma_P : \widetilde{\Omega}^{\bullet}_{B/A}/\mathcal{I}^p \xrightarrow{\sim} \Omega^{\bullet}_{P/A}/I^p \tag{1.5.5a}$$

for $P \to B$ as above, satisfying $\varepsilon(Q, P)\sigma_P = \sigma_Q$.

The isomorphisms (1.4.2) induce an isomorphism of $\mathcal{D}(B)$

$$\oplus_{i<p} L\Omega^i_{B/A}[-i] \xrightarrow{\sim} \operatorname{gr}_{\mathcal{I}}(\widetilde{\Omega}^{\bullet}_{B/A}/\mathcal{I}^p) \tag{1.5.6}$$

(where $L\Omega^i_{B/A} := L\Lambda^i L_{B/A}$).

1.6. Let A be a $\mathbf{Z}_{(p)}$-algebra, and let $f : B \to C$ a homomorphism of weakly lci A-algebras. Let $u : P \to B$ (resp. $v : Q \to C$) be a surjective homomorphism, with ideal I (resp. J) and P (resp. Q) free over A. We have a commutative diagram

$$\begin{tikzcd} B \arrow[d, "f"'] & P \arrow[l, "u"'] \arrow[d] \\ & P \otimes_A Q \arrow[dl, "w"] \\ C & Q \arrow[l, "v"] \arrow[u] \end{tikzcd} \tag{1.6.1}$$

where the right vertical arrows are the canonical ones, and $w(x \otimes y) := (fu)(x)v(y)$. Let K be the ideal of w. This diagram induces filtered morphisms

$$\Omega^\bullet_{P/A}/I^p \to \Omega^\bullet_{(P\otimes Q)/A}/K^p \leftarrow \Omega^\bullet_{Q/A}/J^p,$$

where the second one is a quasi-isomorphism. We thus get a filtered morphism

$$\Omega^\bullet_{P/A}/I^p \to \Omega^\bullet_{Q/A}/J^p \tag{1.6.2}$$

in $\mathcal{D}(A)$. This morphism is, by construction, compatible with the isomorphisms ε of (1.5.3), hence defines a morphism

$$f : \widetilde{\Omega}^\bullet_{B/A}/\mathcal{I}^p \to \widetilde{\Omega}^\bullet_{C/A}/\mathcal{I}^p \tag{1.6.3}$$

in $\mathcal{D}(A)$ (and an associated graded morphism $\mathrm{gr}f$ in $\mathcal{D}(B)$). One checks that this makes $B \mapsto \widetilde{\Omega}^\bullet_{B/A}/\mathcal{I}^p$ functorial in the weakly lci A-algebra B.

1.7. Let B be an A-algebra. Recall that the derived de Rham complex

$$L\Omega^\bullet_{B/A} \in \mathcal{D}(A)$$

is defined by left Kan extension of the functor $P \mapsto \Omega^\bullet_{P/A}$ on the category of free (or even finitely generated free) A-algebras. It can be calculated as the total complex of the de Rham complex of $P_\bullet$ over A, where $P_\bullet$ is a free simplicial resolution of B/A (i.e., $P_\bullet \to B$ is a quasi-isomorphism, and P_n is a free A-algebra for all n), e.g., the standard simplicial free resolution $P_{\bullet A}(B)$ (cf. ([24], VIII), ([9], 2.1)), and the totalization is calculated by sums:

$$L\Omega^\bullet_{B/A} \xrightarrow{\sim} \mathrm{Tot}(L_{P_\bullet/A}).$$

It comes equipped with a decreasing filtration, the *Hodge filtration*

$$\mathrm{Fil}^i_{\mathrm{Hdg}} L\Omega^\bullet_{B/A} \xrightarrow{\sim} \mathrm{Tot}(\Omega^{\geqslant i}_{P_\bullet/A}). \tag{1.7.1}$$

The associated graded $\mathrm{gr}_{\mathrm{Hdg}} L\Omega^\bullet_{B/A} \in \mathcal{D}(B)$ is given by

$$\mathrm{gr}^i_{\mathrm{Hdg}} L\Omega^\bullet_{B/A} \xrightarrow{\sim} L\Omega^i_{B/A}[-i], \tag{1.7.2}$$

where we have put

$$L\Omega^i_{B/A} := L\Lambda^i L_{B/A}$$

(in particular, $L\Omega^1_{B/A} = L_{B/A}$, and we will use either notation indifferently). We will often omit Hdg from the notation when no confusion can arise.

1.8. Let A be a $\mathbf{Z}_{(p)}$-algebra and let B be a weakly lci A-algebra. We will construct a functorial, filtered isomorphism in $\mathcal{D}(A)$

$$L\Omega^{\bullet}_{B/A}/\mathrm{Fil}^p \xrightarrow{\sim} \widetilde{\Omega}^{\bullet}_{B/A}/\mathcal{I}^p, \tag{1.8.1}$$

such that the isomorphism induced on the graded pieces

$$\oplus_{i<p} L\Omega^i_{B/A}[-i] \xrightarrow{\sim} \mathrm{gr}_{\mathcal{I}}(\widetilde{\Omega}^{\bullet}_{B/A}/\mathcal{I}^p) \tag{1.8.2}$$

coincides with (1.5.6).

Lemma 1.9. Under the assumptions of 1.8, let $u : P_\bullet \to B$ be a free resolution of B/A. Let $I_\bullet \subset P_\bullet$ be the ideal of u.

(a) The map

$$\Omega^{\bullet}_{P_\bullet/A}/\Omega^{\geqslant p}_{P_\bullet/A} \to \Omega^{\bullet}_{P_\bullet/A}/I^p_\bullet\Omega^{\bullet}_{P_\bullet/A} \tag{1.9.1}$$

deduced from the inclusion $\Omega^{\geqslant p} \subset I^p_\bullet \Omega^{\bullet}_{P_\bullet/A}$ induces a filtered quasi-isomorphism on the total complexes. The associated graded map gr(1.9.1) induces a quasi-isomorphism on the total complexes.

(b) The map

$$\Omega^{\bullet}_{P_0/A}/I_0^p \to \Omega^{\bullet}_{P_\bullet/A}/I^p_\bullet, \tag{1.9.2}$$

deduced from the inclusion $P_0 \to P_\bullet$, where P_0 is considered as a constant simplicial ring, induces a quasi-isomorphism of the total complexes.

Proof. (a) It suffices to show the last assertion. By (1.4.1) $\mathrm{gr}_{I_\bullet}(\Omega^{\bullet}_{P_\bullet/A}/I^p_\bullet)$ is identified to $(\Gamma_B(I_\bullet/I^2_\bullet) \otimes \Lambda(B \otimes_{P_\bullet} \Omega^1_{P_\bullet/A}), d)_{<p}$, and the map gr(1.9.1) is identified with $L\Lambda^{<p}$ of its component of degree 1

$$(B \otimes_{P_\bullet} \Omega^1_{P_\bullet/A})[-1] \to (I_\bullet/I^2_\bullet \to B \otimes_{P_\bullet} \Omega^1_{P_\bullet/A}). \tag{$*$}$$

To show that (*) induces a quasi-isomorphism on the total complexes, it suffices to show:

(i) $\Omega^1_{P_\bullet/A} \to B \otimes \Omega^1_{P_\bullet/A}$ is a quasi-isomorphism;

(ii) the complex $I_\bullet/I^2_\bullet$ is acyclic.

Assertion (i) follows from the fact that $P_\bullet \to B$ is a quasi-isomorphism and $\Omega^1_{P_\bullet/A}$ is flat over $P_\bullet$ ([23], I 3.3.2.1) (this gives the case $i = 1$ of (1.7.2)). For (ii) we observe that, as $I_\bullet$ is weakly regular in each degree, the transitivity triangle relative to $A \to P_\bullet \to B$ reads

$$I_\bullet/I^2_\bullet \to B \otimes_{P_\bullet} \Omega^1_{P_\bullet/A} \to L_{B/A} \to .$$

Hence $I_\bullet/I_\bullet^2$ is acyclic, as by definition $B\otimes_{P_\bullet}\Omega^1_{P_\bullet/A} \to L_{B/A}$ is an isomorphism in $\mathcal{D}(B)$.

1.10. Let us construct (1.8.1). By (1.7.1) we have

$$(1.10.1). \qquad L\Omega^\bullet_{B/A}/\mathrm{Fil}^p \xrightarrow{\sim} \mathrm{Tot}(\Omega^\bullet_{P_\bullet/A}/\Omega^{\geqslant p}_{P_\bullet/A})$$

By 1.9 (a) the map (1.9.1) induces an isomorphism

$$(1.10.2) \qquad \mathrm{Tot}(\Omega^\bullet_{P_\bullet/A}/\Omega^{\geqslant p}_{P_\bullet/A}) \to \mathrm{Tot}(\Omega^\bullet_{P_\bullet/A}/I^p_\bullet\Omega^\bullet_{P_\bullet/A}).$$

By 1.9 (b) the map (1.9.2) induces an isomorphism

$$(1.10.3) \qquad \Omega^\bullet_{P_0/A}/I_0^p \xrightarrow{\sim} \mathrm{Tot}(\Omega^\bullet_{P_\bullet/A}/I^p_\bullet).$$

Finally, we have the isomorphism of (1.5.5)

$$(1.10.4). \qquad \sigma_{P_0} : \widetilde{\Omega}^\bullet_{B/A}/\mathcal{I}^p \xrightarrow{\sim} \Omega^\bullet_{P_0/A}/I_0^p.$$

The composition

$$L\Omega^\bullet_{B/A}/\mathrm{Fil}^p \xrightarrow{\sim} \widetilde{\Omega}^\bullet_{B/A}/\mathcal{I}^p$$

of (1.10.1), (1.10.2), $(1.10.3)^{-1}$, $(1.10.4)^{-1}$ is the desired isomorphism (1.8.1).

One must check that it doesn't depend on the choice of the resolution $P_\bullet$. If $P'_\bullet \to B$ is a second free resolution of B, and if we have a morphism $P'_\bullet \to P_\bullet$ of resolutions, the independence is clear. One can reduce to this case in the following way. Let $P_{\bullet A}(B)$ be the standard free resolution of B. By ([23], II 1.2.6.2 (a)), applied to $P_\bullet \to B$ there exists morphisms of free resolutions of B:

$$P_\bullet \leftarrow Q_\bullet \to P_{\bullet A}(B)$$

where $Q_\bullet$ is the diagonal object of the bisimplicial algebra $P_{\bullet A}(P_\bullet)$.

It follows from the description of gr(1.9.1) in the proof of 1.9 (a) that (1.8.1) induces (1.5.6) on the graded pieces.

The isomorphism (1.8.1) is functorial in the A-algebra B. To see this, one can use the standard free resolution $P_{\bullet A}(B)$, which is functorial in B.

