A new approach to de Rham-Witt complexes, after Bhatt-Lurie-Mathew[1]

Luc Illusie

1. Historical background: décalage of filtrations, Ogus' quasi-isomorphism, η_p functor

Around 1965 Deligne considered the strange renumbering $E_r^{p,q} \mapsto E_{r+1}^{2p+q,-p}$ occurring in the spectral sequences of bicomplexes when one uses a naive truncation instead of a canonical one. In order to explain it, he introduced a new operation on filtrations of complexes, that he called the *décalage*. He described it in handwritten notes he gave to Grothendieck at the time, but it's only a few years later that he discussed it at length, namely in [14], where he makes a critical use of it. Let me briefly recall his construction.

Let $\mathcal{A}$ be an abelian category. Denote by $C(\mathcal{A})$ the category of complexes of $\mathcal{A}$, and by $CF(\mathcal{A})$ the category of filtered complexes of $\mathcal{A}$, i.e., pairs (K, F), where $K \in C(\mathcal{A})$ and F is a decreasing filtration on K, $(K \supset \cdots F^n K \supset F^{n+1} K \supset \cdots)$. For $K = (K, F)$ in $CF(\mathcal{A})$, the "décalée" (= shifted) filtration $\mathrm{Dec}(F)$ is defined by

$$\mathrm{Dec}(F)^p K^n = F^{p+n} K^n \cap d^{-1}(F^{p+n+1} K^{n+1}).$$

The filtered complex $(K, \mathrm{Dec}(F))$ is denoted $\mathrm{Dec}(K)$. By definition of $\mathrm{Dec}(F)$, there is a natural map

$$\mathrm{Dec}(F)^p K^n \to H^n(\mathrm{gr}_F^{p+n} K^n)(= E_1^{p+n,-p}(K, F)),$$

which factors through $\mathrm{gr}^p_{\mathrm{Dec}(F)} K^n = E_0^{p,n-p}(\mathrm{Dec}(K))$, inducing a *map of complexes*

$$E_0^{p,n-p}(\mathrm{Dec}(K)) \to E_1^{p+n,-p}(K), \tag{1.1}$$

the left (resp. right) hand side being equipped with the usual differential d_0 (induced by d) (resp. d_1 (a boundary map of *Bockstein type*) induced by the exact sequence of complexes

$$0 \to \mathrm{gr}_F^{p+n+1} K^\bullet \to (F^{p+n} K^\bullet / F^{p+n+2} K^\bullet) \to \mathrm{gr}_F^{p+n} K^\bullet \to 0.)$$

Deligne's crucial observation is the following lemma:

[1] These notes are a slightly expanded version of a talk given at the Conference *Thirty years of Berkovich spaces*, IHP, Paris, July 9, 2018.

Lemma 1.2. ([14], 1.3.4). The morphism (1.1) is a quasi-isomorphism, and it inductively induces isomorphisms

$$E_r^{p,n-p}(\mathrm{Dec}(K)) \to E_{r+1}^{p+n,-p}(K)$$

for $r \geqslant 1$.

The renumbering mentioned at the beginning is explained by taking for K the simple complex associated to a biregular bicomplex M, and observing that, if F is the filtration on K induced by the naive filtration of M by the first degree, then $\mathrm{Dec}(F)$ is the filtration induced by the canonical filtration on M by the second degree. Lemma 1.2 plays a key role in the so-called "lemma of two filtrations" ([14], 1.3.16), itself a basic ingredient in the construction of mixed Hodge structures on the cohomology of complex algebraic varieties.

In the early 1970's, a particular case of the construction Dec appeared in a totally different context, in the work of Ogus ([2], §8) on the so-called *Katz inequality* between Newton and Hodge polygons in crystalline cohomology. Let k be a perfect field of characteristic $p > 0$, $W = W(k)$ the Witt ring on k, $W_m = W/p^mW$, and X/k a proper and smooth variety. Let $H^n(X/W)$ denote the Berthelot-Grothendieck crystalline cohomology of X in degree n, i.e. $H^n(X/W) := \varprojlim_m H^n(X/W_m)$, where $H^n(X/W_m)$ is the cohomology, in degree n, of the crystalline site of X/W_m with value in the structural sheaf $\mathcal{O}_{X/W_m}$. This group is an F-crystal (a finitely generated W-module equipped with a σ-linear isogeny φ, σ being the automorphism of W defined by the Frobenius), and as such, has a *Newton polygon* $\mathrm{Nwt}_n(X)$, the convex polygonal line starting at $(0,0)$ having slope $\lambda \in \mathbf{Q}_{\geqslant 0}$ with horizontal length equal to the rank of the summand of pure slope λ in the Dieudonné-Manin decomposition of $H^n(X/W) \otimes_W K$, where K is the fraction field of W. On the other hand, X has a Hodge polygon $\mathrm{Hdg}_n(X)$, the convex polygonal line starting at $(0,0)$ having slope i with multiplicity $h^{i,n-i} = \dim H^{n-i}(X, \Omega^i_{X/k})$. Then we have the following basic inequality, conjectured by Katz:

Theorem 1.3. (Mazur-Ogus) For all n, $\mathrm{Nwt}_n(X)$ lies on or above $\mathrm{Hdg}_n(X)$.

This was first proved by Mazur [32] assuming that X has a smooth projective lifting Y over W, whose Hodge groups $H^j(Y, \Omega^i_{Y/W})$ are torsion-free. In a letter dated 9/21/1973, Deligne suggested to Mazur a way to use his techniques of gauges to get rid of these restrictive hypotheses, via a local form of the theorem. A copy of the letter was sent to Ogus, who worked out the idea and proved 1.3 in full generality in ([2], §8).

The formulation of Ogus' main local result uses a functor η_p, whose definition is based on the décalage described above. Let Ab denote the category of abelian groups. For $K \in C(\mathrm{Ab})$, with K^i p-torsion-free for all $i \in \mathbf{Z}$, so that we have an inclusion $K \subset K[1/p]$, one defines the subcomplex

$$\eta_p K \subset K[1/p]$$

by

$$(\eta_p K)^i = p^i K^i \cap d^{-1}(p^{i+1} K^{i+1}).$$

In other words, if Fil denotes the p-adic filtration on $K[1/p]$, i.e., $\mathrm{Fil}^n = p^n K$ ($n \in \mathbf{Z}$), then

$$\eta_p K = \mathrm{Dec}(\mathrm{Fil})^0 K.$$

A crucial special case of Ogus' main theorem is the following. Suppose X/k is smooth and admits a formal smooth lifting Z/W, equipped with a σ-linear endomorphism F lifting the absolute Frobenius (i.e., $Fa \equiv a^p \bmod p\mathcal{O}_Z$ for any local section a of $\mathcal{O}_Z$). Then the (p-completed) de Rham complex $\Omega^\bullet_Z = \Omega^\bullet_{Z/W}$ has p-torsion-free components, and F induces an endomorphism $\varphi = F^*$ of it which is divisible by p^i in degree i, so that we get an endomorphism F of the graded algebra Ω^*_Z, with the property that it coincides with F in degree zero, and satisfies $dF = pFd$. It follows that the morphism of complexes φ factors uniquely as

$$\Omega^\bullet_Z \xrightarrow{\widetilde{\varphi}} \eta_p \Omega^\bullet_Z \hookrightarrow \Omega^\bullet_Z.$$

Ogus' result is the following:

Lemma 1.4. ([2], 8.8). The morphism $\widetilde{\varphi}$ is a quasi-isomorphism.

The morphism φ realizes the Frobenius endomorphism of the crystalline cohomology complex $Ru_*\mathcal{O}_{X/W}$, and actually Ogus proves a more general similar result ([2], 8.20), independent of any lifting, and involving certain subsheaves of $\mathcal{O}_{X/W}$. Such generalization is needed to derive the global theorem 1.3.

Though it seems that both Deligne and Ogus were unaware of it, 1.2 yields, via the Cartier isomorphism, an immediate proof of 1.4. We will return to this in 3.2.9.

Lemma 1.4 was a crucial ingredient in the reconstruction of the de Rham-Witt complex $W\Omega^\bullet_X$ of a smooth scheme X/k via its crystalline cohomology groups $R^i u_* \mathcal{O}_{X/W_n}$ in ([25], III 1.5), as suggested by Katz. In the context of logarithmic geometry, this reconstruction was used by Hyodo [22] to *define* the de Rham-Witt complex of a log smooth log scheme X of Cartier type over the standard logarithmic point over k. This de Rham-Witt complex was a basic tool in the formulation by Hyodo-Kato [23] of the Fontaine-Jannsen C_{st} conjecture, first proved by Tsuji [37], and later by several other authors.

In 2016 variants and generalizations of the η_p functor (and its derived version $L\eta_p$) appeared in the work of Bhatt-Morrow-Scholze [7] on integral p-adic Hodge theory. This inspired Bhatt, Lurie, and Mathew in [8] to further analyze the $L\eta_p$ functor and the homological algebra behind Ogus' lemma 1.4 and the reconstruction of $W\Omega^\bullet_X$ alluded to above. This is wrapped up in the form of a general fixed point theorem for the functor $L\eta_p$ (see §3). On the other hand, they propose a new, simple definition of de Rham-Witt complexes for schemes X/k, which turn out to coincide with the classical one when X/k is smooth, and is of interest for some singular X/k. It should also be emphasized that, if one ignores the classical constructions ([24], [27]), definitions in [8] give an alternate approach to them, and lead to simple proofs of the main structure and comparison theorems of [24]. They also yield a simplified proof of the crystalline comparison theorem for Bhatt-Morrow-Scholze's complex $A\Omega$ ([7], Th. 1.10 (i)). However, for lack of time, I will not discuss this proof in the talk. See 5.3 (b) for a brief sketch. Let me also mention quite recent developments closely related to [8], on which it is too early to report:

(a) Using a logarithmic variant of the constructions in [8], given a log scheme X over the standard log point $\underline{k}$ over k, one can hope to define a de Rham-Witt complex $\mathcal{W}\omega^\bullet_X$, which, for $X/\underline{k}$ log smooth of Cartier type coincides with the Hyodo-Kato complex [23] and the one constructed by Matsuue [31]. One can also hope that this approach will lead to simplified proofs of results on $\mathbf{A}_{\rm inf}$-cohomology in the semistable case, proved earlier by Čęsnavičius-Koshikawa [12]. There is work in progress on this by Z. Yao ([39], [40]).

(b) Liftings of Frobenius play a central role in [8]. In [10] they are used to define a new site, the *prismatic site*, whose cohomology is linked to crystalline cohomology, on the one hand, and to $\mathbf{A}_{\rm inf}$-cohomology, on the other hand. However, it seems that the relation of this new theory with that of [8] (not to speak of [39]) is not yet well understood.

2. Saturated de Rham-Witt complexes.

Let me start with some basic definitions from [8].

A *Dieudonné complex* (M, F) is a complex

$$M = (\cdots \to M^i \xrightarrow{d} M^{i+1} \to \cdots)$$

of abelian groups together with the datum of homomorphisms $F : M^i \to M^i$ for all $i \in \mathbf{Z}$, such that

$$dF = pFd.$$

Morphisms are defined in the obvious way. We thus get a category $\mathbf{DC}$.

Examples: (a) If R is an $\mathbf{F}_p$-algebra, the de Rham-Witt complex $W\Omega^\bullet_R$ of [24], with the operator F of *loc. cit.*, is a Dieudonné complex.

(b) If R is smooth over a perfect field k, and if A is a smooth formal lifting of R over $W = W(k)$, together with a σ-linear lifting $F : A \to A$ of the absolute Frobenius, then $\Omega^\bullet_{A/W}$ together with the operator $F : \Omega^i_{A/W} \to \Omega^i_{A/W}$ defined in 1.4 is a Dieudonné complex.

One says that a Dieudonné complex (M, F) is *saturated* if: (i) M is p-torsion-free (i.e., each M^i is p-torsion-free), and (ii) $F : M^i \to M^i$ factors as

$$M^i \xrightarrow{\sim} M^i \cap d^{-1}(pM^{i+1}) \hookrightarrow M^i.$$

In particular, if (M, F) is saturated, F is injective.

If R is of positive dimension, the Dieudonné complex of Example (b) is not saturated. When R is smooth over a perfect field k, that of Example (a) is, by ([24], I 3.21.1.5).

Let (M, F) be a Dieudonné complex, with M p-torsion-free. We have a morphism of complexes $\varphi : M \to M$, defined by $p^i F$ in degree i. It uniquely factors through a morphism

$$\alpha_F : M \to \eta_p M \subset M. \tag{2.0}$$

For (M, F) to be saturated, it is necessary and sufficient that α_F be an isomorphism. Note that $\eta_p M$ (with the endomorphism of $(\eta_p M)^i$ induced by F for each i), is itself a Dieudonné complex, and that α_F is a morphism of Dieudonné complexes.

