Pre-notes for Sapporo seminar, March 2011
De Rham-Witt complexes and p-adic Hodge theory

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1. Historical sketch

1956 : • Cartier isomorphism
- Serre's Witt vector cohomology,
- Dieudonné's theory of Dieudonné modules

1963-65 : • Manin's work on formal groups,
- Gauss-Manin connection

1967 : • Cartier et al. : big Witt vectors, Cartier modules
- Tate : p-divisible groups, Hodge-Tate decomposition
- Monsky-Washnitzer's cohomology
- Grothendieck : crystalline cohomology

1970 : • Berthelot's thesis
- Grothendieck's crystalline Dieudonné theory, problem of the mysterious functor
- Mazur-Ogus : slopes of Frobenius (Katz inequality)

1974 : • Bloch : complex of typical curves on K-groups

1975 : • Deligne-Illusie : de Rham-Witt complex

1980 : • Fontaine's p-adic period rings B_{cris}, B_{dR}

1980-85 : • fine study of de Rham-Witt (Nygaard, Illusie-Raynaud, Ekedahl)
- Bloch-Kato's proof of Hodge-Tate decompositions (good ordinary case)
- Fontaine-Messing's proof of C_{cris} (dim $X < p$, $e \leq p-1$), syntomic cohomology
- Faltings's almost étale theory, tentative proofs of C_{cris}, C_{dR} in general

1988 : • Fontaine-Jannsen's C_{st} conjecture
- Fontaine-Illusie-Kato : log schemes
- Hyodo-Kato log crystalline cohomology, log de Rham-Witt complex
- Kato's proof of C_{st} (2 dim $X < p-1$)

1988 - ... : • Berthelot's rigid cohomology, arithmetic $\mathcal{D}$-modules

1997 : • Tsuji : proof of C_{st} in the general case
- Faltings : sketch of corrected proof of almost purity lemma and C_{st} (details worked out by Gabber-Ramero)

1998 : • Niziol's proof of C_{cris} using K-theory

2000 : • Fontaine, Colmez, André, Kedlaya, Christol-Mebkhout, ... : proofs of main conjectures on p-adic representations (weakly admissible $\Leftrightarrow$ admissible, dR $\Leftrightarrow$ pst, p-adic local monodromy conjecture, finiteness of rigid cohomology)

2004 : • Hesselholt-Madsen's absolute de Rham-Witt complex / $\mathbf{Z}_{(p)}$
• Langer-Zink's relative de Rham-Witt complex / $\mathbf{Z}_{(p)}$
• Zink's theory of displays

2007 : • Olsson : stack theoretic variants of de Rham-Witt

2008 : • Niziol's K-theoretic proof of C_{st}
• Davis-Langer-Zink : overconvergent de Rham-Witt complex

2011 : • Beilinson : new proof of C_{dR} using derived de Rham complexes

$$\begin{tikzcd}
 & \mathbf{Witt\,vector\,H^*} \arrow[d, dash] & \\
 & \mathbf{de\,Rham - Witt\,complex} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbf{de\,Rham\,and\,crystalline\,H^*} \arrow[dr, dash] & & \mathbf{Hodge\,H^*} \arrow[dl, dash] \\
 & p - \mathbf{adic\,étale\,H^*} &
\end{tikzcd}$$

2. Witt vectors

2.1. *Witt polynomials, ghost components*
p = prime number
$w_n(X_0, \cdots, X_i, \cdots) := \sum_{0 \leq i \leq n} p^i X^{p^{n-i}}$:
$w_o = X_0$
$w_1 = X_0^p + pX_1$
$w_2 = X_0^{p^2} + pX_1^p + p^2 X_2$,
$\cdots$

Theorem 2.1.1. *For a set A, let*

$$W(A) := A^{\mathbf{N}} = \{(a_0, \cdots, a_n, \cdots), a_i \in A\}.$$

There exists a unique functor $A \mapsto W(A)$ from rings to rings such that

$$w : W(A) \to A^{\mathbf{N}}$$

is a homomorphism of rings, where $A^{\mathbf{N}}$ is equipped with the product structure.

Proof. [CL, II, §§ 5, 6]. Alternate proof : use *Dwork's lemma* : If $f : A \to A$, $f(a) \equiv a^p \bmod p$, $(x = (x_0, \cdots) \in w(A^{\mathbf{N}})) \Leftrightarrow (x_i = f(x_{i-1}) \bmod p^i \ \forall i > 0)$. See also : [Demazure, III].

Ghost map, ghost components. $1 = (1, 0, \cdots, 0, \cdots)$, $0 = (0, \cdots, 0)$, $S_n(a, b)$, $P_n(a, b)$, $S_0 = a_0 + b_0$, $S_1 = a_1 + b_1 - \sum_{0<i<p} p^{-1}(p!/i!(p-i)!)a_0^i b_0^{p-i}$, $P_0 = a_0 b_0$, $P_1 = b_0^p a_1 + b_1 a_0^p + p a_1 b_1$.

2.2. *Operators R, F, V*

$W_n(A)$, R, V, short exact sequences, $[x] = (x, 0, \cdots)$

There exists a unique $F : W(A) \to W(A)$ functorial in A such that $w(Fa) = (w_1(a), w_2(a), \cdots)$.

$Fa = (f_0(a), \cdots, f_n(a), \cdots)$, $f_n(a) = f_n(a_0, \cdots, a_{n+1})$, $f_0(a) = a_0^p + pa_1$, $f_n(a) \equiv a_n^p \bmod p$

$F : W_n(A) \to W_{n-1}(A)$

$FV = p$, $xVy = V((Fx)y)$, $F[x] = [x^p]$, $(VF = p) \Leftrightarrow (p = 0 \text{ in } A)$.

$p = 0$ in $A \Rightarrow Fa = (a_0^p, \cdots, a_n^p; \cdots)$.

$m \in \mathbf{Z}$ invertible in $A \Rightarrow m$ invertible in $W_n(A)$; in particular, if A is a $\mathbf{Z}_{(p)}$-algebra, so is $W_n(A)$.

2.3. *Examples*

- $W_n(A)$, A perfect of char. p

$V = pF^{-1}$, $W_n(A) = W(A)/p^n W(A)$, $W(A) =$ the (unique) strict p-ring B of residual ring A ($W(A) \overset{\sim}{\to} B$, $a \mapsto \sum r(a_n)^{p^{-n}} p^n$, $r : A \to B$ (the) system of multiplicative representatives)

k perfect field of char. $p \Rightarrow W(k) =$ (the) Cohen ring of k ; $W(\mathbf{F}_p) = \mathbf{Z}_p$.

- $W_n(\mathbf{F}_p[t])$

$$W_n(\mathbf{F}_p[t]) = E^0/V^n E^0,$$

where $E^0 \subset \mathbf{Z}_p[t^{p^{-\infty}}]$ is the set of $\sum_{k \in \mathbf{N}[1/p]} a_k t^k$ such that the denominator of k divides a_k for all k, with F, V induced by F, V on $\mathbf{Q}_p[t^{p^{-\infty}}]$ given by $Ft = t^p$, $V = pF^{-1}$.

(see [DRW, I 2.3] : $E^0 = \sum V^n \mathbf{Z}_p[t]$; there's a unique $\mathbf{Z}_p$-algebra homomorphism $E^0 \to W(\mathbf{F}_p[t])$ compatible with V, sending t to [t] ; it is injective and induces an isomorphism on gr_V.)

Gives a decomposition

$$W_n(\mathbf{F}_p[t]) = \bigoplus_{k \,\text{integral}} (\mathbf{Z}/p^n\mathbf{Z})[t]^k \oplus \bigoplus_{k \,\text{not integral}} V^{u(k)}(\mathbf{Z}/p^{n-u(k)}\mathbf{Z})[t]^{p^{u(k)}k},$$

($p^{u(k)}$ being the denominator of k, and $[t]$ the Teichmüller representative).

A similar description holds for $\mathbf{F}_p[t_1, \cdots, t_r]$ (loc. cit.).

- $W_n(\mathbf{Z}_{(p)})$

$$W_n(\mathbf{Z}_{(p)}) = \prod_{0 \leq i \leq n-1} \mathbf{Z}_{(p)} V^i 1$$

(as a $\mathbf{Z}_{(p)}$-module), with $V^i 1 . V^j 1 = p^i V^j 1$ $(0 \leq i \leq j < n$.

(see [Hesselholt-Madsen, 1.2.4] : $\mathrm{gr}_V W_n(\mathbf{Z}_{(p)})$ free over $\mathbf{Z}_{(p)}$, $(V^i 1)$ split the filtration ; $\sum_{0 \leq i < n} V^i[a_i] = \sum_{0 \leq i < n} b_i V^i 1$, with a_i, b_i in $\mathbf{Z}_{(p)}$ (and the 1-1 correspondence $(a_i) \leftrightarrow (b_i)$ given by complicated functions))

2.4. *Link with big Witt vectors*

$\mathbf{W}(A) := (1 + A[[t]])^*$, $u +_{\mathbf{W}} v := uv$, $(1-at)^{-1} \cdot_{\mathbf{W}} (1-bt)^{-1} := (1-abt)^{-1}$

$A/\mathbf{Z}_{(p)} \Rightarrow W(A) \subset \mathbf{W}(A)$, $W(A) = \pi \mathbf{W}(A)$, $\pi x = E(t)x$,

$E(t) = \exp(\sum_{n \geq 0} t^{p^n}/p^n) = \prod_{n \in I(p)} (1-t^n)^{-\mu(n)/n} \in \mathbf{W}(\mathbf{Z}_{(p)})$ (Artin-Hasse exponential)

$a = (a_0, \cdots) \mapsto \prod_{n \geq 0} E(a_n t^{p^n})$, $W(A) \xrightarrow{\sim} \pi \mathbf{W}(A)$

(see [DRW 0 1.2], [Demazure], [Bloch]).

2.4. *Sheafification*

For A a ring in a topos T, and $n \in \mathbf{N}$, $n > 0$, the presheaf $U \mapsto W(A(U))$ (resp. $U \mapsto W_n(A(U))$) is a sheaf of rings, denoted $W(A)$ (resp. $W_n(A)$. If X is a scheme, the underlying space of X together with the sheaf $W_n(\mathcal{O}_X)$ is a scheme, denoted $W_n(X)$ (LZ, Appendix). If p is nilpotent in A, VW_nA is nilpotent (since it's a DP-ideal, see 3.2). If p is nilpotent on X, $W_n(X)$ is a thickening of X.