2. Comparing the Hodge filtration and the Nygaard filtration in degree $< p$

2.1. In this section we fix a perfect field k of characteristic $p > 0$. Let $W = W(k)$ be its Witt ring. We denote by σ the Frobenius automorphism of W. Recall that if R is a smooth k-algebra, the de Rham-Witt complex $W\Omega^\bullet_R$ is endowed with the *Nygaard filtration*

$$(2.1.1) \quad \mathcal{N}^0W\Omega^\bullet_R = W\Omega^\bullet_R \supset \mathcal{N}^1W\Omega^\bullet_R \supset \cdots \supset \mathcal{N}^iW\Omega^\bullet_R \supset \mathcal{N}^{i+1}W\Omega^\bullet_R \supset \cdots$$

(([9], 8.1), ([10], 8.1.1)), where $\mathcal{N}^r W\Omega^\bullet_R$ is the subcomplex defined by

$$\mathcal{N}^r W\Omega^n_R = p^{r-n-1} V W\Omega^n_R$$

for $n < r$, and $\mathcal{N}^r W\Omega^n_R = W\Omega^n_R$ for $n \geqslant r$ (we use the notation of ([10], 8.1.1)). This is a decreasing, $\mathbf{N}$-indexed, multiplicative filtration. By a theorem of Nygaard ([28], th. 1.5) (see ([10], 8.2.1, 4.3.2, 2.7.2) for an alternate proof in a more general setting), the map $\mathcal{N}^r W\Omega^\bullet_R \to \Omega^\bullet_{R/k}$ sending $p^{r-n-1}Vx$ to x for $n < r$, x to Fx for $n = r$ and x to 0 for $n > r$ induces a σ-linear quasi-isomorphism of complexes of R-modules

$$\mathrm{gr}^r_{\mathcal{N}} W\Omega^\bullet_R \to \tau_{\leqslant r}\Omega^\bullet_{R/k}. \tag{2.1.2}$$

2.2. Let R be a k-algebra. In addition to the Hodge filtration (1.7.1), $L\Omega^\bullet_{R/k}$ is endowed with an increasing filtration, the *conjugate filtration* ([6], Prop. 2.3)

$$\mathrm{Fil}^{\mathrm{conj}}_i L\Omega^\bullet_{R/k} \xrightarrow{\sim} \mathrm{Tot}(\tau_{\leqslant i}\Omega^\bullet_{P_\bullet/k}). \tag{2.2.1}$$

The derived *Cartier isomorphism* ([6], Prop. 3.5) induces a σ-R-linear isomorphism

$$C^{-1} : L\Omega^i_{R/k}[-i] \xrightarrow{\sim} \mathrm{gr}^{\mathrm{conj}}_i L\Omega^\bullet_{R/k}, \tag{2.2.2}$$

For R/k smooth, this is the usual Cartier isomorphism. It is actually better to view the conjugate filtration as a filtration of the (R'-linear) complex $F_* L\Omega^\bullet_{R/k} \in \mathcal{D}(R')$, where R' is the pull-back of R by σ and $F : \mathrm{Spec}(R) \to \mathrm{Spec}(R')$ is the relative Frobenius. Then C^{-1} is an R'-linear isomorphism from $L\Omega^i_{R'/k}[-i]$ to the right hand side of (2.2.2).

The *derived de Rham-Witt complex*

$$LW\Omega^\bullet_R \in \widehat{\mathcal{D}}(W) \tag{2.2.3}$$

is defined by left Kan extension of the functor $B \mapsto W\Omega^\bullet_B$ from the category of finitely generated free k-algebras B to the ∞-category $\widehat{\mathcal{D}}(W)$ of derived p-complete objects of $\mathcal{D}(W)$ ([9], 8.2). It comes equipped with the *Nygaard filtration* (a decreasing, $\mathbf{N}$-indexed filtration in $\widehat{\mathcal{D}}(W)$)

$$\mathcal{N}^r LW\Omega^\bullet_R \subset LW\Omega^\bullet_R. \tag{2.2.4}$$

The isomorphism (2.1.2) is derived into a σ-R-linear isomorphism

$$\mathrm{gr}^r_{\mathcal{N}} LW\Omega^\bullet_R \xrightarrow{\sim} \mathrm{Fil}^{\mathrm{conj}}_r L\Omega^\bullet_{R/k} \tag{2.2.5}$$

([9], 8.2 (3)).

2.3. Let B be a finitely generated free k-algebra. Thus B is lci over W. Let $u : P \to B$ a surjective homomorphism of W-algebras, with P finitely generated and free over W. Let I be the ideal of u. Choose a lift $F : P \to P$ of the absolute Frobenius of $P \otimes_W k$, compatible with σ (e.g., if $P = W[x_1, \cdots, x_n]$, F defined by $F(x_i) = x_i^p$). As in ([25], 0 (1.3.16)), let $s_F : P \to W(P)$ denote the unique section of $W(P) \to P$ which is compatible with F and the canonical Frobenius endomorphism of $W(P)$. Consider the (W-linear) composite morphism

$$t_F : P \xrightarrow{s_F} W(P) \xrightarrow{W(u)} W(B). \tag{2.3.1}$$

It induces a homomorphism of W-dga

$$\Omega^\bullet_{P/W} \to \Omega^\bullet_{W(B)/W}. \tag{2.3.2}$$

Composing with the canonical maps

$$\Omega^\bullet_{W(B)/W} \to \varprojlim \Omega^\bullet_{W_n(B)/W_n} \to \varprojlim W_n\Omega^\bullet_B = W\Omega^\bullet_B$$

(where the second map is induced by the canonical (surjective) maps $\Omega^\bullet_{W_n(B)/W_n} \to W_n\Omega^\bullet_B$ ([25], I 1.3)), we get a homomorphism of W-dga

$$t^\bullet_F : \Omega^\bullet_{P/W} \to W\Omega^\bullet_B. \tag{2.3.3}$$

As s_F is a section of the projection $W(P) \to P$, we have $t_F(I) \subset VW(B)$, hence

$$t^\bullet_F(I\Omega^\bullet_{P/W}) \subset \mathcal{N}^1 W\Omega^\bullet_B,$$

and consequently, by multiplicativity of the Nygaard filtration,

$$t^\bullet_F(I^r\Omega^\bullet_{P/W}) \subset \mathcal{N}^r W\Omega^\bullet_B$$

for all $r \in \mathbf{N}$. In other words, $t^\bullet_F$ (2.3.3) is a filtered morphism, with respect to the I-adic filtration on the left hand side and the Nygaard filtration on the right hand side. In particular, $t^\bullet_F$ induces a filtered morphism

$$t^\bullet_F : \Omega^\bullet_{P/W}/I^p \to W\Omega^\bullet_B/\mathcal{N}^p \tag{2.3.4}$$

If $v : Q \to B$ is a second surjective homomorphism, with ideal J, with Q a finitely generated free W-algebra endowed with a σ-compatible lifting G of

Frobenius, and $f : P \to Q$ is a morphism of W-algebras such that $vf = u$ and $fF = Gf$, then the diagram

$$\begin{tikzcd} \Omega^{\bullet}_{P/W} \arrow[d, "f"'] \arrow[dr, "t^{\bullet}_F"] & \\ \Omega^{\bullet}_{Q/W} \arrow[r, "t^{\bullet}_G"] & W\Omega^{\bullet}_B \end{tikzcd}$$

is commutative, and f is compatible with the I and J-adic filtrations, hence induces a commutative diagram

$$\begin{tikzcd} \Omega^{\bullet}_{P/W}/I^p \arrow[d, "f"'] \arrow[dr, "t^{\bullet}_F"] & \\ \Omega^{\bullet}_{Q/W}/J^p \arrow[r, "t^{\bullet}_G"] & W\Omega^{\bullet}_B/\mathcal{N}^p \end{tikzcd} \quad . \tag{2.3.5}$$

In the situation of (1.5.3), with $A = W$, and P_i free finitely generated over W, let F_i be a (σ-compatible) lift of Frobenius on P_i. Endow $P_1 \otimes P_2$ with the lifting $F = F_1 \otimes F_2$ of Frobenius. Then j_i in (1.5.3) is compatible with F_i and F, and therefore the diagram

$$\begin{tikzcd} \Omega^{\bullet}_{P_1/W}/I_1^p \arrow[d, "\varepsilon(P_2,P_1)"'] \arrow[dr, "t^{\bullet}_{F_1}"] & \\ \Omega^{\bullet}_{P_2/W}/I_2^p \arrow[r, "t^{\bullet}_{F_2}"] & W\Omega^{\bullet}_B/\mathcal{N}^p \end{tikzcd}$$

commutes. It follows that the composition

$$t_B : \widetilde{\Omega}^{\bullet}_{B/W}/\mathcal{I}^p \xrightarrow{\sim} \Omega^{\bullet}_P/I^p \xrightarrow{t^{\bullet}_F} W\Omega^{\bullet}_B/\mathcal{N}^p, \tag{2.3.6}$$

where the first map is the isomorphism σ_P (1.5.5a), is independent of the choice of $(u : P \to B, F)$.

Proposition 2.4. The morphism t_B (2.3.6) is an isomorphism. On the associated graded pieces it induces, for $r < p$, an isomorphism in $\mathcal{D}(B)$

$$\mathrm{gr}^r_{\mathcal{I}}(\widetilde{\Omega}^{\bullet}_{B/W}/\mathcal{I}^p) \xrightarrow{\sim} \mathrm{gr}^r_{\mathcal{N}}(W\Omega^{\bullet}_B/\mathcal{N}^p), \tag{2.4.1}$$

hence, composing with (2.1.2), a σ-linear isomorphism

$$\mathrm{gr}^r_{\mathcal{I}}(\widetilde{\Omega}^{\bullet}_{B/W}/\mathcal{I}^p) \xrightarrow{\sim} \tau_{\leqslant r}\Omega^{\bullet}_{B/k}. \tag{2.4.2}$$

Proof. Choose P to be a *lifting* of B. Then $I = pP$, and the map $t_F^\bullet$ (2.3.3) is the canonical map

$$\Omega^\bullet_{P/W} \to \mathcal{W}\mathrm{Sat}(\Omega^\bullet_{P/W}), \tag{2.4.3}$$

where the right hand side calculates $W\Omega^\bullet_B$ via the Dieudonné algebra structure on $\Omega^\bullet_{P/W}$ given by the lifting F of Frobenius on P ([10], 4.2.3). By ([10], 8.3.5), (2.4.3) is a filtered quasi-isomorphism, the left hand side being equipped with the I-filtration (1.3.1), and the right hand side with the Nygaard filtration. In particular, (2.3.4), is a filtered quasi-isomorphism, hence (2.3.6) is a filtered isomorphism.

2.5. Let $f : B \to C$ a homomorphism of finitely generated free k-algebras, and, as in 1.6, let $u : P \to B$ (resp. $v : Q \to C$) be a surjective homomorphism of W-algebras, with ideal I (resp. J) and P (resp. Q) finitely generated and free over W. Choose liftings of Frobenius F on P and G on Q, and let $F \otimes G$ be the associated lifting of Frobenius on $P \otimes_W Q$. The right vertical maps of (1.6.1) are thus compatible with the Frobenius lifts. By ([25], 0 (1.3.19)) and the commutativity of (1.6.1), the diagram

$$\begin{tikzcd}
P \arrow[r, "t_F"] \arrow[d] & W(B) \arrow[dd, "W(f)"'] \\
P \otimes_W Q \arrow[dr, "t_{F\otimes G}"] & \\
Q \arrow[u] \arrow[r, "t_G"] & W(C)
\end{tikzcd}$$

commutes. With the notation of 1.6, it generates a commutative diagram

$$\begin{tikzcd}
\widetilde{\Omega}^\bullet_B/\mathcal{I}^p \arrow[r, "\sim"] \arrow[dd] & \Omega^\bullet_{P/W}/I^p \arrow[r, "t_F^\bullet"] \arrow[d] & W\Omega^\bullet_B/\mathcal{N}^p \arrow[dd, "W\Omega^\bullet_f"'] \\
& \Omega^\bullet_{(P\otimes_W Q)/W}/K^p \arrow[dr, "t^\bullet_{F\otimes G}"] & \\
\widetilde{\Omega}^\bullet_C/\mathcal{I}^p \arrow[r, "\sim"] \arrow[ur, "\sim"] & \Omega^\bullet_{Q/W}/J^p \arrow[r, "t_G^\bullet"] \arrow[u] & W\Omega^\bullet_C/\mathcal{N}^p
\end{tikzcd}$$

where the left vertical arrow is (1.6.3). One checks that this makes t_B (2.3.6) functorial in B.

2.6. Composing t_B with (1.8.1) we get a filtered isomorphism in $\mathcal{D}(W)$

$$L\Omega^\bullet_{B/W}/\mathrm{Fil}^p \xrightarrow{\sim} W\Omega^\bullet_B/\mathcal{N}^p \tag{2.6.1}$$

By 2.5 and the functoriality of (1.8.1), this isomorphism is functorial in B.

As $LW\Omega^\bullet_{-/W}$ (resp. $L\Omega^\bullet_{-/W}$) is deduced by left Kan extension from $W\Omega^\bullet_{-/W}$ on the category $\mathcal{P}_k$ of finitely generated free k-algebras (resp. from its restriction to $\mathcal{P}_k$), (2.6.1) extends to a filtered isomorphism

$$L\Omega^\bullet_{R/W}/\mathrm{Fil}^p \xrightarrow{\sim} LW\Omega^\bullet_R/\mathcal{N}^p \tag{2.6.2}$$

for any k-algebra R, functorial in R. It induces, for $i < p$, an isomorphism in $\mathcal{D}(R)$ on the graded pieces

$$L\Omega^i_{R/W}[-i](\xrightarrow{\sim} \mathrm{gr}^i_{\mathrm{Hdg}} L\Omega^\bullet) \xrightarrow{\sim} \mathrm{gr}^i_{\mathcal{N}} LW\Omega^\bullet_R \xrightarrow{\sim} \mathrm{Fil}^{\mathrm{conj}}_i L\Omega^\bullet_{R/k}, \tag{2.6.3}$$

where the second isomorphism is the σ-linear isomorphism (2.2.5).