If (M, F) is a Dieudonné complex, one defines its saturation $\mathrm{Sat}(M)$ as

$$\mathrm{Sat}(M) = \varinjlim_{n \geq 0} \eta_p^n (M/M_{\mathrm{tors}}),$$

where M^i_{tors} is the submodule of p-torsion in M^i (sometimes called p^∞-torsion), $\eta_p^n = \eta_p \circ \cdots \circ \eta_p$ (n times), and the transition maps are defined inductively by α_F. It follows from the above remark that $\mathrm{Sat}(M)$, with the induced operator F is indeed saturated, and that the functor

$$\mathrm{Sat} : \mathbf{DC} \to \mathbf{DC}_{\mathrm{sat}}$$

is left adjoint to the inclusion functor

$$\mathbf{DC}_{\mathrm{sat}} \hookrightarrow \mathbf{DC}$$

of the full subcategory consisting of saturated complexes.

Example (c): On the polynomial algebra $P = \mathbf{Z}_p[t]$ ($t = (t_1, \cdots, t_n)$, $n \geqslant 1$) consider the lifting F of Frobenius sending t_i to t_i^p. As in example (b) above, it induces an endomorphism φ of the de Rham complex $\Omega^\bullet_P := \Omega^\bullet_{P/\mathbf{Z}_p}$, such that $\varphi|\Omega^i = p^i F$, with F an endomorphism of the graded algebra $\Omega^\bullet_P$ satisfying $dF = pFd$. Then one can show ([8], 4.2.5) that

$$\mathrm{Sat}(\Omega^\bullet_P)$$

is Deligne's *complex of integral forms* $E^\bullet$ of ([24]), I 2.1.3), which plays a crucial role in the constructions of [24] (and, in a more general form, of [27]).

Let (M, F) be a saturated Dieudonné complex. For any $x \in M^i$, px can be written (uniquely) Fy, for $y \in M^i$. Set $y = Vx$. Then $V : M^i \to M^i$ is an endomorphism satisfying $FV = p$, hence, by injectivity of F, $VF = p$, and

$$(*) \qquad FdV = d.$$

This formula, first discovered to hold on de Rham-Witt complexes ([24], I 2.18.3), was at the origin of the definition of the *Raynaud ring* and the corresponding theory of coherent complexes over it [25], widely developed by Ekedahl [18], [19], [20], (see [26] for a survey).

Let M be a saturated Dieudonné complex. Then, for any $n \geqslant 1$, $V^nM + dV^nM : i \mapsto V^nM^i + dV^nM^{i-1}$ is a subcomplex of M, and we get a projective system of quotients

$$\mathcal{W}_\bullet M = (\mathcal{W}_n M := M/(V^nM + dV^nM)_{n\geqslant 1}).$$

It follows from (*) that F (resp. V) induces maps $F : \mathcal{W}_{n+1}M \to \mathcal{W}_n M$ (resp. $V : \mathcal{W}_n M \to \mathcal{W}_{n+1}M$) satisfying again $FV = VF = p$ and (*). One checks that

$$\mathcal{W}M := \varprojlim M_n,$$

equipped with the induced maps F is again a saturated Dieudonné complex. One says that M is *strict* if the canonical map

$$M \to \mathcal{W}M$$

is an isomorphism. If $\mathbf{DC}_{\mathrm{str}}$ denotes the full subcategory of $\mathbf{DC}_{\mathrm{sat}}$ consisting of strict complexes, the functor $M \mapsto \mathcal{W}M$ is left adjoint to the inclusion.

The construction of de Rham-Witt complexes requires additional multiplicative structures. A *Dieudonné algebra* is a strictly commutative differential graded algebra

$$A = (A^0 \xrightarrow{d} A^1 \xrightarrow{d} \cdots)$$

concentrated in nonnegative degrees ("strictly commutative" meaning that in addition to $xy = (-1)^{ij}yx$ for x (resp. y) homogeneous of degree i (resp.

j), one has $x^2 = 0$ for x homogeneous of odd degree), together with homomorphisms $F : A^i \to A^i$ satisfying $dF = pFd$, $F(xy) = Fx.Fy$ for all $x \in A^i$, $y \in A^j$, and $Fx \equiv x^p \bmod pA^0$ for $x \in A^0$. In particular, the underlying complex of abelian groups is a Dieudonné complex. One says that A is *saturated* (resp. *strict*) if this underlying complex is. If A is saturated, then, in addition to the formulas $FV = VF = p$ and (*), we have

$$xVy = V(Fx.y) \qquad (**)$$

for all $x \in A^i$, $y \in A^j$.

Dieudonné algebras form a category $\mathbf{DA}$, whose full subcategory consisting of saturated (resp. strict) algebras is denoted $\mathbf{DA}_{\mathrm{sat}}$ (resp. $\mathbf{DA}_{\mathrm{str}}$). If A is a Dieudonné algebra, A/A_{tors}, $\eta_p(A/A_{\mathrm{tors}})$, $\mathrm{Sat}(A)$, $\mathcal{W}\mathrm{Sat}(A)$ are again Dieudonné algebras in a natural way, and one checks that the functor $A \mapsto \mathrm{Sat}(A)$ (resp. $A \mapsto \mathcal{W}(A)$) is left adjoint to the inclusion $\mathbf{DA}_{\mathrm{sat}} \hookrightarrow \mathbf{DA}$ (resp. $\mathbf{DA}_{\mathrm{str}} \hookrightarrow \mathbf{DA}_{\mathrm{sat}}$).

The de Rham-Witt complex $W\Omega^\bullet_R$ of a smooth algebra R over a perfect field of characteristic $p > 0$ is a strict Dieudonné algebra (([24], I 3.21.1.5), ([25] II 1.3))[2]. The complex $E^\bullet$ of integral forms of the example above is a saturated Dieudonné algebra. It is not strict.

Note the following two important properties ([8], 3.6.1, 3.6.2):

(2.1) For $A \in \mathbf{DA}_{\mathrm{sat}}$, the $\mathbf{F}_p$-algebra A^0/VA^0 is *reduced*;

(2.2) For $A \in \mathbf{DA}_{\mathrm{str}}$, the unique F-compatible lifting[3]

$$A^0 \to W(A^0/VA^0)$$

of the projection $A^0 \to A^0/VA^0$ is an *isomorphism.*

Let R be an $\mathbf{F}_p$-algebra. One defines a *strict Dieudonné algebra over R* as a pair (A, u), where $A \in \mathbf{DA}_{\mathrm{str}}$ and u is a homomorphism (of $\mathbf{F}_p$-algebras) $R \to A^0/VA^0$. Strict Dieudonné algebras over R form a category $\mathbf{DA}_{\mathrm{str}/R}$ in an obvious way. The first main result of [8] is the following theorem ([8], 4.1.1):

Theorem 2.3. The category $\mathbf{DA}_{\mathrm{str}/R}$ admits an initial object, denoted $\mathcal{W}\Omega^\bullet_R$, $e : R \to \mathcal{W}_1\Omega^0_R (= \mathcal{W}\Omega^0_R/V\mathcal{W}\Omega^0_R)$, called the *saturated de Rham-Witt complex of R.* In other words, for $A \in \mathbf{DA}_{\mathrm{str}/R}$, we have

$$\mathrm{Hom}(R, A^0/VA^0) = \mathrm{Hom}_{\mathbf{DA}_{\mathrm{str}/R}}(\mathcal{W}\Omega^\bullet_R, A).$$

[2]Note that if R is an arbitrary $\mathbf{F}_p$-algebra, we have $Fx \equiv x^p \bmod pW(R)$ for any $x \in W(R)$, so that $W\Omega^\bullet_R$ is a Dieudonné algebra; saturation, however, requires smoothness of R/k.

[3]This unique lifting comes from the so-called universal property of Witt vector rings, taking into account that, by assumption, A^0 is p-torsion-free.

The proof is quite simple. It is based on the observation that if B is a p-torsion-free ring, equipped with an endomorphism F satisfying $Fx \equiv x^p \bmod pB$ for all $x \in B$, then there exists a unique endomorphism F of the algebra $\Omega^\bullet_B := \Omega^\bullet_{B/\mathbf{Z}_p}$ which extends F, and satisfies $dF = pFd$, and $Fdx = x^{p-1}dx + d((Fx - x^p)/p)$ for $x \in B$, hence makes $\Omega^\bullet_B$ into a Dieudonné algebra. The ring $W(R_{\mathrm{red}})$ is such a B, so that $\Omega^\bullet_{W(R_{\mathrm{red}})}$ is a Dieudonné algebra. Then, using (2.1), one checks that

$$\mathcal{W}\Omega^\bullet_R := \mathcal{W}\mathrm{Sat}(\Omega^\bullet_{W(R_{\mathrm{red}})})$$

is the desired initial object, the map $e : R \to \mathcal{W}_1\Omega^0_R$ being the obvious one: by definition, $\mathcal{W}_1\mathrm{Sat}(\Omega^\bullet_{W(R_{\mathrm{red}})})^0 = \mathrm{Sat}(\Omega^\bullet_{W(R_{\mathrm{red}})})^0/V$, and as the natural map $W(R_{\mathrm{red}}) \to \mathrm{Sat}(\Omega^\bullet_{W(R_{\mathrm{red}})})^0$ is compatible with F, hence with V, we have a map $R_{\mathrm{red}} = W(R_{\mathrm{red}})/V \to \mathrm{Sat}(\Omega^\bullet_{W(R_{\mathrm{red}})})^0/V$, whose composition with $R \to R_{\mathrm{red}}$ is e.

If k is a perfect $\mathbf{F}_p$-algebra, then $\mathcal{W}\Omega^\bullet_k = \mathcal{W}\Omega^0_k = W(k)$, $V = pF^{-1}$, and e is the isomorphism $k \xrightarrow{\sim} W(k)/V$. So, if R is an algebra over a perfect ring k, then $W\Omega^\bullet_R$ is a differential graded algebra over $W(k)$.

The definition of saturated de Rham-Witt complexes is globalized on schemes as follows. Let X be an $\mathbf{F}_p$-scheme. If U is an open affine subscheme of X, put $\mathcal{O}(U) := \Gamma(U, \mathcal{O}_X)$. It is shown in ([8], 5.2.2, 5.2.3) that, for each $n \geqslant 1$, the presheaf

$$U \mapsto \mathcal{W}_n\Omega^\bullet_{\mathcal{O}(\mathcal{U})}$$

on the category $\mathcal{U}_{\mathrm{aff}}(X)$ of open affine subschemes of X is a sheaf for the Zariski topology, and that the corresponding sheaf on X

$$\mathcal{W}_n\Omega^\bullet_X$$

is quasi-coherent on the scheme $W_n(X)$ having the same underlying space as X and $W_n\mathcal{O}_X$ as structural sheaf, with

$$\Gamma(U, \mathcal{W}_n\Omega^\bullet_X) = \mathcal{W}_n\Omega^\bullet_{\mathcal{O}(U)}$$

for $U \in \mathcal{U}_{\mathrm{aff}}(X)$. The inverse limit

$$\mathcal{W}\Omega^\bullet_X := \varprojlim_{n \geqslant 1} \mathcal{W}_n\Omega^\bullet_X$$

is called the *saturated de Rham-Witt complex* of X. We have

$$\mathcal{W}_n\Omega^\bullet_X := \mathcal{W}\Omega^\bullet_X/(V^n W\Omega^\bullet_X + dV^n W\Omega^\bullet_X),$$

and for $U \in \mathcal{U}_{\mathrm{aff}}(X)$,

$$\Gamma(U, \mathcal{W}\Omega^\bullet_X) = \mathcal{W}\Omega^\bullet_{\mathcal{O}(U)}.$$

The notation $\mathcal{W}$ and the adjective "saturated" (sometimes dropped by Bhatt-Lurie-Mathew) are meant to avoid confusion with the object $W\Omega^\bullet_X$ defined in [24] – which is also the same as the complex $W\Omega^\bullet_{X/\mathbf{F}_p}$ of [27].

By (2.2) we have $\mathcal{W}\Omega^0_X = W(\mathcal{W}_1\Omega^0_X)$, hence the structural map $e : \mathcal{O}_X \to \mathcal{W}_1\Omega^0_X$ induces a map

$$W(e) : W\mathcal{O}_X \to \mathcal{W}\Omega^0_X,$$

which is compatible with F and V, but is not an isomorphism in general (already because $\mathcal{W}_1\Omega^0_X$ has to be reduced).

The saturated de Rham-Witt complex $\mathcal{W}\Omega^\bullet_X$ is functorial in X, and satisfies the same properties as $W\Omega^\bullet_X$ with respect to étale localization. Let $X \to Y$ be an étale morphism. Recall that W_nX is étale over W_nY ([24], 0 1.5.8). It is shown in ([8], 5.3.5) that, for all i, the map

$$W_n\mathcal{O}_X \otimes_{W_n\mathcal{O}_Y} \mathcal{W}_n\Omega^i_Y \to \mathcal{W}_n\Omega^i_X$$

is an isomorphism. In particular, $\mathcal{W}_n\Omega^0_X$ is étale over $\mathcal{W}_n\Omega^0_Y$.

