Riemann Conference

December 27-30, 2015 – Sanya – China

Glimpses on vanishing cycles, from Riemann to today

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1. The origins

Vanishing cycles in Riemann ?

No, but ...

Riemann (1857) studied the hypergeometric equation $E(\alpha, \beta, \gamma)$

$$t(1-t)f'' + (\gamma - (\alpha+\beta+1)t)f' - \alpha\beta = 0$$

($\alpha, \beta, \gamma \in \mathbf{C}$), and the monodromy of its solutions around its singular points $(0, 1, \infty)$.

$E(\alpha, \beta, \gamma)$ has regular singularities at these points (moderate growth of solutions).

The hypergeometric function

$$F(\alpha, \beta, \gamma, t) = \sum_{n \geq 0} \frac{(\alpha, n)(\beta, n)}{(\gamma, n)} \frac{t^n}{n!}$$

($|t| < 1$), where $(u, n) = \prod_{0 \leq i \leq n-1}(u+i)$, is the unique solution which is holomorphic at 0 with value 1.

Solutions form a complex local system $\mathcal{H}_{\mathbf{C}}$ of rank 2 over $S = \mathbf{P}^1_{\mathbf{C}} - \{0, 1, \infty\}$. For a chosen base-point $t_0 \in S$, it is given by

$$\rho : \pi_1(S, t_0) \to \mathrm{GL}((\mathcal{H}_{\mathbf{C}})_{t_0}) \simeq \mathrm{GL}_2(\mathbf{C}).$$

Suitable standard loops around $s = 0, 1, \infty$ give local monodromy operators $T_s \in \mathrm{GL}_2(\mathbf{C})$, satisfying $T_0 T_1 T_\infty = 1$, generating the global monodromy group

$$\Gamma := \rho(\pi_1(S, t_0))) \subset \mathrm{GL}_2(\mathbf{C}).$$

What are the T_s's ? What is Γ ?

An example : the Legendre family

Consider the family X/S of elliptic curves on $S = \mathbf{P}^1_{\mathbf{C}} - \{0, 1, \infty\}$:

$$X_t : y^2 = x(x-1)(x-t).$$

For $\alpha = \beta = 1/2$, $\gamma = 1$,

$$E(1/2, 1/2, 1) : t(1-t)f'' + (1-2t)f' - \frac{f}{4} = 0$$

is the DE satisfied by the periods of holomorphic differential forms on X_t.

The relative de Rham cohomology group $\mathcal{H}_{\mathrm{dR}} := \mathcal{H}^1_{\mathrm{dR}}(X/S)$ is a free $\mathcal{O}_S$-module of rank 2, equipped with the Gauss-Manin connection ∇.

$$\mathcal{H}_{\mathrm{dR}} = \mathcal{O}_S e_1 \oplus \mathcal{O}_S e_2,$$

$$e_1 = [dx/y],\ e_2 = \nabla(d/dt)(e_1),$$

with

$$\nabla(d/dt)e_2 = \frac{(2t-1)e_2}{t(1-t)} + \frac{e_1}{4t(1-t)}.$$

Horizontal solutions $f_1 e_1^\vee + f_2 e_2^\vee$ of the dual of $\mathcal{H}_{\mathrm{dR}}$ are given by $f_1 = f,\ f_2 = f_1'$, where f, a local section of $\mathcal{O}_S$, satisfies

$$E(1/2, 1/2, 1) : t(1-t)f'' + (1-2t)f' - \frac{f}{4} = 0.$$

We have

$$\mathcal{H}_{\mathrm{dR}}^{\nabla=0} = \mathcal{H}_{\mathbf{Z}} \otimes \mathbf{C},$$

where $\mathcal{H}_{\mathbf{Z}} := \mathcal{H}^1(X/S, \mathbf{Z})$, a rank 2 $\mathbf{Z}$-local system, equipped with the (symplectic, unimodular) intersection form $\langle , \rangle$.

If γ is a local horizontal section of $\mathcal{H}_{\mathbf{Z}}^{\vee} = \mathcal{H}_1(X/S, \mathbf{Z})$, the period $\int_\gamma \frac{dx}{y}$ is a solution of $E(1/2, 1/2, 1)$. For example, the hypergeometric function

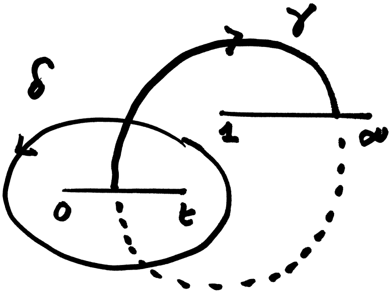
$$F(1/2, 1/2, 1, t) = \frac{1}{\pi} \int_1^\infty \frac{dx}{y}$$

is a solution.

The representation $\rho : \pi_1(S, t_0) \to \mathrm{GL}((\mathcal{H}_{\mathbf{C}})_{t_0})$ is deduced from

$$\rho : \pi_1(S, t_0) \to \mathrm{Sp}((\mathcal{H}_{\mathbf{C}})_{t_0}) \simeq SL_2(\mathbf{Z}).$$

Local monodromies around 0 and 1 can be calculated by choosing suitable symplectic bases (γ, δ) of $(\mathcal{H}_{\mathbf{Z}})_t$, using the description of X_t as a 2-sheeted cover of $\mathbf{P}^1_{\mathbf{C}}$.