Remark 2.7. The quotients by Fil^p and $\mathcal{N}^p$ in (2.6.1) cannot be removed. Bhatt (letter to the author, Jan. 19, 2019) has shown that

$$R\varprojlim_n (L\Omega^\bullet_{k/W} \otimes^L W_n) \xrightarrow{\sim} \widehat{W\langle x\rangle}/(x-p), \tag{2.7.1}$$

where $(-)\widehat{}$ means p-adic completion, and $W\langle x\rangle$ is the W-divided power algebra on x. In particular, $\widehat{W\langle x\rangle}/(x-p)(\xrightarrow{\sim} \widehat{W\langle y\rangle}/(y))$ has non-trivial p-torsion (e.g., $y^{[p]}$).

Here is Bhatt's proof of (2.7.1). Consider the morphism of short exact sequences of pro-objects

$$\begin{tikzcd}
0 \arrow[r] & W_\bullet[x] \arrow[r, "x-p"] \arrow[d, "x\mapsto 0"'] & W_\bullet[x] \arrow[r] \arrow[d, "x\mapsto 0"'] & W_\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W_\bullet \arrow[r, "-p"] & W_\bullet \arrow[r] & k \arrow[r] & 0,
\end{tikzcd}$$

where $(-_\bullet)$ denotes the pro-object $(-_n)_{n\geqslant 1}$. It implies that the right square is tor-independent, i.e.,

$$(W_\bullet[x] \to W_\bullet) \otimes^L_{W_\bullet[x]} W_\bullet \to (W_\bullet \to k)$$

is an isomorphism. Therefore

$$L\Omega^\bullet_{W_\bullet/W_\bullet[x]} \otimes^L_{W_\bullet[x]} W_\bullet \to L\Omega^\bullet_{k/W_\bullet}$$

is an isomorphism, too. By ([6], Th. 3.27, or 3.40), we have

$$L\Omega^\bullet_{W_\bullet/W_\bullet[x]} \xrightarrow{\sim} W_\bullet\langle x\rangle,$$

hence

$$L\Omega^\bullet_{k/W_\bullet} \xrightarrow{\sim} W_\bullet\langle x\rangle/(x-p),$$

and (2.7.1) follows by taking completions. The above argument is essentially that of ([6], 3.40).

One can show (Bhatt) that (2.7.1) underlies a filtered isomorphism, the left hand side being endowed with the Hodge filtration, and the right hand side by the images of the ideals $(x)^{[n]}$. In particular, one gets

$$L\Omega^\bullet_{k/W}/\mathrm{Fil}^p \xrightarrow{\sim} \widehat{W\langle x\rangle}/((x)^{[p]}+(x-p)) \xrightarrow{\sim} W[[x]]/((x)^p+(x-p) \xrightarrow{\sim} W/(p)^p$$

thus recovering (2.6.1) in this case ($R=k$).

2.8. Let X be a k-scheme. As in ([10] 5.2), denote by $\mathcal{U}_{\mathrm{aff}}(X)$ the collection of all affine open subschemes of X. The derived de Rham complex

$$L\Omega^\bullet_{X/W} \in \mathcal{D}(X,W)$$

is the sheaf associated to the presheaf on $\mathcal{U}_{\mathrm{aff}}(X)$ defined by $U \mapsto L\Omega^\bullet_{\mathcal{O}_X(U)/W}$. The derived de Rham-Witt complex

$$LW\Omega^\bullet_X \in \widehat{\mathcal{D}}(X,W)$$

is (defined as) the sheaf associated to the presheaf on $\mathcal{U}_{\mathrm{aff}}(X)$ defined by $U \mapsto LW\Omega^\bullet_{\mathcal{O}_X(U)}$ (where $\widehat{\mathcal{D}}(X,W)$ denotes the category of derived p-complete objects of $\mathcal{D}(X,W)$).

The Hodge filtration (1.7.1) globalizes to

$$\mathrm{Fil}^i_{\mathrm{Hdg}} L\Omega^\bullet_{X/W} \subset L\Omega^\bullet_{X/W},$$

with associated graded

$$\mathrm{gr}^i_{\mathrm{Hdg}} L\Omega^\bullet_{X/W} \xrightarrow{\sim} L\Omega^i_{X/W}[-i].$$

Let X' be the pull-back of X by the Frobenius of k and $F: X \to X'$ be the relative Frobenius. Then $F_*L\Omega^\bullet_{X/k}$ is defined as an object of $\mathcal{D}(X') := \mathcal{D}(X', \mathcal{O}_{X'})$, and the conjugate filtration (2.2.1) globalizes to

$$\mathrm{Fil}^{\mathrm{conj}}_i F_*L\Omega^\bullet_{X/k} \subset F_*L\Omega^\bullet_{X/k},$$

with associated graded given by the derived Cartier isomorphism (in $\mathcal{D}(X')$)

$$C^{-1} : L\Omega^i_{X'/k}[-i] \xrightarrow{\sim} \mathrm{gr}^{\mathrm{conj}}_i F_*L\Omega^\bullet_{X/k}.$$

The Nygaard filtration (2.2.4) globalizes to

$$\mathcal{N}^r LW\Omega^\bullet_X \subset LW\Omega^\bullet_X,$$

with associated graded given by the isomorphism (in $\mathcal{D}(X')$)

$$\mathrm{gr}^r_{\mathcal{N}} LW\Omega^\bullet_{X'} \xrightarrow{\sim} \mathrm{Fil}^{\mathrm{conj}}_r F_* L\Omega^\bullet_{X/k}.$$

Finally, the isomorphisms (2.6.2), (2.6.3) globalize. We have obtained:

Theorem 2.9. Let X be a k-scheme. The isomorphisms (2.6.2) globalize to a filtered isomorphism in $\widehat{\mathcal{D}}(X, W)$

$$L\Omega^\bullet_{X/W}/\mathrm{Fil}^p \xrightarrow{\sim} LW\Omega^\bullet_X/\mathcal{N}^p, \tag{2.9.1}$$

where the left hand side is filtered by the Hodge filtration and the right hand side by the Nygaard filtration. On the associated graded pieces (2.9.1) induces, for $r < p$, an isomorphism in $\mathcal{D}(X')$

$$L\Omega^r_{X'/W}[-r] \xrightarrow{\sim} \mathrm{Fil}^{\mathrm{conj}}_r F_* L\Omega^\bullet_{X/k}. \tag{2.9.2}$$

These isomorphisms are functorial in X/k in a natural way.

One can ask which filtration on the left hand side does the conjugate one on the right hand side correspond to. This is the subject of the next section.

3. The Koszul filtration

3.1. Recall that for a short exact sequence $0 \to E' \to E \to E'' \to 0$ of flat modules over a ringed topos, and any integer $r \geqslant 0$, we have the (finite increasing) *Koszul filtration* on $\Lambda^r E$,

$$0 \subset \mathrm{Kos}_0 \Lambda^r E \subset \mathrm{Kos}_1 \Lambda^r E \subset \cdots \subset \mathrm{Kos}_r \Lambda^r E = \Lambda^r E,$$

$$\mathrm{Kos}_i \Lambda^r E = \mathrm{Im}(\Lambda^{r-i} E' \otimes \Lambda^i E \to \Lambda^r E)$$

with associated graded

$$\mathrm{gr}^{\mathrm{Kos}}_i \Lambda^r E = \Lambda^{r-i} E' \otimes \Lambda^i E''.$$

This extends to complexes, the Λ^i being derived. In particular, the short exact sequence of cotangent complexes associated with a composition of morphisms of schemes $X \xrightarrow{f} Y \to S$ ([23], 2.1.5.2) defines a Koszul filtration on $L\Omega^i_{X/S} := L\Lambda^i L_{X/S}$,

$$\mathrm{Kos}^{X/Y/S}_i L\Omega^r_{X/S} \tag{3.1.1}$$

$(0 \leqslant i \leqslant r)$, denoted Kos_i when no confusion can arise, with associated graded

$$\mathrm{gr}_i^{\mathrm{Kos}} L\Omega^r_{X/S} = f^* L\Omega^{r-i}_{Y/S} \otimes L\Omega^i_{X/Y}. \tag{3.1.2}$$

Proposition 3.2. With the notation and under the assumptions of 2.9, for $r < p$, the isomorphism (2.9.2) underlies a filtered isomorphism, where the right hand side is filtered by the conjugate filtration, and the left hand one by the Koszul filtration Kos_i relative to $X'/\mathrm{Spec}(k)/\mathrm{Spec}(W)$. In particular, we have

$$(\mathrm{Kos}_i L\Omega^r_{X'/W})[-r] \xrightarrow{\sim} L\Omega^i_{X'/W}[-i], \tag{3.2.1}$$

and the square

$$\begin{tikzcd} L\Omega^i_{X'/W}[-i] \arrow[r, "\sim"] \arrow[d] & \mathrm{Fil}_i^{\mathrm{conj}} F_* L\Omega^\bullet_{X/k} \arrow[d] \\ L\Omega^r_{X'/W}[-r] \arrow[r, "\sim"] & \mathrm{Fil}_r^{\mathrm{conj}} F_* L\Omega^\bullet_{X/k}, \end{tikzcd} \tag{3.2.2}$$

where the horizontal isomorphisms are (2.9.2), and the left vertical map is $\mathrm{Kos}_i \hookrightarrow \mathrm{Kos}_r$, commutes. We have

$$\mathrm{gr}_i^{\mathrm{Kos}} L\Omega^r_{X'/W}[-r] \xrightarrow{\sim} L\Omega^i_{X'/k}[-i], \tag{3.2.3}$$

and on the associated graded pieces, (2.9.2) induces the derived Cartier isomorphism (2.2.2)

$$C^{-1} : L\Omega^i_{X'/k}[-i] \xrightarrow{\sim} \mathrm{gr}_i^{\mathrm{conj}} F_* L\Omega^\bullet_{X/k}.$$

The case $r = 1$ will be of special interest to us. As $L_{k/W} = k[1]$, we have:

Corollary 3.3. The isomorphisms (3.2.2) induce an isomorphism of distinguished triangles

$$\begin{tikzcd} \mathcal{O}_{X'} \arrow[r] \arrow[d, "\sim"'] & L\Omega^1_{X'/W}[-1] \arrow[r] \arrow[d, "\sim"'] & L\Omega^1_{X'/k}[-1] \arrow[r] \arrow[d, "\sim"'] & {} \\ \mathrm{Fil}_0^{\mathrm{conj}} F_* L\Omega^\bullet_{X/k} \arrow[r] & \mathrm{Fil}_1^{\mathrm{conj}} F_* L\Omega^\bullet_{X/k} \arrow[r] & \mathrm{gr}_1^{\mathrm{conj}} F_* L\Omega^\bullet_{X/k} \arrow[r] & {}. \end{tikzcd} \tag{3.3.1}$$

Proof of 3.2. It suffices to treat the case where $X = \mathrm{Spec}(B)$, with B is a finitely generated free k-algebra. As in the proof of 2.4, choose a finitely generated free W-algebra lifting B together with a (σ-compatible) endomorphism F of P lifting Frobenius. By ([23], II 2.2.3), the lifting P decomposes $L_{B/W}$ into

$$L_{B/W} \xrightarrow{\sim} L_{B/P} \oplus L_{B/k} = B[1] \oplus \Omega^1_{B/k}.$$

