Celebrating the Mathematics of Pierre Deligne

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Pierre Deligne's secret garden: looking back at some of his letters

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Deligne's garden at the Ormaille

Deligne at his desk

Plan

1. 1 letter of Oct. 28, 1976 : Hodge theory
2. 2 letters of Oct. 28, 1976 : Euler-Poincaré characteristics
3. A letter of June 1, 1988 : from “divisors" to log structures

1. Hodge theory

First letter of Oct. 28, 1976

28/10/76

Cher Luc,

Voici une preuve globale de l'existence d'un complexe de DR donnant lieu à la théorie de Hodge mixte.

(...)

<u>Conclusion</u> : pour $U/\mathbb{C}$, on a un complexe (K^*, d, F), fonctoriel (à quasi-iso près)

avec
1) $\mathcal{H}\, K^{*an} = \mathbb{C}_U$, $\quad H(U, K^*) \xrightarrow{\sim} H(U^{an}, K^{*an}) = H(U, \mathbb{C})$
2) pour U propre, la suite spectrale de F dégénère en E_1, et aboutit à la filtration de Hodge.

Hodge theory

- $X/\mathbb{C}$ proper smooth $\mapsto$ pure Hodge structure
  - (H^n, F), $H^n = \oplus_{p+q=n} H^{pq}$,
  - $H^{pq} = F^p \cap \overline{F}^q$
- $X/\mathbb{C}$ separated, finite type $\mapsto$ mixed Hodge structure (H^n, F, W)
- Deligne's letter : (sketch of) proof of conjecture on F, made in letter of Oct. 9, 1973

First half of letter of Oct. 9, 1973

Berne, le 9 octobre 1973

Cher Illusie,

Soit X un espace analytique complexe, et

$$\varepsilon : X_* \longrightarrow X$$

un espace analytique (lisse) simplicial qui soit un hyperrecouvrement propre de X

‖ Conjecture : $R\varepsilon_{**}(\Omega^p_{X_*}) \in D^+_{coh}(X, \mathcal{O})$ et indépendant du choix de X_*

[Notation: $\underline{\underline{\Omega}}^p_{X_*}$]

Rmq : ceci clarifie beaucoup ce que je fais dans Hodge III. On a vu X un double complexe, avec d_1 linéaire et d_2 opérateur différentiel du premier ordre, et bien défini à quasi-isomorphisme filtré près (c'est $R\varepsilon_{**}\Omega^*_{X_*}$, de gradué les $R\varepsilon_{**}\Omega^p_{X_*}$).

Pour X projectif, la suite spectrale dégénérée en E_1 qui aboutit à la filtration de Hodge

$$H^q(X, \Omega^p) \Rightarrow H^{p+q}(X^{an}, \mathbb{C})$$

$X = \mathbb{C}$-analytic space

$\varepsilon : Y_\bullet \to X$ proper hyper-covering, $Y_n/\mathbb{C}$ smooth

Conjecture (Deligne)

(1) *$(R\varepsilon_*\Omega^\bullet_{Y_\bullet/\mathbb{C}}, F)$ independent of ε. In particular :*

$$R\varepsilon_*\Omega^p_{Y_\bullet/\mathbb{C}} = \mathrm{gr}^p_F R\varepsilon_*\Omega^\bullet_{Y_\bullet/\mathbb{C}} \in D^+(X, \mathcal{O})$$

independent of ε

(2) *For $X/\mathbb{C}$ projective,*

$$E_1^{pq} = H^q(X, \Omega^p) \Rightarrow H^{p+q}(X, \mathbb{C})$$

degenerates at E_1.

