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1. The old result

Theorem 1 (DI, 1987). $k$ perfect field, char. $p > 0$, $X/k$ smooth. Let $X' = X \otimes_k (k, F_k)$, and $F : X \to X' = \text{relative Frobenius}$. 

Smooth liftings of $X$ to $W_2(k)$ correspond to decompositions

\begin{equation}
\mathcal{O}_{X'} \oplus \Omega^1_{X'/k}[-1] \sim \tau^{\leq 1} F_* \Omega^\bullet_{X/k}
\end{equation}

in $D(X', \mathcal{O}_{X'})$, inducing $C^{-1}$ (Cartier isomorphism) on $H^i$.

Gives an affine bijection on isomorphism classes of objects, inducing identity on translation group $H^1(X', T_{X'})$.

Moreover, any decomposition (1.1) uniquely extends multiplicatively to a decomposition

\begin{equation}
\bigoplus_{i \leq p-1} \Omega^i_{X'/k}[-i] \sim \tau^{\leq p-1} F_* \Omega^\bullet_{X/k}
\end{equation}

inducing $C^{-1}$ on $H^i$. 
Idea of proof

- **local liftings** of $X'$ to $W_2(k)$: a gerbe on $X'$

$$\text{Lift}(X'/W_2),$$

banded by $T_{X'}$ (sheaf of automorphisms of any object).

- **local splittings** of $\tau_{\leq 1} F_* \Omega^\bullet_{X/k}$ ($= \text{local sections of } F_* Z \Omega^1_{X/k} \to \Omega^1_{X'/k}$): a gerbe on $X'$

$$\text{Split}(\tau_{\leq 1} F_* \Omega^\bullet_{X/k}),$$

banded again by $T_{X'/k}$.

Using local liftings of $X'$ plus local liftings $\tilde{F}$ of $F$ (and associated $p^{-1} \tilde{F}^*$ on $\Omega^1$), can construct an equivalence of gerbes

$$\text{(1.3)} \quad \text{Lift}(X'/W_2) \sim \text{Split}(\tau_{\leq 1} F_* \Omega^\bullet_{X/k})$$

inducing identity on $T_{X'/k}$. (NB. more general (1.3) holds over bases $\mathbb{F}_p$ flatly lifted mod $p^2$.)
2. Another strategy

Local liftings of $X'$ to $W_2$ controlled by $\tau^{\geq -1}L_{X'/W_2}$
(NB. $X/k$ smooth $\Rightarrow L_{X/W} \sim \tau^{\geq -1}L_{X/W_2}$).

Goal: directly construct isomorphism in $D(X', \mathcal{O}_{X'})$

(2.1) $L_{X'/W}[-1] \sim \tau^{\leq 1}F_\ast \Omega^\bullet_{X'/k}$

inducing $C^{-1}$ on $H^1 = \Omega^1_{X'/k}$ and $H^0 = \mathcal{O}_{X'}$.

Basics on cotangent complex and deformations show that (2.1) implies the isomorphism

(1.3) $\text{Lift}(X'/W_2) \sim \text{Split}(\tau^{\leq 1}F_\ast \Omega^\bullet_{X/k})$
Proof:

Lift($X'/W_2$) = fiber at $1 \in \mathcal{O}_{X'}$ of map

(Picard stack associated to) $R\mathcal{H}om(L_{X'/W}, \mathcal{O}_{X'})[1] \to H^0$

• Split($\tau^{\leq 1}F_*\Omega^\bullet_{X/k}$) = fiber at $1 \in \mathcal{O}_{X'}$ of map

(Picard stack associated to) $R\mathcal{H}om(\tau^{\leq 1}F_*\Omega^\bullet_{X/k}, \mathcal{O}_{X'}) \to H^0$,

both stacks having $H^0 = \mathcal{O}_{X'}$ and $H^{-1} = T_{X'}$, $H^i = 0$ otherwise.)
Will deduce

\[(2.1) \quad L_{X'/W}[−1] \sim \tau_{≤1} F_* \Omega^*_{X/k}\]

from:

**Theorem 2** (I., 2019). There exists a **filtered isomorphism** (i.e., in $DF(X, W)$), with $\mathcal{O}$-linear associated graded:

\[(2.2) \quad L\Omega^*_{X/W}/\text{Fil}^p \sim W\Omega^*_{X}/\mathcal{N}^p,\]

where

$L\Omega^*$: derived de Rham complex

(if $X = \text{Spec}(R)$, $L\Omega^*_{R/W} := \text{Tot}(\Omega^*_{P_/W})$,
$P_* \to R$ a simplicial resolution by polynomial algebras over $W$.)