Then

$$L\Omega^r_{B/W} \xrightarrow{\sim} \oplus_{0\leqslant i\leqslant r} L\Lambda^{r-i}(B[1]) \otimes^L \Omega^i_{B/k} \xrightarrow{\sim} \oplus_{0\leqslant i\leqslant r} B[r-i] \otimes \Omega^i_{B/k},$$

and the Koszul filtration relative to $W \to k \to B$ is given by

$$(*) \qquad \mathrm{Kos}_i L\Omega^r_{B/W} \xrightarrow{\sim} \oplus_{0\leqslant j\leqslant i} B[r-j] \otimes \Omega^j_{B/k} \xrightarrow{\sim} L\Omega^i_{B/W},$$

which proves (3.2.1). On the other hand, we have the isomorphism (1.8.2)

$$L\Omega^r_{B/W}[-r] \xrightarrow{\sim} \mathrm{gr}^r_{pP}\Omega^\bullet_{P/W},$$

and the identification

$$(**) \qquad \mathrm{gr}^r_{(p)}\Omega^\bullet_{P/W} \xrightarrow{\sim} \oplus_{0\leqslant i\leqslant r}\Omega^i_{B/k}[-i],$$

given by division by p^{r-i} in degree i. By (*), the description of (1.8.2) via (1.4.2) shows that the composite isomorphism

$$(***) \qquad L\Omega^r_{B/W}[-r] \xrightarrow{\sim} \oplus_{0\leqslant i\leqslant r}\Omega^i_{B/k}[-i]$$

is compatible with the Koszul filtration, i.e., the square (with horizontal maps (***))

$$(3.3.2) \qquad \begin{array}{ccc} (\mathrm{Kos}_i L\Omega^r_{B/W})[-r] = L\Omega^i_{B/W}[-i] & \longrightarrow & \oplus_{0\leqslant j\leqslant i}\Omega^j_{B/k}[-j] \\ \downarrow & & \downarrow \\ L\Omega^r_{B/W}[-r] & \longrightarrow & \oplus_{0\leqslant i\leqslant r}\Omega^i_{B/k}[-i], \end{array}$$

commutes. The composition of the quasi-isomorphism (2.4.3)

$$\mathrm{gr}^r t^\bullet_F : \mathrm{gr}^r_{(p)}\Omega^\bullet_{P/W} \xrightarrow{\sim} \mathrm{gr}^r_{\mathcal{N}} W\Omega^\bullet_{B/k}$$

and the quasi-isomorphism (2.1.2)

$$\mathrm{gr}^r_{\mathcal{N}} W\Omega^\bullet_B \to \tau_{\leqslant r}\Omega^\bullet_{B/k}$$

can be rewritten, via (**), as the quasi-isomorphism

$$F_r : \oplus_{0\leqslant i\leqslant r}\Omega^i_{B/k}[-i] \to \tau_{\leqslant r}\Omega^\bullet_{B/k}$$

induced by $p^{-i}F$ in degree i. As the square

$$
\begin{tikzcd}
\oplus_{0\leqslant j\leqslant i}\Omega^j_{B/k}[-j] \arrow[r, "F_i"] \arrow[d] & \tau_{\leqslant i}\Omega^\bullet_{B/k} \arrow[d] \\
\oplus_{0\leqslant j\leqslant r}\Omega^j_{B/k}[-j] \arrow[r, "F_r"] & \tau_{\leqslant r}\Omega^\bullet_{B/k}
\end{tikzcd}
\tag{3.3.3}
$$

trivially commutes, the composition of (3.2.2) and (3.2.3) gives (3.2.2). The remaining assertions of 3.2 are immediate, F_r inducing by definition the Cartier isomorphism in each degree.

4. Liftings mod p^2 and partial decompositions

4.1. We keep the notation of 2.1. If S is a scheme, and X is an S-scheme, we say that X is *weakly lci* if

$$\mathrm{toramp}(L_{X/S}) \subset [-1,0].$$

For $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$, this is equivalent to saying that B is weakly lci over A (1.1). If X is lci over S in the sense of ([5], VIII 1.1), i.e., locally embeddable by a regular immersion into a smooth S-scheme, then X is weakly lci over S, and $L_{X/S}$ is of *perfect amplitude* in $[-1,0]$ ([23], III 3.2.6). If $f : X \to Y$ is an S-morphism, and if f is weakly lci, and Y weakly lci over S, then X is weakly lci over S ([23], III 3.3.5). In particular, if X is a weakly lci k-scheme, X is weakly lci over W.

Proposition 4.2. Let X/k be weakly lci.

(i) The transitivity triangle for $X \to \mathrm{Spec}(k) \to \mathrm{Spec}(W_2)$ induces, by truncation, a distinguished triangle

$$\mathcal{O}_X[1] \to \tau_{\geqslant -1}L_{X/W_2} \to L_{X/k} \to . \tag{4.2.1}$$

(ii) The canonical morphism $L_{X/W} \to \tau_{\geqslant -1}L_{X/W_2}$ is an isomorphism, and sits in an isomorphism of distinguished triangles

$$
\begin{tikzcd}
\mathcal{O}_X[1] \arrow[r] \arrow[d, "\mathrm{Id}"'] & L_{X/W} \arrow[r] \arrow[d, "\sim"'] & L_{X/k} \arrow[r] \arrow[d, "\mathrm{Id}"'] & {} \\
\mathcal{O}_X[1] \arrow[r] & \tau_{\geqslant -1}L_{X/W_2} \arrow[r] & L_{X/k} \arrow[r] & {} .
\end{tikzcd}
\tag{4.2.2}
$$

Proof. As $L_{k/W} = k[1]$, the upper triangle in (4.2.2) is the transitivity triangle for $X \to \mathrm{Spec}(k) \to \mathrm{Spec}(W)$. The transitivity triangle for

$\mathrm{Spec}(k) \to \mathrm{Spec}(W_2) \to \mathrm{Spec}(W)$ gives $L_{k/W_2} = k[2] \oplus k[1]$. Therefore $\mathrm{Spec}(W_2) \to \mathrm{Spec}(W)$ induces a morphism of distinguished triangles

$$
\begin{tikzcd}
\mathcal{O}_X[1] \arrow[r] \arrow[d] & L_{X/W} \arrow[r] \arrow[d] & L_{X/k} \arrow[r] \arrow[d, "\mathrm{Id}"'] & {} \\
\mathcal{O}_X[2] \oplus \mathcal{O}_X[1] \arrow[r] & L_{X/W_2} \arrow[r] & L_{X/k} \arrow[r] & {} ,
\end{tikzcd}
\tag{$*$}
$$

in which the left vertical arrow is the identity on $\mathcal{O}_X[1]$. As $H^{-2}(L_{X/k}) = 0$ since X/k is weakly lci, the cohomology sequence of the lower triangle in (*) gives the short exact sequence

$$0 \to \mathcal{O}_X \to H^{-1}(L_{X/W_2}) \to H^{-1}(L_{X/k}) \to 0,$$

and the morphism of cohomology sequences defined by (*) induces an isomorphism $H^{-1}(L_{X/W}) \xrightarrow{\sim} H^{-1}(L_{X/W_2})$. It follows that the vertical map of (*) induces an isomorphism $L_{X/W} \xrightarrow{\sim} \tau_{\geqslant -1} L_{X/W_2}$, and (*) gives (4.2.2), and, in particular, (i).

4.3. Let X/k be weakly lci. Let $\widetilde{X}$ be a W_2-extension of X by $\mathcal{O}_X$ as an ideal of square zero. The cartesian square

$$
\begin{tikzcd}
X \arrow[r] \arrow[d] & \widetilde{X} \arrow[d] \\
\mathrm{Spec}(k) \arrow[r] & \mathrm{Spec}(W_2)
\end{tikzcd}
\tag{4.3.1}
$$

defines an $\mathcal{O}_X$-linear map

$$u(\widetilde{X}) : \mathcal{O}_X \to \mathcal{O}_X.$$

For $\widetilde{X}$ to be a *lifting* of X to W_2, i.e., to be *flat* over W_2, it is necessary and sufficient that $u(\widetilde{X})$ is an isomorphism. Recall ([23], III 1.2.3) that $\widetilde{X}$ defines a morphism

$$c(\widetilde{X}) \in \mathrm{Hom}(L_{X/W_2}, \mathcal{O}_X[1]) = \mathrm{Hom}(\tau_{\geqslant -1} L_{X/W_2}, \mathcal{O}_X[1]) \tag{4.3.2}$$

and that

$$u(\widetilde{X}) = \delta(c(\widetilde{X})),$$

where $\delta : \mathrm{Hom}(L_{X/W_2}, \mathcal{O}_X[1]) \to \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X)$ is the coboundary map deduced from the map $\mathcal{O}_X[1] \to L_{X/W_2}$ in the transitivity triangle of $X \to \mathrm{Spec}(k) \to \mathrm{Spec}(W_2)$. We shall say that $\widetilde{X}$ is a *normalized* lifting if $u(\widetilde{X}) =$

Id. Unless otherwise stated we will consider only normalized liftings. Thus, for such a lifting, $c(\widetilde{X})$ (4.3.2) is a *retraction* of the left arrow in (4.2.1). As $\mathcal{O}_X[1] \xrightarrow{\sim} L_{X/\widetilde{X}}$, the corresponding decomposition

$$\tau_{\geqslant -1} L_{X/W_2} \xrightarrow{\sim} \mathcal{O}_X[1] \oplus L_{X/k} \tag{4.3.3}$$

is also that defined by (4.3.1) ([23], II 2.2.3), and $c(\widetilde{X})$ is the first projection. Thanks to 4.2 (ii), we will identify (4.3.3) with a decomposition

$$L\Omega^1_{X/W}[-1] \xrightarrow{\sim} \mathcal{O}_X \oplus L\Omega^1_{X/k}[-1] \tag{4.3.4}$$

of the upper triangle of (3.3.1). Combining with 3.3, we obtain:

Theorem 4.4. Let X/k be weakly lci. Isomorphism classes of liftings of X to W_2 (or, equivalently, of liftings of X' to W_2) correspond bijectively to isomorphism classes of splittings of $\mathrm{Fil}^{\mathrm{conj}}_1 F_* L\Omega^\bullet_{X/k}$:

$$\mathrm{Fil}^{\mathrm{conj}}_1 F_* L\Omega^\bullet_{X/k} \xrightarrow{\sim} \mathrm{gr}^{\mathrm{conj}}_0 F_* L\Omega^\bullet_{X/k} \oplus \mathrm{gr}^{\mathrm{conj}}_1 F_* L\Omega^\bullet_{X/k}. \tag{4.4.1}$$

4.5. Assume X/k smooth. Then $L\Omega^\bullet_{X/k} \xrightarrow{\sim} \Omega^\bullet_{X/k}$, and $\mathrm{Fil}^{\mathrm{conj}}_i F_*\Omega^\bullet_{X/k} = \tau_{\leqslant i} F_*\Omega^\bullet_{X/k}$. In this case, 4.4 recovers ([18], 3.6 (a)) as splittings (4.4.1) are decompositions of $\tau_{\leqslant 1} F_*\Omega^\bullet_{X/k}$ in $D(X', \mathcal{O}_{X'})$ in the sense of ([18], 3.1). In particular, the class

$$e(K) \in \mathrm{Ext}^2_{\mathcal{O}_{X'}}(\Omega^1_{X'/k}, \mathcal{O}_{X'})$$

of $K = \tau_{\leqslant 1} F_*\Omega^\bullet_{X/k}$ is the obstruction to the existence of a lifting of X to W_2. Conditions for the vanishing of such extension classes were recently examined by Schröer ([32], 9.1).

Local liftings of X to W_2 form a gerbe $\mathrm{Lift}(X/W_2)$ banded by $T = T_{X/k} = (\Omega^1_{X/k})^\vee$ (([20], VII 1.2), ([18], 3.4)). Local splittings (4.3.3) of $\tau_{\geqslant -1} L_{U/W_2}$ form a (T-banded) gerbe $\mathrm{Split}(\tau_{\geqslant -1} L_{X/W_2})$. Associating to a lifting of an open subscheme U of X to W_2 (or, equivalently, X') the corresponding decomposition (4.3.3) yields an equivalence

$$\mathrm{Lift}(X/W_2) \xrightarrow{\sim} \mathrm{Split}(\tau_{\geqslant -1} L_{X/W_2}) \tag{4.5.1}$$

Similarly, local splittings (4.4.1) (or decompositions of K) form a gerbe $\mathrm{Split}(\tau_{\leqslant 1} F_*\Omega^\bullet_{X/k})$ banded by $T_{X'/k}$. Thus (3.3.1) yields an equivalence

$$\mathrm{Lift}(X/W_2) (\xrightarrow{\sim} \mathrm{Lift}(X'/W_2)) \xrightarrow{\sim} \mathrm{Split}(\tau_{\leqslant 1} F_*\Omega^\bullet_{X/k}). \tag{4.5.2}$$

This is ([18], 3.5) in the case $S = \mathrm{Spec}(k)$, $\widetilde{S} = W_2$.