3. Crystalline cohomology

3.1. *Inputs from complex analytic geometry : Poincaré lemma, Gauss-Manin connection*

- *Poincaré lemma*

analytic : $X/\mathbf{C}$ smooth analytic space : $\mathbf{C} \to \Omega^{\cdot}_{X/\mathbf{C}}$ = quasi-isomorphism

formal : k = field of char. 0, $t = (t_1, \cdots, t_n)$: $k \to \Omega^{\cdot}_{k[[t]]/k}$ = quasi-isomorphism

algebraic : k = field of char. 0, $t = (t_1, \cdots, t_n)$: $k \to \Omega^{\cdot}_{k[t]/k}$ = quasi-isomorphism

($n = 1$; $0 \to k \to k[t] \to k[t]dt \to 0$ exact, $t^i \mapsto it^{i-1}dt$ $(i \geq 1)$

$\mathrm{char}(k) = p > 0 \Rightarrow \Omega^{\cdot}_{k[t]/k}$ quasi-isomorphic to $k[t^p] \otimes (k \oplus kt^{p-1}dt[-1])$ (generalization : *Cartier isomorphism*)

- *Gauss-Manin*

relative Poincaré lemma : $f : X \to Y$ smooth morphism of complex analytic spaces $\Rightarrow f^{-1}\mathcal{O}_Y \to \Omega^{\cdot}_{X/Y}$ quasi-isomorphism.

If f *proper*, then $R^i f_* \mathbf{C}$ = local system, and

$$\mathcal{H}^i_{dR}(X/Y) := R^i f_* \Omega^{\cdot}_{X/Y} = \mathcal{O}_Y \otimes R^i f_* \mathbf{C}.$$

$\Rightarrow$ For $Y/\mathbf{C}$ smooth, get integrable connection $\nabla = d \otimes Id : \mathcal{H}^i_{dR}(X/Y) \to \Omega^1_Y \otimes \mathcal{H}^i_{dR}(X/Y)$, with horizontal sections $R^i f_* \mathbf{C}$.

If Y = *smooth* $\mathbf{C}$*-scheme*, $f : X \to Y$ proper smooth, by GAGA

$$\mathcal{H}^i_{dR}(X/Y)^{an} = \mathcal{H}^i_{dR}(X^{an}/Y^{an}),$$

and by Manin there exists a canonical integrable connection

$$\nabla_{GM} : \mathcal{H}^i_{dR}(X/Y) \to \Omega^1_Y \otimes \mathcal{H}^i_{dR}(X/Y)$$

such that $(\nabla_{GM})^{an} = \nabla$. Purely alg. construction. Variants : Katz-Oda, Grothendieck.

$\Rightarrow$ Grothendieck's observation : k = perfect field of char. $p > 0$, $W = W(k)$, $t = (t_1, \cdots, t_n)$, $X/S = \mathrm{Spec} W[[t]]$ proper smooth such that $\mathcal{H}^i_{dR}(X/S)$ free of finite type $\forall i$. Let $u : \mathrm{Spec} W \to S$, $v : \mathrm{Spec} W \to S$ such that $u \equiv v$ mod p. Get : $X_u := u^* X$, $X_v := v^* X$ such that $X_u \otimes k = X_v \otimes k = Y$, and $H^i_{dR}(X_u/W) = u^* \mathcal{H}^i(X/S)$, $H^i_{dR}(X_v/W) = v^* \mathcal{H}^i(X/S)$. By $\nabla = \nabla_{GM}$, get *isomorphism*

$$\chi(u,v) : H^i_{dR}(X_u/W) \xrightarrow{\sim} H^i_{dR}(X_v/W),$$

$$u^*(x) \mapsto \sum_{m \geq 0} (1/m!)(u^*(t) - v^*(t))^m v^*(\nabla(D)^m x)$$

($x \in \mathcal{H}^i_{dR}(X/S)$, $D = (D_1, \cdots, D_n)$, $D_i = \partial/\partial t_i$), with $\chi(v,w)\chi(u,v) = \chi(u,w)$, $\chi(u,u) = \mathrm{Id}$ (NB. $(1/m!)(u^*(t) - v^*(t))^m \in W$; series converge p-adically : $p > 2$ easy, by Berthelot in general).

$\Rightarrow$ question (Grothendieck) : for Y/k proper, smooth, X_1, X_2 proper smooth liftings $/W$, can one hope for an isomorphism (generalizing $\chi(u,v)$)

$$\chi_{12}; H^i_{dR}(X_1/W) \xrightarrow{\sim} H^i_{dR}(X_2/W)$$

with $\chi_{23}\chi_{12} = \chi_{13}$? (Monsky-Washnitzer : analogue in the affine case OK)

Answer : Yes : solution : *crystalline cohomology* $H^i(Y/W)$ (depending only on Y, with no assumption of existence of lifting), providing can. iso :

$$\chi : H^i(Y/W) \xrightarrow{\sim} H^i_{dR}(X/W)$$

for any proper smooth lifting X/W of Y, such that for X_1, X_2 as above, $\chi_2 = \chi_{12}\chi_1$.

Berthelot-Grothendieck's definition : $H^i(Y/W) = \text{proj.lim}_n H^i(Y/W_n)$, $H^i(Y/W_n) = H^i((Y/W_n)_{cris}, \mathcal{O})$, $(Y/W_n)_{cris}$: *crystalline site*, $\mathcal{O} =$ structural sheaf of rings.

Later : $H^i(Y/W) = H^i(Y_{zar}, W\Omega^{\cdot}_Y)$, $W\Omega^{\cdot}_Y =$ *de Rham-Witt complex.*

3.2. *Divided powers*

$I \subset A =$ ideal ; *divided powers* on $I =$ family $\gamma_n : I \to A$, $n \in \mathbf{N}$, satisfying formally the properties of $x^n/n!$:

$\gamma_0(x) = 1$, $\gamma_1(x) = x$, $\gamma_n(x) \in I$ for $n \geq 1$,

$\gamma_n(x+y) = \sum_{p+q=n} \gamma_p(x)\gamma_q(y)$,

$\gamma_n(\lambda x) = \lambda^n \gamma_n(x)$,

$\gamma_p(x)\gamma_q(x) = ((p+q)!/p!q!)\gamma_{p+q}(x)$

$\gamma_p(\gamma_q(x)) = (pq)!/p!(q!)^p \gamma_{pq}(x)$.

In particular,

$$n!\gamma_n(x) = x^n.$$

DP-ideal, *DP-structure.*

Examples

- $I = pW \subset W$ ($W = W(k)$, k perfect, char. $p > 0$). Then : $\forall n \in \mathbf{N}$,

$$p^n/n! \in W.$$

Proof. $v_p(n!) = (n - \sum_{0 \leq i \leq r} a_i)/(p-1)$, with $n = \sum_{0 \leq i \leq r} a_i p^i$, $0 \leq a_i < p$, hence

$$v_p(p^n/n!) = (n(p-2) + \sum a_i)/(p-1) \geq 0,$$

and > 0 if $n > 0$).

Note : $p > 2 \Rightarrow \lim_{n\to\infty} p^n/n! = 0$

$p = 2$: $v_2(2^n/n!) = \sum a_i$ ($= 1$ for $n = 2^m$)

Induced DP on W_m.

A/W finite totally ramified, $[A : W] = e$, $\pi \in A$ uniformizing parameter, then (πA has a DP structure) $\Leftrightarrow$ ($e \leq p-1$).

- M an A-module,

$$\Gamma M = \oplus_{n \geq 0} \Gamma^n M = A \oplus M \oplus \Gamma^2 M \oplus \cdots$$

the *DP-algebra on* M, $\Gamma^+ M = \oplus_{n>0} \Gamma^n M$ (if M is locally free of finite type, $\Gamma^n M = (S^n(M^\vee))^\vee = TS^n M$). [1] There exists a unique DP on $\Gamma^+ M$ extending $M \to \Gamma^n M$, $x \mapsto x^{[n]}$.

$$A < t_1, \cdots, t_r > := \Gamma(\oplus_{1 \leq i \leq r} At_i) = \oplus_{k=(k_1,\cdots,k_r)} At^{[k]}.$$

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[1] $TS^n M = (M^{\otimes n})^{S_n}$ is the submodule of symmetric tensors of degree n.

Divided power Poincaré lemma. There exists a unique integrable connection d on the $A[t]$ module $A < t >$ such that $dt_i^{[n]} = t_i^{[n-1]}dt$ and $d(xy) = dx.y + y.dx$, and $A \to A < t > \otimes \Omega^{\cdot}_{A[t]/A}$ is a quasi-isomorphism.

- A a $\mathbf{Z}_{(p)}$-algebra $\Rightarrow$ $(\gamma_n)_{n \geq 1}$ on I is determined by γ_p (or $(p-1)!\gamma_p$). (see [Grothendieck, p. 74] or [LZ, 1.2]). [2]

- R a $\mathbf{Z}_{(p)}$-algebra $\Rightarrow \gamma_n(Vx) = (p^{n-1}/n!)Vx^n$ is in $VW(R)$ for $x \in W(R)$, $n > 0$, and (γ_n) $(n > 0)$, $\gamma_0 = 1$ is a DP on $VW(R)$, called *canonical.*

Divided power envelope (*Berthelot's construction*). For (B, J), J an ideal in B, there exists a (unique) pair $(D_B(J), \overline{J})$, $\overline{J}$, an ideal in $D_B(J)$ equipped with DP γ and a morphism $(B, J) \to (D_B(J), \overline{J})$ universal for morphisms in (C, K), with K a DP-ideal. Called *DP-envelope* of (B, J).

Variant for B an A-algebra, with a PD-ideal I in A, with γ on $\overline{J}$ made *compatible* with the DP on I (i. e. PD of I extend to $ID_B(J)$ and compatible with the DP of $\overline{J}$ on the intersection). Case of interest : $A = W_n(k)$, $I = (p)$.

Example. $M = A$-module, $B = SM = \oplus_{n \in \mathbf{N}} S^n M$ the symmetric algebra on M , $J = S^+ M \Rightarrow (D_B(J), \overline{J}) = (\Gamma M, \Gamma^+ M)$.

3.3. *The crystalline site.*

X/W_n, $W_n = W_n(k)$, k perfect of char. $p > 0$

Crys(X/W_n) *crystalline site* : objects : (U, T, γ), U Zariski open (or étale) in X, $U \to T$ closed immersion $/W_n$, with DP γ on $I = \mathrm{Ker}(\mathcal{O}_T \to \mathcal{O}_U)$ compatible with the canonical DP on pW_n (NB. $p^n = 0 \Rightarrow I =$ nilideal : $U \to T$ a *thickening*) ; morphisms : obvious ; covering families : $(U_i, T_i) \to (U, T)$ such that $(T_i \to T)$ covering (Zar or étale). Zariski (resp. étale) crystalline site.

