From Pierre Deligne's secret garden : looking back at some of his letters

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It's no secret that Pierre Deligne loves gardening. For many years he kept a small garden at the Ormaille, in Bures-sur-Yvette, in which he had planted trees, and grew flowers, herbs, and vegetables. But today I'd like to show you parts of another beautiful garden of his, namely the collection of his letters. From the mid 60's up to now Pierre has had a huge correspondence with a great number of mathematicians. Some of his letters were published (or gave rise to publications), most of them were not, and many stories he started telling in them generated developments which are still active today. I will give you a few examples. The choice was difficult !

1. Three letters of 1976 : Hodge theory, Euler numbers

On October 28, 1976, Deligne wrote me three letters, on two different topics.

Letter 1. Here he sketched a proof of a (particular case of a) conjecture in Hodge theory he had made in a letter to me dated October 9, 1973. The conjecture was the following.

Conjecture 1.1. Let X be a complex analytic space. Let $\varepsilon : Y_\bullet \to X$ be a proper hypercovering, with Y_n smooth over $\mathbf{C}$ for all n. Consider the total complex $R\varepsilon_*\Omega^\bullet_{Y_\bullet/\mathbf{C}}$, filtered by the Hodge filtration $\Omega^{\geq p}_{Y_\bullet/\mathbf{C}}$ on $\Omega^\bullet_{Y_\bullet/\mathbf{C}}$, an object of the derived category $D_{\mathrm{diff}}(X)$ of filtered complexes of sheaves of $\mathbf{C}$-vector spaces, with differential operators of order ≤ 1, and $\mathcal{O}_X$-linear associated graded. Then, in $D_{\mathrm{diff}}(X)$, $R\varepsilon_*\Omega^\bullet_{Y_\bullet/\mathbf{C}}$ is independent of the choice of ε, namely there should exist a transitive system of isomorphisms between these objects when ε varies. He proposed to denote this object by $\underline{\Omega}^\bullet_X$. In particular, for all p, $\underline{\Omega}^p_X := \mathrm{gr}^p\underline{\Omega}^\bullet_X$ is a well defined object of $D(X) := D(X, \mathcal{O}_X)$. Moreover, for X projective, the spectral sequence

$$E_1^{pq} = H^q(X, \underline{\Omega}^p_X) \Rightarrow H^{p+q}(X, \underline{\Omega}^\bullet_X)$$

degenerates at E_1 and abuts to the Hodge filtration of (the mixed Hodge structure) of $H^*(X, \mathbf{C})$.

As an evidence, he explained the case of a smooth X and ε the 0-coskeleton of the blow-up of a smooth subvariety Z of X. He also described several formal properties that $\underline{\Omega}^\bullet_X$ should enjoy. In his letter of Oct. 28, 1976, he explained how, in the projective case, one could prove the sought for independence by his favourite argument of pencils, a global argument, that he had used to prove finiteness of étale cohomology in ([33], Th. finitude). Details and generalizations were written up by du Bois in his thesis [9]. This complex $\underline{\Omega}^\bullet_X$ was later called the *du Bois complex*, and singularities for which $\mathcal{O}_X \to \mathrm{gr}^0\underline{\Omega}^\bullet_X (= R\varepsilon_*\mathcal{O}_{Y_\bullet})$ is an isomorphism, *du Bois singularities.* They are important in birational geometry. They were studied by various authors after du Bois (Steenbrink, Ishida, Kollar, etc.), see [30] for recent applications. However, basic questions remain :

(a) Deligne's conjecture 1.1 was for complex analytic spaces. As far as I know, it is still open.

(b) For schemes separated and of finite type over $\mathbf{C}$, can one define (in a functorial way) a bifiltered complex (K_X, F, W) on X giving, by application of $R\Gamma(X, -)$, the mixed Hodge structure of X [1] ? A new insight into this question was recently suggested by Beilinson's work on the p-adic de Rham comparison theorem [5] (a suitable variant of his construction of the dga $\mathcal{A}_{\mathrm{dR}}$ on X could give a positive answer).