3. Main properties of saturated de Rham-Witt complexes

3.1. Comparison with previous constructions

Let X be an $\mathbf{F}_p$-scheme. The projective system

$$\mathcal{W}_\bullet\Omega^\bullet_X := (\mathcal{W}_n\Omega^\bullet_X)_{n\geqslant 1},$$

together with the operators $F : \mathcal{W}_{n+1}\Omega^\bullet_X \to \mathcal{W}_n\Omega^\bullet_X$, $V : \mathcal{W}_n\Omega^\bullet_X \to \mathcal{W}_{n+1}\Omega^\bullet_x$, and the homomorphism $W_\bullet(e) : W_\bullet(\mathcal{O}_X) \to \mathcal{W}_\bullet\Omega^0_X$, form an F-V-procomplex over $\mathcal{O}_X$ in the sense of ([27], Definition 1.4). Therefore, by the universal property of the Langer-Zink de Rham-Witt complex, we have a canonical map of F-V-procomplexes over $\mathcal{O}_X$:

$$\mathrm{can}_\bullet : W_\bullet\Omega^\bullet_X \to \mathcal{W}_\bullet\Omega^\bullet_X. \tag{3.1.1}$$

(Recall that, as X is an $\mathbf{F}_p$-scheme, the left hand side coincides with the object defined in [24].)

Theorem 3.1.2 ([8], Th. 4.4.12). *If X is regular, then (3.1.1) is an isomorphism. In particular,*

$$\mathrm{can}_1 : \Omega^\bullet_X \to \mathcal{W}_1\Omega^\bullet_X$$

is an isomorphism.

By Popescu's theorem and étale localization, one is reduced to $X = \mathrm{Spec}(\mathbf{F}_p[t])$, $(t = (t_1, \cdots, t_n))$, as in (2, Example (c)). By ([24], I 2.5), we have

$$W_\bullet\Omega^\bullet_{\mathbf{F}_p[t]} = \mathcal{W}_\bullet\mathrm{Sat}(\Omega^\bullet_{\mathbf{Z}_p[t]}).$$

However, it is formal (cf. ([8], 4.2.3)) that $\mathcal{W}\mathrm{Sat}(\Omega^\bullet_{\mathbf{Z}_p[t]})$ (with the map $\mathbf{F}_p[t] \to \mathcal{W}_1\mathrm{Sat}(\Omega^\bullet_{\mathbf{Z}_p[t]})^0$ defined by the identity of $\mathbf{F}_p[t]$) is an initial object of $\mathbf{DA}_{\mathrm{str}/\mathbf{F}_p[t]}$.

One can show that for a cusp ($R = \mathbf{F}_p[x,y]/(x^2 - y^3)$) or an ordinary double point ($R = \mathbf{F}_p[x,y]/xy$), $\mathrm{can}_1 : \Omega^\bullet_R \to \mathcal{W}_1\Omega^\bullet_R$ is not an isomorphism. We will come back to this in 4.1, 4.2.

3.2. Comparison with crystalline cohomology

Let X be a scheme over a perfect field k of characteristic $p > 0$, and write W for $W(k)$. For $n \geqslant 1$, consider the crystalline topos $(X/W_n)_{\mathrm{crys}}$, with its structural sheaf of rings $\mathcal{O}_{X/W_n}$, and the Berthelot canonical map to the Zariski topos $u : (X/W_n)_{\mathrm{crys}} \to X_{\mathrm{Zar}}$. In ([24], II (1.1.1)), for X/k separated[4] and of finite type, there is defined a canonical map

$$Ru_*\mathcal{O}_{X/W_n} \to W_n\Omega^\bullet_X \tag{3.2.1}$$

of projective systems of objects of $D^+(X, W_n)$, compatible with the multiplicative structures, and the Frobenius endomorphism (defined by p^iF on $W_\bullet\Omega^i_X$). It is shown in ([24], II 1.4) that if X/k is smooth, (3.2.1) is an isomorphism (a more general comparison theorem is established in [27]). The proof depends on what Bhatt, Lurie and Mathew rightly call "laborious" local calculations. If one replaces the de Rham-Witt complex by the saturated one, one can give an independent, much simpler proof of this last result. This is indeed one of the main points of [8]. We recall the argument, which is unfortunately scattered in various places of [8].[5]

First of all, for X/k separated and of finite type, we will define a map similar to (3.2.1), with $W_\bullet\Omega^\bullet_X$ replaced by $\mathcal{W}_\bullet\Omega^\bullet_X$. The construction is essentially the same as in ([24], II 1.1). As in *loc. cit.*, choose an *embedding system*

$$X \xleftarrow{\varepsilon_\bullet} U_\bullet \xrightarrow{i_\bullet} Y_\bullet,$$

where $\varepsilon_\bullet$ is a Zariski open affine hypercovering, and, for all $n \geq 0$, $i_.$ a closed immersion into a formal smooth simplicial scheme over W, where in addition, $Y_\bullet$ is equipped with a compatible system $F_\bullet$ of liftings of Frobenius. Let

$$Ru_*\mathcal{O}_{X/W} := R\varprojlim Ru_*\mathcal{O}_{X/W_n}$$

(for an alternate definition in terms of a crystalline topos over W, see ([2], §7)). Let $D_\bullet$ be the PD-envelope of $U_\bullet$ in $Y_\bullet$, $D_{\bullet n}$ its reduction over W_n, and

[4]This assumption is missing in *loc. cit.*. It is implicitly used in the construction of an embedding system. It is probable that it is superfluous, see ([23], p. 237).

[5]See Remark 3.2.9 (iii) for an update.

let

$$\Omega^\bullet_{D_\bullet/W,[]} := \varprojlim_{n\geq 1} \Omega^\bullet_{D_{\bullet n}/W_n,[]}$$

where $\Omega^\bullet_{D_{\bullet n}/W_n,[]}$ is the PD-de Rham complex of $D_{\bullet n}$. As recalled in ([24], II 1.1), we have a canonical isomorphism $Ru_*\mathcal{O}_{U_\bullet}/W_n \xrightarrow{\sim} \Omega^\bullet_{D_{\bullet n}/W_n,[]}$, hence, by cohomological descent, and inverse limit, an isomorphism

$$Ru_*\mathcal{O}_{X/W} \xrightarrow{\sim} R\varepsilon_{\bullet *}\Omega^\bullet_{D_\bullet/W,[]}. \tag{3.2.2a}$$

On the other hand, consider the completed de Rham complex

$$\Omega^\bullet_{W\mathcal{O}_{Y_\bullet}/W} := \varprojlim \Omega^\bullet_{W_n\mathcal{O}_{Y_\bullet}/W_n}.$$

As explained in *loc. cit.*, $F_\bullet$ defines a composite homomorphism

$$\mathcal{O}_{Y_\bullet} \to W\mathcal{O}_{Y_\bullet} \to W\mathcal{O}_{U_\bullet}, \tag{3.2.2b}$$

hence a morphism of differential graded algebras

$$\Omega^\bullet_{Y_\bullet/W} \to \Omega^\bullet_{W\mathcal{O}_{U_\bullet}/W}. \tag{3.2.2c}$$

As $W\mathcal{O}_{U_\bullet}$ is a λ_p-ring, there exists a unique structure of Dieudonné algebra on $\Omega^\bullet_{W\mathcal{O}_{U_\bullet}/W}$ ([8], 3.7.6). Consider the morphism of differential graded algebras

$$\Omega^\bullet_{Y_\bullet/W} \to \mathcal{W}\Omega^\bullet_{U_\bullet}, \tag{3.2.2d}$$

deduced from (3.2.2c) by composing with the canonical map to $\mathcal{W}$Sat, and the canonical isomorphism (of Dieudonné algebras)

$$\mathcal{W}\mathrm{Sat}(\Omega^\bullet_{W\mathcal{O}_{U_\bullet}/W}) \xrightarrow{\sim} \mathcal{W}\Omega^\bullet_{U_\bullet}$$

([8], 4.1.4). As (3.2.2b) sends the ideal $\mathcal{I}_\bullet$ of $U_\bullet$ in $Y_\bullet$ to $VW\mathcal{O}_{U_\bullet}$, (3.2.2d) sends $\mathcal{I}_\bullet$ to the ideal $V\mathcal{W}\Omega^0_{U_\bullet}$. As $V\mathcal{W}\Omega^0_{U_\bullet}$ has canonical divided powers $x \mapsto x^{[m]}$, and the relation $dx^{[m]} = x^{[m-1]}dx$ holds, since $\mathcal{W}\Omega^\bullet_{U_\bullet}$ is p-torsion-free. Therefore (3.2.2d) factors uniquely through a morphism of differential graded algebras

$$\Omega^\bullet_{D_\bullet/W,[]} \to \mathcal{W}\Omega^\bullet_{U_\bullet}. \tag{3.2.2e}$$

By cohomological descent, the canonical map $\iota : \mathcal{W}\Omega^\bullet_X \to R\varepsilon_{\bullet *}\mathcal{W}\Omega^\bullet_{U_\bullet}$ is an isomorphism. Now apply $R\varepsilon_{\bullet *}$ to (3.2.2e), and compose with the inverse of ι and the isomorphism (3.2.2a): we get a morphism (in $D(X,W)$)

$$Ru_*\mathcal{O}_{X/W} \to \mathcal{W}\Omega^\bullet_X. \tag{3.2.2}$$

One checks as in ([24]) that (3.2.2) is independent of the choice of the embedding system, and that it is compatible with the product structure and Frobenius endomorphisms on both sides (given by p^iF in degree i on the right hand side). Applying $\otimes^L \mathbf{Z}/p^n\mathbf{Z}$ on both sides, and composing with the canonical map $\mathcal{W}\Omega^\bullet_X \otimes \mathbf{Z}/p^n\mathbf{Z} \to \mathcal{W}_n\Omega^\bullet_X$ (a quasi-isomorphism by ([8], 2.7.3), see (3.2.7) below), we get a projective system of morphisms in $D(X, W_n)$

$$Ru_*\mathcal{O}_{X/W_n} \to \mathcal{W}_n\Omega^\bullet_X, \tag{3.2.2n}$$

(from which (3.2.2) can be recovered by applying $R\varprojlim$). This is the desired analogue of (3.2.1).

Assume now X/k smooth. Then (3.2.2) (or, equivalently, (3.2.2n) for all $n \geqslant 1$) is an isomorphism. To prove it, we may assume that X is affine, $X = \mathrm{Spec}(R)$, and has a formal smooth lifting Z/W, $Z = \mathrm{Spf}(B)$, together with a lifting F of Frobenius, as in 1.4. Then $\Omega^\bullet_B$, together with the operator F deduced from F on B by functoriality and division by p^i in degree i is a p-torsion-free Dieudonné algebra, and we have a tautological map

$$\Omega^\bullet_B \to \mathcal{W}\mathrm{Sat}(\Omega^\bullet_B). \tag{3.2.3}$$

As observed earlier (cf. ([8], 4.2.3)), we have

$$\mathcal{W}\Omega^\bullet_R = \mathcal{W}\mathrm{Sat}(\Omega^\bullet_B).$$

The composition

$$\Omega^\bullet_B \otimes \mathbf{Z}/p^n \to \mathcal{W}_n\mathrm{Sat}(\Omega^\bullet_B) \tag{3.2.4}$$

of the reduction mod p^n of (3.2.3)

$$\Omega^\bullet_B \otimes \mathbf{Z}/p^n \to \mathcal{W}\mathrm{Sat}(\Omega^\bullet_B) \otimes \mathbf{Z}/p^n \tag{3.2.5}$$

and the canonical projection

$$\mathcal{W}\mathrm{Sat}(\Omega^\bullet_B) \otimes \mathbf{Z}/p^n \to \mathcal{W}_n\mathrm{Sat}(\Omega^\bullet_B) \tag{3.2.6}$$

realizes (3.2.2n). So it suffices to show that (3.2.5) and (3.2.6) are quasi-isomorphisms. Deligne's lemma 1.2 is used in both cases, but for (3.2.5) an additional input is needed, namely, the Cartier isomorphism.

(i) *(3.2.6) is a quasi-isomorphism.* As observed above, this is a particular case of the following lemma ([8], 2.7.3):

Lemma 3.2.7. Let M be a saturated Dieudonné complex. Then, for all $n \geqslant 1$, the projection

$$M/p^nM \to \mathcal{W}_nM$$

is a quasi-isomorphism.

See *loc. cit.* for the proof. It relies on an easy lemma ([8], 2.7.1) (see (5.1.1)), and Deligne's lemma, which ensures that the map (of type (1.1))

$$\eta_{p^n} M/p^n \eta_{p^n} M \to H^\bullet(M/p^n M)$$

(where η_f is Dec^0 applied to the f-adic filtration) is a quasi-isomorphism.

(ii) *(3.2.5) is a quasi-isomorphism.* By definition, the map $F : \Omega^i_B \to \Omega^i_B$ lifts the Cartier isomorphism $C^{-1} : \Omega^i_R \xrightarrow{\sim} H^i(\Omega^\bullet_R)$. In [8] one says that a Dieudonné complex (M, F) is *of Cartier type* if, for each i, M^i is p-torsion-free and the map $M^i/pM^i \to H^i(M/pM)$ induced by F is an isomorphism. Therefore, assertion (ii) is a particular case of the following lemma ([8], 2.8.5):

Lemma 3.2.8. Let M be a Dieudonné complex of Cartier type. Then:

(a) The map $\alpha_F \otimes \mathbf{Z}/p : M/pM \to \eta_p M/p\eta_p M$ (cf. (2.0)) is a quasi-isomorphism.

(b) The canonical map $M/pM \to \mathrm{Sat}(M)/p\mathrm{Sat}(M)$ is a quasi-isomorphism.