Fil$: Hodge filtration,

$W\Omega^*_{X}$: de Rham-Witt complex

$\mathcal{N}^i$: Nygaard filtration:
\[
\text{Fil}^i L\Omega^\bullet_{R/W} := \text{Tot}(\Omega^\geq_i_{P/\mathcal{W}})
\]

(2.3) \[\text{gr}^i_{\text{Fil}} = L\Omega^i_{X/W}[-i] := (L\Lambda^i L_{X/W})[-i]\]

(in particular \(\text{gr}^1 = L\Omega^1_{X/W}[-1] = L_{X/W}[-1]\)).

\[
\mathcal{N}^i W\Omega^n_X = p^{i-n-1} V W\Omega^n_X \\
\text{(for } n < i, \text{ and } \mathcal{N}^i W\Omega^n_X = W\Omega^n_X \text{ for } n \geq i \text{)} \text{ with }
\]

(2.4) \[\text{gr}^i_{\mathcal{N}} W\Omega^\bullet_{X'} = \tau^{\leq i} F_* \Omega^\bullet_{X/k}.
\]

(in particular \(\text{gr}^1 = \tau^{\leq 1} F_* \Omega^\bullet_{X/k}\)).
• graded piece of degree 1 of

\[(2.2) \quad L\Omega^\bullet_{X/W}/\text{Fil}^p \sim W\Omega^\bullet_X/\mathcal{N}^p,\]

plus formulas for gr\(^1\) imply (2.1), i.e.,

\[L_{X'/W}[-1] \sim \tau_{\leq 1} F_* \Omega^\bullet_{X/k} \]

• Any smooth lifting \(\widetilde{X}'\) of \(X'\) to \(W_2\) gives a decomposition

\[L_{X'/W}[-1] = \tau_{\geq 1} L_{X'/\widetilde{X}'}[-1] \oplus \Omega^1_{X'/k}[-1] = \mathcal{O}_{X'} \oplus \Omega^1_{X'/k}[-1],\]

hence, by applying \(L\Lambda^{p-1}\) and formulas for gr\(^{p-1}\), a decomposition

\[\bigoplus_{i \leq p-1} \Omega^i_{X'/k}[-i] \sim \tau_{\leq p-1} F_* \Omega^\bullet_{X/k}.\]
Remark. Not possible to remove $/\text{Fil}^p$ and $/\mathcal{N}^p$ from (2.2), because of Example (Bhatt):

$$L\Omega^\bullet_{k/W} = \widehat{W\langle x\rangle}/(x - p) = \widehat{W\langle y\rangle}/(y),$$

where $\widehat{(\quad)}$ means $p$-adic completion, $W\langle x\rangle = \text{divided power envelope of } W[x]$. 
Proof of Th. 2

Use (local) embeddings $X \hookrightarrow Z$, ideal $J$, $Z/W$ smooth.

Gives

\[(*) \quad LX/W[-1] = (J/J^2 \to O_X \otimes \Omega^1_{Z/W}), \]

from which one deduces a filtered isomorphism

\[(**\) \quad L\Omega^\bullet_{X/W}/\text{Fil}^p = \Omega^\bullet_{Z/W}/J^p\Omega^\bullet_{Z/W}, \]

\[J^r\Omega^\bullet_{Z/W} := (J^r \to \Omega^1_{Z/W} \to \cdots) \]

Key points:

\[\text{gr}^1_J\Omega^\bullet_{Z/W} = L\Omega^1_{X/W}[-1] := LX/W[-1]) \]

by \((*)\), and

\[L\Gamma^r(M[-1]) = L\Lambda^r[M][-r], \]

\[\Gamma^r = S^r \text{ for } r < p. \]
Additional Frobenius lift $F$ on $Z$ gives $(F,\text{Frobenius})$-compatible 

$$\mathcal{O}_Z \to \mathcal{W}\mathcal{O}_X,$$

sending $J$ to $\mathcal{W}\mathcal{O}_X$, hence filtered $(J,\mathcal{N})$-map 

$$\Omega^\bullet_{Z/W} \to \mathcal{W}\Omega^\bullet_X,$$

inducing a filtered quasi-isomorphism 

$$\Omega^\bullet_{Z/W}/J^p\Omega^\bullet_{Z/W} \sim \mathcal{W}\Omega^\bullet_X/N^p,$$

as checked locally by taking for $Z$ a lifting of $X$, and applying Nygaard’s formula for $\text{gr}^r$.