Sheaf on Zar (resp.ét) Crys(X/W_n) $\leftrightarrow$ compatible family of Zar (resp. ét) sheaves $F_{(U,T)}$ and maps $a_f : f^*F_{(V,Z)} \to F_{U,T)}$ for $f : (U, T) \to (V, Z)$ such that $a_f =$ iso if $f : T \to Z$ open (resp. étale). Topos of sheaves on Crys(X/W_n) denoted $(X/W_n)_{crys}$. Functorial in X/W_n. In particular, the absolute Frobenius of X and σ : SpecW_n $\to$ SpecW_n, $\sigma(a_0, \cdots, a_{n-1}) = (a_0^p, \cdots, a_{n-1}^p)$, induce a morphism $F : (X/W_n)_{crys} \to (X/W_n)_{crys}$.

Example : $(U, T) \mapsto \mathcal{O}_T$ is a sheaf of rings, called *structural sheaf*, denoted $\mathcal{O}_{X/W_n}$.

Canonical maps.

$$i : X \to (X/W_n)_{crys}$$

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[2] S. Yasuda observes that in fact the datum of a dp-structure is equivalent to that of a single function g $(= (p-1)!\gamma_p)$ satisfying $g(\lambda x) = \lambda^p g(x)$, $pg(x) = x^p$, and $g(x+y) = g(x) + g(y) + \sum_{0<i<p}(1/p)(p!/i!(p-i)!)x^i y^{p-i}$.

($X = X_{zar}$ or $X_{\acute{e}t}$), a closed immersion of ringed toposes,

$$0 \to J_{X/W_n} \to \mathcal{O}_{X/W_n} \to i_*\mathcal{O}_X \to 0,$$

and a morphism of toposes (ringed by the constant ring W_n)

$$u = u_{X/W_n} : (X/W_n)_{crys} \to X,$$

$\Gamma(U, u_*F) := \Gamma((U/W_n)_{crys}, F)$.

Crystalline cohomology

$$H^i(X/W_n) := H^i((X/W_n)_{crys}, \mathcal{O}_{X/W_n}),$$

a W_n-module. In derived style

$$R\Gamma(X/W_n) := R\Gamma((X/W_n)_{crys}, \mathcal{O}_{X/W_n}) = R\Gamma(X, Ru_*\mathcal{O}_{X/W_n}).$$

Remark. Crystalline site, topos, structural sheaf $\mathcal{O}$, canonical map u generalize to $X \to (S, I, \gamma)$, p nilpotent on S, $I \subset \mathcal{O}_S$ ideal with DP γ extendable to X.

3.4. *Calculation of $H^*(X/W_n)$*

Assume we have a *closed embedding* $i : X \to Z$, of ideal I, with Z/W_n *smooth.* Let $(\mathcal{O}_D, \overline{I})$ be the DP-envelope of I (compatible with the DP on (p)), so that $X \to Z$ factors as

$$X \to D \to Z,$$

with $X \to D$ a thickening. Then $\mathcal{O}_D$ has a canonical *integrable connection* $d : \mathcal{O}_D \to \mathcal{O}_D \otimes \Omega^1_{Z/W_n}$ such that $d(x^{[m]}) = x^{[m-1]}dx$ for $x \in I$. Consider the corresponding de Rham complex of Z/W_n with coefficients in $\mathcal{O}_D$:

$$\mathcal{O}_D \otimes \Omega^{\cdot}_{Z/W_n}.$$

Theorem 3.4.1. (Berthelot-Grothendieck) *There exists a canonical isomorphism*

$$Ru_*\mathcal{O}_{X/W_n} \xrightarrow{\sim} \mathcal{O}_D \otimes \Omega^{\cdot}_{Z/W_n}$$

in $D(X, W_n)$.

(In fact, there is constructed a transitive system of isomorphisms for variable embeddings $X \subset Z$.)

Corollary 3.4.2.

$$H^*(X/W_n) \xrightarrow{\sim} H^*(Z, \mathcal{O}_D \otimes \Omega^{\cdot}_{Z/W_n}).$$

In particular, for X/k smooth, Z/W_n a smooth lifting,

$$H^*(X/W_n) \tilde{\to} H^*_{dR}(Z/W_n).$$

Proof of 3.4.1. The (sheaf defined by the) single DP-thickening $X \subset D$ covers the final object of $(X/W_n)_{crys}$, its powers D^r (= DP-envelope of X diagonally embedded in $(Z/W_n)^r$) are acyclic for u_*, and $u_*(\mathcal{O}_{X/W_n}|D^r) = \mathcal{O}_{D^r}$. Therefore

$$Ru_*\mathcal{O}_{X/W_n} \tilde{\to} \check{\mathcal{C}}(D, \mathcal{O})$$

with

$$\check{\mathcal{C}}(D, \mathcal{O}) = (\mathcal{O}_D \to \mathcal{O}_{D^2} \to \cdots \mathcal{O}_{D^r} \to \cdots).$$

Using the DP-Poincaré lemma one shows that the above complex (called the *Čech-Alexander complex*) is isomorphic in $D(X, W_n)$ to the de Rham complex $\mathcal{O}_D \otimes \Omega^{\cdot}_{Z/W_n}$.

Remark. Th. 3.4.1 generalizes to $X \to (S, I, \gamma)$, with an embedding $X \to Z$ into Z smooth over S (see [B], [BO]).

3.5. Crystalline cohomology for X/k proper and smooth

For X/k proper and smooth,

$$H^i(X/W) := \text{proj.lim}_n H^i(X/W_n)$$

is a finitely generated W-module for all i. In fact, $H^i(X/W) = H^i$ of the *perfect complex* $R\Gamma(X/W) := R\text{proj.lim}_n R\Gamma(X/W_n)$. If Z/W is a *proper, smooth lifting* of X/k, then

$$R\Gamma(X/k) \tilde{\to} R\Gamma_{dR}(Z/W) := R\Gamma(Z, \Omega^{\cdot}_{Z/W}).$$

For A/W finite, totally ramified, with $e = [A : W]$, and Z/A a proper, smooth lifting of X (i. e. $Z \otimes_A k = X$), one still has

$$H^*(X/W) \otimes_W A \tilde{\to} H^*_{dR}(Z/A)$$

if $e \leq p - 1$; in general, only

$$H^*(X/W) \otimes_W K \tilde{\to} H^*_{dR}(Z/A) \otimes_A K,$$

for $K = \text{Frac}(A)$ (Berthelot-Ogus).

For X/k proper, smooth, $X \mapsto H^*(X/W) \otimes K_0$ ($K_0 = \text{Frac}(W)$) is a *Weil cohomology* : Künneth, Poincaré duality, cycle class, with “correct” Betti numbers, i. e. $\dim H^i(X/W) \otimes K_0 = \dim H^i(X_{\overline{k}}, \mathbf{Q}_\ell)$ ($\overline{k}$ an algebraic closure of k, $\ell \neq p$), at least if X/k is projective (Katz-Messing) or liftable to char. 0 (i. e. to A as above) (Berthelot-Ogus + Artin-Grothendieck).

For $k = \mathbf{F}_q$, $q = p^a$, by Berthelot,

$$Z(X/\mathbf{F}_q, t) = \prod \det(1 - F^a t, H^i(X/W) \otimes K_0)^{(-1)^{i+1}},$$

with $\det(1 - F^a t, H^i(X/W)) \otimes K_0) = \det(1 - F^a t, H^i(X_{\overline{k}}, \mathbf{Q}_\ell))$ if X/k is projective (Katz-Messing). [3]

3.6. Slopes of Frobenius

Assume k algebraically closed, let X/k be proper, smooth, fix $i \in \mathbf{Z}$, and let $H := H^i(X/W) \otimes K_0$. Let $\varphi : H \to H$ be the σ-linear endomorphism defined by $F : (X/W_n)_{crys} \to (X/W_n)_{crys}$. Poincaré duality $\Rightarrow$ φ is *bijective*, i. e. H is an *F-isocrystal.* By Dieudonné-Manin,

$$H = \oplus H_\lambda, \tag{3.6.1}$$

with H_λ *pure of slope* λ, i. e. a direct sum of m_λ copies of $M_\lambda := K_{0,\sigma}[F]/(F^s - p^r)$, $\lambda = r/s \geq 0$, $(r, s) = 1$, $F\lambda = \sigma(\lambda)F$ (the *slopes* $0 \leq \lambda_1 < \cdots < \lambda_r$ of H are the λ for which $m_\lambda \neq 0$) (= p-adic valuations of "eigenvalues" of φ). *Newton polygon* $\mathrm{Nwt}_i(X) = \mathrm{Nwt}(H)$: slope λ_i with horizontal length $m_{\lambda_i} s$ ($r/s = \lambda_i$). *Hodge polygon* $\mathrm{Hdg}_i(X)$ = slope r with multiplicity the *Hodge number* $h^{r,i-r}$, $h^{r,s} := \dim H^s(X, \Omega^r_{X/k})$. Basic inequality :

Theorem 3.6.2. (Mazur-Ogus) $\mathrm{Nwt}_i(X)$ *lies above* $\mathrm{Hdg}_i(X)$.

In particular, for $k = \mathbf{F}_q$, if $H^i(X, \mathcal{O}) = 0$, all eigenvalues of F^a on $H^i(X/W)$ are divisible by q.

The proof of 3.6.2 uses the Cartier isomorphism as an essential tool. See 4.5.3 for a key lemma.

Remark. Assuming only k perfect, H decomposes as in (3.6.1) with H_λ the largest sub-F-crystal such that the slopes of $H_\lambda \otimes K_0(\overline{k})$ are all λ, and 3.6.2 is still valid.

Remark. Suppose $X = Z \otimes_A k$, Z/A proper, smooth as above. Then $h^{r,s}(X) \geq h^{r,s}(Z_K)$ ($Z_K = Z \otimes K$) (semi-continuity). Hence $\mathrm{Hdg}_i(Z_K)$ is above $\mathrm{Hdg}_i(X)$. *p-adic Hodge theory* (C_{cris} theorem) implies : $\mathrm{Nwt}_i(X)$ *lies above* $\mathrm{Hdg}_i(Z_K)$.