In the general setting of *loc. cit.*, we have an equivalence similar to (4.5.1)

$$\mathrm{Lift}(X'/\widetilde{S}) \xrightarrow{\sim} \mathrm{Split}(\tau_{\geqslant -1} F_*(L_{X'/\widetilde{S}})). \tag{4.5.3}$$

Using ([29], (2.26.3), and A7), one can construct a canonical equivalence

$$\mathrm{Split}(\tau_{\geqslant -1} L_{X'/\widetilde{S}}) \xrightarrow{\sim} \mathrm{ch}(R\mathcal{H}om(\tau_{\geqslant -1} L_{X'/\widetilde{S}}, \mathcal{O}_{X'})[1]), \tag{4.5.4}$$

where $\mathrm{ch}(K)$ is the $\mathcal{O}_{X'}$-linear Picard stack associated with $K \in D^{[-1,0]}(X', \mathcal{O}_{X'})$ in the notation of ([2], XVIII 1.4.11) and ([16], 1 (B)). On the other hand, as $\tau_{\leqslant 1} F_* \Omega^\bullet_{X/S}$ is of perfect amplitude in $[0,1]$, with $H^0(F_*\Omega^\bullet_{X/S}) = \mathcal{O}_{X'}$, we have a canonical equivalence

$$\mathrm{Split}(\tau_{\leqslant 1} F_* \Omega^\bullet_{X/S}) \xrightarrow{\sim} \mathrm{ch}(R\mathcal{H}om(\tau_{\leqslant 1} F_* \Omega^\bullet_{X/S}, \mathcal{O}_{X'})). \tag{4.5.5}$$

Thus, in view of (4.5.3) – (4.5.5), ([18], 3.5) yields an isomorphism

$$R\mathcal{H}om(\tau_{\geqslant -1} L_{X'/\widetilde{S}}, \mathcal{O}_{X'})[1] \xrightarrow{\sim} R\mathcal{H}om(\tau_{\leqslant 1} F_* \Omega^\bullet_{X/S}, \mathcal{O}_{X'}),$$

hence, by duality, an isomorphism

$$\tau_{\geqslant -1}(L_{X'/\widetilde{S}})[-1] \xrightarrow{\sim} \tau_{\leqslant 1} F_* \Omega^\bullet_{X/S}. \tag{4.5.6}$$

Though there is no longer any de Rham-Witt complex available here, one can directly construct such an isomorphism, independently of ([18], 3.5) – and thus reproving it – by using, in the affine case, embeddings of X into smooth $\widetilde{S}$-schemes equipped with liftings of their relative Frobenius maps.

4.6. Let's come back to the hypotheses of 4.4. In general, local liftings of X to W_2 no longer form a gerbe, but they form an $\mathcal{O}_X$-linear Picard stack $\mathrm{Lift}(X/W_2)$. By ([29], A7) this stack can be shown to correspond to the stack of local decompositions (4.3.3), which we will denote by $\mathrm{Split}(\tau_{\geqslant -1}(L_{X/W_2}))$

$$\mathrm{Lift}(X/W_2) \xrightarrow{\sim} \mathrm{Split}(\tau_{\geqslant -1}(L_{X/W_2}))$$

(note that the right hand side is in general different from the stack of local decompositions of $\tau_{\geqslant -1}(L_{X/W_2})$ as a sum of its $H^i[-i]$ in the derived category). We thus get a canonical equivalence

$$\mathrm{Lift}(X/W_2) \xrightarrow{\sim} \mathrm{ch}(R\mathcal{H}om(\tau_{\leqslant -1}(L_{X/W_2}), \mathcal{O}_X)[1]),$$

and in the right hand side, $\tau_{\leqslant -1}(L_{X/W_2})$ can be replaced by $L_{X/W}$. Local splittings (4.4.1) of $\mathrm{Fil}^{\mathrm{conj}}_1 F_*(L\Omega^\bullet_{X/k})$ also form a Picard stack $\mathrm{Split}(\mathrm{Fil}^{\mathrm{conj}}_1 F_*(L\Omega^\bullet_{X/k}))$, and we have a canonical equivalence

$$\mathrm{Split}(\mathrm{Fil}^{\mathrm{conj}}_1 F_*(L\Omega^\bullet_{X/k})) \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{O}_{X'}}(\mathrm{Fil}^{\mathrm{conj}}_1 F_*(L\Omega^\bullet_{X/k}), \mathcal{O}_{X'})$$

as $\mathrm{gr}_0^{\mathrm{conj}} F_*(L\Omega^\bullet_{X/k})) \xrightarrow{\sim} \mathcal{O}_{X'}$. Putting this together, we get a canonical equivalence

$$\mathrm{Lift}(X/W_2) \xrightarrow{\sim} \mathrm{Split}(\mathrm{Fil}_1^{\mathrm{conj}} F_*(L\Omega^\bullet_{X/k})),$$

which refines 4.4, and generalizes (the case $S = \mathrm{Spec}(k)$, $\widetilde{S} = W_2$ of) ([18], 3.5).

4.7. Let X/k be weakly lci. Assume that X admits a lifting $\widetilde{X}$ to W_2. Then $\widetilde{X}$ defines a decomposition (4.3.4). For $r < p$, applying $L\Lambda^r$ to $L\Omega^1_{X/W}$, we get a decomposition

$$L\Omega^r_{X/W}[-r] \xrightarrow{\sim} \oplus_{0 \leqslant i \leqslant r} L\Omega^i_{X/k}[-i]. \tag{4.7.1}$$

Combining with (2.9.2) and 3.2, we get:

Theorem 4.8. Let X/k be weakly lci and admit a lifting $\widetilde{X}$ to W_2. Then $\widetilde{X}$ defines, for $r < p$, an isomorphism (in $\mathcal{D}(X')$)

$$\oplus_{0 \leqslant i \leqslant r} L\Omega^i_{X'/k}[-i] \xrightarrow{\sim} \mathrm{Fil}_r^{\mathrm{conj}} F_*(L\Omega^\bullet_{X/k}), \tag{4.8.1}$$

compatible with the conjugate filtration on the right hand side and the filtration of the graduation on the left hand one, inducing the derived Cartier isomorphism on the graded pieces.

For X/k *smooth*, (4.8.1) recovers ([18], 2.1). For X/k lci, in the affine case, assuming the existence of a lifting of Frobenius on $\widetilde{X}$, a decomposition of the form (4.8.1), with no restriction on r, was proved by Bhatt in ([6], 3.17), see also ([8] 1.5 (4)).

4.9. Variants of 2.9, 4.4, 4.8 could be considered:

(a) in the *logarithmic* setup: around th. 2.3 of [33] (generalizing results of Kato [26]);

(b) in the *prismatic* setup: around theorems of Bhatt, Bhatt-Scholze and Anschütz-Le Bras ([13], 15.6), ([1], 3.2.1).

We may return to these questions later.

5. Cohomological amplitude estimates (after B. Bhatt)

The results in this section are due to Bhatt. We extract them from [12].

5.1. We keep the notation of 2.1. If X is a k-scheme, we denote by X' its pull-back by the Frobenius of k, and by $F : X \to X'$ the relative Frobenius morphism. The complex $F_*\Omega^\bullet_{X/k}$ is $\mathcal{O}_{X'}$-linear. We denote by $Z\Omega^i_{X/k}$ (resp. $B\Omega^i_{X/k}$) the sheaf of cycles (resp. boundaries) of $\Omega^\bullet_{X/k}$ in degree i. We will view it as a sub-$\mathcal{O}_{X'}$-module of $F_*\Omega^i_{X/k}$. If X/k is smooth, then $\Omega^i_{X/k}$ is locally free of finite type over $\mathcal{O}_X$, and $F_*\Omega^i_{X/k}$, $Z\Omega^i_{X/k}$, $B\Omega^i_{X/k}$ are locally free of finite type over $\mathcal{O}_{X'}$ ([25], 0 2.2.8), and we have exact sequences

$$0 \to Z\Omega^i_{X/k} \to F_*\Omega^i_{X/k} \to B\Omega^{i+1}_{X/k} \to 0, \tag{5.1.1}$$

$$0 \to B\Omega^i_{X/k} \to Z\Omega^i_{X/k} \to \mathcal{H}^i(F_*\Omega^\bullet_{X/k}) \to 0, \tag{5.1.2}$$

with the Cartier isomorphism

$$C^{-1} : \Omega^i_{X'/k} \xrightarrow{\sim} \mathcal{H}^i(F_*\Omega^\bullet_{X/k}). \tag{5.1.3}$$

5.2. The previous local freeness assertions do not extend to the singular case, as, by Kunz' theorem, if X is of finite type over k, X is regular if and only if F is flat. However, they imply that, by left Kan extension, the functors $P \mapsto Z\Omega^i_{P/k}$, $P \mapsto B\Omega^i_{P/k}$ on finitely generated free algebras can be derived into functors $R \to LZ\Omega^i_{R/k}$, $R \mapsto LB\Omega^i_{R/k}$ on arbitrary k-algebras. These constructions globalize on k-schemes, and the sequences (5.1.1), (5.1.2) globalize to exact sequences (in the derived ∞-category $\mathcal{D}(X', \mathcal{O}_{X'})$)

$$0 \to LZ\Omega^i_{X/k} \to F_*L\Omega^i_{X/k} \to LB\Omega^{i+1}_{X/k} \to 0, \tag{5.2.1}$$

$$0 \to LB\Omega^i_{X/k} \to LZ\Omega^i_{X/k} \to \mathrm{gr}^{\mathrm{conj}}_i(F_*\Omega^\bullet_{X/k}) \to 0, \tag{5.2.2}$$

and we have the derived Cartier isomorphism (2.2.2)

$$C^{-1} : L\Omega^i_{X'/k} \xrightarrow{\sim} \mathrm{gr}^{\mathrm{conj}}_i(F_*\Omega^\bullet_{X/k}). \tag{5.2.3}$$

In particular,

$$\mathcal{O}_{X'} = L\Omega^0_{X'/k} \xrightarrow{\sim} LZ\Omega^0_{X/k} \xrightarrow{\sim} \mathrm{gr}^{\mathrm{conj}}_0(F_*\Omega^\bullet_{X/k}). \tag{5.2.4}$$

Finally, (5.2.1) gives the exact sequence

$$0 \to \mathrm{Fil}^{\mathrm{conj}}_i F_*L\Omega^\bullet_{X/k} \to F_*(L\Omega^\bullet_{X/k}/\mathrm{Fil}^{i+1}_{\mathrm{Hdg}}) \to LB^{i+1}_{X/k}[-i] \to 0, \tag{5.2.5}$$

which will not be used until 5.5.

Proposition 5.3. ([12], Prop. 2.3) Let X be an lci k-scheme of finite type, of pure dimension d. Let s be the dimension of the singular locus of X.

(a) The complex $L\Omega^i_{X/k}[-i]$ is of perfect amplitude in $[0, i]$.

(b) The complexes $F_*L\Omega^i_{X/k}[-i]$, $LZ\Omega^i_{X/k}[-i]$, $LB\Omega^i_{X/k}[-i]$ lie in $\mathcal{D}^{\geqslant 0}_{\mathrm{coh}}(X')$ [2]. For $i > d$, they all lie in $\mathcal{D}^{\geqslant d-s}(X')$ (and are zero if X/k is smooth).

[2]The susbscript "coh" means that the cohomology sheaves are coherent.

For the proof we need the following lemma:

Lemma 5.4. ([12], Lemma 2.4) Let A be a regular local noetherian ring of dimension d. Let $K \in D(A)$ be a non-zero perfect complex with $\mathrm{toramp}(K) \subset [0, +\infty)$. Let $j \geqslant 0$ be such that $H^i(K) = 0$ for $i < j$ and $H^j(K) \neq 0$. Then, for any associated prime $\mathfrak{p}$ of $H^j(K)$, we have $\dim(A/\mathfrak{p}) \geqslant d - j$. In particular, any irreducible component of $\mathrm{Supp}(H^j(K))$ has dimension $\geqslant d - j$.