Conclude by applying 

\begin{equation}
(\ast\ast) \quad L\Omega^\bullet_{X/W}/\text{Fil}^p = \Omega^\bullet_{Z/W}/J^p\Omega^\bullet_{Z/W}.
\end{equation}
The singular case

By left Kan extension from finite polynomial rings over $k$, (2.2) extends to any scheme $X/k$, provided that $\mathcal{W}\Omega_X^\bullet$ is replaced by its derived variant $L\mathcal{W}\Omega_X^\bullet$:

\[(2.5) \quad L\Omega_{X/W}^\bullet/\text{Fil}^p \sim L\mathcal{W}\Omega_X^\bullet/\mathcal{N}^p\]

Again,

\[\text{gr}^i_{\text{Fil}} L\Omega_{X/W}^\bullet = L\Omega_X^\bullet/W[-i],\]

but

\[(2.6) \quad \text{gr}^i_{\mathcal{N}} L\mathcal{W}\Omega_X^\bullet = \text{Fil}^\text{conj}_i F_* L\Omega_X^\bullet/k,\]

where $\text{Fil}^\text{conj}_\bullet$ is the (increasing) conjugate filtration, with

\[(2.7) \quad L\Omega_{X'/k}^i[-i] \sim \text{gr}^i_{\text{conj}} F_* L\Omega_X^\bullet/k.\]
By left Kan extension from finite polynomial rings over $W_2$, any (flat) lifting $\tilde{X}'$ of $X'$ over $W_2$ gives a decomposition

$$(2.8) \quad \bigoplus_{i \leq p-1} L\Omega^i_{X'/k}[-i] \xrightarrow{\sim} \text{Fil}^{\text{conj}}_{p-1} F_* L\Omega^\bullet_{X/k}$$

generalizing (1.2). Indeed:

Key observation (A. Mathew): though, for $X/k$ not lci, $L_{X/W} \to \tau \geq -1 L_{X/W_2}$ no longer an isomorphism, still the datum of $\tilde{X}'$ gives functorial map $L_{X'/W} \to L_{X'/W_2} \to L_{X'/\tilde{X}'} \to \mathcal{O}_{X'}[1]$ splitting the triangle

$$\mathcal{O}_{X'}[1] \to L_{X'/W} \to L_{X'/k} \to .$$

For $X/k$ lci, (2.8) yields partial degeneration and vanishing theorems, both in char. $p$ and in char. 0, see section 4.
3. A prismatic generalization (after Bhatt et al.)

The RHS of the isomorphism

\[(3.0) = (2.2) \quad L\Omega_{X/W}^\bullet/Fil^p \sim W\Omega_{X/N}^\bullet,\]

can be re-written in terms of prismatic cohomology relative to the prism \((W = W(k), (p))\): one has a canonical isomorphism (special case of the prismatic-crystalline comparison theorem) (Bhatt-Scholze, Li-Liu)

\[(3.1) \quad W\Omega_{X}^\bullet \sim \varphi_W^* \Delta_{X/W},\]

where \(\varphi_W = \text{Frobenius of } W,\)

\[\Delta_{X/W} := R\nu^* \mathcal{O}(X/W)_\Delta,\]

\[\nu : (X/W)_\Delta \to X_{et}\]

the canonical map from the prismatic site to the étale one.

Isomorphism (3.1) is compatible with Nygaard filtrations on both sides.
More generally:

Let \((A, I)\) be a prism (examples: \((W, (p))\), \((W[[u]], (E(u)))\) (with \(W[[u]]/(E(u)) = \mathcal{O}_K\)), \((A_{\text{inf}}, \xi)\) (with \(A_{\text{inf}}/(\xi) = \mathcal{O}_C\)), \(\varphi_A : A \to A\) given by

\[
\varphi_A(x) = x^p + p\delta_A(x)
\]

Assume \((A, I)\) bounded (i.e. \((A/I)[p^{\infty}] = (A/I)[p^n]\) for some \(n\)).