4. The de Rham-Witt complex

4.1. Witt complexes : the Langer-Zink construction

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[3] 2011/3/14 : I just received a preprint by J. Suh, *Symmetry and parity of slopes of Frobenius on proper smooth varieties*, in which he shows that this result and the one above still hold in the proper smooth, not necessarily projective case.

Definitions. (1) B an A-algebra (in some topos T), $I \subset B$ an ideal with DP γ_n, M a B-module. An *A-dp-derivation* $D : B \to M$ is an A-derivation such that $D\gamma_n(x) = \gamma_{n-1}(x)Dx$ for $x \in I$ (i. e. local section of I). Denote by $d : B \to \tilde{\Omega}^1_{B/A,\gamma}$ (or $\tilde{\Omega}^1_{B/A}$) the universal A-dp-derivation

$$\mathrm{Hom}(\tilde{\Omega}^1_{B/A}, M) = \mathrm{Der}_{A,\gamma}(B, M).$$

(2) A *B/A-dga* is a strictly anticommutative graded B-algebra $P = \oplus_{n\in\mathbf{N}} P^n$, equipped with an A-linear map $d : P^n \to P^{n+1}$ such that $d^2 = 0$ and $d(xy) = dx.y + (-1)^i x.dy$ for $x \in P^i$, $y \in P^j$. A *B/A-dp-dga* is a B/A-dga such that $B \to P^0 \to P^1$ is a dp-derivation. Initial B/A-dp-dga denoted

$$\tilde{\Omega}^{\cdot}_{B/A},$$

with $\tilde{\Omega}^i = \Lambda^i \tilde{\Omega}^1$, a quotient of $\Omega^{\cdot}_{B/A}$.

(3) For A a $\mathbf{Z}_{(p)}$-algebra, a *Witt complex over B/A* is a projective system of $W_n(B)/W_n(A)$-dga P_n for $n \geq 1$

$$\cdots \to P_{n+1} \to P_n \to \cdots \to P_1$$

equipped with maps $F : P_{n+1} \to P_n$, $V : P_n \to P_{n+1}$, satisfying :

$W_n B \to P^0_n$ compatible with F, V ;

$Fx.Fy = F(xy)$;

$xVy = V(Fx.y)$:

$FV = p$;

$FdV = d$;

$Fd[x] = [x^{p-1}]d[x]$ for $x \in B$

(here $[x] = [x].1_{P^0_{\cdot}}$ by abuse).

A map of Witt complexes is a map of projective systems compatible with all the structures.

(NB. The terminology *Witt complex* is borrowed from [HM] ; a Witt complex is called an F-V-procomplex) in [LZ].)

Standard formulas in any Witt complex : $dF = pFd$, $Vd = pdV$,

$V(xdy_1 \cdots dy_r) = Vx.dVy_1. \cdots .dVy_r$,

(e. g. $Vdx = VFdVx = V1.dVx = d(V1.Vx) = d(V(FVx)) = pdVx$).

Theorem 4.1.1. (Langer-Zink). *For A a $\mathbf{Z}_{(p)}$-algebra, the category of Witt complexes over B/A admits an initial object, denoted*

$$W_{\cdot}\Omega^{\cdot}_{B/A},$$

called the de Rham-Witt (pro)-complex of B/A. Moreover :

(a) $W_n\Omega^0_{B/A} = W_nB$ *for all* n ;

(b) *The de Rham-Witt complex of* B/A *is a projective system of dp-dga, for the canonical DP structure on* $VW_{n-1}B$. *The (unique) map of dp-dga*

$$\tilde{\Omega}^{\cdot}_{W_nB/W_nA} \to W_n\Omega^{\cdot}_{B/A}$$

is surjective, and an isomorphism for $n = 1$:

$$\Omega^{\cdot}_{B/A} \xrightarrow{\sim} W_1\Omega^{\cdot}_{B/A}.$$

(c) *If* $p = 0$ *in* A, *then* $VF = p$.

Proof. One first checks the following two key points :

(i) If $P_{\cdot}$ is a Witt complex, then, for all n, $d : W_nB \to P^1_n$ is a dp-derivation (and hence P_n is a dp-dga)

(e. g., for $x \in B$, $d\gamma_p(V[x]) = \gamma_{p-1}(V[x])dV[x] \Leftrightarrow p^{p-2}dV[x]^p = p^{p-2}V[x]^{p-1}dV[x]$, and already $dV[x]^p = d([x]V1) = V1d[x] = VFd[x] = V([x]^{p-1}d[x]) = V[x]^{p-1}dV[x]$)

(ii) If $D : W_nA \to M$ is a dp-derivation into a W_nA-module M, then $FD : W_{n-1}A \to F_*M$ defined by

$$FDx = [a^{p-1}]D[a] + DVb$$

for $x = [a] + Vb$, is a dp-derivation.

It follows from (ii) that the projective system $\tilde{\Omega}^{\cdot}_{W_nB/W_nA}$ acquires maps (of graded algebras) $F : \tilde{\Omega}^{\cdot}_{W_nB/W_nA} \to \tilde{\Omega}^{\cdot}_{W_{n-1}B/W_{n-1}A}$ satisfying some of the formulas in (3) ($FdVx = dx$ for $x \in W_nB$, $Fd[x] = [x^{p-1}]d[x]$ for $x \in B$, $dFx = pFdx$, for $x \in W_{n+1}B$). The projective system $W_{\cdot}\Omega^{\cdot}_{B/A}$ is then constructed inductively as a quotient of $\tilde{\Omega}^{\cdot}_{W_nB/W_nA}$.

In (ii), the fact that FD is a derivation (already is additive) makes crucial use of the fact that D is a dp-derivation. Compare with the definition of the Cartier operator C^{-1}, sending dx to the class of $x^{p-1}dx$, which is additive (modulo boundaries). For A of char. p, $F : W_2\Omega^1_{B/A} \to \Omega^1_{B/A}$ lifts the Cartier operator $C^{-1} : \Omega^1_{B/A} \to \Omega^1_{B/A}/dB$.

For a morphism $f : X \to S$ of schemes over $\mathbf{Z}_{(p)}$,

$$W_{\cdot}\Omega^{\cdot}_{X/S} := W_{\cdot}\Omega^{\cdot}_{\mathcal{O}_X/f^{-1}(\mathcal{O}_S)}$$

is called the *de Rham-Witt (pro)-complex of* X/S.

Obvious functoriality in B/A and X/S. We are mainly interested in the case where p is *nilpotent* in S, and even $S = \operatorname{Spec} k$, k a perfect field of char. p.

4.2. Other constructions

- If A is a perfect ring of char. p, $W_{\cdot}\Omega^{\cdot}_{B/A}$ coincides with *Illusie's de Rham-Witt complex* constructed in [DRW] (if $I_{\cdot}$ is the latter, $I.$ is a Witt complex over B/A, and the corresponding map $W_{\cdot}\Omega^{\cdot}_{B/A} \to I_{\cdot}$ is an isomorphism, as the universal property of $I.$ as a V-pro-complex yields an inverse to it). This isomorphism is compatible with F, V. Langer-Zink's approach simplifies the construction of F on $I.$.

- For k a perfect field of char. $p > 2$ and X/k smooth of dim. $< p$, it is shown in [DRW] that $W.\Omega^{\cdot}_{X/k}$ coincides with Bloch's complex of typical curves on SK_{i+1}, $\cdots \to C^iX \to \cdots$. (Kato [K1] sketched how to remove the restrictions $p > 2$ and $\dim X < p$ in Bloch's construction, and presumably the isomorphism extends.)

- For X/k smooth as above, it is shown in [DRW] that

$$W\Omega^{\cdot}_X := \text{proj.lim} W_n\Omega^{\cdot}_{X/k}$$

is the quotient of $\text{proj.lim}\Omega^{\cdot}_{W_n\mathcal{O}_X}$ by the closure (for the canonical filtration) of the p-torsion, a quotient considered first by Lubkin.

- For B a $\mathbf{Z}_{(p)}$-algebra, Hesselholt-Madsen [HM] define a *Witt complex* over B as a projective system of strictly anticommutative W_nB-graded algebras E_n, with operators F, d, V as in (3) above, (with $d^2 = 0$ and $d(xy) = dx.y + (-1)^i x.dy$), *forgetting the W_nA-linearity of d*. They show that the category of Witt complexes over B has an initial object, called the *(absolute) de Rham-Witt complex of B*,

$$W_{\cdot}\Omega^{\cdot}_B.$$

They study it for $p > 2$. The Langer-Zink complex $W_{\cdot}\Omega^{\cdot}_{B/A}$ is a quotient of $W_{\cdot}\Omega_B$, studied in [He].

- *Other variants* : Olsson's variant of the Langer-Zink construction for certain morphisms of algebraic stacks [O], Davis-Langer-Zink overconvergent de Rham-Witt complex for X/k smooth [DLZ].

4.3. Local description of $W\Omega^{\cdot}_{X/S}$ (smooth case)

- *Étale extensions*

(1) For X/S, $W_n\Omega^i_{X/S}$ is *quasi-coherent* on $W_n(X)$ for all i, n.

(2) Assume p nilpotent on S. Then, for Y an S-scheme and $X \to Y$ étale, $W_n(X) \to W_n(Y)$ is étale, and

$$W_n\mathcal{O}_X \otimes_{W_n\mathcal{O}_Y} W_n\Omega^i_{Y/S} \to W_n\Omega^i_{X/S}$$

is an isomorphism.

Proof. The main point is to show the first assertion of (2). See [LZ, appendix]. Much easier if $p = 0$ (cf. [DRW]). It is shown in [LZ] that (2) holds if, instead of assuming p nilpotent on S, one assumes that Y is F-finite, i. e. the absolute Frobenius of $Y \otimes \mathbf{F}_p$ is finite.

- *Canonical bases*

For X/S smooth, the determination of the local structure of $W_n\Omega^{\cdot}_{X/S}$ is reduced by (2) to that of $W_n\Omega^{\cdot}_{B/A}$ for a polynomial algebra $B = A[T_1, \cdots, T_r]$.

Case $A = \mathbf{F}_p$. We have the following description of $W_n\Omega^{\cdot}_B := W_n\Omega^{\cdot}_{B/\mathbf{F}_p}$, due to Deligne :

$$W_n\Omega^{\cdot}_B = E^{\cdot}/(V^nE^{\cdot} + dV^nE^{\cdot}),$$

where $E^{\cdot}$ is the so-called *complex of integral forms*, defined by

$$E^{\cdot} \subset \Omega^{\cdot}_{C/\mathbf{Q}_p},\ C = \mathbf{Q}_p[T_1^{p^{-\infty}}, \cdots, T_r^{p^{-\infty}}],$$

with

$$V = pF^{-1}, FT_i = T_i^p,$$

where $(\omega \in E^i) \Leftrightarrow (\omega$ and $d\omega$ integral) (i. e. coefficients in $\mathbf{Z}_p$).