Deligne's letter of Oct. 28, 1976 : (sketch) of proof of (1), (2) for $X/\mathbb{C}$ projective, by a global method (from SGA 4 1/2, Th. finitude)

du Bois complex

- $\underline{\Omega}^{\bullet}_{X/\mathbb{C}} := (R\varepsilon_* \Omega^{\bullet}_{Y_\bullet/\mathbb{C}}, F)$
- $\underline{\Omega}^{p}_{X/\mathbb{C}} := \mathrm{gr}^p_F \underline{\Omega}^{\bullet}_{X/\mathbb{C}} = R\varepsilon_* \Omega^{p}_{Y_\bullet/\mathbb{C}}$
- du Bois singularities : $\mathcal{O}_X \to \underline{\Omega}^0_{X/\mathbb{C}}$ an isomorphism
- Example : rational singularities $\Rightarrow$ du Bois (Kovacs, 1999)
- Problems : analytic case ? W ? link with Beilinson's $\mathcal{A}_{\mathrm{dR}}$?

2. 2nd, 3rd letters of Oct. 28, 1976

Euler-Poincaré characteristics

Set-up

- *k alg. closed, char. p, $\ell \neq p$,*
- *X/k separated, finite type,*
- *$\mathcal{F}$ = (constructible) ℓ-adic sheaf on X*

Problem

Understand

$$\chi_c(X, \mathcal{F}) = \sum (-1)^i \dim H^i_c(X, \mathcal{F})$$

Known at the time of Deligne's letters

- for $p = 0$, $\chi_c(X, \mathcal{F})$ depends only on rank function $x \mapsto \mathrm{rk}\,\mathcal{F}_x$

  RR formula (MacPherson) for χ

- $p > 0$, X/k a curve : *Grothendieck-Ogg-Shafarevich formula*

  $j : U \hookrightarrow X$ dense open, X/k proper smooth curve,
  $\mathcal{F}$ lisse on U

$$\chi_c(U, \mathcal{F}) = \mathrm{rk}(\mathcal{F})\chi_c(U) - \sum_{x \in X-U} \mathrm{Sw}_x(j_!\mathcal{F})$$

  $\mathrm{Sw}_x(j_!\mathcal{F})$: Swan conductor of $\mathcal{F}$ at x (integer measuring wild ramification of $\mathcal{F}$ at x)

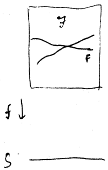
Second letter of Oct. 28, 1976 (first half)

28/10/76

Cher Luc,

Voici une semi-continuité que j'avais conjecturée dans le temps sur les conducteurs de Swan.

• Th 1 Soit $f : X \to S$ un morphisme lisse séparé de dim relative 1, $F \subset X$ fini sur S et $\mathcal{F}$ un faisceau loc^t const. sur $U = X - F$, prolongé par 0 sur X.



(Pour ce qui nous intéresse, $\mathcal{F}$ peut être simplement un faisceau de $\mathbb{Z}/\ell$-modules). On suppose $\mathcal{F}$ de rang constant r.

On suppose $F \to S$ universellement ouvert.

Pour $s \in S$, soient $\bar{s}$ un point géométrique en s, et

$$\varphi(s) = \sum_{f \in F_{\bar{s}}} \left(Sw_f(\mathcal{F} \text{ sur } X_{\bar{s}}) + r \right)$$

Alors (i) la fonction $\varphi(s)$ est semi-continue inférieurement, et constructible

(ii) si elle est constante, $(X, \mathcal{F})$ est universellement localement

Swan conductor jumps

In short :

- Swan decreases by specialization
- if dim(base) = 1, jump measured by nearby cycles group:

Theorem

$f : X \to S$ smooth curve, (S, s, η) strictly local trait
$\sigma : S \xrightarrow{\sim} \sigma(S) = Y \subset X$ section of f, $j : U = X - Y \hookrightarrow X$
$\mathcal{F}$ lisse $\mathbb{F}_\ell$-sheaf on U
Then :

$$\mathrm{Sw}_{\sigma(s)}(j_!\mathcal{F}|X_s) - \mathrm{Sw}_{\sigma(\overline{\eta})}(j_!\mathcal{F}|X_{\overline{\eta}}) = -\dim R^1\Psi(j_!\mathcal{F})_{\sigma(s)}$$

Ingredients of proof

- local-to-global argument to reduce, by base change, to “jump formula"
- (compactification + deformation) argument
- Grothendieck-Ogg-Shafarevich formula