Letter 2. It starts like this : "Cher Luc, Voici une semi-continuité que j'avais conjecturée dans le temps sur le conducteur de Swan." The statement is the following :

Theorem 1.2. Let $f : X \to S$ be a smooth morphism, separated and of finite type, with relative dimension 1, where S is an excellent noetherian scheme. Let $Y \subset X$ be a closed subscheme, finite and flat over S, and let $u : U = X - Y \subset X$ be the open complement. Let ℓ be a prime number invertible on S. Let $\mathcal{F}$ be a lisse sheaf of $\mathbf{F}_\ell$-modules on U, of constant rank r. Consider the function $\varphi : S \to \mathbf{N}$ defined by

$$\varphi(s) = \sum_{y \in Y_{\overline{s}}} (\mathrm{Sw}_y(u_!\mathcal{F}|X_{\overline{s}}) + r),$$

where $\overline{s}$ is a geometric point over s, and Sw_y denotes the Swan conductor at y. Then :

(i) φ is constructible and lower semi-continuous ;

(ii) if φ is locally constant, then f is universally locally acyclic with respect to $u_!\mathcal{F}$.

An immediate corollary is that if moreover f is proper, then the local constancy of φ implies that $R^n f_*(u_!\mathcal{F})$ is lisse for all n.

In his letter Deligne sketched the main steps of the proof. The details were written up by Laumon [19]. Very roughly, the proof goes like this. By

[1] The filtration on $R\varepsilon_*\Omega^\bullet_{Y_\bullet/\mathbf{C}}$ by $R\varepsilon_*\Omega^\bullet_{Y_{\geq p}/\mathbf{C}}$ fails to achieve this, for trivial reasons, already for $X = \mathrm{Spec}\,\mathbf{C}$.

a *local-to-global* argument, which has often been imitated since then, one reduces to the case where S is a strictly local trait. In this case, one proves a more precise result, to the effect that the jump of φ from the generic point η to the special point s of S is measured by a vanishing cycles group, namely, assuming that Y_s consists of a single point x, then, we have

$$\varphi(s) - \varphi(\eta) = -\dim_{\mathbf{F}_\ell} R^1\Psi(u_!\mathcal{F})_x. \tag{1.2.1}$$

To prove this formula, by an ingenious deformation argument one constructs a finite cover $\pi : \tilde{X} \to X$, étale around x, and a compactification $\tilde{Z}$ of $\tilde{X}$ such that the inverse image of $\mathcal{F}$ on $\tilde{X} - \pi^{-1}(Y)$ extends to a lisse sheaf on $\tilde{Z} - \pi^{-1}(Y)$ (the construction given in [19] at this point differs from that proposed by Deligne in his letter ; it is taken from another letter of Deligne to me, of Dec. 29, 1978). One concludes by applying the Grothendieck-Ogg-Shafarevich formula.

Letter 3. "Cher Luc, Troisième lettre (j'étais en forme hier)". This letter again deals with Euler-Poincaré characteristics. It has two distinct parts.

(a) Let k is an algebraically closed field and ℓ a prime number invertible in k. Let X be a proper scheme over k.

It had been known (probably since the early 60's) that, if k is of characteristic zero, the (étale) Euler-Poincaré characteristic $\chi(X, \mathcal{F})$ of a constructible sheaf of $\mathbf{F}_\ell$-vector spaces $\mathcal{F}$ on X depends only on the *rank function*, $x \mapsto \dim \mathcal{F}_x$, a constructible function on X. Two years before Deligne's letter MacPherson [23] had even given a Riemann-Roch type formula for $\chi(X, \mathcal{F})$ in terms of certain characteristic classes associated to this rank function - a formula which was to be revisited from a quite different perspective by Brylinski-Dubson-Kashiwara [7] and many authors afterwards, expressing $\chi(X, \mathcal{F})$ (for X smooth) as the intersection number, in the cotangent bundle T^*X, of the zero section and a *characteristic cycle* associated to $\mathcal{F}$.

It was also well known that such a property fails if k is of characteristic $p > 0$, as shown by the Grothendieck-Ogg-Shafarevich formula. Because of the additivity property of χ, $\chi(X, \mathcal{F})$ depends only on the class of $\mathcal{F}$ in the Grothendieck group $K(X, \mathbf{F}_\ell)$ of constructible sheaves of $\mathbf{F}_\ell$-vector spaces on X. In his letter Deligne proves the following striking result :

Theorem 1.3. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are constructible $\mathbf{F}_\ell$-sheaves on X which *étale locally* have the same image in the Grothendieck group, then $\chi(X, \mathcal{F}_1) = \chi(X, \mathcal{F}_2)$.