Proof. As $E^0/V^nE^0 = W_n(B)$, $E_{\cdot} := (E^{\cdot}/(V^nE^{\cdot} + dV^nE^{\cdot}))_{n \geq 1}$ is a Witt complex over $B/\mathbf{F}_p$, so we have a natural map $W_{\cdot}\Omega^{\cdot}_{B/\mathbf{F}_p} \to E_{\cdot}$ of Witt complexes. To show that it's an isomorphism, one uses :

As a complex of $\mathbf{Z}_p$-modules, E has a natural grading by the group

$$\Gamma = (\mathbf{Z}[1/p]_{\geq 0})^r,$$

$$E = \oplus_{k \in \Gamma}\, {}_kE,$$

where $x = \sum a_i(T)\mathrm{dlog}T_i$ belongs to ${}_kE$, i. e. is of homogeneous of degree k, if and only if the polynomials $a_i(T)$ are (here $i = (i_1 < \cdots < i_m)$, $\mathrm{dlog}T_i = \mathrm{dlog}T_{i_1} \cdots \mathrm{dlog}T_{i_r}$).

Each ${}_kE^m$ has a canonical basis consisiting of elements $e_i(k)$ ($i = (i_1 < \cdots < i_m)$) sent to specific elements in the de Rham-Witt complex.

Example : $r = 1$, $B = \mathbf{F}_p[T]$, ${}_kE^0 = \mathbf{Z}_pe_0(k)$, ${}_kE^1 = \mathbf{Z}_pe_1(k)$, with $e_0(k) = p^{u(k)}T^k$ if $k \notin \mathbf{Z}$ where $p^{u(k)}$ is the denominator of k, $e_0(k) = T^k$ otherwise, $e_1(k) = T^k\mathrm{dlog}T$ $(k > 0)$. Then $e_0(k)$ is sent to $[T]^k$ if $k \in \mathbf{Z}$, to $V^{u(k)}[T]^{p^{u(k)}k}$ if $k \notin \mathbf{Z}$, $e_1(k)$ to $[T]^k\mathrm{dlog}[T] := [T]^{k-1}d[T]$ if $k \in \mathbf{Z}$ $(k > 0)$, $dV^{u(k)}[T]^{p^{u(k)}k}$ if $k \notin \mathbf{Z}$. One gets direct sum decompositions

$$W_n(B) = \bigoplus_{k \,\text{integral}} (\mathbf{Z}/p^n\mathbf{Z})[T]^k \oplus \bigoplus_{k \,\text{not integral}} V^{u(k)}(\mathbf{Z}/p^{n-u(k)}\mathbf{Z})[T]^{p^{u(k)}k},$$

$$W_n\Omega^1_{B/\mathbf{F}_p} = \bigoplus_{k>0,\ k \text{ integral}} (\mathbf{Z}/p^n\mathbf{Z})[T]^k \mathrm{dlog}[T]$$

$$\oplus \bigoplus_{k \text{ not integral}} dV^{u(k)}(\mathbf{Z}/p^{n-u(k)}\mathbf{Z})[T]^{p^{u(k)}k},$$

$$W_n\Omega^i_{B/\mathbf{F}_p} = 0,\ i > 1.$$

Key observation (Deligne) : *$W_n\Omega^\cdot_{B/\mathbf{F}_p}$ contains the de Rham complex $\Omega^\cdot_{(\mathbf{Z}/p^n\mathbf{Z})[T]}$ as a direct summand*:

$$W_n\Omega^\cdot_{B/\mathbf{F}_p} = \Omega^\cdot_{(\mathbf{Z}/p^n\mathbf{Z})[T]} \oplus (W_n\Omega^\cdot_{B/\mathbf{F}_p})_{\text{not integral}},$$

and the complement $(W_n\Omega^\cdot_{B/\mathbf{F}_p})_{\text{not integral}}$ is acyclic.

The limit

$$W\Omega^\cdot_B := \text{proj.lim.}\, W_n\Omega^\cdot_B,$$

can be described as

$$WB = \{ \sum_{k\in\mathbf{N}[1/p]} a_kT^k, a_k \in \mathbf{Z}_p, \mathrm{den}(k)|a_k \forall k, \lim_{k\to\infty} \mathrm{a_k} = 0\}$$

$$W\Omega^1_B = \{ \sum_{k>0, k\in\mathbf{N}[1/p]} a_kT^k(dT/T), a_k \in \mathbf{Z}_p, \lim_{k\to\infty} \mathrm{den(k).a_k} = 0\}$$

$W\Omega^i_B = 0, i > 1$.

All this is generalized to any r in [DRW] and to any A in [LZ]. In particular :

$$W_n\Omega^\cdot_{A[T_1,\cdots,T_r]/A} = \Omega^\cdot_{W_n(A)[T_1,\cdots,T_r]/W_n(A)} \oplus (W_n\Omega^\cdot_{A[T_1,\cdots,T_r]/A})_{\text{not integral}},$$

with the not integral part acyclic. And for X/S smooth of relative dimension d :

$$W_n\Omega^\cdot_{X/S} = (0 \to W_n\mathcal{O}_X \to W_n\Omega^1_{X/S} \to \cdots \to W_n\Omega^{d-1}_{X/S} \to W_n\Omega^d_{X/S} \to 0).$$

- *The canonical filtration*

$W\Omega^\cdot_{X/S} := \text{proj.lim}_n W_n\Omega^\cdot_{X/S}$,
$\mathrm{Fil}^n W\Omega^\cdot_{X/S} := \mathrm{Ker} W\Omega^\cdot_{X/S} \to W_n\Omega^\cdot_{X/S}$
Then ([LZ]) : For X/S smooth,

$$\mathrm{Fil}^n W\Omega^i_{X/S} = V^n W\Omega^i_{X/S} + dV^n W\Omega^{i-1}_{X/S}.$$

Moreover ([DRW] for S perfect, [BER] in general) : *For $S/\mathbf{F}_p$, X/S smooth,* $\mathrm{gr}^n W\Omega^i_{X/S}$ *is an extension of $\Omega^{i-1}_{X/S}/Z_n\Omega^{i-1}_{X/S}$ by $\Omega^i_{X/S}/B_n\Omega^i_{X/S}$* :

$$0 \to \Omega^i_{X/S}/B_n\Omega^i_{X/S} \to \mathrm{gr}^n W\Omega^i_{X/S} \to \Omega^{i-1}_{X/S}/Z_n\Omega^{i-1}_{X/S} \to 0$$

In particular, gr^n is locally free of finite type, of formation compatible with base change.

Here, Z_n and B_n are the iterated cycles and boundaries of $\Omega^{\cdot}_{X/S}$ defined inductively by the Cartier isomorphism, from $Z_0 = \Omega^i$, $B_0 = 0$, $C^{-1} : B_n\Omega^i_{X^{(p)}/S} \xrightarrow{\sim} B_{n+1}\Omega^i_{X/S}/B_1$, $C^{-1} : Z_n\Omega^i_{X^{(p)}/S} \xrightarrow{\sim} Z_{n+1}\Omega^i_{X/S}/B_1$.

4.3. De Rham-Witt complex and crystalline cohomology

Theorem 4.3.1. *k perfect field of char. p, X/k smooth. There exists a canonical isomorphism of projective systems of $D(X, W_n)$:*

$$Ru_*\mathcal{O}_{X/W_n} \xrightarrow{\sim} W_n\Omega^{\cdot}_{X/k}$$

(notations of 3.4.1).

This isomorphism is compatible with the multiplicative structures, and functorial in X/k. It induces isomorphisms

$$R\Gamma(X/W_n) \xrightarrow{\sim} R\Gamma(X, W_n\Omega^{\cdot}_{X/k}),$$

$$H^*(X/W_n) \xrightarrow{\sim} H^*(X, W_n\Omega^{\cdot}_{X/k}).$$

Proof. First, suppose X affine. Choose an embedding $i : X \to Z$ into a smooth W-scheme Z. Let $Z_n := Z \otimes W_n$. Construct inductively a compatible system of W_n-extensions $u_n : W_nX \to Z_n$ of the inclusion $i_n : X \hookrightarrow Z_n$. Let $X \hookrightarrow D_n \to Z_n$ be the dp-envelope of i_n. As the ideal of $X \hookrightarrow W_nX$ has divided powers, u_n uniquely factors through D_n. We get maps $\Omega^{\cdot}_{Z_n/W_n} \to \Omega^{\cdot}_{W_nX/W_n} \to W_n\Omega^{\cdot}_{X/k}$, whose composite factors through $D_n \otimes \Omega^{\cdot}_{Z_n/W_n} = \tilde{\Omega}^{\cdot}_{D_n/W_n}$ as $d : W_n\mathcal{O}_X \to W_n\Omega^1_{X/k}$ is a dp-derivation. The resulting map

$$Ru_*\mathcal{O}_{X/W_n} \xrightarrow{\sim} D_n \otimes \Omega^{\cdot}_{Z_n/W_n} \to W_n\Omega^{\cdot}_{X/k}$$

does not depend on the choice of the embedding. To check it's an isomorphism, we may assume Z_n lifts X, and even reduce to $X = \mathrm{Spec}k[t_1, \cdots, t_r]$, $Z_n = \mathrm{Spec}W_n[t_1, \cdots, t_r]$. Then the result follows from the fact that the inclusion

$$\Omega^{\cdot}_{Z_n/W_n} \subset W_n\Omega^{\cdot}_{X/k}$$

is a quasi-isomorphism (cf. 4.3, end of *Canonical bases*).

General case : hypercover by open affines, use cohomological descent.

Comparison th. 4.3.1 extended by Langer-Zink to X/S smooth, p nilpotent on S :

$$Ru_*\mathcal{O}_{X/W_n(S)} \xrightarrow{\sim} W_n\Omega^{\cdot}_{X/S}.$$

Same proof.

Remark. The proof actually gives an isomorphism in the derived category of projective systems of W_n-modules over X (this is finer, and needed to apply $R\lim$ functors).