γ
δ
1
∞
0
t

- In a suitable symplectic base, T_0 and T_1 are given by

$$T_0 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad T_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

- The global monodromy group is conjugate in $\mathrm{SL}_2(\mathbf{Z})$ to the subgroup $\Gamma = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\}$ of index 2 of the congruence subgroup $\Gamma(2)$ defined by $a \equiv d \equiv 1 \bmod 4$. It acts freely on the Poincaré upper half plane $D = \{\mathrm{Im} z > 0\}$.
- Riemann's period mapping $t \mapsto (\int_\gamma \omega, \int_\delta \omega)$, where $\omega = \frac{dx}{y} \in H^0(X_t, \Omega^1)$, induces an isomorphism

$$S = \mathbf{P}^1_{\mathbf{C}} - \{0, 1, \infty\} \simeq D/\Gamma.$$

which extends to an isomorphism

$$\mathbf{P}^1_{\mathbf{C}} \simeq M_2 \ (= (D \cup \mathbf{P}^1(\mathbf{Q}))/\overline{\Gamma}(2))$$

sending 0, 1, ∞ to the 3 cusps of M_2 ($\overline{\Gamma}(2)$ = image of $\Gamma(2)$ in $\mathrm{PSL}_2(\mathbf{Z})$).

In particular, as $\chi(S) = -1$, and $[\mathrm{SL}_2(\mathbf{Z}) : \Gamma] = 12$, the Galois cover

$$D \to S = D/\Gamma$$

implies that $S = B\Gamma$, hence $\chi(\Gamma) = -1$, and

$$\chi(\mathrm{SL}_2(\mathbf{Z})) = -\frac{1}{12},$$

as is well known.

It was discovered by Picard (around 1880) that the form of T_0 is "explained" by the fact that δ vanishes when $t \to 0$, and that the singularity of the surface X at $(x = 0, y = 0)$ is equivalent to $u^2 + v^2 = t^2$ (Picard-Lefschetz formula).

2. The Milnor fibration

Let $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ be a germ of holomorphic function having an isolated critical point at 0 with $f(0) = 0$.

Milnor (1967) proved that, for $\varepsilon > 0$ small, and $0 < \eta << \varepsilon$, if $B = \{z | \sum_0^n |z_i|^2 \leq \varepsilon\}$, $D = \{|t| \leq \eta\}$, the restriction of f to $B \cap f^{-1}(D)$,

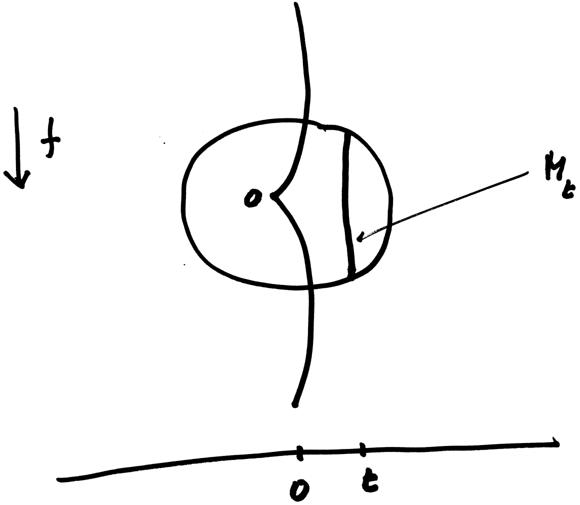
$$f : B \cap f^{-1}(D) \to D,$$

induces over $D - \{0\}$ a locally trivial C^∞ fibration in (real) $2n$-dimensional manifolds with boundary

$$M_t = f^{-1}(t) \cap B,$$

trivial along the boundary ∂M_t.

This is now called the Milnor fibration, and M_t is called a Milnor fiber.


f
0
M_t
0
t

Moreover, Milnor proved:

- M_t has the homotopy type of a bouquet of μ n-dimensional spheres:

$$S^n \vee \cdots \vee S^n \ (\mu \ \text{terms}),$$

hence, if $\widetilde{H}^i = \mathrm{Coker}(H^i(\mathrm{pt}) \to H^i)$,

$$\widetilde{H}^i(M_t, \mathbf{Z}) = \begin{cases} \mathbf{Z}^\mu & \text{if } i = n \\ 0 & \text{if } i \neq n. \end{cases}$$

- The Milnor number $\mu = \mu(f)$ is given by

$$\mu = \dim_{\mathbf{C}} \mathbf{C}\{z_0, \cdots, z_n\}/(\partial f/\partial z_0, \cdots, \partial f/\partial z_n).$$

Letting t turn once around zero clockwise in D gives an automorphism of $H^n(M_t, \mathbf{Z})$, the monodromy automorphism

$$T \in \mathrm{Aut}(H^n(M_t, \mathbf{Z})).$$

Milnor conjectured:

- The eigenvalues of T are roots of unity (i.e., T is quasi-unipotent).