His proof is by induction on the dimension of X, using a pencil $X' \to \mathbf{P}^1_k$ (after a modification $X' \to X$), and the theory of vanishing cycles. By a similar method Laumon [20] proved that, assuming only X/k separated and of finite type, for any constructible $\overline{\mathbf{Q}}_\ell$-sheaf $\mathcal{F}$ on X, $\chi(X, \mathcal{F}) = \chi_c(X, \mathcal{F})$, where $\chi_c = \sum(-1)^i \dim H^i_c$, a result proved earlier by Grothendieck assuming resolution of singularities, and unconditionally (and independently) par Gabber (unpublished).

A couple of years later, in the course of his work with Lusztig, using a totally different method, based on Brauer theory and the Lefschetz-Verdier trace formula, Deligne gave a strong refinement of 1.3, namely that if on the strata of a suitable stratification of X, $\mathcal{F}_1$ and $\mathcal{F}_2$ have the same rank and the same wild ramification at infinity, then $\chi(X, \mathcal{F}_1) = \chi(X, \mathcal{F}_2)$ (see ([12], 2.9, 2.12) for precise statements). In particular, if X is (proper) and normal, $j : U \hookrightarrow X$ is a dense open subset, and $\mathcal{F} = j_!\mathcal{G}$ for a locally constant sheaf $\mathcal{G}$ of rank r on U, tamely ramified along $X - U$, then

$$\chi(X, \mathcal{F}) = r\chi_c(U) \ (= r\chi(U))$$

This question was much later revisited by Vidal ([31], [32]), who proved relative variants of this refinement. Another generalization was given by Kato and T. Saito, using their construction of a *Swan class* ([17], 4.3.10). See also [13] for further developments.

(b) In this second part, Deligne tackles the problem of generalizing the Grothendieck-Ogg-Shafarevich formula to higher dimension. He considers the next case, namely : X is a proper surface over an algebraically closed field k, that he assumes normal ("for simplicity"), ℓ a prime number invertible in k, D a Weil divisor on X, $j : U = X - D \hookrightarrow X$ the complementary open subset, and $\mathcal{F}$ a locally constant sheaf of $\mathbf{F}_\ell$-vector spaces on U, of constant rank r. The problem is to write a formula for $\chi(X, j_!\mathcal{F})$ in terms of r, the Euler-Poincaré characteristics of U and of the components of D (or suitable dense open subsets of them), and the ramification of $\mathcal{F}$ along D. Wild ramification is admitted, but here he assumes that it is not too bad, namely that at each generic point δ of D, if X_δ denotes the trait obtained by localization of X at δ, with fraction field K, the normalization of X_δ in a finite Galois extension of K trivializing $\mathcal{F}|\operatorname{Spec} K$ doesn't make appear any inseparable extension of the residue field (this condition was later expressed by Laumon [21] as $\mathcal{F}$ *having non fierce ramification along* D). Under this condition Deligne shows the following :

(i) If $\overline{\delta}$ is a geometric point above a generic point of D, one can define the *Swan conductor* $\mathrm{Sw}(\mathcal{F}, \overline{\delta})$ of $\mathcal{F}$ at $\overline{\delta}$, an integer characterized by the property that there exists a dense open subset D^0 of D, smooth over k, such that for any smooth curve Y cutting D transversally at a point a of D^0 specialization of δ, then $\mathrm{Sw}(\mathcal{F}, \overline{\delta})$ is equal to the Swan conductor at a of $\mathcal{F}|Y$.

(ii) For X projective, if $(D_i)_{1 \leq i \leq N}$ are the irreducible components of D, $D_i^0 := D_i \cap D^0$, and $\mathrm{Sw}_i(\mathcal{F})$ is the Swan conductor at a geometric generic point of D_i, then one has

$$\chi(X, j_!\mathcal{F}) = \chi_c(U)\mathrm{rk}(\mathcal{F}) - \sum_i \chi(D_i^0)\mathrm{Sw}_i(\mathcal{F}) - \sum_{x \in D - D^0} \mathrm{Sw}_x(\mathcal{F}), \tag{1.3.1}$$

where $\mathrm{Sw}_x(\mathcal{F})$ is a nonnegative integer, of local nature at x, defined in terms of vanishing cycles at x for a suitable local pencil $X \to \mathbf{A}^1$ at x.

Deligne sketches the proof and gives two key examples ((1) D a section of a smooth pencil $X \to \mathbf{A}^1$, an example closely related to 1.2, (2) $\mathcal{F}$ of rank 1 associated to an Artin-Schreier cover $T^p - T = y/x^d$ of $\mathrm{Spec}\, k[x, y]$ (minus $x = 0$) and a nontrivial additive character of $\mathbf{F}_p$, with $p = \mathrm{char}(k)$ and d prime to p. A complete proof was written up by Laumon in his thesis [22]. Similar results were obtained by S. Saito [26] by a different method.