Third letter of Oct. 28, 1976 (beginning)

28/10/76

Cher Luc,

Troisième lettre (j'étais en forme hier)

Th. Soit X propre sur k alg clos, et $\mathcal{F}_1, \mathcal{F}_2$ deux éléments du groupe de Grothendieck des faisceaux constructibles de $\mathbb{F}_\ell$-vectoriels. Si $\mathcal{F}_1$ et $\mathcal{F}_2$ sont localement égaux, alors $\chi(\mathcal{F}_1) = \chi(\mathcal{F}_2)$

Contents of letter

General theme: $\chi(X, \mathcal{F})$, X/k proper, k alg. closed,

$\mathcal{F}$ constructible $\mathbb{F}_\ell$-sheaf on X

Two parts in the letter :

- $\chi(X, \mathcal{F}_1) = \chi(X, \mathcal{F}_2)$ if $\mathcal{F}_1$, $\mathcal{F}_2$ have étale locally same image in Grothendieck group
- (partial) generalization of Grothendieck-Ogg-Shafarevich formula to surfaces

First part

- Proof of $\chi(X, \mathcal{F}_1) = \chi(X, \mathcal{F}_2)$ by induction on $\dim X$, using pencil $X' \to \mathbb{P}^1_k$ (after modification $X' \to X$) and vanishing cycles
- Similar method used by Laumon to prove $\chi = \chi_c$ (1981)
- Deligne (1978) : (Brauer theory + Lefschetz-Verdier) gives stronger result ("same wild ramification at infinity" suffices) ; in particular,

  $$\chi_c(X, \mathcal{F}) = r\chi_c(X)$$

  for X normal, $\mathcal{F}$ locally constant of rank r, tamely ramified at infinity
- revisited by Vidal (2001), Kato-Saito (2009)

Second part

Set-up

- *X/k proper smooth surface, $D = \sum D_i$ sncd on X, $j : U = X - D \hookrightarrow X$*
- *$\mathcal{F}$ lisse $\mathbb{F}_\ell$-sheaf on U*
- *assume $\mathcal{F}$ has no fierce ramification along D : for any generic pt δ of D, if K = fraction field of $\mathcal{O}_{X,\delta}$, and K'/K = Galois extension trivializing $\mathcal{F}|\mathrm{Spec} K$, no inseparable extension of $k(\delta)$ appears in normalization of $\mathcal{O}_{X,\delta}$ in K'*
- *Example : $T^p - T = yx^{-m}$: non fierce $\Leftrightarrow p \nmid m$*

Theorem

(i) *Can define generic Swan conductors* $\mathrm{Sw}_i(\mathcal{F})$ *and open subsets* $D_i^0 \subset D$ *where* $\mathrm{Sw}(\mathcal{F}|C) = \mathrm{Sw}_i(\mathcal{F})$ *for any curve* C *transverse to* D_i *at a point of* D_i^0

(ii) *One has*

$$\chi(X, j_!\mathcal{F}) = \chi_c(U)\mathrm{rk}(\mathcal{F}) - \sum_i \chi(D_i^0)\mathrm{Sw}_i(\mathcal{F}) - \sum_{x \in D - D^0} \mathrm{Sw}_x(\mathcal{F})$$

for certain integers $\mathrm{Sw}_x(\mathcal{F})$ *depending on local behavior of* $\mathcal{F}$ *at* x*.*

Proof by method of pencils. Details written up by Laumon in his thesis (1983).

Fierce case ? Deligne's letter of Nov. 4, 1976

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Cher Luc,

J'ai réfléchi à ce que donne la méthode des pinceaux de Lefschetz pour $\chi(S, \mathcal{F})$, S une surface, dans le cas général :

Soit S surface, D diviseur union de diviseurs irréductibles D_i, $\mathcal{F}$ faisceau loc c^t de $\mathbb{Z}/\ell$-modules sur $U = S - D$, prolongé par 0 sur S.

On suppose S lisse, D_i lisse.