Grothendieck proved it, using Hironaka's resolution of singularities and his theory of $R\Psi$ and $R\Phi$.

3. Grothendieck and Deligne

Given a 1-parameter family $(X_t)_{t \in S}$ of (algebraic, or analytic varieties), and a point $s \in S$, Grothendieck (1967) constructed in SGA 7 a complex of sheaves on X_s, called complex of vanishing cycles, measuring the difference between $H^*(X_s)$ and $H^*(X_t)$ for t "close" to s (special fiber X_s vs general fiber X_t), and a closely related one, called nowadays complex of nearby cycles.

Set-up : complex analytic, or étale.

Will discuss only the étale one.

Étale set-up

$S = (S, s, \eta)$, a strictly local trait
η: the generic point
$\overline{\eta}$: a separable closure of η.

For $f : X \to S$, get cartesian squares

$$\begin{array}{ccccc} X_s & \xrightarrow{i} & X & \xleftarrow{\bar{j}} & X_{\overline{\eta}} \\ \downarrow & & \downarrow f & & \downarrow \\ s & \longrightarrow & S & \longleftarrow & \overline{\eta} \end{array}.$$

Work with coefficients ring $\Lambda = \mathbf{Z}/\ell^{\nu}\mathbf{Z}$ (ℓ prime, invertible on S) (or $\mathbf{Z}_\ell$, $\mathbf{Q}_\ell$, $\overline{\mathbf{Q}}_\ell$, ℓ prime, invertible on S), write $D(-)$ for $D(-, \Lambda)$.

For $K \in D^+(X_\eta)$, the complex of nearby cycles is:

$$R\Psi_f(K) := i^* R\bar{j}_*(K | X_{\overline{\eta}}) \in D^+(X_s).$$

Comes equipped with an action of the inertia group $I = \mathrm{Gal}(\overline{\eta}/\eta)$ (complex of sheaves of I-modules on X_s).

For $K \in D^+(X)$, get an (I-equivariant) exact triangle

$$K|X_s \to R\Psi_f(K|X_\eta) \to R\Phi_f(K) \to,$$

where $R\Phi_f(K)$ is called the complex of vanishing cycles.

A generalization

$S = (S, s, \eta)$ henselian trait, not necessarily strictly local. Take strict localization of S at a separable closure $\widetilde{s}$ of s:

$$\widetilde{S} = (\widetilde{S}, \widetilde{s}, \widetilde{\eta}) \to (S, s, \eta).$$

For $f : X \to S$, base changed $\widetilde{f} : \widetilde{X} \to \widetilde{S}$, and $K \in D^+(X_\eta)$ (resp. $K \in D^+(X)$), define

$$R\Psi_f K \ \ (\text{resp. } \ R\Phi_f K) \in D^+(X_{\widetilde{s}})$$

as $R\Psi_{\widetilde{f}}(K|\widetilde{X}_{\widetilde{\eta}})$ (resp. $R\Phi_{\widetilde{f}}(K|\widetilde{X})$). Get action of full Galois group $\mathrm{Gal}(\overline{\eta}/\eta)$ ($\overline{\eta} \to \widetilde{\eta}$), not just of inertia $I = \mathrm{Gal}(\overline{\eta}/\widetilde{\eta}) \subset \mathrm{Gal}(\overline{\eta}/\eta)$.

General properties

- Functoriality Consider a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle g} & \\ S & & \end{array}$$

  If h is smooth, the natural map

$$h^* R\Psi_Y \to R\Psi_X h^*$$

  is an isomorphism. In particular, if f is smooth, $R\Phi_f(\Lambda) = 0$.
  If h is proper, the natural map

$$Rh_* R\Psi_X \to R\Psi_Y Rh_*$$

  is an isomorphism. In particular (taking $Y = S$), if f is proper, for $K \in D^+(X_\eta)$, we have a canonical isomorphism (compatible with the Galois actions)

$$R\Gamma(X_{\tilde{s}}, R\Psi_X K) \xrightarrow{\sim} R\Gamma(X_{\overline{\eta}}, K).$$

For X/S proper, the triangle $K|X_{\widetilde{s}} \to R\Psi_f(K|X_\eta) \to R\Phi_f(K) \to$ gives an exact sequence

$$\cdots \to H^{i-1}(X_{\widetilde{s}}, R\Phi_X(K)) \to H^i(X_{\widetilde{s}}, K) \xrightarrow{\mathrm{sp}} H^i(X_{\overline{\eta}}, K)$$

$$\to H^i(X_{\widetilde{s}}, R\Phi_X(K)) \to \cdots ,$$

where sp is the specialization map:

$$\mathrm{sp} : H^i(X_{\widetilde{s}}, K) \simeq H^i(X_{\widetilde{S}}, K) \to H^i(X_{\overline{\eta}}, K).$$

When $K = \Lambda$, $R\Phi_X(\Lambda)$ is concentrated on the points $x \in X_{\widetilde{s}}$ where X/S is not smooth.