4.4. The slope spectral sequence

4.4.1. Suppose now X/k *proper and smooth.* Then 4.3.1 gives :

$$R\Gamma(X/W) \xrightarrow{\sim} R\Gamma(X, W\Omega^{\cdot}_{X/k})$$

and $R\Gamma(X/W)$ is a *perfect complex*, with $R\Gamma(X/W) \otimes^L_W k \to R\Gamma(X, \Omega^{\cdot}_{X/k})$. Moreover :

- The (σ-linear) endomorphism φ of $R\Gamma(X/W)$ induced by the absolute Frobenius of X is induced by the endomorphism Φ of $W\Omega^{\cdot}_{X/k}$ such that $\Phi = p^i F$ in degree i.
- $F : W\Omega^d_{X/k} \to W\Omega^d_{X/k}$ is bijective, which yields a σ^{-1}-linear endomorphism v of $R\Gamma(X/W)$ such that $\varphi v = v\varphi = p^d$.

The next result is deeper :

Theorem 4.4.2. *For any* (i, j)*, the canonical map*

$$H^j(X, W\Omega^i_{X/k}) \to \text{proj.lim}_n H^j(X, W_n\Omega^i_{X/k})$$

is an isomorphism, $H^j(X, W\Omega^i_{X/k})$ *is separated and complete for the* V*-topology, its subgroup* $T^{i,j}$ *of* p*-torsion is killed by a power of* p*, and*

$$H^j(X, W\Omega^i_{X/k})/T^{i,j}$$

is a free W*-module of finite rank.*

Proof. The argument in [DRW], imitated from Bloch, consists in studying $H^*(X, W\Omega^{\leq i})$, with the operator V_i given on $W\Omega^{\leq i}$ by $p^{i-j}V$ in degree j. Using the structure of $\text{gr}^n W_{.}\Omega^{\cdot}$, one shows that $H^*(X, W\Omega^{\leq i})$ is finitely generated over $W_\sigma[[V]]$ and of finite length modulo V. Using Φ (with $\Phi V_i = V_i \Phi = p^{i+1}$, this implies that $H^*(X, W\Omega^{\leq i})$ is sum of a free W-module of finite rank and a p-torsion module killed by a power of p, and 4.4.2 follows by dévissage.

Remark. As observed in [BBE], the proof shows that the conclusion of 4.4.2 holds for $i = 0$ and X/k proper, not necessarily smooth.

Corollary 4.4.3. $H^j(X, W\Omega^i_{X/k})/T^{i,j}$*, with the operators* F, V *induced by* F, V *on* $W\Omega^i$*, is the Cartier module of a smooth formal* p*-divisible group. Equipped with the operator* $p^i F$*, it's an* F*-crystal of slopes in* $[i, i+1[$.

Corollary 4.4.4. *The* (Φ*-equivariant*) *spectral sequence*

$$E_1^{ij} = H^j(X, W\Omega^i_{X/k}) \Rightarrow H^{i+j}(X, W\Omega^{\cdot}_{X/k}) \, (= H^{i+j}(X/W))$$

degenerates at E_1 modulo torsion and gives isomorphisms

$$H^j(X,\Omega^i_{X/k})\otimes K_0 \xrightarrow{\sim} (H^{i+j}(X/W)\otimes K_0)_{[i,i+1[},$$

where $(H^{i+j}(X/W)\otimes K_0)_{[i,i+1[}$ is the part of the F-isocrystal $H^{i+j}(X/W)\otimes K_0$ of slopes in $[i,i+1[$

The spectral sequence of 4.4.4 is called the *slope spectral sequence.*

In particular :

Corollary 4.4.5. *There is a natural isomorphism, for all j,*

$$H^j(X,W\mathcal{O}_X)\otimes K_0 \xrightarrow{\sim} (H^i(X/W)\otimes K_0))_{[0,1[}$$

Remark. It was recently shown by Berthelot, Bloch and Esnault [BBE] that 4.4.5 extends to the proper, possibly singular case, provided that $H^i(X/W)\otimes K_0$ is replaced by Berthelot's *rigid cohomology* $H^i_{\mathrm{rigid}}(X/K_0)$.

Remark. The slope spectral sequence is studied in more detail in [DRW], [IR], and by Ekedahl [E]. See also the survey [I]. One application, described in [DRW, II 5.12], is the (refined) *Igusa-Artin-Mazur inequality* : if k is algebraically closed, and X/k projective, smooth, then

$$\rho = b_2 - 2h - r,$$

where $\rho = \mathrm{rkNS}(X/k)$, $b_2 = \dim H^2(X/W)\otimes K_0$, $h = \dim(H^2(X/W)\otimes K_0)_{[0,1[}$, and $r = \mathrm{rk}T_pH^2(X,\mathbf{G}_m)$. When Artin-Mazur's formal Brauer group Φ^2 of X is representable by a smooth formal group, h is the dimension of its p-divisible part. The projectiveness assumption is used in loc. cit. to ensure a symmetry property of slopes of Frobenius on H^2. This property has been shown by J. Suh to actually hold in the general proper smooth case as well (see footnote 2).

4.5. Higher Cartier isomorphisms, alternate construction of the de Rham-Witt complex

For X/S smooth, $S/\mathbf{F}_p$, the *Cartier isomorphism* is an isomorphism of graded algebras

$$C^{-1}_{X/S} : \oplus\Omega^i_{X^{(p)}/S} \xrightarrow{\sim} \oplus\mathcal{H}^i F_*\Omega^{\cdot}_{X/S},$$

where $X^{(p)}$ = pull-back of X by the absolute Frobenius of S, $F : X \to X^{(p)}$ the relative Frobenius, such that C^{-1} sends $a\otimes 1 \in \mathcal{O}_{X^{(p)}}$ to a^p and $da\otimes 1$ to the class of $a^{p-1}da$.

Suppose $S = \mathrm{Spec}k$, k perfect of char. p. Then $F : W_2\Omega^i_X \to \Omega^i_X$ lifts the absolute Cartier isomorphism C^{-1} (composed of $C^{-1}_{X/S}$ and the canonical isomorphism $\Omega^i_X \xrightarrow{\sim} \Omega_{X^{(p)}})$ (cf. 4.1.1 (ii)). (We drop $_{/k}$ for short.) More generally :

Theorem 4.5.1. *For $n \geq 1$, $F^n : W_{2n}\Omega^i_X \to W_n\Omega^i_X$ induces an isomorphism*

$$W_n\Omega^i_X \xrightarrow{\sim} \mathcal{H}^i W_n\Omega^{\cdot}_X,$$

compatible with products, and equal to C^{-1} for $n = 1$.

Proof. Main point : show : $F^n W_{2n}\Omega^i_X = ZW_n\Omega^i_X$. The proof given in [DRW] is insufficient, corrected in [IR]. Makes crucial use of the description of $W_n\Omega^{\cdot}_X$ for $X = \mathrm{Spec}k[t_1, \cdots, t_r]$ in terms of the complex of integral forms (4.3) and, of course, of the Cartier isomorphism.

By 4.3.1, F^n induces W_n-linear isomorphisms

$$W_n\Omega^i_X \xrightarrow{\sim} \sigma^n_* \mathcal{H}^i(X/W_n), \tag{4.5.2}$$

where $\mathcal{H}^i(X/W_n); = R^i u_* \mathcal{O}_{X/W_n}$.

Assume X lifted to formal smooth Z/W, let $Z_n := Z \otimes W_n$. Then $\mathcal{H}^i(X/W_n) = \mathcal{H}^i_{dR}(Z_n/W_n)$ (3.4.1), and (4.5.2), for $i = 0$ and $i = 1$ are given by :

$i = 0 :$ $a = (a_0, \cdots, a_{n-1}) \in W_n\mathcal{O}_X$ sent to $b_0^{p^n} + pb_1^{p^{n-1}} + \cdots + p^{n-1}b^p_{n-1}$ in $\mathcal{H}^0_{dR}(Z_n/W_n)$, where b_i in $\mathcal{O}_Z$ lifts a_i,

$i = 1 :$ $d(a_0, \cdots, a_{n-1)}$ in $W_n\Omega^1_X$ sent to $\sum b_i^{p^{n-i}-1} db_i$ in $\mathcal{H}^1_{dR}(Z_n/W_n)$.

For $i = 0$, (4.5.2) factors the n-th ghost component $w_n : W_{n+1}(\mathcal{O}_{Z_{n+1}}) \to \mathcal{O}_{Z_{n+1}}$, and, for $i = 1$, the composite map (4.5.2)$dR : W_{n+1}\mathcal{O}_X \to \Omega^1_{Z_n}/d\mathcal{O}_{Z_n}$ lifts $F^n d : W_{n+1}\mathcal{O} \to \Omega^1_X/d\mathcal{O}_X$.

$\Rightarrow$ *reconstruction* of $W_{\cdot}\Omega^{\cdot}_X$ (suggested by Katz) :

$$W_n\Omega^i_X := \sigma^n_* \mathcal{H}^i(X/W_n),$$

$$F : W_{n+1}\Omega^i_X \to W_n\Omega^i_X$$

given by the *restriction* $\mathcal{H}^i(X/W_{n+1}) \to \mathcal{H}^i(X/W_n)$,

$$d : W_n\Omega^i_X \to W_{n+1}\Omega^{i+1}_X,$$

given locally by the *Bockstein* operator associated with the exact sequence

$$0 \to \Omega^{\cdot}_{Z_n/W_n} \to \Omega^{\cdot}_{Z_{2n}/W_{2n}} \to \Omega^{\cdot}_{Z_n/W_n} \to 0,$$

where the first map is multiplication by p^n,

$$V : W_n\Omega^i_X \to W_{n+1}\Omega^i_X$$

induced by multiplication by p on $\Omega^{\cdot}_{Z_{n+1}/W_{n+1}}$.

To reconstruct $R : W_{n+1}\Omega^i_X \to W_n\Omega^i_X$, suppose Z/W admits a formal lifting Φ of Frobenius (exists if X/k affine). Then, Φ^* is divisible by p^i on $\Omega^i_{Z/W}$, let $f = p^{-i}\Phi$ on $\Omega^i_{Z/W}$. For $x \in \mathcal{H}^i(X/W_{n+1}) = \mathcal{H}^i_{dR}(Z_{n+1}/W_{n+1})$, there exists $y \in \Omega^i_{Z/W}$, unique modulo $p^n\Omega^i_{Z/W} + d\Omega^{i-1}_{Z/W}$, such that $x = fy \bmod p^{n+1}\Omega^i_{Z/W} + d\Omega^{i-1}_{Z/W}$. Then, for y_n the image of y in $\Omega^i_{Z_n/W_n}$, $dy_n = 0$, and $x \mapsto$ class of y_n in $\mathcal{H}^i_{dR}(Z_n/W_n)$ defines R.