1.4. Posterity.

Letters 2 and 3 marked the beginning of a systematic study of Euler-Poincaré characteristics, both in geometric (over algebraically closed fields) and arithmetic situations (over local fields). The ultimate goal would be to state and prove Grothendieck Riemann-Roch type formulas for ℓ-adic sheaves in a relative setting, a goal which is far from having been reached today, despite considerable progress. It is well beyond the scope of these notes to give a comprehensive historical account of the development of this topic. I will only briefly sketch a few points.

1.4.1. At the end of letter 3, Deligne raises the question of defining "generic Swan conductors" in the same setting, but in the fierce case. He comes back to this in a long letter to me, dated Nov. 4, 1976. His method is to analyze the ramification of $\mathcal{F}$ along D (X and D are assumed to be smooth) using high order jet bundles of curves transverse to D and suitable pencils. He studies the Artin-Schreier case in detail, where crucial differential forms appear. The program proposed in this letter has so far not been implemented (except by Laumon in the non fierce case, as I mentioned above), though it spurred a lot of research from various angles.

1.4.2. In the case of fierce ramification, but for $\mathcal{F}$ of rank 1, a breakthrough was made by K. Kato [15], using local class field theory and analogies with the theory of $\mathcal{D}$-modules in characteristic zero. He discovered that the ramification was controlled by certain differential forms (reminiscent of those considered by Deligne in 1.4.1), and he introduced the key notions of *cleanliness* and *refined Swan conductor*. He gave a Riemann-Roch type formula for $\chi(X, \mathcal{F})$ involving this invariant.

In the early 2000's new methods appeared.

1.4.3. *Ramification filtrations.* In a series of papers ([1], [2], [4], [28]) Abbes and T. Saito, using (at first) techniques of rigid geometry, and also techniques of log geometry, defined and studied upper numbering ramification filtrations of Galois groups on traits with not necessarily perfect residue fields (generalizing the classical ones in the perfect case). The graded quotients are annihilated by p and define certain 1-forms on the residue field, generalizing Kato's refined Swan conductor. A prime-to-p Hasse-Arf theorem is proved.

1.4.4. *Characteristic class and characteristic cycle.* A new approach, suggested by logarithmic geometry, consisting in "blowing-up the ramification locus R in the diagonal of $X \times X$" (i. e., roughly speaking, first blowing-up all $D_i \times D_i$'s in $X \times X$, for D_i running through the components of D, and then further blowing-up a suitable rational linear combination R of the D_i's in the resulting diagonal embedding), was introduced and extensively studied by T. Saito and his collaborators. The upshot is a Brylinski-Dubson-Kashiwara type formula for χ. The strategy goes like this. Let X/k be proper and smooth, and let $j : U = X - D \hookrightarrow X$ be the complement of a divisor with simple normal crossings on X. Let $\mathcal{F}$ be a lisse $\mathbf{F}_\ell$-sheaf on U. We want to "calculate" $\chi(X, j_!\mathcal{F})$. Let $a : X \to \operatorname{Spec} k$ be the projection.

(i) By the Lefschetz-Verdier trace formula, we have

$$\chi(X, j_!\mathcal{F}) = a_* C(j_!\mathcal{F}),$$

where $C(j_!\mathcal{F}) \in H^0(X, K_X)$ is the class of the identity morphism of $j_!\mathcal{F}$, and a_* the trace map ($K_X = a^!\mathbf{F}_\ell$ being the dualizing complex ; here, as X/k is smooth, $H^0(X, K_X) = H^{2d}(X, \mathbf{F}_\ell(d))$ if $d = \dim X$). Abbes and Saito [3] call $C(j_!\mathcal{F})$ the *characteristic class* of $\mathcal{F}$.

(ii) Under certain assumptions on the ramification of $\mathcal{F}$, T. Saito [27] defines a codimension d-cycle $CC(\mathcal{F})$ on the (logarithmic) cotangent bundle of X (a vector bundle of rank d on X), called the *characteristic cycle* of $\mathcal{F}$, whose intersection with the zero section is the characteristic class

$$C(j_!\mathcal{F}) = (CC(\mathcal{F}).X)$$

The construction of $CC(\mathcal{F})$ relies on the properties of the ramification filtration of $\mathcal{F}$ at the generic points of D, and heavily uses the blow-up technique alluded to above. A non log variant of this construction is given in [29]. On the other hand, the above mentioned *Swan class*, defined in [17], gives rise to a generalization of the Grothendieck-Ogg-Shafarevich formula in higher dimension (and relative situations).