- Finiteness (Deligne, 1974) Nearby cycles are constructible: $R\Psi_X$ induces

$$R\Psi_X : D_c^b(X_\eta) \to D_c^b(X_{\widetilde{s}}).$$

- Perversity (Gabber, 1981) $R\Psi$ commutes with Grothendieck-Verdier duality:

$$R\Psi(D_{X_\eta} K) \stackrel{\sim}{\to} D_{X_{\widetilde{s}}} R\Psi K,$$

induces $\mathrm{Per}(X_\eta) \to \mathrm{Per}(X_{\widetilde{s}})$.

In the analytic setup, there are analogous definitions and properties, and a comparison theorem (Deligne, 1968) between the étale $R\Psi$ and the analytic $R\Psi$, similar to Artin-Grothendieck's comparison theorem Betti vs étale.

Over $\mathbf{C}$, nearby cycles have been extensively studied in connection with Hodge theory (Steenbrink et al.), and the theory of $\mathcal{D}$-modules (M. Saito et al.).

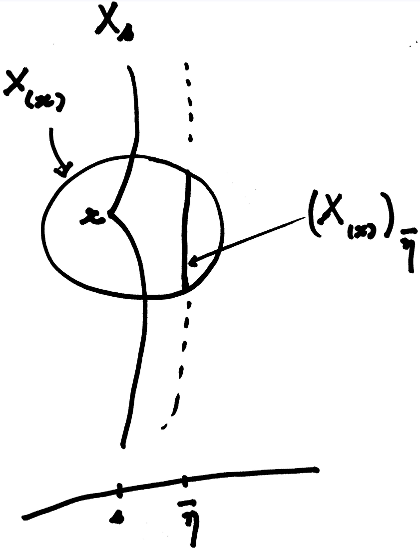
A crucial example

Let X/S be as above, with S strictly local, and $x \to X_s$ be a geometric point.

For $K \in D^+(X)$, by general nonsense on étale cohomology, the stalk of $R\Psi(K)$ ($:= R\Psi_X K$) at x is given by

$$(R\Psi K)_x = R\Gamma((X_{(x)})_{\overline{\eta}}, K).$$

Here $X_{(x)}$ is the strict localization of X at x (a kind of Milnor ball), and $(X_{(x)})_{\overline{\eta}}$ its geometric generic fiber (a kind of Milnor fiber).



X_s
$X_{(x)}$
x
$(X_{(x)})_{\bar\eta}$
s
$\bar\eta$

But $(R^q\Psi K)_x$ is difficult to calculate!

Known for $K = \Lambda$ (constant sheaf), when X has semistable reduction at x, i.e., étale locally at x,

$$X \xrightarrow{\sim} S[t_1, \cdots, t_n]/(t_1 \cdots t_r - \pi)$$

(π a uniformizing parameter in S). Then:

- $$(R^1\Psi\Lambda)_x = \mathrm{Ker}(\mathbf{Z}^r \xrightarrow{\mathrm{sum}} \mathbf{Z}) \otimes \Lambda(-1)$$
- $$(R^q\Psi\Lambda)_x = \Lambda^q(R^1\Psi\Lambda)_x$$

  (Λ^q = q-th exterior power, $\Lambda(m)$ = m-th Tate twist).
- The inertia group I acts trivially on $(R^q\Psi\Lambda)_x$.

For $X = S[t_1, \cdots, t_r]/(t_1 \cdots t_r - \pi)$,
topological model of $(X_{(x)})_{\overline{\eta}}$: fiber of

$$(S^1)^r \to S^1, \ (z_1, \cdots, z_r) \mapsto z_1 \cdots z_r.$$

Proof combines:

- Grothendieck's calculation of tame nearby cycles $(R^q\Psi\Lambda)_{\mathrm{t}} := (R^q\Psi\Lambda)^P$ ($P \subset I$ the wild inertia), modulo validity of Grothendieck's absolute purity conjecture for components of $(X_{(x)})_s$
- validity OK and $(R^q\Psi\Lambda)_{\mathrm{t}} = R^q\Psi\Lambda$ (Rapoport-Zink, 1982).

Recall Grothendieck's absolute purity conjecture:

For regular divisor $D \subset X$, X regular, $\Lambda = \mathbf{Z}/\ell^\nu\mathbf{Z}, \cdots$ as above, ℓ invertible on X,

$$\mathcal{H}^q_D(X, \Lambda) = \begin{cases} \Lambda_D(-1) & \text{if } q = 2 \\ 0 & \text{if } q \neq 2. \end{cases}$$

Modulo absolute purity conjecture (OK if $S/\mathbf{Q}$, and now in general by Gabber (1994)), Grothendieck calculated tame nearby cycles for X étale locally of the form $S[t_1, \cdots, t_n]/(ut_1^{n_1} \cdots t_r^{n_r} - \pi)$ (u a unit):

$$R^q\Psi\Lambda_{\mathrm{t},x} = \mathbf{Z}[\mu_{\ell^m}] \otimes \Lambda^q(\mathrm{Ker}(\mathbf{Z}^r \stackrel{\sum \mathrm{n_i x_i}}{\rightarrow} \mathbf{Z})) \otimes \Lambda(-q)$$

where $\gcd(n_1, \cdots, n_r) = \ell^m d$, $(\ell, d) = 1$.