Existence and uniqueness of y rely on the following key lemma :

Lemma 4.5.3. (Ogus). *With the above notations, let $L \subset \Omega^{\cdot}_{Z/W}$ be the subcomplex defined by*

$$L^i = \{x \in p^i\Omega^i_{Z/W} | dx \in p^{i+1}\Omega^{i+1}_{Z/W}\}.$$

Then $\Phi^* : \Omega^{\cdot}_{Z/W} \to \Omega^{\cdot}_{Z/W}$ factors through L and induces, for each $n \geq 1$, a quasi-isomorphism

$$\Omega^{\cdot}_{Z_n/W_n} \to L_n := L \otimes W_n.$$

(To get y from x, apply 4.5.3 to the class of $p^i\tilde{x}$ in $\mathcal{H}^i(L_n)$, for $\tilde{x} \in \Omega^i_{Z/W}$ lifting x.)

Proof. : [BO, 8.8] : dévissage, reducing to Cartier isomorphism. Lemma 4.5.3 is the crucial ingredient in the proof of the Mazur-Ogus theorem 3.6.2.

Applications.

- Structure (for X/W proper and smooth) of the *conjugate spectral sequence*

$$E_2^{ij} = \text{proj.lim} H^i(X, \mathcal{H}^j(X/W_n)) \Rightarrow H^{i+j}(X/W)$$

(degenerates at E_2 modulo torsion), and analysis of the *log-Hodge-Witt groups*

$$H^j(X, W\Omega^i_{\log}) := \text{proj.lim} H^j(X, W_n\Omega^i_{X,\log}),$$

where $W_n\Omega^i_{X,\log} \subset W_n\Omega^i_X$ is the additive subsheaf étale locally generated by the forms $\mathrm{dlog}[x_1]\cdots\mathrm{dlog}[x_i]$, for $x_m \in \mathcal{O}^*_X$, $1 \leq m \leq i$.

- Construction of $W_{\cdot}\Omega^{\cdot}_X$ via (4.5.2) works in the log context, see §6 (Hyodo-Kato).

5. Review of log schemes

Pre-log structure, log structure, log scheme
Examples : trivial log str., $\mathcal{O}_X \cap j_*\mathcal{O}_U$
Morphisms ; {schemes} $\subset$ {log schemes}
Associated log structure M^a : push-out of

$$\begin{tikzcd}\mathcal{O}^* & \alpha^{-1}(\mathcal{O}^*) \arrow[l] \arrow[r] & M\end{tikzcd}$$

$((u,a) \equiv (v,b) \Leftrightarrow \exists c,d \in \alpha^{-1}(\mathcal{O}^*) | ad = bc, cu = dv$ for (u,a) and (v,b) in $(\mathcal{O}^*, M))$, universal property

$f^*M := (f^{-1}M)^a$, strict morphism

Chart $P \to M$, $X \to \mathrm{Spec}\mathbf{Z}[P]$; chart of a morphism

Examples : $\mathrm{Spec}\mathcal{O}_S[T_1, \cdots, T_r]$, $(t_1 \cdots t_r = 0) \subset \mathrm{Spec}A$, A regular local, (t_i) regular parameters ; trait, standard log point $(\mathbf{N} \to k,\ 1 \to 0)^a$, semistable reduction

$P \to P^{gp}$, integral, fine, fs monoid (resp. log scheme)

Examples : dnc, affine toric variety, toric variety (torus embedding), toroidal embedding

Fiber products, base change, strict case

$\Omega^1_{(X;M)/(S,N)}$, d, dlog, $\alpha(a)\mathrm{dlog}a = d\alpha(a)$

$\Omega^1_{(X;M)/(S,N)} = (\Omega^1_{X/S} \oplus (\mathcal{O}_X \otimes_{\mathbf{Z}} M^{gp})/ < (d\alpha(a), 0) - (0, \alpha(a) \otimes a), (0, 1 \otimes b) > (a \in M^{gp},\ b \in N^{gp})$

$\omega^1_{X/S}$, $\Omega^1_{\underline{X}/\underline{S}}$, $\Omega^i_{(X,M)/(S,L)}$, log dR complex $\Omega^{\cdot}_{(X,M)/(S,L)}$ (or $\omega^{\cdot}_{X/S}$, or $\Omega^{\cdot}_{\underline{X}/\underline{S}}$, or $\Omega^{\cdot}_{X/S}$)

Examples : relative dnc : $\Omega^{\cdot}_{X/S}(\log D)$, semistable reduction : $\Omega^{\cdot}_{X/S}(\log(D/E))$, toric varieties

Exact closed immersion, log thickening

Log smooth, log étale ; strict case ; chart characterization

Examples : toroidal embeddings, relative dnc, semistable reduction, $\mathrm{Spec}k[x, y/x] \to \mathrm{Spec}k[x,y]$, log blow-up

Cartier isomorphism :

• semistable type : $(s = \mathrm{Spec}k, L)$ standard log point, (X, M) of semistable type over (s, L) : étale loc. $X = \mathrm{Spec}\, k[t_1, \cdots, t_d]/(t_1 \cdots t_r)$, with charts

$$\begin{tikzcd}
k[t_1, \cdots, t_d]/(t_1 \cdots t_r) & \mathbb{N}^r \arrow[l] \\
k \arrow[u] & \mathbb{N} \arrow[l, "1 \mapsto 0"'] \arrow[u, "1 \mapsto (1, \cdots, 1)"']
\end{tikzcd}$$

(e. g. special fiber of semistable scheme over trait).

• more generally, log smooth Cartier type : $f : (X, M) \to (S, L)$, $S/\mathbf{F}_p$, log smooth and *saturated* morphism of fs log schemes (saturated = (log) *integral* + reduced geometric fibers). ($\Leftrightarrow$ (log) integral and in the Frobenius diagram (with cartesian square)

$$\begin{tikzcd}
(X, M) \arrow[d, "f"] & (X', M') \arrow[l] \arrow[d] & (X, M) \arrow[l, "F"'] \arrow[dl, "f"] \\
(S, L) & (S, L) \arrow[l, "F_{abs}"'] &
\end{tikzcd}$$

the relative Frobenius F is *exact*, see [K2], [Ts, II 3.1]) ($F_{abs} : a \mapsto a^p$ on $\mathcal{O}_S$ and on L). Examples : (poly) semistable reduction, log smooth saturated toric morphism Spec$A[P] \to$ Spec$A[Q]$; Kummer étale (e. g. $x^n = t$, $(n,p) = 1$) : not Cartier type.

log smooth, Cartier type ⇒ Cartier isomorphism

$$C^{-1} : \Omega^i_{(X',M')/(S,L)} \xrightarrow{\sim} F_*\mathcal{H}^i\Omega^{\cdot}_{(X,M)/(S,L)},$$

$$(a \otimes 1)\mathrm{dlog}x_1 \cdots \mathrm{dlog}x_r \mapsto a^p\mathrm{dlog}x_1 \cdots \mathrm{dlog}x_r,$$

$a \in \mathcal{O}_{\mathcal{X}}$, $x_i \in M$.

(⇒ decompositions of Deligne-Illusie type of $F_*\mathcal{H}^i\Omega^{\cdot}_{(X,M)/(S,L)}$ in situations lifted mod p^2 and dim$f < p$. Applications to (classical) Hodge theory (e. g. [IKN]).

Definitions of integral and exact : P, Q fine monoids, $h : Q \to P$ integral if $\mathbf{Z}[Q] \to \mathbf{Z}[P]$ flat ; h exact if $Q = (h^{gp})^{-1}(P)$ in Q^{gp} ; $f : (X,M) \to (Y,N)$ integral (resp. exact) if $(f^*N)_x \to M_x$ integral (resp. exact) $\forall x \in X$.

6. De Rham-Witt complex and log crystalline cohomology

See slides.

7. The Hyodo-Kato isomorphism

See [HK] and slides Illusie-Sapporo-Hyodo-Kato.pdf. See also [Nak, §7] for complements and corrections to [HK]. For a new approach to the Hyodo-Kato isomorphism, see [Be].

8. Rational points over finite fields for regular models of algebraic varieties of Hodge type ≥ 1, after P. Berthelot, H. Esnault and K. Rülling

8.1. *Slopes of Frobenius and rational points*

Recall : For $q = p^a$, $k = \mathbf{F}_q$, Y/k separated, finite type,

$$Z(Y,t) = \exp(\sum_{n \geq 1} |Y(\mathbf{F}_{q^n})| t^n/n) = \prod (1 - t^{\deg(x)})^{-1} \in (1 + t\mathbf{Z}[[t]]) \cap \mathbf{Q}(t),$$

(Dwork), hence

$$Z(Y,t) = \prod (1 - \alpha_i t) / \prod (1 - \beta_j t),$$

α_i, β_j algebraic integers, $\alpha_i \neq \beta_j$ for all (i,j). By Grothendieck,

$$Z(Y,t) = \prod \det(1 - F^a t, H^i_c(Y_{\overline{k}}, \mathbf{Q}_\ell))^{(-1)^{i+1}}.$$

with inverse roots of $\det(1 - F^a t, H^i_c(Y_{\overline{k}}, \mathbf{Q}_\ell))$ algebraic integers (Deligne), but we won't use these results in this section. The next statement is an easy consequence of the slope spectral sequence :

Proposition 8.1.1. *Assume : (i) Y/k geometrically connected,*

(ii) Y/k proper and smooth,

(iii) $H^i(Y, W\mathcal{O}_Y) \otimes \mathbf{Q} = 0$ for all $i > 0$.

Then :

(iv) For all finite extensions $k' = \mathbf{F}_{q^n}$ of k, $|Y(k')| \equiv 1$ mod q^n.

Proof. Recall Berthelot's formula

$$(*) \qquad Z(Y, t) = \prod P_i(t)^{(-1)^{i+1}},$$

$$P_i(t); = \det(1 - F^a t, H^i(Y/W)).$$

As $H^i(Y, W\mathcal{O}_Y) \otimes \mathbf{Q} = (H^i(Y/W) \otimes \mathbf{Q})_{[0,1[}$, (iii) $\Rightarrow$ all slopes of Frobenius on $H^m(Y/W)$ for $m > 0$ are ≥ 1, hence (Dieudonné-Manin) all α_i, β_j above appearing in P_m, $m > 0$ are divisible by q. As $P_0(t) = 1 - t$ by (i),

$$Z'/Z = \sum_{n \geq 1} |Y(\mathbf{F}_{q^n})| t^{n-1} = \sum_{n \geq 1} a_n t^{n-1},$$

with $a_n = |Y(\mathbf{F}_{q^n}| \equiv 1 \text{ mod } q^n$.