1.4.5. *Bloch's conductor formula.* For a scheme X separated and of finite type over a complete discrete valuation field K, with perfect residue field k, and a constructible sheaf $\mathcal{F}$ of $\mathbf{Q}_\ell$-vector spaces on X, one can consider the Swan conductor

$$\mathrm{Sw}(X, \mathcal{F}) := \sum_i (-1)^i \mathrm{Sw}(H^i_c(X_{\overline{K}}, \mathcal{F})),$$

where $\overline{K}$ is a separable closure of K and $H^i_c(X_{\overline{K}}, \mathcal{F})$ is viewed as a representation of $\mathrm{Gal}(\overline{K}/K)$. Finding a formula for this integer in terms of characteristic classes associated to $\mathcal{F}$ has been a longstanding problem in ramification theory.

Assume that X/K is proper and smooth and is the generic fiber of proper, regular, flat model $\mathcal{X}$ over $S = \mathrm{Spec}\, \mathcal{O}_K$. In this case, one can define the *Artin conductor* of $\mathcal{X}$,

$$\mathrm{Art}(\mathcal{X}/S) := \chi(X_{\overline{K}}) - \chi(X_{\overline{k}}) + \mathrm{Sw}(X, \mathbf{Q}_\ell),$$

where χ denotes an ℓ-adic Euler-Poincaré characteristic. In [6] Bloch proposed a formula for $\mathrm{Art}(\mathcal{X}/S)$, as the degree of certain zero cycle on the special fiber $\mathcal{X}_k$, a *localized Chern class* of $\Omega^1_{\mathcal{X}/S}$, and proved it for $\dim X = 1$. Bloch's conjecture was proven by Kato and Saito [16] in any dimension, assuming that the reduced special fiber has normal crossings. Quite recently, Kato and Saito [18] proved variants of this formula with the constant sheaf $\mathbf{Q}_\ell$ on X replaced by a constructible sheaf $\mathcal{F}$ (assuming $\mathrm{char}(K) = 0$), and as a by-product, established the 2-dimensional (arithmetic) case of Serre's conjecture on the Artin character of a finite group of automorphisms of a regular local ring having the closed point as an isolated fixed point.

2. A letter of 1988 : from "divisors" to logarithmic structures

Deligne likes to use quotation marks to denote certain mathematical objects. For example, in the mid 60's he introduced the notation "$\varprojlim_{i \in I}$"K_i (resp. "$\varinjlim_{i \in I}$"K_i) to denote a pro- (resp. ind-) object, a notation more suggestive than Grothendieck's notation $(K_i)_{i \in I}$, and which quickly became standard. In a letter to me dated June 1, 1988, whose modest aim is to give "quelques généralités sur les diviseurs à croisements normaux verticaux mais relatifs", he defines a *"divisor"* on a scheme X as the datum of an invertible sheaf $\mathcal{L}$ on X and an $\mathcal{O}_X$-linear map $u : \mathcal{L} \to \mathcal{O}_X$. He emphasizes that, if e is a local basis of $\mathcal{L}$, one does not assume that $u(e)$ is a nonzero divisor. One could have $u(e) = 0$. Thus the notion of "divisor" is a natural generalization of that of (effective) Cartier divisor.

Next, as suggested by semistable reduction, given a morphism $f : X \to S$ and a "divisor" $E = (\mathcal{L}, u)$ on S, Deligne defines a *"relative vertical normal crossings divisor on X above E"*, $D = \sum_{i \in I} D_i$, as a finite collection $D_i = (\mathcal{L}_i, u_i)$ of "divisors" $D_i = (\mathcal{L}_i, u_i)$, $i \in I$, on X, together with an isomorphism $f^*\mathcal{L} \xrightarrow{\sim} \otimes_i \mathcal{L}_i$, compatible in an obvious way with the data u and u_i. Given such a "relative divisor" D, he defines a module of relative logarithmic differentials

$$\Omega^1_{X/S}(\log D),$$

generated by $\Omega^1_{X/S}$ and symbols $\mathrm{dlog}\, s_i$ for s_i a local basis of $\mathcal{L}_i$, subject to the relations : $\mathrm{dlog}(as_i) = da/a + \mathrm{dlog}\, s_i$ for $a \in \mathcal{O}^*_X$; $\sum_i \mathrm{dlog}\, s_i = 0$ if $f^*(s) = \otimes_i s_i$ for s a local basis of $\mathcal{L}$; $u_i(s_i)\mathrm{dlog}\, s_i = du_i(s_i)$.