Proof (*loc. cit.*). By ([5], II 2.2.9 (b), 2.2.10), K can be represented by a bounded complex of finite free A-modules concentrated in degree $\geqslant 0$. Let $Q := K^j/\mathrm{Im}(K^{j-1})$. Then the complex

$$0 \to K^0 \to K^1 \to \cdots \to K^{j-1} \to K^j \to Q \to 0$$

is acyclic. Therefore $\mathrm{proj.dim}(Q) \leqslant j$. As $Q \neq 0$ since $H^j(K) \subset Q$, by Auslander-Buchsbaum's formula [3] ([21], 0_{IV} 17.3.4) in this case), this gives $\mathrm{depth}(Q) \geqslant d - j$. Thus, by ([21], 0_{IV} 16.4.6.2), for any associated prime $\mathfrak{p}$ of Q, we have $\dim(A/\mathfrak{p}) \geqslant d - j$. Then the same holds for any associated prime $\mathfrak{p}$ of $H^j(K)$.

Proof of 5.3 (*loc. cit.*). (a) As X/k is lci, by ([23], III 3.2.6) we have

$$\mathrm{perf.amp}(L\Omega^1_{X/k}) \subset [-1, 0].$$

Hence, as $L\Omega^i_{X/k} = L\Lambda^i L\Omega^1_{X/k}$, by ([23], I 4.2.2.5) we have

$$\mathrm{perf.amp}(L\Omega^i_{X/k}) \subset [-i, 0].$$

(b) The first assertion follows from (a) by induction on i via (5.2.2) – (5.2.4). Let us prove the second one. Fix $i > d$. The question is local, so we may assume that X is affine. Choose a finite surjective morphism $\pi : X' \to Y := \mathbf{A}^d_k$. As X (resp. X') is lci, X (resp. X') is Cohen-Macaulay. Therefore, by ([21], 0_{IV} 17.3.5 (ii)), both $\pi : X' \to Y$ and $\pi F : X \to Y$ are finite, locally free. As $\mathrm{perf.amp}(L\Omega^i_{X/k}[-i]) \subset [0, i]$, we thus have, for all i,

$$(i) \qquad \mathrm{perf.amp}(\pi_* L\Omega^i_{X/k}[-i]) \subset [0, i].$$

in other words,

$$(ii) \qquad \mathrm{perf.amp}(\pi'_*(F_* L\Omega^i_{X/k})[-i]) \subset [0, i].$$

Let us now prove, by induction on i that

$$(iii) \qquad \mathrm{perf.amp}(\pi'_* LZ\Omega^i_{X/k}[-i]) \subset [0, i],$$

(iv) $$\mathrm{perf.amp}(\pi'_* LB\Omega^i_{X/k}[-i]) \subset [0,i].$$

for all i. For $i=0$, $LB\Omega^i_{X/k}=0$. By (5.2.4), $LZ\Omega^0_{X/k} \xrightarrow{\sim} \mathcal{O}_{X'}$, hence $\pi'_* LZ\Omega^0_{X/k}$ is finite and flat, i.e., of perfect amplitude in $[0]$. Fix $j \geqslant 0$. Assume that (iii) and (iv) have been proved for $i \leqslant j$, and let us prove them for $i = j+1$. As $\pi'_* LZ\Omega^j_{X/k}$ and $\pi'_*(F_* L\Omega^j_{X/k})$ are of perfect amplitude in $[-j,0]$, by applying π'_* to (5.2.1) we get $\mathrm{perf.amp}(\pi'_* LB\Omega^{j+1}_{X/k}) \subset [-j-1,0]$, i.e., (iv). By (a) applied to X', we have $\mathrm{perf.amp}(L\Omega^{j+1}_{X'/k}) \subset [-j-1,0]$, hence $\mathrm{perf.amp}(\pi'_* L\Omega^{j+1}_{X'/k}) \subset [-j-1,0]$. Therefore, by (iv) and π'_* applied to (5.2.2) and (5.2.3) (for $i=j+1$) we get $\mathrm{perf.amp}(\pi'_* LZ\Omega^{j+1}_{X/k}) \subset [-j-1,0]$, i.e. (iii).

Now, fix $i > d$, and let K be any of the complexes

$$F_* L\Omega^i_{X/k}[-i],\ LZ\Omega^i_{X/k}[-i],\ LB\Omega^i_{X/k}[-i].$$

Assume that $K \neq 0$, and let j be the minimum of the integers r such that $H^r(K) \neq 0$. As, by (ii) – (iv), $\pi'_* K$ is of perfect amplitude in $[0,+\infty)$, by 5.4, any irreducible component of $\mathrm{Supp}(H^j(\pi'_* K))$, hence of $\mathrm{Supp}(H^j(K))$ is of dimension $\geqslant d-j$. But, if $S := \mathrm{Sing}(X')$ is the singular locus of X', $K|(X'-S) = 0$ as $i > d$, hence $\mathrm{Supp}(H^j(K)) \subset S$, and therefore $d - j \leqslant s$, i.e., $j \geqslant d-s$, in other words, $K \in \mathcal{D}^{\geqslant d-s}(X')$. The last assertion is trivial.

5.5. Let X be a k-scheme. In order to exploit 5.3, it will be convenient to use the *Hodge completed* derived de Rham complex

(5.5.1) $$L\widehat{\Omega}^\bullet_{X/k} := R\varprojlim_{n\geqslant 0}(L\Omega^\bullet_{X/k}/\mathrm{Fil}^n_{\mathrm{Hdg}}).$$

It inherits the Hodge filtration of $L\Omega^\bullet_{X/k}$,

(5.5.2) $$\mathrm{Fil}^i_{\mathrm{Hdg}} L\widehat{\Omega}^\bullet_{X/k} = R\varprojlim_{n\geqslant 0}(\mathrm{Fil}^i_{\mathrm{Hdg}} L\Omega^\bullet_{X/k}/\mathrm{Fil}^{i+n}_{\mathrm{Hdg}} L\Omega^\bullet_{X/k}),$$

with associated graded

$$\mathrm{gr}^i_{\mathrm{Hdg}} L\widehat{\Omega}^\bullet_{X/k} \xrightarrow{\sim} L\Omega^i_{X/k}[-i].$$

Again, we will omit the subscript Hdg when no confusion can arise. We will view $F_* L\widehat{\Omega}^\bullet_{X/k}$, with its filtration $F_*\mathrm{Fil}^i$, as a filtered object of $\mathcal{D}(X')$. It also inherits the conjugate filtration

(5.5.3) $$\mathrm{Fil}^{\mathrm{conj}}_i F_* L\widehat{\Omega}^\bullet_{X/k} := R\varprojlim_{n>0} \mathrm{Fil}^{\mathrm{conj}}_i (L\Omega^\bullet_{X/k}/\mathrm{Fil}^{i+1+n}_{\mathrm{Hdg}}),$$

deduced from

$$\mathrm{Fil}_i^{\mathrm{conj}} F_*(L\Omega^\bullet_{R/k}/\mathrm{Fil}^{i+1+n}_{\mathrm{Hdge}}) := \mathrm{Tot}(\tau_{\leqslant i} F_*(\Omega^\bullet_{P_\bullet/k}/\Omega^{\geqslant i+1+n}_{P_\bullet/k})),$$

in the notation of 1.7, (2.2.1), but as the transition maps in the projective system on the right hand side of (5.5.3) are (trivially) isomorphisms, we have

$$\mathrm{Fil}_i^{\mathrm{conj}} F_* L\widehat{\Omega}^\bullet_{X/k} \xrightarrow{\sim} \mathrm{Fil}_i^{\mathrm{conj}} F_* L\Omega^\bullet_{X/k}.$$

Corollary 5.6. Under the assumptions and with the notation of 5.3, consider the canonical maps (in $\mathcal{D}(X')$)

$$\alpha : F_* L\widehat{\Omega}^\bullet_{X/k} \to F_*(L\widehat{\Omega}^\bullet_{X/k}/\mathrm{Fil}^{d+1})$$

and

$$\beta : \mathrm{Fil}_d^{\mathrm{conj}} F_* L\Omega^\bullet_{X/k} \to F_*(L\widehat{\Omega}^\bullet_{X/k}/\mathrm{Fil}^{d+1})$$

(cf. ([12], proofs of 0.1 and 3.1)). Then Cone(α) and Cone(β) belong to $\mathcal{D}^{\geqslant d-s-1}(X')$.

Proof. We have

$$\mathrm{Cone}(\alpha) \xrightarrow{\sim} F_*\mathrm{Fil}^{d+1}[1].$$

By 5.3 (b), $F_* L\Omega^i_{X/k}[-i+1] \in \mathcal{D}^{d-s-1}(X')$ for all $i \geqslant d+1$. By (5.5.2) this implies $F_*\mathrm{Fil}^{d+1}[1] \in \mathcal{D}^{d-s-1}(X')$, hence the assertion for α. By (5.2.5) we have

$$\mathrm{Cone}(\beta) \xrightarrow{\sim} LB\Omega^{d+1}_{X/k}[-d].$$

By 5.3 (b), we have $LB\Omega^{d+1}_{X/k}[-d] \in D^{d-s-1}(X')$, which implies the assertion for β.

6. Partial degeneration theorems

Theorem 6.1. Under the hypotheses and with the notation of 5.3, assume moreover that X/k is proper.

(a) For all n and all i, $H^n(X, L\Omega^i_{X/k}[-i])$ is of finite dimension over k.

(b) Assume in addition that $d < p$ and X is liftable to W_2. Then for all $n < d - s - 1$,

$$\dim_k H^n(X, L\widehat{\Omega}^\bullet_{X/k}) = \sum_{0 \leqslant i \leqslant d} \dim_k H^n(X, L\Omega^i_{X/k}[-i]), \tag{6.1.1}$$

and

$$H^n(X, L\Omega^i_{X/k}[-i]) = 0 \tag{6.1.2}$$

for $i > d$.

Proof. As X/k is proper, (a) follows from 5.3 (a). Let us prove (b). By 5.6, $R\Gamma(X', \mathrm{Cone}(\alpha))$ and $R\Gamma(X', \mathrm{Cone}(\beta))$ belong to $D^{\geqslant d-s-1}(k)$. Therefore $H^n(X', \alpha)$ and $H^n(X', \beta)$ are isomorphisms, hence we have a composite isomorphism

$$\alpha^{-1}\beta : H^n(X', \mathrm{Fil}_d^{\mathrm{conj}} F_* L\Omega^\bullet_{X/k}) \xrightarrow{\sim} H^n(X', F_* L\widehat{\Omega}^\bullet_{X/k}).$$

As $d < p$ and X is liftable to W_2, we have the decomposition isomorphism (4.8.1) (depending on the choice of a lifting)

$$c : \oplus_{0 \leqslant i \leqslant d} L\Omega^i_{X'/k}[-i] \xrightarrow{\sim} \mathrm{Fil}_d^{\mathrm{conj}} F_* L\Omega^\bullet_{X/k}.$$

Composing $H^n(c)$ with $\alpha^{-1}\beta$ we get an isomorphism

$$\oplus_{0 \leqslant i \leqslant d} H^n(X', L\Omega^i_{X'/k}[-i]) \xrightarrow{\sim} H^n(X', F_* L\widehat{\Omega}^\bullet_{X/k}),$$

which gives (6.1.1), as $H^n(X', F_* L\widehat{\Omega}^\bullet_{X/k}) = H^n(X, L\widehat{\Omega}^\bullet_{X/k})$ and

$$\dim_k H^n(X', L\Omega^i_{X'/k}[-i]) = \dim_k H^n(X, L\Omega^i_{X/k}[-i]).$$

As $F_* L\Omega^i_{X/k}[-i]$ is in $\mathcal{D}^{\geqslant d-s}(X')$ for $i > d$, and $n < d - s - 1$ ($n < d - s$ would actually suffice), we have

$$H^n(X', F_* L\Omega^i_{X/k}[-i]) = H^n(X, L\Omega^i_{X/k}[-i]) = 0,$$

which proves (6.1.2).

6.2. In 6.1 (b) assume that X/k is smooth. Then $s = -\infty$, hence $d - s - 1 = +\infty$, and (6.1.1) holds for all n. On the other hand, $L\widehat{\Omega}^\bullet_{X/k} \xrightarrow{\sim} L\Omega^\bullet_{X/k} \xrightarrow{\sim} \Omega^\bullet_{X/k}$, and $H^{n-i}(X, \Omega^i_{X/k}) = 0$ for $i > n$. Thus (6.1.1) reads

$$\dim_k H^n(X, \Omega^\bullet_{X/k}) = \sum_{0 \leqslant i \leqslant n} \dim_k H^{n-i}(X, \Omega^i_{X/k}),$$

i.e., the Hodge to de Rham spectral sequence of X/k degenerates at E_1. This is ([18], 2.4) for $d < p$.