(c) Assume that, in addition, each M^i is p-adically complete (which is the case, for example, for $M = \Omega^\bullet_B$ as above). Then the canonical map $M \to \mathcal{W}\mathrm{Sat}(M)$ is a quasi-isomorphism, and induces, for each $n \geqslant 1$, a quasi-isomorphism $M \otimes \mathbf{Z}/p^n \to \mathcal{W}\mathrm{Sat}(M) \otimes \mathbf{Z}/p^n$.

Assertions (b) and (c) are easy consequences of (a). For (c) one has to observe that any strict Dieudonné complex is derived p-complete[6] (as equal to the limit of a strict inverse system of $\mathbf{Z}/p^n$-modules), hence its components, which are p-torsion-free, are p-adically complete. The proof of (a) is simple and beautiful. Consider the commutative diagram of complexes (where the slanted arrow is the composition, the first horizontal arrow is $\alpha_F \otimes \mathbf{Z}/p$, the map Del is Deligne's quasi-isomorphism of Lemma 1.2, for $r = 0$, and the vertical arrow is the trivial isomorphism):

$$\begin{array}{ccccc} M^n/pM^n & \longrightarrow & (\eta_p M)^n/p(\eta_p M)^n & \xrightarrow{\mathrm{Del}} & H^n(p^n M/p^{n+1} M) \\ & \searrow & & & \downarrow{\scriptstyle p^{-n}} \\ & & & & H^n(M/pM). \end{array}$$

The slanted arrow is nothing but the Cartier isomorphism C^{-1}. Therefore $\alpha_F \otimes \mathbf{Z}/p$ is a quasi-isomorphism.

Remarks 3.2.9. (i) Assume that each M^i is p-adically complete. Then 3.2.8 (a) implies that

$$\alpha_F : M \to \eta_p M$$

[6] See 5.2.

is a quasi-isomorphism. Indeed, by induction on n, $\alpha_F \otimes \mathbf{Z}/p^n : M/p^nM \to \eta_pM/p^n\eta_pM$ is a quasi-isomorphism for all $n \geqslant 1$. As the components of M are p-adically complete and torsion-free, M is derived p-complete, hence η_pM is also derived p-complete ([7], Lemma 6.19), so its components are p-complete, and $\alpha_F = R\varprojlim \alpha_F \otimes^L \mathbf{Z}/p^n : M \to \eta_pM$ is a quasi-isomorphism.

(ii) Remark (i) applied to $\widetilde{\varphi}$ gives Ogus' Lemma 1.4.

Remark 3.2.10.[7] The proof given above for the construction of (3.2.1) and the fact that it is an isomorphism for X/k smooth, though it simplifies that of ([24], II 1.4), still proceeds along the same lines, using embeddings into formally smooth schemes over W endowed with a lifting of Frobenius. Another proof in the same vein is given by Ogus in ([35], §5). Quite recently, Bhatt, Lurie, and Mathew found a totally new one, involving no embeddings in schemes with a lifting of Frobenius ([8], 10.1.2). As a bonus, it gives a *uniqueness* property for the comparison isomorphism. The only input from the theory of the saturated de Rham-Witt complex they use is the following:

(a) For X/k smooth, the canonical map

$$\Omega^\bullet_{X/k} \to \mathcal{W}_1\Omega^\bullet_X \tag{3.2.10.1}$$

is an isomorphism.

(b) For X/k smooth, the canonical map

$$\mathcal{W}\Omega^\bullet_X \to \mathcal{W}_1\Omega^\bullet_X (\xrightarrow{\sim} \Omega^\bullet_{X/k})$$

induces a quasi-isomorphism

$$\mathcal{W}\Omega^\bullet_X/p\mathcal{W}\Omega^\bullet_X \to \Omega^\bullet_{X/k}. \tag{3.2.10.2}$$

Note that, by Popescu's theorem, (a) and (b) extend to X being merely assumed to be a regular $\mathbf{F}_p$-scheme.

Both (a) and (b) follow easily from Lemmas 3.2.7 and 3.2.8. Indeed, for (a), we may assume that X is lifted to Y formally smooth over W, with a lifting of Frobenius, so that, as observed above, we have $\mathcal{W}\Omega^\bullet_X = \mathcal{W}\mathrm{Sat}(\Omega^\bullet_{Y/W})$. Then we have quasi-isomorphisms

$$\Omega^\bullet_{Y/W} \to \varprojlim \mathrm{Sat}(\Omega^\bullet_{Y/W})/p^n \to \varprojlim \mathcal{W}_n\Omega^\bullet_X = \mathcal{W}\Omega^\bullet_X, \tag{$*$}$$

the first one by (3.2.8 (b)), the second one by (3.2.7). Then the reduction mod p of this composition is a quasi-isomorphism

$$\Omega^\bullet_Y/p \to \mathcal{W}\Omega^\bullet_X/p,$$

[7] added in February, 2020.

and the isomorphism it induces on H^i is the right vertical arrow of the commutative square

$$\begin{array}{ccc} \Omega^i_Y/p & \xrightarrow{C^{-1}} & H^i(\Omega^\bullet_Y/p) \\ \downarrow & & \downarrow \\ \mathcal{W}_1\Omega^i_X & \xrightarrow{F} & H^i(\mathcal{W}\Omega^\bullet_X/p), \end{array}$$

where the upper horizontal map is the Cartier isomorphism. As the bottom horizontal arrow is an isomorphism (by ([8], 2.7.1) (see (5.1.1)), the left vertical arrow, which is (3.2.10.1) is an isomorphism as well. For (b), we may assume again that X has the lifting (Y, F), and (3.2.10.2) is the inverse of the reduction mod p of the composition of the quasi-isomorphisms (*).

The new result, mentioned above, is the following:

Theorem 3.2.11. ([8], 10.1.2) There exists a unique equivalence of presheaves on the category of affine regular $\mathbf{F}_p$-schemes $X = \mathrm{Spec}(R)$ with values in the derived ∞-category $\mathcal{D}(\mathbf{F}_p)$

(3.2.11.1) $$R\Gamma(X/\mathbf{Z}_p) \xrightarrow{\sim} R\Gamma(X, W\Omega^\bullet_X)(= W\Omega^\bullet_R),$$

which preserves the $\mathbf{F}_p$-algebra structures on both sides, and lifts the isomorphism in $D(\mathbf{F}_p)$

$$R\Gamma(X, \Omega^\bullet_X)(= \Omega^\bullet_R) \xrightarrow{\sim} R\Gamma(X, W\Omega^\bullet_X) \otimes^L \mathbf{F}_p$$

inverse to (3.2.10.2).

The isomorphism (3.2.2) is deduced from (3.2.11.1) by sheafification (and passing to the homotopy category). One uses Lurie's equivalence

$$\mathcal{D}^+(X, \mathbf{F}_p) \xrightarrow{\sim} Sh^+(X, \mathcal{D}(\mathbf{F}_p))$$

given by [DAG VIII, Proposition 2.1.8], where the right hand side denotes the full subcategory of the category of sheaves on X with values in $\mathcal{D}(\mathbf{F}_p)$ consisting of objects L such that $\mathcal{H}^i(L) = 0$ for $i << 0$.

The proof of 2.11 makes heavy use of (quasi-)syntomic techniques developed in [9].

3.3. The Frobenius isogeny

Let k be a perfect field k (of characteristic p). If X/k is smooth of dimension d, $W\Omega^i_X = 0$ for $i > d$ ([24], I 3.7 (a)). As $W\Omega^\bullet_X$ is saturated, it follows that F induces an automorphism of $W\Omega^d_X$, so that we have an endomorphism v of $W\Omega^\bullet_X$ defined by $p^{d-i-1}V$ in degree i (with $p^{-1}V = F^{-1}$

in degree d). If φ is the endomorphism of $W\Omega^\bullet_X$ defined by p^iF in degree i, then the endomorphisms φ and v satisfy

$$v\varphi = \varphi v = p^d. \tag{3.3.1}$$

This implies that, if in addition X/k is proper, then $H^*(X/W)/\text{tors}$ is an F-crystal of level $\leqslant d$ (cf. ([24], II 2.8.3)). That had been known since the early days of crystalline cohomology (as a consequence of Poincaré duality, proved by Berthelot [1]). Refinements and generalizations were proved in ([2], 8.20) and ([3], 1.6).

A relation similar to (3.3.1) holds for saturated de Rham-Witt complexes and possibly singular varieties. We have the following theorem ([8], 9.3.6):

Theorem 3.3.2. Let X/k be of finite type, and let N be an integer such that the local embedding dimension[8] of X/k is at most N. Then

$$\mathcal{W}\Omega^i_X = 0$$

for $i > N$.

As above, we get that F is bijective on $\mathcal{W}\Omega^N_X$, and:

Corollary 3.3.3. Let φ (resp. v) be the endomorphism of $\mathcal{W}\Omega^\bullet_X$ defined by p^iF (resp $p^{N-i-1}V$) in degree i. Then we have:

$$v\varphi = \varphi v = p^N.$$

The proof of 3.3.2 is quite indirect. It relies on the theory of the *derived de Rham-Witt complex* $LW\Omega^\bullet_R$ for $\mathbf{F}_p$-algebras R, and a rather surprising theorem ([8], Th. 9.3.1) to the effect that $\mathcal{W}\Omega^\bullet_R$ is the "saturation" of $LW\Omega^\bullet_R$ in a suitably derived sense. The assumption on X appears in the use of Quillen's standard décalage formula relating derived functors of $\bigwedge$ and Γ.

Corollary 3.3.4. Let $f : X \to Y$ be a morphism of k-schemes of finite type, and let K denote the fraction field of W.

(a) If f is a universal homeomorphism, then $\mathcal{W}\Omega^\bullet_Y \otimes K \to f_*\mathcal{W}\Omega^\bullet_X \otimes K$ is an isomorphism.

(b) If f is a universal homeomorphism with trivial residue extensions, then $\mathcal{W}\Omega^\bullet_Y \to f_*\mathcal{W}\Omega^\bullet_X$ is an isomorphism.

Proof. (a) ([8], 9.3.9) Applying 3.3.3 to the absolute Frobenius endomorphisms of X and Y, we get that $\mathcal{W}\Omega^\bullet_X \otimes K \to \mathcal{W}\Omega^\bullet_{X^{\text{perf}}} \otimes K$ is an

[8]Recall that this is the minimum d such that X can be locally embedded in a smooth scheme of dimension d over k, and is also the minimum d such that $\Omega^1_{X/k}$ can be locally generated by d elements.

isomorphism, where $X^{\mathrm{perf}} \to X$ is the perfection of X, and similarly for Y. But $f^{\mathrm{perf}} : X^{\mathrm{perf}} \to Y^{\mathrm{perf}}$ is again a universal homeomorphism, hence an isomorphism ([6], Lemma 3.8), so $\mathcal{W}\Omega^\bullet_{Y^{\mathrm{perf}}} \to \mathcal{W}\Omega^\bullet_{X^{\mathrm{perf}}}$ is an isomorphism.

(b) Let $f^{\mathrm{sn}} : X^{\mathrm{sn}} \to Y^{\mathrm{sn}}$ be the seminormalization of f (cf. 4.2). Then f^{sn} is a universal homeomorphism with trivial residue extensions, hence an isomorphism (cf. [The Stacks project, Tag 0EUK]). Hence the conclusion follows from ([8], 6.5.2)) (cf. 4.2).

4. Examples

In the next sections, unless otherwise stated, k denotes a perfect field of characteristic $p > 0$, and $W = W(k)$ as usual.

4.1. Nodes

Let $R = k[x, y]/(xy)$. Using the lifting $W[x, y]/(xy)$, together with the lifting of Frobenius given by $x \mapsto x^p$, $y \mapsto y^p$, one can show that one has a Mayer-Vietoris-like short exact sequence

$$0 \to \mathcal{W}\Omega^\bullet_R \to W\Omega^\bullet_{k[x]} \oplus W\Omega^\bullet_{k[y]} \to W \to 0$$

(where $(a, b) \in W\Omega^i_{k[x]} \oplus W\Omega^i_{k[y]}$ is sent to $\overline{a} - \overline{b}$, with $\overline{a}$ (resp. $\overline{b}$) the image of a (resp. b) in $W\Omega^i_k$). It is F and V compatible, and remains exact if $\mathcal{W}$ (resp. W) is replaced by $\mathcal{W}_n$ (resp. W_n), $n \geqslant 1$. In particular, the map $\Omega^\bullet_R \to \mathcal{W}_1\Omega^\bullet_R$ is bijective in degree 0 but not in degree 1.

More generally, assume that X/k is a finite union of smooth schemes X_i/k $(1 \leqslant i \leqslant N)$ crossing transversally. For $J \subset \{1, \cdots, N\}$, let $X_J := \cap_{j \in J} X_j$, and let $X_r := \coprod_{\sharp J = r+1} X_J$, and $\varepsilon_r : X_r \to X$ be the projection. Then one can show that one has an exact sequence

$$0 \to \mathcal{W}\Omega^\bullet_X \to \varepsilon_{0*} W\Omega^\bullet_{X_0} \to \varepsilon_{1*} W\Omega^\bullet_{X_1} \to \cdots \to \varepsilon_{r*} W\Omega^\bullet_{X_r} \to \cdots,$$

compatible with F and V, and which remains exact when $\mathcal{W}$ (resp. W) is replaced by $\mathcal{W}_n$ (resp. W_n), $n \geqslant 1$. In a sense, $\mathcal{W}\Omega^\bullet_X$ plays the role of a p-adic analogue of the *du Bois complex* [17]. See reference [35] in 4.3 for generalizations.