Considérons la ramification sauvage de $\mathcal{F}$ restreint à un arc de courbe Y transverse à D_i en un point du lieu lisse de D. Le conducteur de Swann ne dépend que du jet d'ordre k (k assez grand) de Y, et a en général sa valeur maximale Sw_i

Soit donc J_i le fibré sur D_i des k-jets de courbes lisses partant d'un point de D_i. Soit D_i° un ouvert dense de D_i tel que, en tout point de D_i°, presque tout jet transverse passant par ce point donne à Sw sa valeur Sw_i, et que $D_i^\circ \cap D_j = \emptyset$ pour $i \neq j$.

Posterity

- Kato (1994) : $\dim X$ arbitrary, $\mathrm{rk}(\mathcal{F}) = 1$, any ramification : cleanliness, refined Swan conductor (using local class field, analogy with $\mathcal{D}$-modules)

- from 2000 on : new methods (and results)
  - ramification filtrations (non necessarily perfect residue field) : Abbes-Saito ; defined by techniques of rigid and log geometry ; graded quotients $\mapsto$ generalized refined Swan conductors
  - characteristic class and characteristic cycle
  - Bloch's conductor formulas

Characteristic class and characteristic cycle

- general method (T. Saito) : for $\mathcal{F}$ lisse on $U = X - D$ (X/k proper smooth, $D = \sum D_i$: sncd), blow-up ramification locus R of $\mathcal{F}$ in the diagonal of $X \times X$

  (or in the diagonal of the blow-up of $X \times X$ along the $D_i \times D_i$'s)

- Characteristic class (Abbes-T. Saito, 2007) : by Lefschetz-Verdier trace formula

$$\chi(X, j_!\mathcal{F}) = \mathrm{Tr}\, C(j_!\mathcal{F})$$

  where $j : U \hookrightarrow X$, $\mathrm{Tr} : H^{2d}(X, \mathbb{F}_\ell(d)) \to \mathbb{F}_\ell$ = trace map, $d = \dim X$, and class of $\mathrm{Id}(j_!\mathcal{F})$

$$C(j_!\mathcal{F}) \in H^{2d}(X, \mathbb{F}_\ell(d)),$$

  characteristic class of $\mathcal{F}$

- Characteristic cycle (T. Saito, 2009)

$$CC(\mathcal{F}) \in Z^d(T^*_X(\log D)) \otimes \mathbb{Q}$$

giving $C(j_!\mathcal{F})$ by *Brylinski-Dubson-Kashiwara* type formula

$$C(j_!\mathcal{F}) = (CC(\mathcal{F}).X)$$

- Swan class (Kato-T. Saito, 2008) ; relative G-O-S formulas
- non log variants (T. Saito, 2013) ; revisits Deligne's pencils (non characteristic $\Rightarrow$ locally acyclic)

Bloch's conductor formulas

Set-up

- *K complete discrete valuation field, perfect residue field k, X/K separated, finite type*
- *$\mathcal{F}$ constructible $\mathbb{Q}_\ell$-sheaf on X, ($\ell \neq \mathrm{char}(k)$)*
- *Problem : find a formula for*

$$\mathrm{Sw}(X, \mathcal{F}) := \sum_i (-1)^i \mathrm{Sw}(H^i_c(X_{\overline{K}}, \mathcal{F}))$$

in terms of characteristic classes of $\mathcal{F}$.

Conjecture (Bloch, 1987)

For $X = \mathcal{X}_K$, $\mathcal{X}$ *regular, proper and flat over* $\mathcal{O}_K$, *of relative dimension* d, *with* X/K *smooth,*

$$\mathrm{Sw}(X, \mathbb{Q}_\ell) = \chi(\mathcal{X}_{\overline{k}}) - \chi(X_{\overline{K}}) + (-1)^d \deg c_{d+1,\mathrm{loc}}(\Omega^1_{\mathcal{X}/\mathcal{O}_K}),$$

where $c_{d+1,\mathrm{loc}}$ = *"localized Chern class" (in* $CH_0(\mathcal{X}_k)$*)*

Theorem (Kato-Saito, 2005)