Let \(X/(A/I)\) a smooth formal scheme.

Define

\[
\Delta_{X/A} := R\nu_* \mathcal{O}_{(X/A)_\Delta},
\]

\[
\nu : (X/A)_\Delta \to X_{\text{et}}
\]

the canonical map from the prismatic site to the étale one.

Then:
Theorem 3 (Li-Liu [LL, Th. 4.24]). There exists a (canonical) filtered isomorphism in the derived $\infty$-category $\mathcal{D}F(X, A)$:

\[
\varphi^*_A \Delta_{X/A} \otimes^L_A L\Omega^\bullet_{(A/I)/A} \sim L\Omega^\bullet_{X/A}
\]  

(derived tensor product and derived de Rham complexes are $p$-completed).

The filtrations are the $I$-adic filtration on $A$, the Nygaard filtration $\mathcal{N}$ on $\varphi^*_A \Delta_{X/A}$, and the Hodge filtration $\text{Fil}$ on derived de Rham complexes. The associated graded of (3.2) is an isomorphism in $\mathcal{D}(X, \mathcal{O}_X)$. 
Examples

- For \((A, I)\) transversal (i.e., \(A/I\) \(p\) torsion free),
  \[
  L\Omega^\bullet_{(A/I)/A} = \widehat{D_A(I)}
  \]
  (\(p\)-completed PD-envelope of \(I\) in \(A\)), and (3.2) is rewritten
  \[
  (3.2.1) \quad \varphi_A^* \Delta_X/A \otimes_A D_A(I) \sim L\Omega^\bullet_X/A,
  \]
  which, in this case, due to a classical result on \(L\Omega^\bullet_X/A\), is a form of
  the prismatic-crystalline comparison theorem.

- Take \((A, I) = (W(k), (p))\), \(k\) perfect. It is not transversal. Then, by (3.1), (3.2) reads
  \[
  (3.2.2) \quad W\Omega^\bullet_X \otimes^L_W L\Omega^\bullet_{k/W(k)} \sim L\Omega^\bullet_X/W(k).
  \]
  For \(X = \text{Spec}(k)\), \(W\Omega^\bullet_X = W(k)\), and (3.2.2) is tautologically the
  identity. Recall (Bhatt)
  \[
  \widehat{L\Omega^\bullet_{k/W}} = \widehat{W\langle y\rangle/(y)},
  \]
  and Hodge filtration = filtration on \(\widehat{W\langle y\rangle/(y)}\) by the \((y)^[n]\).
Application

- Dividing (3.2) by $p$-th steps of the filtrations gives

$$(3.3) \quad \varphi_A^* \Delta_{X/A}/\mathcal{N}^p \sim L\Omega^\bullet_{X/A}/\text{Fil}^p,$$

which, for $(A, I) = (W, (p))$ is the inverse of the isomorphism (3.0).

- Applying $\text{gr}^1$ to (3.3) gives Bhatt-Scholze [BS, 15.6]:

$$(3.4) \quad L\Omega^1_{X/A}[-1]\{-1\} \sim \tau^{\leq 1}(\Delta_{X'/A} \otimes^L_A A/I)$$

$$(\{-1\} := \otimes(I/I^2)^{-1}, X' := X \otimes_A, \varphi_A A).$$

- (3.4) generalized by Anschütz-Le Bras [AL, 3.2.1) to any formal $X/(A/I)$, with $\Delta$ denoting derived prismatic cohomology, and $\tau^{\leq 1}$ replaced by first step of conjugate filtration.

- (3.3) proposed by Bhatt (email to I., 21 Feb. 2019) with sketch of proof.
Techniques of proof

Same as in the Bhatt-Scholze prismatic-crystalline comparison theorem and construction of the Nygaard filtration.

- Use (corrected) Čech-Alexander complex calculating prismatic cohomology to define the map (3.2)(in the other direction of (3.0)).

- To analyze compatibility of (3.5) with filtrations, use quasisyntomic descent and large quasisyntomic \((A/I)\)-algebras. Work first in the transversal case.