6.3. Let K be a field of characteristic zero, and let X be a K-scheme of finite type. One defines the *Hodge completed* derived de Rham complex

$$L\widehat{\Omega}^\bullet_{X/K} := R\varprojlim_{n \geqslant 0} (L\Omega^\bullet_{X/K} / \mathrm{Fil}^n_{\mathrm{Hdg}}). \tag{6.3.1}$$

with its Hodge filtration $\mathrm{Fil}^i_{\mathrm{Hdg}} L\Omega^\bullet_{X/K}$ as in (5.5.1), (5.5.2). It is an object of $\mathcal{D}(X,K)$. Bhatt proved that, if $K = \mathbf{C}$, its hypercohomology calculates the Betti cohomology of X, i.e., there exists a canonical isomorphism

$$R\Gamma(X, L\widehat{\Omega}^\bullet_{X/\mathbf{C}}) \xrightarrow{\sim} R\Gamma(X^{\mathrm{an}}, \mathbf{C}), \tag{6.3.2}$$

where X^{an} is the analytic space associated to $X(\mathbf{C})$. For an arbitrary field K of characteristic zero, he constructed, more generally, isomorphisms in $\mathcal{D}(X,K)$ ([7], Prop. 5.2):

$$L\widehat{\Omega}^\bullet_{X/K} \xrightarrow{\sim} Ru_*\widehat{\mathcal{O}}_{X/K} \xrightarrow{\sim} R\varepsilon_*\Omega^\bullet_{Y_\bullet/K}, \tag{6.3.3}$$

where: $u : (X/K)_{\mathrm{crys}} \to X_{\mathrm{Zar}}$ is the canonical morphism from the crystalline topos of X to the Zariski topos, and $\mathcal{O}_{X/K}$ is the structural sheaf, $Ru_*\widehat{\mathcal{O}}_{X/K} := R\varprojlim_n \mathcal{O}_{X/K}/J^n_{X/K}$ (with $J_{X/K} := \mathrm{Ker}(\mathcal{O}_{X/K} \to \mathcal{O}_X)$), and $\varepsilon : Y_\bullet \to X$ is a proper hypercovering, with Y_n/K smooth for all n. For $K = \mathbf{C}$, the comparison map

$$R\varepsilon_*\Omega^\bullet_{Y_\bullet/\mathbf{C}} \to R\varepsilon^{\mathrm{an}}_*\Omega^\bullet_{Y^{\mathrm{an}}_\bullet/\mathbf{C}}$$

is an isomorphism by Grothendieck's theorem [22], hence (6.3.2) follows from (6.3.3) by the Poincaré lemma and cohomological descent. The last term in (6.3.3) underlies the filtered *du Bois complex* [19], however (6.3.3) is an isomorphism only on the underlying unfiltered objects.

Bhatt deduced from 6.1 the following theorem:

Theorem 6.4. ([12], Th. 0.1) Let K be a field of characteristic zero, and let X be a proper, lci K-scheme of pure dimension d. Let s be the dimension of its singular locus. Then, for $n < d - s - 1$, we have

$$H^n(X, L\Omega^i_{X/K}[-i]) = 0 \tag{6.4.1}$$

for all $i > d$, and

$$\dim_K H^n(X, L\widehat{\Omega}^\bullet_{X/K}) = \sum_{0 \leqslant i \leqslant d} \dim_K H^n(X, L\Omega^i_{X/K}[-i]). \tag{6.4.2}$$

We need a variant of 5.3:

Lemma 6.5. Let X be an lci K-scheme of finite type, of pure dimension d, with singular locus of dimension s. Then:

(a) The complex $L\Omega^i_{X/K}[-i]$ is of perfect amplitude in $[0, i]$.

(b) For $i > d$, $L\Omega^i_{X/K}[-i]$ belongs to $\mathcal{D}^{\geqslant d-s}(X, \mathcal{O}_X)$, and is zero if X/K is smooth.

Proof. The proof of (a) is the same as that of (5.3 (a)). For (b), as in the proof of (5.3 (b)), we may assume X affine and choose a finite surjective morphism $\pi : X \to Y := \mathbf{A}^d_K$. Like in *loc. cit.* π is automatically finite, locally free, so that, by (a), $\pi_* L\Omega^i_{X/K}[-i]$ is of perfect amplitude in $[0, +\infty)$. As in *loc. cit.* one finishes the proof using 5.4.

Proof of 6.4. (6.4.1) follows from (6.5 (b)) (and in fact $n < d - s$ would suffice). As in 5.6, consider the canonical map

$$\alpha : L\widehat{\Omega}^\bullet_{X/K} \to L\widehat{\Omega}^\bullet_{X/K}/\mathrm{Fil}^{d+1} (\xrightarrow{\sim} L\Omega^\bullet_{X/K}/\mathrm{Fil}^{d+1})$$

As in *loc. cit.*, it follows from (6.5 (b)) that $\mathrm{Cone}(\alpha)$ belongs to $\mathcal{D}^{\geqslant d-s-1}(X, K)$. Thus, for $n < d - s - 1$,

$$H^n(X, \alpha) : H^n(X, L\widehat{\Omega}^\bullet_{X/K}) \to H^n(X, L\Omega^\bullet_{X/K}/\mathrm{Fil}^{d+1})$$

is an isomorphism. Therefore (6.4.2) is equivalent to

$$\dim_K H^n(X, L\Omega^\bullet_{X/K}/\mathrm{Fil}^{d+1}) = \sum_{0 \leqslant i \leqslant d} \dim_K H^n(X, L\Omega^i_{X/K}[-i]). \tag{6.4.3}$$

One proves (6.4.3) by reducing to (6.1 (b)) by standard spreading out techniques. Imitating ([18], proof of 2.7), choose a domain A of finite type over $\mathbf{Z}$, a proper, syntomic (i.e., lci and flat) morphism $f : \mathcal{X} \to \mathrm{Spec}(A)$, of pure relative dimension d, whose geometric fibers have a singular locus of dimension s, and a homomorphism $A \to K$ such that $X = \mathcal{X} \otimes_A K$. Then we have

$$L\Omega^\bullet_{\mathcal{X}/A}/\mathrm{Fil}^{d+1} \otimes^L_A K \xrightarrow{\sim} L\Omega^\bullet_{X/K}/\mathrm{Fil}^{d+1}, \tag{$*$}$$

$$L\Omega^i_{\mathcal{X}/A}[-i] \otimes^L_A K \xrightarrow{\sim} L\Omega^i_{X/K}[-i]. \tag{$**$}$$

As f is lci, the same argument as for 5.3 (a) shows that $L\Omega^i_{\mathcal{X}/A}[-i]$ is of perfect amplitude in $[0, i]$. Therefore $L\Omega^\bullet_{\mathcal{X}/A}/\mathrm{Fil}^{d+1}$ is a finite successive extension of perfect complexes on $\mathcal{X}$, so up to replacing A by $A[a^{-1}]$ for a suitable $a \in A$, we may assume that $R^n f_* L\Omega^\bullet_{\mathcal{X}/A}/\mathrm{Fil}^{d+1}$ and $R^n f_* L\Omega^i_{\mathcal{X}/A}[-i]$ are locally free of finite type over A, of formation compatible with any base change. Proceeding as in ([18], *loc. cit.*), one takes a closed point s of the schematic closure T, in $\mathrm{Spec}(A)$, of $\mathrm{Spec}(A \otimes \mathbf{Q})$, such that T is étale over $\mathbf{Z}$ at s, and $p = \mathrm{char}(k(s)) > d$. Let $\mathcal{O}_s$ be the local ring of T at s. Then $\mathcal{X} \otimes_A \mathcal{O}_s/\mathfrak{m}_s^2$ is a lifting of $\mathcal{X}_s$ to $\mathcal{O}_s/\mathfrak{m}^2 = W_2(k)$, where $k = k(s)$. By (*), (**), and the assumptions made above on f, we have

$$\dim_k H^n(\mathcal{X}_s, L\Omega^\bullet_{\mathcal{X}_s/k}/\mathrm{Fil}^{d+1}) = \dim_K H^n(X, L\Omega^\bullet_{X/K}/\mathrm{Fil}^{d+1}),$$

$$\dim_k H^n(\mathcal{X}_s, L\Omega^i_{\mathcal{X}_s/k}[-i]) = \dim_K H^n(X, L\Omega^i_{X/K}[-i]),$$

so it remains to show

$$(***) \qquad \dim_k H^n(\mathcal{X}_s, L\Omega^\bullet_{\mathcal{X}_s/k}/\mathrm{Fil}^{d+1}) = \sum_{0\leqslant i\leqslant d} \dim_k H^n(\mathcal{X}_s, L\Omega^i_{\mathcal{X}_s/k}[-i]).$$

By 5.6,

$$H^n(\beta) : H^n(\mathcal{X}'_s, \mathrm{Fil}^{\mathrm{conj}}_d F_* L\Omega^\bullet_{\mathcal{X}_s/k}) \to H^n(\mathcal{X}'_s, F_*(L\Omega^\bullet_{\mathcal{X}_s/k}/\mathrm{Fil}^{d+1}))$$

is an isomorphism. By the decomposition isomorphism (4.8.1), we have

$$\oplus_{0\leqslant i\leqslant d} H^n(\mathcal{X}'_s, L\Omega^i_{\mathcal{X}'_s/k}[-i]) \xrightarrow{\sim} H^n(\mathcal{X}'_s, \mathrm{Fil}^{\mathrm{conj}}_d F_* L\Omega^\bullet_{\mathcal{X}_s/k}).$$

The combination of these two isomorphisms yields (***), and finishes the proof.

7. Kodaira type vanishing theorems

Theorem 7.1. Let X/k satisfy the assumptions of 6.1 (b). Let L be an ample invertible sheaf on X. Then, for $n < \min(d, d-s-1)$ and all i,

$$(7.1.1) \qquad H^n(X, L\Omega^i_{X/k}[-i] \otimes L^{-1}) = 0.$$

We will need the following lemmas.

Lemma 7.2. Let X/k be a proper, lci k-scheme of pure dimension d, with singular locus of dimension s. There exists $N \geqslant 0$ such that for all $n < \min(d, d-s-1)$ and all i,

$$(7.2.1) \qquad H^n(X, L\Omega^i_{X/k}[-i] \otimes L^{\otimes -p^N}) = 0.$$

Proof. By 5.3 (b), for $i > d$, $L\Omega^i_{X/k}[-i] \in \mathcal{D}^{\geqslant d-s}(X, \mathcal{O}_X)$, hence the same is true of $L\Omega^i_{X/k}[-i] \otimes L^{\otimes -p^N}$, so the left hand side of (7.2.1) vanishes for $i > d$ and any N. We may therefore limit ourselves to ensuring (7.2.1) for $0 \leqslant i \leqslant d$.

Let $(L\Omega^i_{X/k})^\vee := R\mathcal{H}om(L\Omega^i_{X/k}, \mathcal{O}_X)$. As $L\Omega^i_{X/k}$ is of perfect amplitude in $[-i, 0]$ (5.3 (a)), by ([5], I 7.1) $L\Omega^i_{X/k})^\vee$ is of perfect amplitude in $[0, i]$. Let ω_X be the dualizing sheaf on X (an invertible sheaf, such that $\omega_X[d] \xrightarrow{\sim} a^! k$, where $a : X \to \mathrm{Spec}(k)$ is the projection). Let $0 \leqslant i \leqslant d$, $0 \leqslant n < \min(d, d-s-1)$, and let $j := n - i$. Let $N \geqslant 0$. By Grothendieck's duality, $H^j(X, L\Omega^i_{X/k} \otimes L^{\otimes -p^N})$ is dual over k to $H^{d-j}(X, (L\Omega^i_{X/k})^\vee \otimes L^{\otimes p^N} \otimes \omega_X)$. As $n < d$, hence $d - j > i$, and $L\Omega^i_{X/k})^\vee$ has coherent cohomology sheaves contained in the interval $[0, i]$, by Serre's vanishing theorem, there exists an $N = N(n,i) \geqslant 0$ such that $H^{d-j}(X, (L\Omega^i_{X/k})^\vee \otimes L^{\otimes p^N} \otimes \omega_X) = 0$ for all $N \geqslant N(n,i)$. Taking for N the maximum of the $N(n,i)$ for $0 \leqslant i \leqslant d$ and $0 \leqslant n < \min(d, d-s-1)$, we have $H^{d-j}(X, (L\Omega^i_{X/k})^\vee \otimes L^{\otimes p^N} \otimes \omega_X) = 0$ for $0 \leqslant i \leqslant d$ and $0 \leqslant n < \min(d, d-s-1)$, hence by duality, $H^n(X, L\Omega^i_{X/k}[-i] \otimes L^{\otimes -p^N}) = 0$.