Here I acts on $\mathbf{Z}[\mu_{\ell^m}]$ by permutation through its tame quotient $\mathbf{Z}_\ell(1)$, in particular, acts on $R^q\Psi\Lambda_{\mathrm{t},x}$ through a finite quotient, hence quasi-unipotently on $R\Psi\Lambda_{\mathrm{t},x}$.

Combined with Hironaka's resolution of singularities, and functoriality of $R\Psi$ for proper maps, calculation yields a proof of Milnor's conjecture on the monodromy of isolated singularities.

4. Grothendieck's local monodromy theorems

Grothendieck's arithmetic local monodromy theorem is the following:

Theorem

$S = (S, s, \eta)$ henselian, $k = k(s)$, ℓ prime different from $p = \mathrm{char}(k)$. Assume that no finite extension of k contains all roots of unity of order a power of ℓ (e. g., k finite). Let

$$\rho : \mathrm{Gal}(\overline{\eta}/\eta) \to \mathrm{GL}(V)$$

be a continuous representation into a finite dimensional $\mathbf{Q}_\ell$-vector space V. Then, there exists an open subgroup $I_1 \subset I$, such that, for all $g \in I_1$, $\rho(g)$ is unipotent.

Proof.

Exercise ! (Use strong action of $\mathrm{Gal}(\overline{k}/k)$ on tame inertia I_t: $g\sigma g^{-1} = \sigma^{\chi(g)}$, χ = cyclotomic character.) □

A corollary is that there exists a unique nilpotent morphism

$$N : V(1) \to V,$$

called the monodromy operator, such that, for all $\sigma \in I_1$ and $x \in V$,

$$\sigma x = \exp(N(t_\ell(\sigma)x)),$$

where $t_\ell : I \to \mathbf{Z}_\ell(1)$ is the ℓ-component of the tame character.

The operator N is $\mathrm{Gal}(\overline{\eta}/\eta)$-equivariant. In particular, for $k = \mathbf{F}_q$, if $F \in \mathrm{Gal}(\overline{\eta}/\eta)$ is a lifting of the geometric Frobenius ($a \to a^{1/q}$), then

$$NF = qFN.$$

Led to the Weil-Deligne representation.

The geometric local monodromy theorem is the following result, due to Grothendieck in a weaker form, later improved by various authors:

Theorem

Let S be an (arbitrary) henselian trait. Let X_η be separated and of finite type over η. Then, there exists an open subgroup $I_1 \subset I$, independent of ℓ, such that for all $i \in \mathbf{Z}$ and all $g \in I_1$,

$$(g-1)^{i+1} = 0$$

on $H^i(X_{\overline{\eta}}, \Lambda)$ (resp. $H^i_c(X_{\overline{\eta}}, \Lambda)$).

History

- Existence of I_1 (*a priori* ℓ-dependent) for H^i_c with $i+1$ replaced by uncontrolled bound, proved by Grothendieck, as a consequence of the arithmetic local monodromy theorem (reduction to k small). Method generalized to H^i once finiteness of H^i was proved (Deligne, 1974).

- Existence of I_1 (*a priori* ℓ-dependent), with bound $i+1$, proved by Grothendieck for X_η/η proper and smooth, modulo validity of absolute purity and resolution of singularities, as a consequence of local calculation of $R^q\Psi\mathbf{Z}_\ell$ in the (quasi-) semistable case. Unconditional for $i \leq 1$, or $p = 0$.
- Existence of I_1, independent of ℓ, but with $i+1$ replaced by uncontrolled bound, proved by Deligne (1996), using Rapoport-Zink's calculation of $R\Psi\mathbf{Z}_\ell$ in the semistable case, and de Jong's alterations. Final result obtained by refinement of this method (Gabber - I., 2014).

Why care for exponent $i+1$?
Grothendieck's motivation: for $i = 1$, exponent 2 is a crucial ingredient in his proof of the semistable reduction theorem for abelian varieties:

Theorem

With S as before, let A_η be an abelian variety over η. There exists a finite extension η_1 of η such that A_{η_1} acquires semistable reduction over the normalization (S_1, s_1, η_1) of S in η_1, i.e., the connected component $A^0_{s_1}$ of the special fiber of the Néron model of A_{η_1} is an extension of an abelian variety by a torus:

$$0 \to (\text{torus}) \to A^0_{s_1} \to (\text{abelian variety}) \to 0.$$

Deligne-Mumford (1969) deduced from it the semistable reduction theorem for curves:

Corollary

Let X_η be a proper, smooth curve over η. There exists a finite extension η_1 of η such that X_{η_1} has semistable reduction over the normalization S_1 of S in η_1, i.e., is the generic fiber of a proper, flat X_1/S_1, with X_1 regular, and special fiber $(X_1)_{s_1}$ a reduced curve having simple nodes.

- Corollary is the key tool in Deligne-Mumford's proof of the irreducibility of the coarse moduli space M_g (over any algebraically closed field k).
- Proofs of corollary independent of theorem found later (Artin-Winters, 1971; T. Saito, 1987).
- For $\mathrm{char}(k) = 0$, a generalization of corollary to arbitrary dimension proved by Mumford et al. (1973).
- Over S excellent (any char.), a generalization of corollary in a weaker form given by de Jong (1996). Recently improved by Gabber, Temkin.