\( R \) large quasisyntomic \((A/I)\)-algebra: \( A/I \to R \) quasisyntomic (i.e. \( p \)-completely flat and \( \text{tor.amp}(L_R/(A/I) \subset [-1,0]) \)), and we have a surjection of a Tate algebra \((A/I)\langle X_s^{1/p^\infty} \rangle_{s \in \Sigma} \to R, \Sigma \) a set.
4. The ICI case: partial degeneration and vanishing theorems (after Bhatt)

Back to \( k = \) perfect field of char. \( p \), \( W = W(k) \), \( W_n := W_n(k) \).

Recall that, for any \( X/k \), a flat lifting \( \widetilde{X}'/W_2 \) of \( X' \) gives a decomposition in \( D(X', \mathcal{O}_{X'}) \)

\[
\bigoplus_{i \leq p-1} L\Omega_{X'/k}^i \sim \rightarrow \text{Fil}_{p-1}^{-1} F_* L\Omega_{X/k}^\bullet
\]

(2.8) 

This decomposition is compatible with the obvious filtrations on both sides, and induces the (generalized) Cartier isomorphism

\[
C^{-1} : L\Omega_{X'/k}^i \sim \rightarrow \text{gr}^{-1}_i F_* L\Omega_{X/k}^\bullet
\]

on \( \text{gr}^i \).

From (2.8) Bhatt deduced the following theorem:
Theorem 4.1 (Bhatt). \( X/k \) proper, lci, of pure dimension \( d < p \), liftable to \( W_2 \). 

\( s := \) dimension of singular locus of \( X \).

\[
\hat{\Omega}^\bullet_{X/k} := R \lim_{\leftarrow r} L\Omega^\bullet_{X/k}/\text{Fil}^r
\]

(Hodge completed derived de Rham complex).

Then, for \( n < d - s - 1 \),

\[
\dim_k H^n(X, \hat{\Omega}^\bullet_{X/k}) = \sum_{0 \leq i \leq d} \dim_k H^n(X, L\Omega^i_{X/k}[-i]),
\]

and

\[
H^n(X, L\Omega^i_{X/k}[-i]) = 0
\]

for \( i > d \).
Remarks. 1. If $X/k$ smooth, then $\text{Sing}(X) = \emptyset$, $s = -\infty$, $d - s - 1 = +\infty$, $L\hat{\Omega}^\bullet_{X/k} = \Omega^\bullet_{X/k}$, so for all $n$,

$$\dim_k H^n_{\text{dR}}(X/k) = \sum_i \dim_k H^{n-i}(X, \Omega^i_{X/k}),$$

i.e., one recovers [DI]'s result that Hodge to de Rham spectral sequence degenerates at $E_1$.

2. In [DI], degeneration (even, decomposition of the de Rham complex with no properness assumption) holds for $d = p$. Unknown if conclusion of Th. 4.1 holds assuming only $d \leq p$. 
By standard spreading out arguments Th. 4.1 implies:

**Theorem 4.2** (Bhatt). \( K \): field of char. 0, \( X/K \) proper, lci, of (arbitrary) pure dimension \( d \).

\( s \): dimension of singular locus of \( X \)

Then, for \( n < d - s - 1 \),

\[
\dim_K H^n(X, L\widehat{\Omega}_{X/K}^\bullet) = \sum_{0 \leq i \leq d} \dim_k H^n(X, L\Omega_{X/K}^i[-i]),
\]

and

\[
H^n(X, L\Omega_{X/K}^i[-i]) = 0
\]

for \( i > d \).
Remarks. 1. As above, for \( X/K \) smooth, Th. 4.2 recovers the classical \( E_1 \)-degeneration of Hodge to de Rham spectral sequence in char. 0.

2. For \( K = \mathbb{C} \), and any (separated and of finite type) \( X/K \), one has (Bhatt, 2012):

\[
R\Gamma(X, L\hat{\Omega}^\bullet \mathcal{X}/\mathbb{C}) \sim \xrightarrow{\sim} R\Gamma(X(\mathbb{C}), \mathbb{C})
\]

(Betti cohomology).
Proof of Th. 4.1

Main ingredient: cohomological amplitude estimates on $L\Omega^i_{X/k}$, $LZ\Omega^i_{X/k}$, $LB\Omega^i_{X/k}$ (derived cycles, boundaries), using perf.amp$(L_{X/k}) \subset [−1, 0]$ and Cartier isomorphism.