The next lemma is a variant of Raynaud's lemma ([18], 2.9):

Lemma 7.3. Under the assumptions of 7.1, let M be an invertible sheaf on X. Assume that

$$H^n(X, L\Omega^i_{X/k}[-i] \otimes M^{\otimes p}) = 0$$

for $n < \min(d, d-s-1)$ and $i \leqslant d$, then

$$H^n(X, L\Omega^i_{X/k}[-i] \otimes M) = 0$$

for $n < \min(d, d-s-1)$ and $i \leqslant d$.

Proof. By the projection formula, we have

$$H^n(X, L\Omega^i_{X/k}[-i] \otimes M^{\otimes p}) \xrightarrow{\sim} H^n(X', F_* L\Omega^i_{X/k}[-i] \otimes M'), \tag{7.3.1}$$

where M' is the pull-back of M on X', so that $M^{\otimes p} = F^* M'$. Consider the spectral sequence of $R\Gamma(X', -)$ applied to $F_*(L\Omega^\bullet_{X/k}/\mathrm{Fil}^{d+1}) \otimes M'$, filtered by $F_*(\mathrm{Fil}^i L\Omega^\bullet_{X/k}/\mathrm{Fil}^{d+1}) \otimes M'$, whose E_1 term is

$$E_1^{ij} = H^{i+j}(X', L\Omega^i_{X'/k}[-i] \otimes M')$$

for $i \leqslant d$, and $E_1^{ij} = 0$ for $i > d$. By (7.3.1), the assumption implies

$$H^n(X', F_*(L\Omega^\bullet_{X/k}/\mathrm{Fil}^{d+1}) \otimes M') = 0 \tag{7.3.2}$$

for $n < \min(d, d-s-1)$. Consider the map β of 5.6, and

$$\beta \otimes M' : \mathrm{Fil}^{\mathrm{conj}}_d F_* L\Omega^\bullet_{X/k} \otimes M' \to F_*(L\Omega^\bullet_{X/k}/\mathrm{Fil}^{d+1}) \otimes M'.$$

Assume first that $s \geqslant 0$, so $\min(d, d-s-1) = d-s-1$. As $\mathrm{Cone}(\beta)$, hence $\mathrm{Cone}(\beta \otimes M')$, is in $\mathcal{D}^{\geqslant d-s-1}(X')$, $H^n(X', \beta \otimes M')$ is an isomorphism for $n < d-s-1$. As in the proof of 6.1, we now use the decomposition isomorphism c, which gives

$$c \otimes M' : \otimes_{0 \leqslant i \leqslant d} L\Omega^i_{X'/k}[-i] \otimes M' \xrightarrow{\sim} \mathrm{Fil}^{\mathrm{conj}}_d F_* L\Omega^\bullet_{X/k} \otimes M'.$$

Thus, for $n < d - s - 1$, composing $H^n(X', \beta \otimes M')$ and $H^n(X', c \otimes M')$ we get an isomorphism

$$\oplus_{0 \leqslant i \leqslant d} H^n(X', L\Omega^i_{X'/k}[-i] \otimes M') \xrightarrow{\sim} H^n(X', F_*(L\Omega^\bullet_{X/k}/\mathrm{Fil}^{d+1}) \otimes M'),$$

hence, by (7.3.2), we have

$$H^n(X', L\Omega^i_{X'/k}[-i] \otimes M') = 0$$

for $0 \leqslant i \leqslant d$, but this is equivalent to $H^n(X, L\Omega^i_{X/k}[-i] \otimes M) = 0$. If $s = -\infty$, i.e., X/k is smooth, and $\min(d, d-s-1) = d$, then $L\Omega^\bullet_{X/k} = \Omega^\bullet_{X/k}$, $\mathrm{Fil}^{d+1} = 0$, $\beta \otimes M'$ is an isomorphism, and the preceding argument gives the desired vanishing for $n < d$ (in this case, the proof is exactly that of ([18], 2.9)).

Proof of 7.1. By 7.2 choose $N \geqslant 0$ such that (7.1.1) holds (for all $n < \min(d, d-s-1)$ and all $i \leqslant d$). If $N = 0$, we are done. If not, by 7.3 we can replace N by $N - 1$. Iterating, we arrive at $N = 0$, which finishes the proof.

7.4. Assume that, in 7.1, X/k is smooth. Then $\min(d, d-s-1) = d$, and (7.1.1) gives ([18], (2.8.2)) for $d < p$. As observed in ([14], 3.4), the Serre dual formulation ([18], (2.8.1)) fails when X is singular (even if $s = 0$).

By spreading out arguments similar to those used in the proof of 6.4 we deduce:

Theorem 7.5. Let K be a field of characteristic zero. Let X be a proper, lci K-scheme of pure dimension d, with singular locus of dimension s. Let L be an ample invertible sheaf on X. Then, for $n < \min(d, d-s-1)$ and all i,

$$H^n(X, L\Omega^i_{X/K}[-i] \otimes L^{-1}) = 0. \tag{7.5.1}$$

7.6. This is essentially ([14], Th. 3.2), except that we need $d - s - 1$ instead of $d - s$. I don't know how to recover *loc. cit.* by characteristic p methods.

Acknowledgements. I thank Bhargav Bhatt for illuminating discussions and correspondence, and communicating to me [12], whose results are crucial in sections 6 and 7. I thank Kęstutis Česnavičius for comments on an early version. Parts of this paper were written while I was visiting the Weizmann Institute in Rehovot, the Einstein Institute at the Hebraic University of Jerusalem, the Morningside Center of Mathematics and the AMSS in Beijing, and the Chennai Mathematical Institute. I thank these institutions for their generous support and stimulating atmosphere.

References

[1] Anschütz, Johannes; Le Bras, Arthur-César. *Prismatic Dieudonné theory.* arXiv:1907.10525v1, 2019.

[2] Artin, Michael; Grothendieck, Alexandre; Verdier, Jean-Louis. *Séminaire de Géométrie Algébrique du Bois Marie - 1963-64 - Théorie des topos et cohomologie étale des schémas* (SGA 4), vol. 3, Berlin, New York, Springer-Verlag, Lecture Notes in Mathematics 305, 1972.

[3] Auslander, Maurice; Buchsbaum, David A. *Homological dimension in local rings.* Transactions of the American Mathematical Society, 85: 390–405, 1957.

[4] Avramov, Luchezar L. *Locally complete intersection homomorphisms and a conjecture of Quillen on the vanishing of cotangent homology.* Ann. of Math. (2) 150 (1999), no. 2, 455–487.

[5] Berthelot, Pierre; Grothendieck, Alexandre; Illusie, Luc. *Séminaire de Géométrie Algébrique du Bois Marie - 1966-67 - Théorie des intersections et théorème de Riemann-Roch* (SGA 6), Berlin, New York, Springer-Verlag, Lecture Notes in Mathematics 225, 1971.

[6] Bhatt, Bhargav. *p-adic derived de Rham cohomology.* arXiv:1204.6560v1, 2012.

[7] Bhatt, Bhargav. *Completions and derived de Rham cohomology.* arXiv:1207.6193v1, 2012.

[8] Bhatt, Bhargav. *Torsion in the crystalline cohomology of singular varieties.* Doc. Math. 19 (2014), 673–687.

[9] Bhatt, Bhargav; Morrow, Matthew; Scholze, Peter. Bhatt, Bhargav; Morrow, Matthew; Scholze, Peter. *Topological Hochschild homology and integral p-adic Hodge theory.* Publ. Math. Inst. Hautes Études Sci. 129 (2019), 199–310.

[10] Bhatt, Bhargav; Lurie, Jacob; Mathew, Akhil. *Revisiting the de Rham-Witt complex.* arXiv:1805.05501v2 (2018).

[11] Bhatt, Bhargav. *Prismatic cohomology*, http://www-personal.umich.edu/ bhattb/teaching/prismatic-columbia/ 2018.

[12] Bhatt, Bhargav. *Partial degeneration of the derived Hodge-to-de Rham spectral sequence for lci varieties.* preprint, 2019.

[13] Bhatt, Bhargav, Scholze, Peter. *Prisms and Prismatic Cohomology*, preprint, 2019, arXiv:1905.08229v2.

[14] Bhatt, Bhargav; Blickle, Manuel; Lyubeznik, Gennady; Singh, Anurag K.; Zhang, Wenliang. *Stabilization of the cohomology of thickenings.* Amer. J. Math. 141 (2019), no. 2, 531–561.

[15] Charles De Clercq, Mathieu Florence, and Giancarlo Lucchini Arteche. *Lifting vector bundles to Witt vector bundles.* arXiv:1807.04859v3, 2018.

[16] Deligne, Pierre. *Notes sur le complexe cotangent.* Unpublished.

[17] Deligne, P.; Lusztig, G. *Representations of reductive groups over finite fields.* Ann. of Math. (2) 103 (1976), no. 1, 103–161.

[18] Deligne, Pierre; Illusie, Luc. *Relèvements modulo p^2 et décomposition du complexe de de Rham.* Invent. Math. 89 (1987), no. 2, 247–270.

[19] Du Bois, Philippe. *Complexe de de Rham filtré d'une variété singulière.* Bull. Soc. Math. France 109 (1981), no. 1, 41–81.

[20] Giraud, Jean. *Cohomologie non abélienne.* Grundlehren der mathematischen Wissenschaften, Vol. 179. Berlin-Heidelberg-New York: Springer 1971.

[21] Grothendieck, A. *Éléments de géométrie algébrique IV. Étude locale des schémas et des morphismes de schémas, Première partie.*, rédigés avec la collaboration de J. Dieudonné. Pub. Math. IHES 20, 1964.

[22] Grothendieck, Alexander. *On the de Rham cohomology of algebraic varieties.* Inst. Hautes Études Sci. Publ. Math., (29):95–103, 1966.

[23] Illusie, Luc. *Complexe cotangent et déformations.* I. Lecture Notes in Mathematics, Vol. 239. Springer-Verlag, Berlin-New York, 1971.

[24] Illusie, Luc. *Complexe cotangent et déformations. II.* (French) Lecture Notes in Mathematics, Vol. 283. Springer-Verlag, Berlin-New York, 1972.

[25] Illusie, Luc. *Complexe de de Rham-Witt et cohomologie cristalline.* Ann. Sci. École Norm. Sup. (4) 12 (1979), no. 4, 501–661.

[26] Kato, Kazuya. *Logarithmic structures of Fontaine-Illusie.* Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), 191–224, Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[27] Kato, Kazuya; Saito, Takeshi. *On the conductor formula of Bloch.* Publ. Math. Inst. Hautes Études Sci. No. 100 (2004), 5–151.

[28] Nygaard, Niels O. *Slopes of powers of Frobenius on crystalline cohomology.* Ann. Sci. École Norm. Sup. (4) 14 (1981), no. 4, 369–401.

[29] Olsson, Martin. *Deformation theory of representable morphisms of algebraic stacks.* Math. Z. 253 (2006), no. 1, 25–62.

[30] Quillen, Daniel. *On the (co-) homology of commutative rings.* 1970 Applications of Categorical Algebra (Proc. Sympos. Pure Math., Vol. XVII, New York, 1968) pp. 65–87. Amer. Math. Soc., Providence, R.I.

[31] Roby, Norbert. *Lois polynomes et lois formelles en théorie des modules.* Ann. Sci. École Norm. Sup. (3) 80 1963 213–348.

[32] Schröer, Stefan. *The Deligne-Illusie theorem and exceptional Enriques surfaces.* Preprint, 2019.

[33] Sheng, Mao; Shentu, Junchao. *Some examples of DR-indecomposable special fibers of semi-stable reductions over Witt rings.* Preprint, 2019.