5. The Deligne-Milnor conjecture

At the opposite of semistable reduction, we have isolated singularities.

Let $S = (S, s, \eta)$ be a strictly local trait, with $k = k(s)$ algebraically closed. Assume X regular, flat, finite type over S, relative dimension n, smooth outside closed point $x \in X_s$. Then $R\Phi\Lambda$ is concentrated at x, and in cohomological degree n:

$$(R\Phi^q\Lambda)_x = \begin{cases} 0 & \text{if } q \neq n \\ \Lambda^r & \text{if } q = n \end{cases}$$

The coherent module $\mathcal{E}xt^1(\Omega^1_{X/S}, \mathcal{O}_X)$ is concentrated at x, its length

$$\mu := \lg(\mathcal{E}xt^1(\Omega^1_{X/S}, \mathcal{O}_X))$$

generalizes the classical Milnor number.

The action of I on $R^n\Phi\Lambda$ has a Swan conductor $\mathrm{Sw}(R^n\Phi\Lambda) \in \mathbf{Z}$, measuring wild ramification ($= 0$ if S of char. 0).

Deligne conjectured (SGA 7 XVI, 1972):

$$\mu = r + \mathrm{Sw}(R^n\Phi\Lambda).$$

Generalizes Milnor formula over $\mathbf{C}$.
Conjecture proved:

- if X/S finite, or x is an ordinary quadratic singularity, or S is of equal characteristic (Deligne, loc. cit.)
- if $n = 1$ (Bloch, 1987 + Orgogozo, 2003)

General case open. In equal char., generalization by T. Saito (2015) with Λ replaced by a constructible sheaf.

6. The Picard-Lefschetz formula

Let X/S as before, with relative dimension n. Assume x is an ordinary quadratic singularity of X/S, i.e., étale locally at x, X/S is of the form (π a uniformizing parameter):

$$\sum_{1 \leq i \leq m+1} x_i x_{i+m+1} = \pi$$

$(n = 2m+1)$,

$$\sum_{1 \leq i \leq m} x_i x_{i+m} + x_{2m+1}^2 = \pi$$

$(n = 2m,\ p > 2)$,

$$\sum_{1 \leq i \leq m} x_i x_{i+m} + x_{2m+1}^2 + a x_{2m+1} + \pi = 0$$

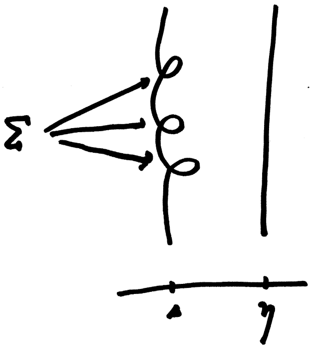
with $a^2 - 4\pi \neq 0$ $(n = 2m,\ p = 2)$.

Then

$$(R^n \Phi \Lambda)_x \xrightarrow{\sim} \Lambda,$$

with action of inertia I trivial is n odd, through a character of order 2 if n even, tame if $p > 2$.

Assume now X/S proper, flat, of relative dimension $n > 0$, smooth outside $\Sigma \subset X_s$ finite, and each $x \in \Sigma$ is an ordinary quadratic singularity.



Σ
η

Then the monodromy of $H^*(X_{\overline{\eta}})$ is described as follows (Deligne, SGA 7 XV, 1972):

- For $i \neq n,\ n+1$, $H^i(X_s) \overset{\text{sp}}{\underset{\sim}{\to}} H^i(X_{\overline{\eta}})$.
- For each $x \in \Sigma$, there exists $\delta_x \in H^n(X_{\overline{\eta}})(m)$ ($n = 2m$ or $2m+1$), well defined up to sign, called the vanishing cycle at x, and the sequence

$$0 \to H^n(X_s) \overset{\text{sp}}{\to} H^n(X_{\overline{\eta}}) \overset{(-,\delta_x)}{\to} \sum_{x \in \Sigma} \Lambda(m-n) \to H^{n+1}(X_s)$$

$$\overset{\text{sp}}{\to} H^{n+1}(X_{\overline{\eta}}) \to 0.$$

is exact. One has $(\delta_x, \delta_y) = 0$ for $x \neq y$, $(\delta_x, \delta_x) = 0$ for n odd, and $(\delta_x, \delta_x) = (-1)^m.2$ for $n = 2m$. Here $(a, b) = \mathrm{Tr}(ab)$, where $\mathrm{Tr} : H^{2n} \to \Lambda(-n)$.