Key Lemma (Bhatt). $X/k$ lci, purely of dimension $d$, $s := \dim(\text{Sing}(X))$. Then all complexes

$$L\Omega^i_{X/k}[−i], F_*L\Omega^i_{X/k}[−i], LZ\Omega^i_{X/k}[−i], LB\Omega^i_{X/k}[−i]$$

live in $D^{\geq}(X')$, and, for $i > d$, live in $D^{\geq d−s}(X')$.

Proof of key lemma relies on (easy points) of a theory of Cohen-Macaulayness for complexes developed by Bhatt and used by him in his proof of Cohen-Macaulayness (modulo powers of a prime $p$) of absolute integral closures of excellent noetherian domains.
Combining decomposition (2.8) with Bhatt’s estimates and Raynaud’s trick [DI, 2.9] one gets Kodaira type vanishing theorems:

**Theorem 4.3.** $X/k$ as in Th. 4.1: proper, lci, liftable to $W_2$, dim. $d < p$, singular locus of dim. $s$. Let $L$ be an ample invertible sheaf on $X$. Then:

For $n < \min(d, d - s - 1)$ and all $i$,

$$H^n(X, L\Omega_{X/k}^i[-i] \otimes L^{-1}) = 0.$$ 

**Remarks.** 1. For $X/k$ smooth (i.e., $s = -\infty$), one gets

$$H^n(X, \Omega_{X/k}^i[-i] \otimes L^{-1}) = 0$$

for $n < d$ and all $i$, i.e., [DI, (2.8.2)].
2. For $X/k$ smooth, the vanishing

$$H^n(X, \Omega^i_{X/k}[-i] \otimes L^{-1}) = 0$$

for $n < d$ and all $i$ is, by Serre duality, equivalent to

$$(*) \quad H^n(X, \Omega^i_{X/k}[-i] \otimes L) = 0$$

for $n > d$ and all $i$.

However, for $X/k$ singular, the analogue of (*) fails (observed by Bhatt-Blickle-Lyubeznik-Singh-Zhang [BBLSZ, 3.4]).

3. For $X/k$ not lci, conclusion of 4.3 fails (by Avramov’s solution of Quillen’s conjecture).
Again, by standard spreading out arguments, Th. 4.3 implies a (slightly weaker form of) [BBLSZ, Th. 3.2] (with $d - s$ replaced by $d - s - 1$):

**Theorem 4.4.** $K$: field of char. 0, $X/K$ proper, lci, of pure dimension $d$.

$s$: dimension of singular locus of $X$

$L$: an *ample* invertible sheaf on $X$.

Then, for $n < \min(d, d - s - 1)$ and all $i$, 

$$H^n(X, L\Omega^i_{X/k}[-i] \otimes L^{-1}) = 0.$$ 

**Remark.** I don’t know how to get $d - s$ instead of $d - s - 1$ by mod $p^2$ techniques.
5. A new perspective:  
the stacky approach (after Bhatt - Lurie, Drinfeld)

For $X/k$ smooth, liftable to $W_2$, of dimension $d = p$, the whole complex $F_*\Omega^\bullet_{X/k}$ is decomposable, not just $\tau^{<p} F_*\Omega^\bullet_{X/k}$ ([DI], 2.3)].

Raises the question:

Question ([DI, 2.6 (iii)], still open): $X/k$ smooth, liftable to $W_2$, of dimension $d > p$, is $F_*\Omega^\bullet_{X/k}$ decomposable (in $D(X', \mathcal{O}_{X'})$) (i.e., $\sim \oplus \mathcal{H}^i[-i]$)? For $X/k$ assumed moreover proper, does Hodge to dR degenerate at $E_1$?

Partial results:

• (Suh, 2006, unpublished). For $X/k$ smooth, liftable to $W_2$, all truncations $\tau^{[a,a+1]} F_*\Omega^\bullet_{X/k}$ are decomposable.