He observes that these notions have the advantage of being stable under base change. In particular, in characteristic $p > 0$, they give rise to a relative Frobenius morphism, possibly entailing a Cartier isomorphism, and under suitable dimension and liftability assumptions, decomposition theorems of the de Rham complex as in [8].

In a post-scriptum Deligne says that he encountered "divisors" while studying extensions of a local field K with ramification $\leq e$ (in the upper numbering set-up) : "they depend only on the "divisor" $(m/m^{e+1}, m/m^{e+1} \to V/m^e)$ on V/m^e", where V is the ring of integers of K and m the maximal ideal. However, the timing of his letter (June 1, 1988) is not insignificant. From February through May, 1988 Fontaine had run a seminar at the IHÉS on p-adic periods [11]. Attention had been focused on a conjecture of his and Jannsen, the so-called C_{st}-conjecture, comparing p-adic étale cohomology and de Rham cohomology in the semistable reduction case. A "hidden" structure on de Rham cohomology of the generic fiber had been discovered, bearing some analogy with the Steenbrink limit Hodge structure of the complex case. It involved a certain de Rham-Witt complex with log poles along the special fiber, constructed by Hyodo. This complex depended on the special fiber plus some (at the time) mysterious extra structure coming from the integral model. It's in trying to pinpoint this extra structure that Fontaine and I conceived the notion of *logarithmic structure* (log structure, for short) on a scheme X : roughly[2], a pair consisting of a sheaf of monoids M on X (for the étale topology) containing $\mathcal{O}^*_X$ and a multiplicative homomorphism $\alpha : M \to \mathcal{O}_X$ extending the inclusion of $\mathcal{O}^*_X$ into $\mathcal{O}_X$. The idea of the definition largely stemmed from Deligne's "divisors". His construction of $\Omega^1_{X/S}(\log D)$ also inspired that of Ω^1 for morphisms of log schemes.

At about the same time, and independently, Faltings considered objects similar to Deligne's "divisors" ([10], §2). Deligne's and Faltings' objects

[2]In the formal definition given later, α has to induce an isomorphism $\alpha^{-1}(\mathcal{O}^*_X) \xrightarrow{\sim} \mathcal{O}^*_X$.

turned out to be indeed particular cases of log structures. The theory was developed by Kato in [14], introducing the key notion of *chart* of a log structure. The precise relation is explained in ([14], Complement 1). A “divisor” $(\mathcal{L}, u)$ on a scheme X is the same as a log structure M together with a homomorphism $t : \mathbf{N}_X \to M/\mathcal{O}_X^*$ which étale locally lifts to a chart of M : $\mathcal{L}$ is recovered as the line bundle associated to the $\mathcal{O}_X^*$-torsor inverse image of $t(1)$ in M, and u is deduced from $\alpha : M \to \mathcal{O}_X$. There is a similar description for “relative divisors”. In [14] Kato developed differential calculus on log schemes. Deligne's $\Omega^1_{X/S}(\log D)$ is the logarithmic differential module associated to the morphism of log schemes defined by $(X, D) \to (S, E)$. Kato also constructed a Cartier isomorphism (under suitable assumptions, satisfied in particular by “relative divisors”) and (under extra assumptions of dimension and lifting mod p^2) a Deligne-Illusie type decomposition, thus confirming Deligne's expectations.

That should have been the end of the story. However, in 2000 L. Lafforgue revisited Deligne's notion of “divisor”. He observed that a “divisor” $D = (\mathcal{L}, u)$ on X is the same as a morphism from X to the quotient stack $[\mathbf{A}^1/\mathbf{G}_m]$ on $\operatorname{Spec} \mathbf{Z}$: such a morphism indeed consists of a $\mathbf{G}_m$-torsor P on X, whose associated line bundle is $\mathcal{L}$, and a $\mathbf{G}_m$-equivariant morphism $f : P \to \mathbf{A}^1$, which defines $u = f \wedge^{\mathbf{G}_m} \mathrm{Id} : \mathcal{L} = P \wedge^{\mathbf{G}_m} \mathcal{O} \to \mathcal{O}$. This remark was seminal in the stack-theoretic viewpoint in log geometry, developed by Olsson in [24]. Among the applications, let me mention the Gabber-Olsson theory of the cotangent complex in log geometry [25] (used by Beilinson in [5]).

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