- The inertia I acts trivially on $H^i(X_{\overline{\eta}})$ for $i \neq n$, and on $H^n(X_{\overline{\eta}})$ through orthogonal (resp. symplectic) transformations for $n = 2m$ (resp. $n = 2m+1$), given by the Picard-Lefschetz formula:

  For $\sigma \in I$, $a \in H^n(X_{\overline{\eta}})$,

$$\sigma a - a = \begin{cases} (-1)^m \sum_{x \in \Sigma} \frac{\varepsilon_x(\sigma)-1}{2} \langle a, \delta_x \rangle \delta_x & \text{if } n = 2m \\ (-1)^{m+1} \sum_{x \in \Sigma} t_\ell(\sigma) \langle a, \delta_x \rangle \delta_x & \text{if } n = 2m+1. \end{cases}$$

  Here $t_\ell : I \to \mathbf{Z}_\ell(1)$ is the tame character, and $\varepsilon_x : I \to \pm 1$ is the unique character of order 2 if $p > 2$ and that defined by the quadratic extension $t^2 + at + \pi = 0$ for X locally at x of the form

$$\sum_{1 \leq i \leq m} x_i x_{i+m} + x_{2m+1}^2 + a x_{2m+1} + \pi = 0.$$

Difficult case in the proof: n odd, $n = 2m+1$. Use factorization:

$$
\begin{CD}
H^n(X_{\overline{\eta}}) @>>> \oplus_{x \in \Sigma} (R^n \Phi \Lambda)_x \\
@V{\sigma - 1}VV @VV{\mathrm{Var}(\sigma)_x}V \\
H^n(X_{\overline{\eta}}) @<<< \oplus_{x \in \Sigma} H^n_x(X_s, R\Psi\Lambda)
\end{CD}
\quad ,
$$

where:

- top row is part of specialization sequence
- bottow row = composition of $H^n_x \to H^n$ and $H^n(X_s, R\Psi\Lambda) = H^n(X_{\overline{\eta}})$.
- $(R^n \Phi \Lambda(m+1))_x$ and $H^n_x(X_s, R\Psi\Lambda)(m)$ are isomorphic to Λ, with respective generators $\underline{\delta}'_x$, $\underline{\delta}_x$ defined up to sign, with $\langle \underline{\delta}'_x, \underline{\delta}_x \rangle = 1$, for a perfect pairing with values in $H^{2n}_x(X_s, R\Psi\Lambda(n)) \overset{\mathrm{Tr}}{\xrightarrow{\sim}} \Lambda$. We have $\underline{\delta}_x \mapsto \delta_x \in H^n(X_{\overline{\eta}})$.

$$\begin{array}{ccc} H^n(X_{\overline{\eta}}) & \longrightarrow & \oplus_{x \in \Sigma}(R^n\Phi\Lambda)_x \\ \downarrow{\scriptstyle \sigma - 1} & & \downarrow{\scriptstyle \mathrm{Var}(\sigma)_x} \\ H^n(X_{\overline{\eta}}) & \longleftarrow & \oplus_{x \in \Sigma} H^n_x(X_s, R\Psi\Lambda) \end{array} ,$$

The map $\mathrm{Var}(\sigma)_x$, called variation, is given by the local Picard-Lefschetz formula:

$$\mathrm{Var}(\sigma)_x(\underline{\delta}'_x) = (-1)^{m+1} t_\ell(\sigma)\underline{\delta}_x,$$

which is the crux of the matter.

- Original proof (Deligne) required lifting to char. 0 and a transcendental argument.
- Purely algebraic proof given later (I., 2000), as a corollary of Rapoport-Zink's theory of nearby cycles in the semistable case.

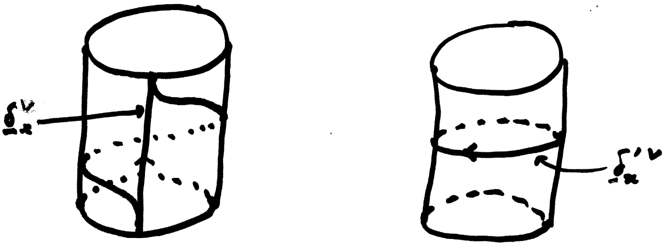
Over $\mathbf{C}$, Milnor fiber M_t of $f : (x_1, \cdots, x_{2m+2}) \mapsto \sum x_i^2$ is fiber bundle in unit balls of tangent bundle to sphere $S^n = \{x \in \mathbf{R}^{n+1} | \sum x_i^2 = 1\}$.

- $R^n\Phi_x$ corresponds to $\widetilde{H}^n(M_t)$,
- $H^n_x(X_s, R\Psi)$ corresponds to $H^n_c(M_t - \partial M_t)$,
- $\underline{\delta}_x$ dual to $\underline{\delta}_x^\vee \in H_n(M_t, \partial M_t)$ given by one fiber of M_t over S^n,
- $\underline{\delta}'_x$ dual to $(\underline{\delta}'_x)^\vee \in \widetilde{H}_n(M_t)$ given by $S^n \subset M_t$.

Next slide: picture, for $n = 1$ $(m = 0)$ of the dual variation map (T the positive generator of $\pi_1(S^1)$)

$$\mathrm{Var}(T)^\vee : H_1(M_t, \partial M_t) \to \widetilde{H}_1(M_t),$$

$$\underline{\delta}_x^\vee \mapsto -(\underline{\delta}'_x)^\vee.$$



$$\underline{\delta}_x^v \longmapsto -\underline{\delta}'^v_x$$

Back to the Legendre family:

$$X_t : y^2 = x(x-1)(x-t).$$

Locally at $x = y = t = 0$, X/S is $x_1^2 + x_2^2 = t^2$, instead of $x_1^2 + x_2^2 = t$, hence variation is doubled, and get

$$T(\delta) = \delta, \quad T(\gamma) = \gamma \pm 2\delta$$

Arithmetic applications

- Grothendieck used the PL formula in his theory of the monodromy pairing for abelian varieties having semistable reduction (SGA 7 IX), with a formula for calculating the group of connected components of the special fiber of the Néron model. Variants, generalizations, and arithmetic applications by Raynaud, Deligne-Rapoport, Mazur, Ribet.
- Most importantly, the PL formula was the key to the cohomological study (by Deligne and Katz, SGA 7 XVIII) of Lefschetz pencils, which led to the fiirst proof, by Deligne, of the Weil conjecture (Weil I).