4.2. Cusps.

Let R be the subring of $k[t]$ generated ty t^2 and t^3 $(R = k[x, y]/(x^2 - y^3))$. Then, using the lifting $W[t^2, t^3] \subset W[t]$, with the lifting of Frobenius $t \mapsto t^p$, it is shown ([8], 6.2.1) that the inclusion $R \subset k[t]$ induces an isomorphism

$$\mathcal{W}\Omega^\bullet_R \xrightarrow{\sim} \mathcal{W}\Omega^\bullet_{k[t]} \; (= W\Omega^\bullet_{k[t]}).$$

In particular the canonical map $R \, (= W\Omega^0_R) \to \mathcal{W}_1\Omega^0_R (= k[t])$ is not an isomorphism.

This is generalized as follows. Recall that a (commutative) ring R is said to be *seminormal* if R is reduced and, for any x, y in R such that $x^2 = y^3$, there exists $t \in R$ such that $x = t^3$ and $y = t^2$. It was proved by Swan ([36], th. 4.1) that any ring R admits a universal map $R \to R^{\mathrm{sn}}$ to a seminormal ring, called the seminormalization of R. This map $R \to R^{\mathrm{sn}}$ is a universal homeomorphism on spectra with trivial residue extensions, and is final among such maps (see [The Stacks project, Tag 0EUK]). For $\mathbf{F}_p$-algebras, the theory of saturated de Rham-Witt complexes yields another description of this seminormalization, namely ([8], 6.5.3), if R is an $\mathbf{F}_p$-algebra, then

$$R^{\mathrm{sn}} = \mathcal{W}_1\Omega^0_R.$$

Moreover, the canonical map $R \to R^{\mathrm{sn}}$ induces an isomorphism

$$\mathcal{W}\Omega^\bullet_R \xrightarrow{\sim} \mathcal{W}\Omega^\bullet_{R^{\mathrm{sn}}}$$

([8], 6.5.2)).

4.3. Toric singularities.[9]

Let X be a log smooth scheme over k endowed with the trivial log structure, and let $j : U \hookrightarrow X$ be inclusion of the smooth locus of X. For example, one can take $X = \mathrm{Spec}(k[P])$, with P a sharp, fine, and saturated monoid. We have adjunction maps

$$\mathcal{W}\Omega^\bullet_X \to j_*j^*\mathcal{W}\Omega^\bullet_X \ (= j_*W\Omega^\bullet_U), \tag{4.3.1}$$

$$\mathcal{W}_n\Omega^\bullet_X \to j_*j^*\mathcal{W}_n\Omega^\bullet_X \ (= j_*W_n\Omega^\bullet_U) \tag{4.3.1 − n}$$

($n \geqslant 1$).

We have the following questions:

(i) Are these maps isomorphisms?

(ii) Is $\mathcal{W}_1\Omega^\bullet_X$ a complex of coherent sheaves?

The complex on the right hand side of (4.3.1-1), i.e., $j_*\Omega^\bullet_U$, is a complex of reflexive, coherent sheaves on the (normal) scheme X. So the answer to (ii) is yes if (4.3.1-1) is an isomorphism. It is asserted in ([11], 3.2) that this complex, called the *Zariski-de Rham complex*, satisfies a Cartier isomorphism $j_*\Omega^i_U \xrightarrow{\sim} \mathcal{H}^i(j_*\Omega^\bullet_U)$. However, according to Ogus (private communication), this may fail in small characteristics, and, hence, by (5.1.3) below, (4.3.1-1) fails to be an isomorphism in such cases. It would be interesting to compare

[9](*Added in July, 2020.*) See [35] for a thorough discussion of the questions mentioned below and several generalizations and complements.

$\mathcal{W}_1\Omega^\bullet_X$ with variants of the Zariski-de Rham complex considered by Ogus in ([34], V 2.2, 2.3).

In the case $X = \mathrm{Spec}(k[P])$ as above, X has the lifting $Y = \mathrm{Spec}(W[P])$, with the lifting F of Frobenius given by multiplication by p on P, which makes $\Omega^\bullet_{W[P]/W}$ into a Dieudonné algebra. Hence, by ([8], 4.2.3), we have the simple description of $\mathcal{W}\Omega^\bullet_X$:

$$\mathcal{W}\Omega^\bullet_X = \mathcal{W}\mathrm{Sat}(\widehat{\Omega}^\bullet_{Y/W}).$$

(where $\widehat{\Omega}$ means the p-completed de Rham complex, cf. the discussion after (5.1.5) below).

5. Fixed points of $L\eta_p$: the fractal nature of strict Dieudonné complexes

5.1. The saturated Cartier isomorphism

Saturated Dieudonné complexes are seldom of Cartier type (in the sense of 3.2.8)[10]. However they give rise to the following analogue of the Cartier isomorphism ([8], 2.7.1), whose proof is immediate from the definitions:

Lemma 5.1.1. Let (M, F) be a saturated Dieudonné complex. Then, for any $n \geqslant 1$ and $i \in \mathbf{Z}$, F^n induces an isomorphism

$$F^n : \mathcal{W}_n M^i \xrightarrow{\sim} H^i(M/p^n M).$$

This isomorphism is compatible with the differential of $\mathcal{W}_n M$ and the Bockstein operator $\delta : H^i(M/p^n M) \to H^{i+1}(M/p^n M)$ induced by the exact sequence of complexes $0 \to M/p^n M \to M/p^{2n} M \to M/p^n M \to 0$, i.e., the square

$$\begin{array}{ccc} \mathcal{W}_n M^i & \xrightarrow{d} & \mathcal{W}_n M^{i+1} \\ \downarrow{\scriptstyle F^n} & & \downarrow{\scriptstyle F^n} \\ H^i(M/p^n M) & \xrightarrow{\delta} & H^{i+1}(M/p^n M) \end{array}$$

is commutative.

Combining with the isomorphism $H^i(M/p^n M) \xrightarrow{\sim} H^i(\mathcal{W}_n M)$ (3.2.7), we get an isomorphism

$$C^{-n} : \mathcal{W}_n M^i \xrightarrow{\sim} H^i(\mathcal{W}_n M), \tag{5.1.2}$$

[10] It follows from (5.1.1), (5.1.2) that if a saturated Dieudonné complex (M, F) is of Cartier type, and its components are p-adically separated, then F is bijective and $d = 0$. See 7.1.

that I propose to call the *saturated Cartier isomorphism*. In particular, if X is a k-scheme, the saturated Cartier isomorphism

$$C^{-n} : \mathcal{W}_n\Omega^\bullet_X \to \mathcal{H}^\bullet(\mathcal{W}_n\Omega^\bullet_X) \tag{5.1.3}$$

is a σ^n-linear isomorphism, and it is compatible with the multiplicative structures on both sides. In the case X/k is smooth, (5.1.3) was discovered in ([25], III 1.4).

Suppose that M is the saturation $\mathrm{Sat}(K)$ of a Dieudonné complex K. Then the relation between Cartier and saturated Cartier isomorphisms is as follows. We have a commutative diagram:

$$\begin{array}{ccc} K^i/pK^i & \xrightarrow{C^{-1}} & H^i(K/pK) \\ \downarrow & & \downarrow \\ \mathcal{W}_1M^i & \xrightarrow{C^{-1}} & H^i(\mathcal{W}_1M), \end{array}$$

where the horizontal maps are induced by F and the vertical maps are the canonical projections. The bottom horizontal map is the isomorphism (5.1.2) for $n = 1$. The right vertical map is the composition of the canonical map $H^i(K/pK) \xrightarrow{\sim} H^i(M/pM)$ (3.2.8 (b)) and the isomorphism (3.2.7) recalled above. If K is if Cartier type, then (by 3.2.8 (b)) the right vertical map is an isomorphism, and therefore so is the left one

$$K^i/pK^i \to \mathcal{W}_1M^i. \tag{5.1.5}$$

Here is a typical example. Let R be a k-algebra of finite type admitting a flat, formal lifting B over W, together with a lifting $F : B \to B$ of Frobenius. Then the (p-completed) de Rham complex $\Omega^\bullet_B$ of B/W is a Dieudonné algebra (by a variant, for formal liftings, of the construction explained at the beginning of the proof of 2.3). By ([8], 4.2.3) we have

$$\mathcal{W}\Omega^\bullet_R = \mathcal{W}\mathrm{Sat}(\Omega^\bullet_B).$$

The map (5.1.5) is the map $\mathrm{can}_1 : \Omega^\bullet_R \to \mathcal{W}_1\Omega^\bullet_R$ of (3.1.1). If R is *smooth* over k, $\Omega^\bullet_B$ is of Cartier type, and can_1 is an isomorphism. In general, $\Omega^\bullet_B$ is not of Cartier type, and can_1 is not an isomorphism (as observed in examples 4.1 and 4.2). However, we still have the saturated Cartier isomorphism (5.1.3).

Let now M be a saturated Dieudonné complex. Consider the family of saturated Cartier isomorphisms (5.1.2) for $n \geqslant 1$. The left hand side $\mathcal{W}_\bullet M^\bullet$ is endowed with various operators: $d : \mathcal{W}_nM^i \to \mathcal{W}_nM^{i+1}$, projections $R : \mathcal{W}_{n+1}M^i \to \mathcal{W}_nM^i$, and operators $F : \mathcal{W}_{n+1}M^i \to \mathcal{W}_nM^i$,

$V : \mathcal{W}_n M^i \to \mathcal{W}_{n+1} M^i$, satisfying various relations and properties making $\mathcal{W}_\bullet M^\bullet$ a *strict Dieudonné tower* in the terminology of ([8], 2.6.1). The saturated Cartier isomorphisms make an isomorphic image of this tower in its cohomology. This is vaguely analogous to the fact that a Mandelbrot set contains a nontrivial homeomorphic image of itself, and is an excuse for the title of the section. It was a challenging problem, however, to concretely describe the corresponding structure on the right hand side, especially in the case where M is the saturation of a Dieudonné complex K of Cartier type, as in the example $K = \Omega^\bullet_B$ above, with R/k smooth. In this case, the natural maps

$$H^\bullet(K/p^nK) \to H^\bullet(M/p^nM) \to H^\bullet(\mathcal{W}_n M)$$

are isomorphisms, and the sought for operators d, R, F, V on the left hand side are mysterious. This is related to the reconstruction of the de Rham-Witt complex $W\Omega^\bullet_X$ of a smooth X/k in terms of the crystalline cohomology of X/W, as mentioned after 1.4.

The solution, however, is simple. The differential $d : H^i(K/p^nK) \to H^{i+1}(K/p^nK)$ is given by the Bockstein operator associated with the exact sequence

$$0 \to K/p^nK \xrightarrow{p^n} K/p^{2n}K \to K/p^nK \to 0.$$

The operator $F : H^i(K/p^{n+1}K) \to H^i(K/p^nK)$ (resp. $V : H^i(K/p^nK) \to H^i(K/p^{n+1}K)$) is induced by the projection (resp. multiplication by p). The definition of the restriction $R : H^i(K/p^{n+1}K) \to H^i(K/p^nK)$ is more subtle. It relies on the Ogus-type isomorphism (3.2.8 (a)):

$$\alpha_F \otimes \mathbf{Z}/p^n : H^i(K/p^nK) \xrightarrow{\sim} H^i(\eta_p K/p^n \eta_p K).$$

Let $x \in K^i$ such that p^{n+1} divides dx, with cohomology class $[x] \in H^i(K/p^{n+1}K)$. Then $p^i x$ gives a cycle in $(\eta_p K)^i/p^n(\eta_p K)^i$. Let $[p^i x]$ denotes its cohomology class in $H^i(\eta_p K/p^n \eta_p K)$. Then

$$R[x] = (\alpha_F \otimes \mathbf{Z}/p^n)^{-1}[p^i x].$$

5.2. The fixed point theorem

The above reconstruction is generalized in [8] in the form of a fixed point theorem for the functor $L\eta_p$. In order to state it, let me recall a few definitions.