• (Achinger, 2020). For $X/k$ smooth, liftable to $W_2$, all truncations $\tau^{[a,a+p-2]} F_*\Omega^\bullet_{X/k}$ ($a > 0$) are decomposable.
Recent improvement by Drinfeld (and, independently, Bhatt-Lurie):

**Theorem 5.1.** (Drinfeld, Bhatt-Lurie, 2020) Let $X/k$ be smooth, liftable to $W_2 = W_2(k)$.

Then a lifting of $X$ to $W_2$ defines a $\mu_p$-action on $F_*\Omega^\bullet_{X/k}$ in $D(X', \mathcal{O}_{X'})$,

i.e., a $\mathbb{Z}/p$-grading

$$F_*\Omega^\bullet_{X/k} = \bigoplus_{\alpha \in \mathbb{Z}/p} (F_*\Omega^\bullet_{X/k})_{\alpha},$$

with nonzero $H^i F_*\Omega^\bullet_{X/k}$ of weight the class of $-i$ in $\mathbb{Z}/p$.

**Corollary 5.2.** Under the assumption of Th. 5.1, all truncations $\tau[a, a+p-1] F_*\Omega^\bullet_{X/k}$ are decomposable.
Glimpses on the proof.
Details haven’t yet appeared. Work in progress.

Main idea (Bhatt-Lurie, 2019): cohomology of prismatic sites underlies richer structure: cohomology of prismatic stacks, giving rise to objects in $D(BG)$, for certain group schemes $G/S$.

The stacks $X^\triangle$.

To any (formal scheme) $X/W(k)$ is associated a ringed, (formal) stack

$$X^\triangle/W(k),$$

called the prismatic stack or prismatization of $X$, functorial in $X$: $f : X \to Y$ gives map of ringed stacks

$$f^\triangle : X^\triangle \to Y^\triangle.$$
**Fundamental Example.** (Drinfeld’s $\Sigma$ [Dr])

$$\text{Spf}(\mathbb{Z}_p)^{\Delta} := [\mathcal{W}_{\text{prim}}/\mathcal{W}^\times](= \Sigma)$$

where:

$\mathcal{W} := (p\text{-typical})$ Witt scheme over $\mathbb{Z}_p$

$\mathcal{W}_{\text{prim}} :=$ formal completion of $\mathcal{W}$ along locally closed subscheme defined by $p = x_0 = 0$, $x_1 \neq 0$, the formal scheme of primitive Witt vectors.

$\mathcal{W}^\times \subset \mathcal{W} := \mathbb{Z}_p$-group scheme of units in $\mathcal{W}$, acting on $\mathcal{W}_{\text{prim}}$ by multiplication.
Definition 5.3. (Bhatt-Lurie)

For $X/W(k)$ formal, $R$ a $p$-nilpotent $W(k)$-algebra,

$$X^\Delta(R)$$

is the groupoid of pairs

$$((I, a), f : \text{Spf}([W(R)/I]) \to X)$$

where

- $I$: an invertible $W(R)$-module, $a : I \to W(R)$: $W(R)$-linear map landing into $W_{\text{prim}}(R)$ ("Cartier-Witt divisor")
- $\text{Spf}([W(R)/I])$: formal derived scheme such that for $X = \text{Spf}(A)$
  $$\text{Mor}(\text{Spf}([W(R)/I]), \text{Spf}(A)) := \text{Mor}_{\text{Ani}}(A, [I \to W(R)])$$
  $\text{Ani} =$ category of animated $W(k)$-algebras ($= \text{derived category of simplicial } W(k)\text{-algebras}$) ($[I \to W(R)]$ is a 1-truncated animated algebra)

In particular,

$$\text{Spf}(\mathbb{Z}_p)^\Delta = \Sigma, \text{Spec}(\mathbb{F}_p)^\Delta = \mathbb{Z}_p.$$
Another key example (Bhatt-Lurie)

\( \text{Spec}(W_2(k))^\Delta \) := stack associated to prestack with values

\[
[\mathcal{F}_{W(R)}(p^2)/W(R)^\times],
\]

on \( p \)-nilpotent \( W(k) \)-algebras \( R \), where

\( \mathcal{F}_{W(R)}(p^2) := \) set of factorizations \( p^2 = db \) in \( W(R) \), with \( d \in W_{\text{prim}}(R), \ b \in W(R) \),

and \( u \in W(R)^\times \) acts by \( (d, b) \mapsto (du, u^{-1}b) \).
5.4. Hodge-Tate point

\[ V(1) = (0, 1, 0, 0, \cdots) : \text{Spf}(\mathbb{Z}_p) \to \text{Spf}(\mathbb{Z}_p)^\Delta \]

induces (unique) “physical” point:

\[ i : \text{Spec}(k) \to \text{Spec}(\mathcal{W}_2(k))^\Delta, \]

corresponding to the factorization \( p^2 = pp \), with stabilizer

\[ \text{Stab}(i) = G := \mathcal{W}_k^\times[F] \]