Conjecture true if $\mathcal{X}_{k,\mathrm{red}}$ = *ncd.*

Generalizations

- Theorem generalized to coefficients, relative versions (for $\mathrm{char}(K) = 0$) (Kato-Saito, 2013),
- with application to arithmetic 2-dimensional case of Serre's conjecture (1960) on finite automorphism groups of regular local ring with isolated fixed point (geometric case proved by K. Kato- S. Saito-T. Saito (1987))

3. “Divisors"

A letter of June 1, 1988

Bruxelles, le 1er juin 1988

Cher Luc,

Voici quelques généralités sur les diviseurs à croisements normaux verticaux mais relatifs

(a) Généralisant la notion de diviseur, on a celle de "diviseur" sur X : un faisceau inversible $\mathcal{L}$ sur X, et $u : \mathcal{L} \to \mathcal{O}_X$. Pour ℓ une trivialisation locale de $\mathcal{L}$, on ne suppose pas que l'équation $u(\ell)$ du "diviseur" soit non diviseur de 0.

From "divisors" to log structures

- effective Cartier divisor on X : line bundle $\mathcal{L}$ + section $s : \mathcal{O}_X \to \mathcal{L}$ nonzero divisor at each point ; equivalently, + $u : \mathcal{L} \to \mathcal{O}_X$, s. t. $u(e)$ nonzero divisor for local base e

- Deligne "divisor" : line bundle $\mathcal{L}$ + $u : \mathcal{L} \to \mathcal{O}_X$ (no condition on u)

- relative variant (inspired by semistable reduction) :

  for $f : X \to S$, $E = (\mathcal{L}, u)$ on S a "divisor" on S,

  a "divisor" D on X above E is

  a finite family $D_i = (\mathcal{L}_i, u_i)$
  with an isomorphism $f^*\mathcal{L} \simeq \otimes\mathcal{L}_i$, compatible with (u, u_i).

- log differential module

$$\Omega^1_{X/S}(\log D)$$

generated by $\Omega^1_{X/S}$ and symbols $\mathrm{dlog}\, e_i$, e_i local base of $\mathcal{L}_i$, with relations

  - $\mathrm{dlog}(ae_i) = da/a + \mathrm{dlog}\, e_i$ for $a \in \mathcal{O}_X^*$;
  - $\sum_i \mathrm{dlog}\, e_i = 0$ if $f^*(e) = \otimes_i e_i$;
  - $u_i(e_i)\mathrm{dlog}\, e_i = du_i(e_i)$.

- “relative divisors" base change compatible

- Deligne conjectured : Cartier isomorphisms, D-I type decompositions (for liftings mod p^2 and dimension $< p$)

- similar notions defined (independently) by Faltings (1990)
- "divisors" led to concept of log structure (developed by K. Kato et al.) :

$$(X, \alpha : M \to \mathcal{O}_X),\ \alpha : \alpha^{-1}(\mathcal{O}_X^*) \xrightarrow{\sim} \mathcal{O}_X^*.$$

- ("divisor" $D = (\mathcal{L}, u)$) $\Leftrightarrow$ (log str. M + $(t : \mathbb{N}_X \to M/\mathcal{O}_X^*)$ étale locally lifting to a chart of M)
- "relative divisor" $(D = \sum D_i) \to (E = (\mathcal{L}, u) \mapsto$ map of log schemes $(X, M) \to (S, N)$, and

$$\Omega^1_{X/S}(\log D) = \Omega^1_{(X,M)/(S,N)}$$

- Kato (1988) : Cartier isomorphism for Cartier type morphisms, D-I decompositions

A new look at "divisors"

- Lafforgue (2000) :

  $$(\text{"divisor"}\ (\mathcal{L}, u)\ \text{on}\ X) \Leftrightarrow (\text{morphism}\ X \to [\mathbb{A}^1/\mathbb{G}_m])$$

- led to stack-theoretic viewpoint in log geometry (Olsson) :
  fine log str. M on X $\Leftrightarrow$ morphism from X to alg. stack locally of toric type