Let $D(\mathbf{Z})$ denote the (unbounded) derived category of $\mathbf{Z}$-modules. It is obtained from the category $C(\mathbf{Z})$ of complexes of $\mathbf{Z}$-modules by formally inverting quasi-isomorphisms. It can also be obtained from the full subcategory

$C(\mathbf{Z})^{\mathrm{tf}}$ of torsion free complexes by formally inverting quasi-isomorphisms. The functor $\eta_p : C(\mathbf{Z})^{\mathrm{tf}} \to C(\mathbf{Z})^{\mathrm{tf}}$ admits an essentially unique extension

$$L\eta_p : D(\mathbf{Z}) \to D(\mathbf{Z}).$$

This is a particular case of ([2], 8.19), but in the case of $\eta = \eta_p$, it just follows from the fact (cf. [8], 7.2.1) that if $K \in C(\mathbf{Z})$ is p-torsion free, then

$$H^i(\eta_p K) = H^i(K)/H^i(K)[p]$$

(where $[p]$ means the kernel of p). A *caveat* here: the image by $L\eta_p$ of a distinguished triangle is not necessarily a distinguished triangle. An object $K \in D(\mathbf{Z})$ is called *derived p-complete* if the natural map

$$K \to R \varprojlim_n K \otimes^L \mathbf{Z}/p^n\mathbf{Z}$$

is an isomorphism (see [7], 6.16). The full subcategory

$$\widehat{D(\mathbf{Z})_p} \subset D(\mathbf{Z})$$

consisting of p-complete objects enjoys many nice properties (see *loc. cit.*). In particular, it is stable under $L\eta_p$. Now, if $T : \mathcal{C} \to \mathcal{C}$ is an endofunctor of a category $\mathcal{C}$, the category

$$\mathcal{C}^T$$

of *fixed points* of T is defined as the category of pairs (K, u), where K is an object of $\mathcal{C}$ and u is an isomorphism $u : K \xrightarrow{\sim} TK$. As a strict complex K is saturated, and derived p-complete, and $L\eta_p K = \eta_p K$, K is via α_F a fixed point of $L\eta_p$. We thus have a tautological inclusion

$$\iota : \mathbf{DC}_{\mathrm{str}} \subset \widehat{D(\mathbf{Z})_p}^{L\eta_p}. \tag{5.2.1}$$

The main result of ([8], 7) is:

Theorem 5.2.2. The inclusion (5.2.1) is an equivalence.

A quasi-inverse ψ of ι is constructed as follows[11]. Let $(K, u : K \xrightarrow{\sim} L\eta_p K)$ be a fixed point of $L\eta_p$ on $\widehat{D(\mathbf{Z})_p}$. For $n \geq 1$, $i \in \mathbf{Z}$, let

$$L^i_n := H^i(K \otimes^L \mathbf{Z}/p^n\mathbf{Z}).$$

[11]This construction was made in an earlier version of [8]. A different argument is given in the last version.

Define $d : L_n^i \to L_n^{i+1}$ by the Bockstein operator associated with $0 \to \mathbf{Z}/p^n \to \mathbf{Z}/p^{2n} \to \mathbf{Z}/p^n \to 0$. Define

$$F : L_{n+1}^i \to L_n^i, \ \ V : L_n^i \to L_{n+1}^i$$

respectively by the obvious projection and the map induced by multiplication by p. Finally, define

$$R : L_{n+1}^i \to L_n^i$$

by the recipe described at the end of 5.1. We may assume that K is p-torsion free, so that $L_n^i = H^i(K/p^nK)$. Given $x \in K^i$ such that p^{n+1} divides dx, then p^ix gives a cocycle of degree i in $\eta_pK/p^n\eta_pK$. Let $[p^ix]$ be its class in $H^i(\eta_pK/p^n\eta_pK)$. Now use the isomorphism

$$H^i(u \otimes^L \mathbf{Z}/p^n) : H^i(K/p^nK) \xrightarrow{\sim} H^i(\eta_pK/p^n\eta_pK),$$

(that we'll still denote by u) and define

$$R[x] := u^{-1}([p^ix])$$

(where $[x]$ is the class of x in L_{n+1}^i). Then one checks that $(L_\bullet^\bullet, d, R, F, V)$ is a strict Dieudonné tower ([8], 2.6.1), so that $L := \varprojlim_R L_n$, equipped with the operators d, F deduced from the tower is a saturated Dieudonné complex ([8], 2.6.5). It is in fact strict, as was shown in an earlier version of [8]). We define

$$\psi K := L.$$

One checks that ψ is indeed a functor from $\widehat{D(\mathbf{Z})_p}^{L\eta_p}$ to $\mathbf{DC}_{\mathrm{str}}$, and that it is quasi-inverse to ι. The isomorphism $\psi \circ \iota \xrightarrow{\sim} \mathrm{Id}$ follows from the saturated Cartier isomorphisms C^{-n} (5.1.1, (5.1.2)). The construction of an isomorphism $\mathrm{Id} \xrightarrow{\sim} \iota \circ \psi$ uses Deligne's lemma 1.2 for the f-adic filtration with $f = p^n$, i.e., the quasi-isomorphism $(\eta_{p^n}K^i/p^n\eta_{p^n}K^i, d) \to (H^i(p^{ni}K/p^{n(i+1)}K), d_1)$.

Remarks 5.2.3. (a) It is surprising that ι could be an equivalence, considering that the left hand side is a category of complexes (where morphisms are morphisms of complexes) while the right hand one is a full subcategory of a derived category, where morphisms are not morphisms of complexes, but classes of certain fractions fs^{-1} (or $t^{-1}g$) with s (resp. t) a quasi-isomorphism. This strange equivalence can even be enhanced to an equivalence on the ∞-category level. Let $\mathcal{D}(\mathbf{Z})$ be the derived ∞-category of $\mathbf{Z}$-modules, i.e., the ∞-category obtained from the category of complexes $C(\mathbf{Z})$ by inverting the quasi-isomorphisms. Let $\widehat{\mathcal{D}(\mathbf{Z})_p}$ be the inverse image of $\widehat{D(\mathbf{Z})_p}$ by the forgetful functor $\mathcal{D}(\mathbf{Z}) \to D(\mathbf{Z})$. The functor

$\eta_p : C(\mathbf{Z})^{\rm tf} \to C(\mathbf{Z})^{\rm tf}$ extends uniquely to a functor $L\eta_p : \mathcal{D}(\mathbf{Z}) \to \mathcal{D}(\mathbf{Z})$, which preserves $\widehat{\mathcal{D}(\mathbf{Z})}_p$. Then we have the following striking refinement of 1.2.2 ([8], Th. 7.4.8): the forgetful functor

$$\widehat{\mathcal{D}(\mathbf{Z})}_p^{L\eta_p} \to \widehat{D(\mathbf{Z})}_p^{L\eta_p}$$

is an equivalence. In particular, the left hand side is equivalent to an ordinary category, and the inclusion $\mathbf{DC}_{\rm str} \hookrightarrow \widehat{\mathcal{D}(\mathbf{Z})}_p^{L\eta_p}$ is an equivalence. Here the notation $\mathcal{C}^T$ for an endofunctor of an ∞-category $\mathcal{C}$ means the homotopy equalizer of the pair of endofunctors (Id, T). This refinement is used in the simplified proof of the crystalline comparison theorem for $A\Omega$ alluded to at the end of 1, see 5.3 (b).

(b) Ogus' globalization of 1.4 is the following. Let X/k be smooth, and consider the map $u : (X/W)_{\rm crys} \to X_{\rm Zar}$ defined in ([2], p. 7.27). Then, by ([2], 8.20) the σ-linear endomorphism φ of $Ru_*\mathcal{O}_{X/W}$ factors as

$$(*) \qquad Ru_*\mathcal{O}_{X/W} \xrightarrow{\widetilde{\varphi}} L\eta_p Ru_*\mathcal{O}_{X/W} \to Ru_*\mathcal{O}_{X/W}$$

where $\widetilde{\varphi}$ is an isomorphism[12]. In particular, on each open subscheme U of X, (*) exhibits $R\Gamma(U/W) = R\Gamma(U, Ru_*\mathcal{O}_{U/W})$ as a fixed point of $L\eta_p$.

As observed after 1.4, Ogus's theorem is more general, involving subsheaves of $\mathcal{O}_{X/W}$ defined by certain *gauges* à la Mazur, needed for his proof of 1.3. However, the factorization (*) – or even 1.4 – suffices to reconstruct the de Rham-Witt complex $W\Omega^\bullet_X$ (and prove its main properties), independently of [24], see 5.3 (a) below. And once one has the machinery of de Rham-Witt at one's disposal, it is rather simple to prove 1.3. This is what Nygaard did in [33]. He even proved generalizations of 1.3 for powers of Frobenius. For this he introduces certain subcomplexes of $W\Omega^\bullet_X$, which provide a replacement for Mazur-Ogus' gauges. They are now called *Nygaard complexes*. They were used in [25] in the study of the conjugate spectral sequence, and more recently by Langer-Zink [29] in their theory of displays. These constructions are revisited in [8] in the framework of Dieudonné complexes and the fixed point theorem 5.2.2. For $r \in \mathbf{Z}$, the Nygaard subcomplex $\mathcal{N}^r M$ of a saturated Dieudonné complex M is the complex

$$\mathcal{N}^r M = (\cdots \to M^{r-2} \xrightarrow{d} M^{r-1} \xrightarrow{dV} M^r \xrightarrow{d} M^{r+1} \to \cdots),$$

where the differential d of M is unchanged except for $d : M^{r-1} \to M^r$, which is replaced by dV. It is a subcomplex of M by the morphism $\mathcal{N}^r M \hookrightarrow M$

[12]Note that, by ([2], 7.22.2), $Ru_*\mathcal{O}_{X/W} = R\varprojlim Ru_*\mathcal{O}_{X/W_n}$ so that in the situation of 1.4, $Ru_*\mathcal{O}_{X/W} = \Omega^\bullet_Z$.

equal to Id for $i \geq r$ and $p^{r-1-i}V$ for $i < r$. One of the main points in [8] is that, if K is a Dieudonné complex of Cartier type, whose components are p-adically complete, then the quasi-isomorphism (3.2.8 (c))

$$K \to \mathcal{W}\mathrm{Sat}(K)$$

can be enhanced into a *filtered quasi-isomorphism*, i.e., a map of filtered complexes inducing quasi-isomorphisms on the associated graded objects, when one puts the Nygaard filtration $\mathcal{N}$ on the right hand side, and the filtration $\mathcal{N}_u$ on the left hand side defined by

$$\mathcal{N}_u^r K := (\cdots \to p^3 M^{r-3} \to p^2 M^{r-2} \to pM^{r-1} \to M^r \to M^{r+1} \to \cdots)$$

(i.e., $\mathcal{N}^r K$ is deduced from K by replacing $d : K^i \to K^{i+1}$ by $pd : K^i \to K^{i+1}$ for $i < r$ (and letting it unchanged otherwise)). In the case $M = \Omega^\bullet_B$, with B as in 3.2, so that the quasi-isomorphism (3.2.8 (c)) is (3.2.3), the left hand side, for $r < p$, calculates $Ru_* J^{[r]}_{X/W}$, and the above filtered quasi-isomorphsim is a particular case of ([29], Th. 4.6). No crystalline interpretation of the Nygaard filtration is known for $r \geqslant p$. See ([16], 2.2 (iv))) for relations between this and decompositions of the de Rham complex.

(c) For various applications, refinements of the fixed point theorem 5.2.2 involving multiplicative structures are needed. Though for strict Dieudonné algebras there is no equivalence similar to (5.2.1), the category $\mathbf{DA}_{\mathrm{str}}$ can be embedded as a full subcategory of the commutative algebra objects of $\widehat{D(\mathbf{Z})_p}^{L\eta_p}$ ([8], 7.6.7).

5.3. Applications

(a) *Reconstruction of the de Rham-Witt complex.* As suggested in ([25], III 1.5), in the smooth, lifted situation ($X = \mathrm{Spec}(R), Z, F$) of (3.2), the quasi-inverse ψ to the equivalence ι of (5.2.2) provides an alternate construction of the de Rham-Witt complex $W\Omega^\bullet_R$, which assumes no prior knowledge of Witt vectors. Indeed, it suffices to apply ψ to $\Omega^\bullet_B$, which by Ogus' lemma (1.4, 3.2.9 (ii)) is an object of $\widehat{D(\mathbf{Z})_p}^{L\eta_p}$. The Dieudonné algebra structure on $A := \varprojlim H^\bullet(\Omega^\bullet_B/p^n)$ (where the inverse limit is taken with respect to the restriction maps described in the proof of 5.2.2) comes from the multiplicative refinements alluded to in (5.2.3 (c)). The map $e : R \to \mathcal{W}_1 A^0 = H^0(\Omega^\bullet_B/p) = H^0(\Omega^\bullet_R)$ is given by Frobenius, and the pair (A, e) possesses the universal property that exhibits A as a saturated dRW complex of R in the sense of th. 2.3 ([8], 4.1.1). Indeed, as $\mathcal{W}\mathrm{Sat}(\Omega^\bullet_B)$, together with the obvious map $R \to \mathcal{W}_1\mathrm{Sat}(\Omega^\bullet_B)$ is a saturated dRW complex of R ([8], 4.2.3), e extends uniquely to a map of strict Dieudonné algebras

$$\underline{e} : \mathcal{W}\Omega^\bullet_R \to A.$$

Its reduction mod $V + dV$,

$$\mathcal{W}_1\Omega^\bullet_R \to \mathcal{W}_1 A$$

has to be the Cartier isomorphism from the de Rham complex of R to the complex $H^\bullet(\Omega^\bullet_B/p)$ with the Bockstein differential

$$C^{-1} : \Omega^\bullet_R \to H^\bullet(\Omega^\bullet_B/p).$$

Therefore, by ([8], 2.5.5), $\mathcal{W}_n\underline{e}$ is an isomorphism for all n, hence $\underline{e}$ is an isomorphism.