(kernel of \( F \) on the \( k \)-group scheme \( \mathcal{W}_k^\times \)) (also denoted \( \mathbb{G}_m^\# \))

**Basic formula** (Drinfeld, Li-Mondal):

\[ \mathcal{W}_k^\times[F] = (\mu_p)_k \times \mathcal{W}_k[F], \]

where \( \mathcal{W}_k[F] = \text{kernel of } F \text{ on } \mathcal{W}_k = \text{PD-envelope of } 0 \text{ in } \mathbb{A}^1_k (= \mathbb{G}^\#_a) \)
Back to Drinfeld’s theorem 5.1

Data of $Y/W_2(k)$ lifting $X/k$ gives maps

$$Y^\Delta \xrightarrow{\nu} X_{Zar} \times \text{Spec}(W_2(k))^\Delta \xleftarrow{i} X_{Zar} \times BG \xleftarrow{} X_{Zar} \times B\mu_p \xleftarrow{} X_{Zar}.$$  

Prismatic Hodge-Tate comparison theorem implies:

$$(\varphi^*R\nu_*(\mathcal{O}/p))|_{X'} \sim F_*\Omega^\bullet_{X/k} \in D(X', \mathcal{O})$$

(for $\varphi = \text{Frobenius on the base } W_2(k)$).

Therefore, $F_*\Omega^\bullet_{X/k}$ underlies an object of $D(X' \times B\mu_p, \mathcal{O})$, and one checks that $H^i$ is of weight $-i$. 

In fact, $F_*\Omega^\bullet_{X/k}$ underlies an object of $D(X' \times BG, \mathcal{O})$.

As quasi-coherent sheaves on $BW_k[F]$ correspond to comodules over the Hopf algebra $k\langle x \rangle$ (PD-envelope of $(x)$ in $k[x]$), i.e. pairs $(E, N)$, where $E$ is a $k$-vector space, and $N$ a nilpotent endomorphism of $E$,

the basic formula $G = (\mu_p)_k \times W_k[F]$ gives that:

Each summand $(F_*\Omega^\bullet_{X/k})_\alpha (\alpha \in \mathbb{Z}/p)$ is endowed with an $\mathcal{O}_{X'}$-linear, nilpotent endomorphism $N_\alpha$. 
Remark (Bhatt-Lurie). Datum

\[(F_* \Omega^\bullet_{X/k} = \oplus_{\alpha \in \mathbb{Z}/p}(F_* \Omega^\bullet_{X/k})_\alpha), (N_\alpha : (F_* \Omega^\bullet_{X/k})_\alpha \to (F_* \Omega^\bullet_{X/k})_\alpha)\]

equivalent to datum of

\[\theta \in \text{End}_{D(X', \mathcal{O}_{X'})}(F_* \Omega^\bullet_{X/k})\]

with “generalized eigenvalues” in \(\mathbb{Z}/p\). Analogous to a Sen operator. For \(X\) proper smooth over \(\text{Spec}(W(k))\) (instead of \(\text{Spec}(k)\)), analogy is upgraded into a comparison theorem, involving a new theory of diffraction (ongoing work by Bhatt and Lurie).
Remarks. 1. Alexander Petrov recently gave examples where \( N_0 \neq 0 \).

2. New approach to action of \( \mathcal{W}_k \times [F] \) by Shizhang Li and Shubhodip Mondal [LM], based on study of endomorphisms of the de Rham cohomology functor. In particular:

**Theorem** ([LM, Th. 4.23])

\[
\text{Aut}(\tilde{R} \text{ smooth}/\mathcal{W}_2(k) \mapsto \Omega^\bullet_{\tilde{R} \otimes k/k} \in \text{CAlg}(D(k))) = G^\#_{m,k}
\]

**Corollary** (LM, Mathew). There is no functorial splitting for \( \tilde{X} \) smooth over \( \mathcal{W}_2(k) \) of \( \Omega^\bullet_{\tilde{X} \otimes k/k} \).


[LM] Li, Shizhang; Mondal, Shubhodip. *On endomorphisms of the de Rham cohomology functor*. In preparation.