Variants and generalizations

- Tame variation

  Recall the case of isolated singularities:
  X regular, flat, finite type over S, relative dimension n, smooth outside closed point $x \in X_s$. Then $R\Phi\Lambda$ is concentrated at x, and in cohomological degree n:

$$(R\Phi^q\Lambda)_x = \begin{cases} 0 & \text{if } q \neq n \\ \Lambda^r & \text{if } q = n \end{cases}$$

  Moreover,

$$H^n_{\{x\}}(X_s, R\Psi\Lambda) = \Lambda^r,$$

  with a perfect intersection pairing

$$R^n\Phi(\Lambda)_x \otimes H^n_{\{x\}}(X_s, R\Psi\Lambda) \to H^{2n}_{\{x\}}(X_s, R\Psi\Lambda) = \Lambda(-n).$$

Finally, if I acts tamely on $R\Psi\Lambda$, i.e., through its quotient $\mathbf{Z}_\ell(1)$, and if σ is a topological generator of it, then $\sigma - 1$ induces an isomorphism

$$\mathrm{Var}(\sigma) : R^n\Phi(\Lambda)_x \xrightarrow{\sim} H^n_{\{x\}}(X_s, R\Psi\Lambda),$$

called the variation at x (I., 2003), a (weak) generalization of the local Picard-Lefschetz formula. The analogue over $\mathbf{C}$ had been known since the 1970's (Brieskorn).

- Thom-Sebastiani theorems

The Picard-Lefschetz theory describes vanishing cycles, monodromy and variation at the isolated critical point $\{0\}$ of the function

$$x_1^2 + \cdots + x_m^2.$$

The classical Thom-Sebastiani theorem $(/\mathbf{C})$ describes the same invariants at the isolated critical point $\{0\}$ of a function of the form

$$f(\underline{x}_1, \cdots, \underline{x}_m) = f_1(\underline{x}_1) + \cdots + f_m(\underline{x}_m),$$

where the $\underline{x}_i$ are independent packs of $n_i + 1$ variables, and $f_i : \mathbf{C}^{n_i+1} \to \mathbf{C}$ has an isolated critical point at $\{0\}$.

If $n = \sum n_i$ (= rel. dim. of f), then (for coefficients $\mathbf{Z}$)

$$R^n\Phi_f = \otimes_{1 \leq i \leq m} R^{n_i}\Phi_{f_i},$$

with monodromy

$$T = \otimes_{1 \leq i \leq m} T_i,$$

and variation

$$\mathrm{Var} = \otimes_{1 \leq i \leq m} \mathrm{Var}_i.$$

Algebraic analogues ?
(over an alg. closed field k, in the étale set-up)

Deligne's observation: analogue wrong in general, tensor product must be replaced by

local convolution product $*$

of Deligne-Laumon.

Formulas in this framework given by Fu Lei (2014), I. (2015).

7. Euler numbers and characteristic cycles

Quite recently, T. Saito, in conjunction with Beilinson's construction of a singular support

$$SS(\mathcal{F}) \subset T^*X$$

for a constructible sheaf $\mathcal{F}$ on a smooth X/k (an equidimensional conic closed subset of T^*X, of dimension $= \dim(X)$), defined a characteristic cycle supported on $SS(\mathcal{F})$, with coefficients in $\mathbf{Z}[1/p]$ (actually, in $\mathbf{Z}$ (Beilinson)):

$$CC(\mathcal{F}) \in Z_{\dim(X)}(T^*X),$$

proved a generalization of the Deligne-Milnor formula (equal characteristic case), and as a corollary, a global index formula for the Euler number of $\mathcal{F}$.

The global index formula reads:

For X/k proper and smooth, k alg. closed, $\Lambda = \mathbf{Q}_\ell$,

$$\chi(X, \mathcal{F}) = (CC(\mathcal{F}), T^*_X X).$$

Here $\chi(X, \mathcal{F}) = \sum_i (-1)^i \dim H^i(X, \mathcal{F})$, $T^*_X(X)$ = 0-section of T^*X.

This work was inspired by Kashiwara-Schapira's analogous theory over $\mathbf{C}$, and various conjectures of Deligne.

Ingredients

- Radon and Legendre transforms (Brylinski), geometric theory of Lefschetz pencils (Katz, SGA 7 XVII)
- Ramification theory for imperfect residue fields (Abbes, T. Saito)
- Deligne's theory of vanishing cycles over general bases (Deligne, Gabber, Orgogozo) (also used in generalized Thom-Sebastiani theorems).

Thank you!