(b) *Crystalline specialization of* $A\Omega$. Let C be an algebraically closed, complete, nonarchimedean valued field of mixed characteristic $(0, p)$, assumed to be perfectoid (i.e. such that the Frobenius map on $\mathcal{O}_C/p\mathcal{O}_C$ is surjective). Let $\mathcal{X}$ be a smooth formal scheme over $\mathcal{O}_C$, with special fiber $\mathcal{X}_k$ (k the residue field of $\mathcal{O}_C$), and rigid generic fiber X. Let

$$A\Omega_{\mathcal{X}} := L\eta_\mu R\nu_* A_{\mathrm{inf},X}$$

be the object of $D^+(\mathcal{X}, A_{\mathrm{inf}})$ constructed in ([7], §9). One of the main results of *loc. cit.* is that there is defined a canonical *specialization isomorphism*

$$A\Omega_{\mathcal{X}} \widehat{\otimes}^L_{A_{\mathrm{inf}}} W \xrightarrow{\sim} W\Omega^\bullet_{\mathcal{X}_k}, \tag{5.3.1}$$

compatible with Frobenius actions on both sides, and multiplicative structures, where $A_{\mathrm{inf}} = W(\mathcal{O}^\flat_C) \to W = W(k)$ is the canonical projection ([7], 14.1.1 (i)). A new proof is given in ([8], §10). It exploits the fact that $A\Omega_{\mathcal{X}}$ is a fixed point of the operator $L\eta_{\widetilde{\xi}}$ for a certain element $\widetilde{\xi}$ of A_{inf} generating the kernel of the standard map $\widetilde{\theta} : A_{\mathrm{inf}} \to \mathcal{O}_C$, and sent to p by the specialization map $A_{\mathrm{inf}} \to W$. The left hand side of (5.3.1) can be thus realized as a fixed point of $L\eta_p$, and one then applies Th. 5.2.2 with its multiplicative refinements, to get a strict Dieudonné algebra, which turns out to be the de Rham-Witt complex of $\mathcal{X}_k$.

6. Open questions

6.1. Comparison with rigid cohomology

As at the beginning of 4, let now k be a perfect field of characteristic $p > 0$, and let X be a projective scheme over k. Let K be the fraction field of $W = W(k)$. Let $R\Gamma_{\mathrm{rig}}(X/K)$ be the object of $D^b(K)$ calculating the rigid cohomology of X/K in the sense of Berthelot (se e. g. [30]). For X/k smooth, there is a canonical isomorphism

$$R\Gamma(X/W) \otimes_W K \xrightarrow{\sim} R\Gamma_{\mathrm{rig}}(X/K),$$

compatible with the product structures and the action of Frobenius, where

$$R\Gamma(X/W) = R\Gamma(X, Ru_*\mathcal{O}_{X/W}) = R\Gamma(X, W\Omega^\bullet_X)$$

calculates the crystalline cohomology of X.

Question 6.1.1. Can one expect a similar isomorphism

$$R\Gamma(X, \mathcal{W}\Omega^\bullet_X) \otimes_W K \xrightarrow{\sim} R\Gamma_{\mathrm{rig}}(X/K)$$

for X/k no longer assumed to be smooth?

When X/k is singular, $R\Gamma(X/W) := R\Gamma(X, Ru_*\mathcal{O}_{X/W})$ is badly behaved (see e. g. [5]), and is *not* calculated by $R\Gamma(X, \mathcal{W}\Omega^\bullet_X)$ but by $R\Gamma(X, LW\Omega^\bullet_X)$ when X/k is locally of complete intersection. Here $LW\Omega^\bullet_X$ is the derived de Rham-Witt complex (see ([8], 9.3.1, 9.3.5) for the relation between $\mathcal{W}\Omega^\bullet_X$ and $LW\Omega^\bullet_X$.

Evidence for a positive answer to the above question is scarce. The only case where the answer is known to be yes is when X has locally strict normal crossings singularities, thanks to the local du Bois shape of $\mathcal{W}\Omega^\bullet_X$ (4.1) and Tsuzuki's proper cohomological descent for rigid cohomology [38]. An example of Bhatt (private communication) shows that one can't expect cdh-descent results for $H^*(X, \mathcal{W}\Omega^\bullet_X)$ in general. On the other hand, 3.3.4 might still give some hope.

6.2. Finiteness

Let X/k be of finite type, with k as in 6.1. When X/k is smooth, $\mathcal{W}\Omega^\bullet_X = W\Omega^\bullet_X$, in particular $\mathcal{W}_1\Omega^\bullet_X$ is the de Rham complex $\Omega^\bullet_X$, the sheaves $W_n\Omega^i_X$ are coherent on the scheme $W_n(X)$, and their structure is well understood [24]. The upshot is that, when in addition X/k is proper, then $R\Gamma(X, W\Omega^\bullet_X)$ $(= R\Gamma(X/W))$ is an object of the derived category $D^b_c(R)$ of coherent complexes over the Raynaud ring $R = R(k)$, mentioned in 2, Example (c). This property implies the fine results on the slope spectral sequence $E_1^{ij} = H^j(X, W\Omega^i_X) \Rightarrow H^{i+j}(X/W)$ treated in ([25], [26]). In general, however, the structure of the sheaves $\mathcal{W}_n\Omega^i_X$ is mysterious. As mentioned at the end of §2, it is known that they are *quasi-coherent* over $W_n(X)$ ([8], 5.2.3), but:

Question 6.2.1. Are the sheaves $\mathcal{W}_1\Omega^i_X$ coherent over X?

Outside of the smooth case, the only known things are:

(i) The answer is yes when X has local normal crossings singularities, and more generally, if X has *ideally toroidal singularities* [35]. In particular, the answer is yes if X/k is a *curve*.[13]

[13] We may assume k algebraically closed and X seminormal, in which case, by (cf. [The

(ii) The answer is yes for $i = 0$, with no extra assumption on X, as $\mathcal{W}_1\Omega^0_X$ is the seminormalization of $\mathcal{O}_X$ (4.2) (which is finite over $\mathcal{O}_X$).

Using the saturated Cartier isomorphism (5.1.3) one can define iterated cycles (resp. boundaries) $Z_n\mathcal{W}_1\Omega^i_X$ (resp. $B_n\mathcal{W}_1\Omega^i_X$) as in the smooth case, and prove analogues of the results of [24] about the structure of $W_n\Omega^i_X$ (see section 8). In particular, we have

$$\mathrm{Ker}(V^n : \mathcal{W}_1\Omega^i_X \to \mathcal{W}_{n+1}\Omega^i_X) = B_n\mathcal{W}_1\Omega^i_X,$$

$$\mathrm{Ker}(dV^n : \mathcal{W}_1\Omega^{i-1}_X \to \mathcal{W}_{n+1}\Omega^i_X) = Z_{n+1}\mathcal{W}_1\Omega^{i-1}_X,$$

and a description of the associated graded $\mathrm{gr}^n\mathcal{W}\Omega^i_X$ for the filtration of $\mathcal{W}\Omega^i_X$ by the subgroups $V^n + dV^n$, as an extension

$$0 \to \mathcal{W}_1\Omega^i_X/B_n \to \mathrm{gr}^n\mathcal{W}\Omega^i_X \to \mathcal{W}_1\Omega^{i-1}_X/Z_n \to 0.$$

Finally, we have an exact sequence

$$0 \to \mathcal{W}\Omega^{i-1}_X \overset{(F^n,-F^nd)}{\to} \mathcal{W}\Omega^{i-1}_X \oplus \mathcal{W}\Omega^i_X \overset{dV^n+V^n}{\to} \mathcal{W}\Omega^i_X \to \mathcal{W}_n\Omega^i_X \to 0$$

similar to ([25], II (1.2.1)), which, in terms of the Raynaud ring R and its quotient $R_n = R/(V^nR + dV^nR)$ is equivalent to an isomorphism

$$R_n \otimes^L_R \mathcal{W}\Omega^\bullet_X \overset{\sim}{\to} \mathcal{W}_n\Omega^\bullet_X. \qquad (6.2.2)$$

In view of the quasi-coherence of the sheaves $\mathcal{W}_n\Omega^i_X$, this implies that

$$R\Gamma(X, \mathcal{W}\Omega^\bullet_X) \overset{\sim}{\to} R\varprojlim R_n \otimes^L_R R\Gamma(X, \mathcal{W}\Omega^\bullet_X).$$

Assume now that the sheaves $\mathcal{W}_1\Omega^i_X$ are coherent. Then, if X/k is *proper*, $R\Gamma(X, \mathcal{W}_1\Omega^\bullet_X)$ belongs to $D^b_c(k[d])$, so that, by Ekedahl's criterion ([26], 2.4.7), ([19], 0 5.13, III 1.1), $R\Gamma(X, \mathcal{W}\Omega^\bullet_X)$ belongs to $D^b_c(R)$, and the results on the slope spectral sequence of ([25], II) hold, with $W\Omega^\bullet_X$ replaced by $\mathcal{W}\Omega^\bullet_X$ ([19], III 1.1). In particular, it degenerates at E_1 modulo torsion, and $H^j(X, \mathcal{W}\Omega^i_X) \otimes K$ gives the part of the F-crystal $H^{i+j}(X, \mathcal{W}\Omega^\bullet_X)$ of slopes in the interval $[i, i+1)$. If X is seminormal, so that $\mathcal{W}\Omega^0_X = W\mathcal{O}_X$ (by ([8], 6.5.2), see 4.2), and if in addition the answer to 6.1.1 is yes, we get (in this case) Berthelot-Bloch-Esnault's result that $H^j(X, W\mathcal{O}_X) \otimes K$ is the part of $H^j_{\mathrm{rig}}(X/K)$ of slopes in $[0, 1)$.

Stacks project, Tag 0EUK]), X is isomorphic to the curve deduced from the normalization $\pi : X^{\mathrm{n}} \to X$ by contracting to a point each finite set $\pi^{-1}(x)$, for $x \in X(k)$ ([15], 10.3.1), hence has singularities which are locally étale isomorphic to a union of coordinate axes in an affine space.

6.3. Relative variants

One can hope for variants of saturated de Rham-Witt complexes in relative situations for morphisms $X \to Y$ over $\mathbf{F}_p$, à la Langer-Zink [27] (or Matsuue [31] for log morphisms). The problem is related to the construction of saturated de Rham-Witt complexes for coefficients in suitable de Rham-Witt connections (cf. [21], [28]).

7. Appendix 1: Saturation and Cartier type

We prove the statement at the beginning of 5.1, namely:

Proposition 7.1. If a saturated Dieudonné complex (M, F) is of Cartier type, and its components are p-adically separated, then F is bijective and $d = 0$.

Consider the commutative diagram

$$\begin{array}{ccc} M^i/pM^i & \xrightarrow{F} & H^i(M/pM) \\ \downarrow & & \downarrow \\ \mathcal{W}_1 M^i & \xrightarrow{C^{-1}} & H^i(\mathcal{W}_1 M) \end{array}$$

where the right vertical arrow is the isomorphism (3.2.7), C^{-1} is the saturated Cartier isomorphism (5.1.2), and the left vertical arrow is the obvious projection. As M is of Cartier type, the top horizontal arrow is an isomorphism. Therefore, the left vertical arrow is an isomorphism

$$(*) \qquad M^i/pM^i \xrightarrow{\sim} \mathcal{W}_1 M^i = M^i/(VM^i + dVM^{i-1}).$$

It is the composition of the canonical projections

$$M^i/pM^i \twoheadrightarrow M^i/VM^i \twoheadrightarrow M^i/(VM^i + dVM^{i-1}).$$

Therefore these two maps are isomorphisms. In particular, $pM^i = VM^i$, hence $pFM^i = pM^i$, and, as M^i is p-torsion-free,

$$FM^i = M^i.$$

In other words, F is surjective, hence bijective. As $dF^n = p^n F^n d$, it follows that p^n divides dM^i for all $n \geq 1$. As M^{i+1} is p-adically separated, $dM^i = 0$.

In general, if $K \in \mathbf{DC}$ is of Cartier type, and $M = \mathrm{Sat}(K)$, we have a commutative diagram of isomorphisms

$$\begin{array}{ccc} K^i/pK^i & \xrightarrow{C^{-1}} & H^i(K/pK) \\ \downarrow & & \downarrow \\ \mathcal{W}_1M^i & \longrightarrow & H^i(M/pM) \\ & \searrow^{C^{-1}} & \downarrow \\ & & H^i(\mathcal{W}_1M). \end{array}$$

8. Appendix 2: Saturated Cartier isomorphism and finiteness

We prove the results in 6.2. Let M be a strict Dieudonné complex. For $n = 1$, the saturated Cartier isomorphism (5.1.2) is an isomorphism

$$C^{-1} : \mathcal{W}_1M^i \xrightarrow{\sim} H^i(\mathcal{W}_1M) \tag{8.1}$$

For a smooth k-algebra R, with k as in 3.2, and $M = \mathcal{W}\Omega^\bullet_R$, by the isomorphism can_1 of 3.1.2 it corresponds to the usual Cartier isomorphism $C^{-1} : \Omega^i_R \xrightarrow{\sim} H^i(\Omega^\bullet_R)$. Using (8.1) one can define analogues of the iterated cycles and boundaries of $\Omega^\bullet_X$ of the smooth case. Namely, for $n \geqslant 0$, one defines inductively $B_n := B_n\mathcal{W}_1M^i$, $Z_n := Z_n\mathcal{W}_1M^i$,

$$0 = B_0 \subset B_1 \subset \cdots \subset B_n \subset \cdots \subset Z_n \subset \cdots \subset Z_1 \subset Z_0 = \mathcal{W}_1M^i, \tag{8.2}$$

by the conditions

$$B_1 := d\mathcal{W}_1M^{i-1}_X,\ Z_1 := \mathrm{Ker}(d : \mathcal{W}_1M^i \to \mathcal{W}_1M^{i+1}),$$
$$C^{-1} : B_n \xrightarrow{\sim} B_{n+1}/B_1,\ C^{-1} : Z_n \xrightarrow{\sim} Z_{n+1}/B_1.$$

In other words,

$$Z_n = \mathrm{Im}(F^n), B_{n+1} = \mathrm{Im}(F^n d) \tag{8.3}$$

where $F^n : \mathcal{W}_{n+1}M^i \to \mathcal{W}_1M^i$, and $d : \mathcal{W}_{n+1}M^{i-1} \to \mathcal{W}_{n+1}M^i$. For R/k smooth, and $M = \mathcal{W}\Omega^\bullet_R$ as above, (8.3) is half of ([24], I (3.11.3), (3.11.4))[14].

The next lemma is a generalization of ([24], I 3.8):

Lemma 8.4. With the above notations, for all $n \geqslant 0$ and all i we have morphisms of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & B_n\mathcal{W}_1M^i & \longrightarrow & \mathcal{W}_1M^i & \xrightarrow{V^n} & \mathcal{W}_{n+1}M^i \\ & & \downarrow & & \downarrow \mathrm{Id} & & \downarrow \\ 0 & \longrightarrow & B_{n+1}\mathcal{W}_1M^i & \longrightarrow & \mathcal{W}_1M^i & \xrightarrow{V^n} & \mathcal{W}_{n+1}M^i/dV^n\mathcal{W}_1M^{i-1}, \end{array} \tag{8.4.1}$$

[14] The other half in the more general situation considered here, i.e., $\mathrm{Ker}(F^n) = V\mathcal{W}_nM^i$, $\mathrm{Ker}(F^n d) = F\mathcal{W}_{n+2}M^{i-1}$ can also be proved, using 8.4.

$$
(8.4.2) \qquad \begin{array}{ccccccc}
0 \longrightarrow & Z_{n+1}\mathcal{W}_1 M^{i-1} & \longrightarrow & \mathcal{W}_1 M^{i-1} & \xrightarrow{dV^n} & \mathcal{W}_{n+1} M^i & \\
& \big\downarrow & & \big\downarrow \mathrm{Id} & & \big\downarrow & \\
0 \longrightarrow & Z_n \mathcal{W}_1 M^{i-1} & \longrightarrow & \mathcal{W}_1 M^{i-1} & \xrightarrow{dV^n} & \mathcal{W}_{n+1} M^i / V^n \mathcal{W}_1 M^i, &
\end{array}
$$

where the left (resp. right) vertical maps are the canonical inclusions (resp. projections).

Proof. The commutativity of the diagrams is trivial. That the compositions of two consecutive horizontal arrows is zero follows from (8.3). Let's prove the reverse inclusions.

(a) Let $x \in M^i$, with image $\overline{x} \in \mathcal{W}_1 M^i$, such that

$$V^n x = V^{n+1} y + dV^{n+1} z$$

for some $y \in M^i$, $z \in M^{i-1}$. Applying F^{n+1} we get:

$$p^n F x = p^{n+1} y + dz.$$

As M is saturated, there exists $t \in M^{i-1}$ such that $z = F^n t$. Then

$$Fx = py + F^n dt.$$

By (8.2), (8.3), this implies $C^{-1}\overline{x} \in B_{n+1}/B_1$, hence $\overline{x} \in B_n$, which proves the exactness of the top row of (8.4.1). Now, if $\overline{x}$ is such that there exists $y \in M^i$, $z \in M^{i-1}$ with

$$V^n x = V^{n+1} y + dV^n z,$$

applying this time F^n instead of F^{n+1}, we get

$$p^n x = p^n V y + dz,$$

hence there exists $t \in M^{i-1}$ such that $z = F^n t$. Then

$$x = Vy + F^n dz,$$

hence, by (8.2), (8.3) again, $\overline{x} \in B_{n+1}$, which proves the exactness of the bottom row of (8.4.1).

(b) Let $x \in M^{i-1}$, with image $\overline{x} \in \mathcal{W}_1 M^{i-1}$, such that

$$dV^n x = V^{n+1} y + dV^{n+1} z$$

for some $y \in M^i$, $z \in M^{i-1}$. Applying F^n we get

$$(*) \qquad d(x - Vz) = p^n V y,$$

hence $x - Vz = F^n t$ for some $t \in M^{i-1}$, and $\overline{x} \in Z_n$. Then (*) gives $F^n dt = Vy$, hence $C^{-n} d\overline{t} = 0$, hence $d\overline{t} = 0$, i.e., $\overline{t} \in Z_1$, and $\overline{x} \in Z_{n+1}$, which proves the exactness of the top row of (8.4.2). Finally, let $x \in M^{i-1}$, with image $\overline{x} \in \mathcal{W}_1 M^{i-1}$, such that

$$dV^n x = V^n y + dV^{n+1} z$$

for some $y \in M^i$, $z \in M^{i-1}$. Applying F^n, we get

$$d(x - Vz) = p^n y,$$

and therefore $x - Vz = F^n t$ for some $t \in M^{i-1}$, so that $\overline{x} \in Z_n$, which proves the exactness of the bottom row of (8.4.2)

For $n \geqslant 0$, let $\mathrm{Fil}^n M$ be the subcomplex of M defined by

(8.5) $$\mathrm{Fil}^n M^i := V^n M^i + dV^n M^{i-1}$$

($\mathrm{Fil}^n M = V^n M + dV^n M$ in short), so that $M/\mathrm{Fil}^n M = \mathcal{W}_n M$. These subcomplexes form a decreasing filtration with $\mathrm{Fil}^0 M = M$, sometimes called the *canonical filtration* (as in ([24], I 3 A). We put $\mathrm{gr}^n := \mathrm{Fil}^n / \mathrm{Fil}^{n+1}$.

The following is a generalization of ([24], I 3.9), which, as in *loc. cit.*, is an immediate consequence of 8.4.

Proposition 8.6. For M as above, we have an exact cross, where we write M_1^i instead of $\mathcal{W}_1 M^i$ for short:

(8.6.1)
$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & M_1^{i-1}/Z_{n+1} & & & & \\
 & & & & \downarrow{\scriptstyle dV^n} & \searrow & & & \\
0 & \longrightarrow & M_1^i/B_n & \xrightarrow{V^n} & \mathrm{gr}^n M^i & \xrightarrow{\beta} & M_1^{i-1}/Z_n & \longrightarrow & 0 \\
 & & & \searrow & \downarrow{\scriptstyle \beta'} & & & & \\
 & & & & M_1^i/B_{n+1} & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array},$$

where β (resp. β') sends $V^n x + dV^n y$ to the class of y (resp. x), and the slanted arrows are the canonical projections.

It follows from 8.6, as in ([24], I 3.10 (d), 3.4) that multiplication by p induces an injective homomorphism

$$\underline{p} : \mathcal{W}_n M \hookrightarrow \mathcal{W}_{n+1} M \tag{8.6.2}$$

and that, for $0 \leqslant r \leqslant n+1$, we have

$$\operatorname{Ker}(p^r : \mathcal{W}_{n+1} M \to \mathcal{W}_{n+1} M) = \operatorname{Fil}^{n+1-i} M / \operatorname{Fil}^{n+1} M. \tag{8.6.3}$$

Various other results of ([24], I 3.10 – 3.21) formally follow, as in *loc. cit.*. Note, though, that the proof of ([24], 3.21) (which, for the saturated de Rham-Witt complex, is part of (5.1.2)) has a gap, which was filled in in ([25], II 1.3).

8.7. When M is a strict Dieudonné *algebra*, the fact that $M^0 = W(M_1^0)$ for $M_1 = \mathcal{W}_1 M$ (2.2) and formula (2 (**)) imply the following. Consider $\operatorname{gr}^n M = \operatorname{Fil}^n \mathcal{W}_{n+1} M$, which is a module over $W_{n+1}(M_1^0)/p$, as an M_1^0-module via $F : M_1^0 = W_{n+1}(M_1^0)/VW_n(M_1^0) \to W_{n+1}(M_1^0)/p$. Then the horizontal and vertical arrows are linear, when the four terms around the central one are considered as M_1^0- modules via $F^{n+1} : M_1^0 \to M_1^0$.

For M the saturated de Rham-Witt complex of a k-algebra R, or more generally of a k-scheme X, we have $\mathcal{W}_1 \Omega^0_X = \mathcal{O}_{X^{\mathrm{sn}}}$, where $X^{\mathrm{sn}} \to X$ is the semi-normalization of X (4.2), and the cross takes the shape

(8.7.1)

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \downarrow & & & \\
 & & & F_*^{n+1}\mathcal{W}_1\Omega_X^{i-1}/Z_{n+1} & & & \\
 & & & \downarrow{\scriptstyle dV^n} & \searrow & & \\
0 \longrightarrow & F_*^{n+1}\mathcal{W}_1\Omega_X^{i}/B_n & \xrightarrow{V^n} & \operatorname{gr}^n \mathcal{W}\Omega_X^i & \xrightarrow{\beta} & F_*^{n+1}\mathcal{W}_1\Omega_X^{i-1}/Z_n & \longrightarrow 0 \\
 & & \searrow & \downarrow{\scriptstyle \beta'} & & & \\
 & & & F_*^{n+1}\mathcal{W}_1\Omega_X^{i}/B_{n+1} & & & \\
 & & & \downarrow & & & \\
 & & & 0 & & &
\end{array},$$

where all horizontal and vertical maps are $\mathcal{O}_{X^{\mathrm{sn}}}$-linear.

Corollary 8.8. Suppose X/k is of finite type (with k as in 3.2). Then:

(a) X^{sn} is finite over X, and in particular, noetherian.

(b) Assume that, for all i, $\mathcal{W}_1 \Omega^i_X$ is a coherent module over X^{sn} (or, equivalently, tover X, thanks to (a)). Then, for all $n \geqslant 1$ and all i, $\operatorname{gr}^n \mathcal{W}\Omega^i_X$ is coherent over X^{sn}.

(c) Under the assumption of (b), for all $n \geqslant 1$ and all i, $\mathcal{W}_n\Omega^i_X$ is a coherent module over the scheme $W_n(X^{\rm sn})$ having $X^{\rm sn}$ as underlying space, and $W_n(\mathcal{O}_{X^{\rm sn}})$ as structural sheaf of rings (by ([27], Prop. A4), this scheme is noetherian).

Proof. (a) As X is of finite type over k, its normalization $X^{\rm n}$ is finite over X. By the universal property of $X^{\rm sn}$, the canonical map $X^{\rm n} \to X$ factors through $X^{\rm sn}$, and, as $X^{\rm sn}$ is reduced, $\mathcal{O}_{X^{\rm sn}} \to \mathcal{O}_{X^{\rm n}}$ is injective, and the conclusion follows.

(b) By (a), $F : X^{\rm sn} \to X^{\rm sn}$ is finite. Therefore, for all $n \geqslant 1$, $F^n_* \mathcal{W}_1\Omega^i_X$ is coherent over $X^{\rm sn}$. As C^{-1} (8.1) is linear from $\mathcal{W}_1\Omega^i_X$ to $F_*(\mathcal{W}_1\Omega^i_X)/B_1)$, it follows, by induction on n, that B_n (resp. Z_n) is an $\mathcal{O}_{X^{\rm sn}}$-submodule of $F^n_*(\mathcal{W}_1\Omega^i_X)$, hence is coherent, as well as the quotient $F^n_*(\mathcal{W}_1\Omega^i_X)/B_n$ (resp. $F^n_*(\mathcal{W}_1\Omega^i_X)/Z_n$). By either the horizontal or vertical line of (8.6.2), it follows that $\mathrm{gr}^n\mathcal{W}\Omega^i_X$ is coherent over $X^{\rm sn}$.

(c) follows from (b) by induction on n, using the exact sequence

$$0 \to \mathrm{gr}^n\mathcal{W}\Omega^i_X \to \mathcal{W}_{n+1}\Omega^i_X \to \mathcal{W}_n\Omega^i_X \to 0.$$

The last results of 6.2, relating $\mathcal{W}\Omega^\bullet_X$ to the Raynaud ring $R = R(k)$, are consequences of the following lemma, which generalizes (6.2.2):

Lemma 8.9. Let M be a strict Dieudonné complex. Then, for all $n \geqslant 1$, we have an exact sequence

$$(8.9.1). \qquad 0 \to M^{i-1} \overset{(F^n, -F^n d)}{\to} M^{i-1} \oplus M^i \overset{dV^n + V^n}{\to} \mathcal{W}_n M^i \to 0$$

Proof. The only non-trivial point is the inclusion

$$(*) \qquad \mathrm{Ker}(dV^n + V^n) \subset \mathrm{Im}((F^n, -F^n d)).$$

Let $x \in M^{i-1}$, $y \in M^i$ such that

$$dV^n x + V^n y = 0.$$

Applying F^n we get $dx = -p^n y$, hence, as M is saturated, there exists $t \in M^{i-1}$ such that $x = F^n t$, and therefore $V^n y = -p^n dt$, and $y = -F^n dt$, which proves (*).

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