In [BBE], Berthelot, Bloch and Esnault show that (i) and (iii) suffice for (iv) to hold. By Étesse-Le Stum, Berthelot's formula (*) holds with crystalline cohomology replaced by Berthelot's compactly supported rigid cohomology $H^i_{c,rig}(Y/K_0)$, and it is proven in [BBE] that a suitably defined cohomology group with compact supports $H^i_c(Y, W\mathcal{O}) \otimes \mathbf{Q}$ is finite dimensional and, again, calculates the part of $H^i_{c,rig}(Y/K_0)$ of slope < 1.

8.2. *Berthelot-Esnault-Rülling's theorem*

Suppose now that $Y = X_k$ is the special fibre of a scheme X over a dvr R of mixed char. $(0, p)$, with perfect residue field k and fraction field K.

Theorem 8.2.2. ([BER]) *Assume :*

(i) X regular, and proper and flat over R ;

(ii) X_K geometrically connected ;

(iii) $H^i(X_K, \mathcal{O}_{X_K}) = 0$ for all $i > 0$.

Then, if $k = \mathbf{F}_q$, $|X_k(\mathbf{F}_{q^n})| \equiv 1$ mod q^n for all $n \geq 1$.

Remarks.

(1) Esnault proved the conclusion of 8.2.2 assuming (i), (ii), and instead of (iii), that X_K is of coniveau ≥ 1 in degree > 0, i. e. for each $i > 0$, there

exists a dense open U in X_K such that the restriction map $H^i(X_{\overline{K}}, \mathbf{Q}_\ell) \to H^i(U_{\overline{K}}, \mathbf{Q}_\ell)$ is zero. By mixed Hodge theory this condition implies (iii), and should be equivalent to it according to Grothendieck's generalized Hodge conjecture.

(2) By Zariski connectedness theorem (i) and (ii) in 8.2.2 imply $Y = X_k$ is geometricall connected. Therefore, by [BBE] 8.2.2 follows from :

Theorem 8.2.3. ([BER]) *Under the assumptions (i), (ii), (iii) of 8.2.2 one has (for $Y = X_k$) :*
(iv) $H^i(Y, W\mathcal{O}_Y) \otimes \mathbf{Q} = 0$ for all $i > 0$.

Actually, an even stronger result is proven in [BER] :

Theorem 8.2.4. ([BER]) *Let X be regular and proper and flat over R. If, for one $q \in \mathbf{Z}$, $H^q(X_K, \mathcal{O}) = 0$, then (for $Y = X_k$) $H^q(Y, W\mathcal{O}_Y) \otimes \mathbf{Q} = 0$.*

Note : base changing by Spec$\widehat{R}$ changes neither assumptions nor conclusions so we may and will assume R complete.

Particular cases.

(a) Assume X/R *smooth.* Then the conclusion of 8.2.4 means that the slopes of Frobenius on $H^q(Y/W)$ are ≥ 1. Assume furthermore :
(a1) $H^q(X, \mathcal{O}) = H^{q+1}(X, \mathcal{O}) = 0$.
Then, by base change, $H^q(Y, \mathcal{O}) = 0$, so, by the Mazur-Ogus inequality, the slopes of $H^q(Y/W)$ are ≥ 1 (One can also show by induction $H^q(Y, W_n\mathcal{O}) = 0$, hence $H^q(Y, W\mathcal{O}) = 0$.)
Without the assumption (a1), it may happen that $H^q(Y, \mathcal{O}) \neq 0$ (Serre's examples of failure of Hodge symmetry in char. p). In this case, the Mazur-Ogus inequality says nothing. However, as observed in 3.6.2, p-adic Hodge theory (the C_{cris} theorem) implies that the Newton polygon of $H^q(Y/W)$ is above the Hodge polygon of $H^q_{Hdg}(X_K)$, hence the slopes of $H^q(Y/W)$ are ≥ 1.
(b) Assume X/R has *semistable reduction.* By the slope spectral sequence for the log de Rham-Witt complex, the conclusion of 8.2.4 still means that the slopes of Frobenius on $H^q(Y/(W, W(L))$ ((Speck, L) the standard log point) are ≥ 1, and this is true by the C_{st} theorem.

8.3. *Strategy of proof of 8.2.4.*

The general idea is to reduce to the semistable case by using de Jong alterations and cohomological descent.

- *Use of de Jong alterations*

Starting point : because X is *integral* and *flat* over R, by de Jong, there exists a finite extension K_1 of K, with ring of integers R_1, and a commutative diagram

$$\begin{tikzcd} X \arrow[d] & Z \arrow[l] \arrow[d] \\ \mathrm{Spec} R & \mathrm{Spec} R_1 \arrow[l] \end{tikzcd} ,$$

with Z integral, semistable over R_1, and $Z \to X$ a projective alteration. The morphism $Z_{K_1} \to X_{K_1}$ may not be surjective, but passing to a Galois extension K' of K containing K_1 and taking a disjoint sum X_0 of translated by the Galois group of pull-backs of $Z/\mathrm{Spec} X_{R_1}$ to $\mathrm{Spec} R'$, $(X_0)_{K'} \to X_{K'}$ is surjective.

Iteration : Fix $m > q$. Iterating the process, one constructs an augmented m-truncated simplicial scheme

$$\varepsilon : X_\bullet \to X_{R'}$$

(R' the ring of integers of a suitable extension K' of K), such that :

- each X_n is a sum of pull-backs of semistable schemes over rings of integers of subextensions of K'

- $\varepsilon_{K'} : (X_\bullet)_{K'} \to X_{K'}$ is a proper m-truncated hypercovering

- X_0 is, as above, the disjoint sum of base changes of a semistable Z/R_1, with $f : Z \to X$ a projective alteration, Z integral.

• *Use of cohomological descent and classical Hodge theory*

Since $q < m$, as each $(X_n)_{K'}$ is smooth over K' and $\varepsilon_{K'}$ is a proper m-truncated hypercovering, it follows from Deligne's mixed Hodge theory that

$$H^q(X_{K'}, \Omega^{\cdot}_{X_{K'}/K'}) \to H^q((X_\bullet)_{K'}, \Omega^{\cdot}_{(X_\bullet)_{K'}/K'})$$

is an isomorphism of *filtered* spaces (for the Hodge filtration). In particular, $H^q((X_\bullet)_{K'}, \mathcal{O}) = 0$.

• *Use of p-adic Hodge theory*

By the C_{st} theorem for truncated simplicial semistable schemes (Tsuji), it follows that the slopes of Frobenius on $H^q((X_\bullet)_{k'}/(W(k'), W(L)))$ are ≥ 1. By a generalization of de Rham-Witt theory to the truncated simplicial semistable case, this means that

$$H^q((X_\bullet)_{k'}, W\mathcal{O}) \otimes \mathbf{Q} = 0. \tag{8.3.1}$$

• *A trace argument*

If the map

$$\varepsilon_{k'} : (X_\bullet)_{k'} \to X_{k'}$$

was a truncated proper hypercovering, cohomological descent for rigid cohomology (Tsuzuki) - and its compatibility with slopes - would give the vanishing of $H^q(X_{k'}, W\mathcal{O}) \otimes \mathbf{Q}$, hence that of $H^q(X_k, W\mathcal{O}) \otimes \mathbf{Q}$. However, $\varepsilon_{k'}$ is not in general a truncated proper hypercovering. Still, the functoriality map

$$H^q(X_k, W\mathcal{O}) \otimes \mathbf{Q} \to H^q((X_0)_{k'}, W\mathcal{O}) \otimes \mathbf{Q} \tag{8.3.2}$$

is zero, as it factors through $H^q((X_\bullet)_{k'}, W\mathcal{O}) \otimes \mathbf{Q} = 0$. Therefore it's enough to show that (8.3.2) is *injective*. By the construction of X_0 as a sum of pull-backs of Z, it's enough to show that

$$f_k^* : H^q(X_k, W\mathcal{O}) \otimes \mathbf{Q} \to H^q(Z_k, W\mathcal{O}) \otimes \mathbf{Q} \tag{8.3.3}$$

is injective. This is achieved by a trace argument. One constructs a trace map

$$\tau_{f_k} : H^q(Z_k, W\mathcal{O}) \otimes \mathbf{Q} \to H^q(X_k, W\mathcal{O}) \otimes \mathbf{Q}$$

such that

$$\tau_{f_k} f_k^* = r.\mathrm{Id}, \tag{8.3.4}$$

where r is the generic degree of the alteration f.

8.4. *The trace map*

As X and Z are regular, integral, with $\dim Z = \dim X$, $f : Z \to X$ is a *complete intersection morphism of virtual relative dimension zero* (i. e. locally defined by a regular immersion of codimension d in a smooth X-scheme of relative dimension d). Moreover, f is projective (in the sense that Z is a closed subscheme of some projective space $\mathbf{P}^d_X$). The construction of τ_{f_k} and the proof of (8.3.4) uses essentially only these facts. There are three steps. Denote by $(-)_n$ the reduction mod p^{n+1}.

- *Step 1*

Construction of (compatible) trace maps

$$\mathrm{Tr}_{f_n} : Rf_{n*}\mathcal{O}_{Z_n} \to \mathcal{O}_{X_n}$$

with

$$\mathrm{Tr}_{f_n} f_n^* = r.\mathrm{Id} \tag{8.4.1}$$

(where $f_n^* = \mathcal{O}_{X_n} \to Rf_{n*}\mathcal{O}_{Z_n}$ is the adjunction map).

This is more or less standard Grothendieck duality [Ha] (with signs made precise by Conrad [C]). In terms of a factorization

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "f"'] & P = \mathbf{P}^d_X \arrow[dl, "\pi"'] \\ X & \end{tikzcd}$$

(with i a regular immersion of codimension d), Tr_{f_n} is the composition

$$\mathrm{Tr}_{f_n} = \mathrm{Tr}_{\pi_n}\mathrm{Tr}_{i_n},$$

with Tr_{π_n} given by the canonical isomorphism $R^d\pi_{n*}\Omega^d_{P_n/X_n} \xrightarrow{\sim} \mathcal{O}_{X_n}$, and Tr_{i_n} by the cohomology class of i_n.

- *Step 2*

Construction of (compatible) trace maps, for $n \geq 1$,

$$(\tau_{f_0})_n : R(f_0)_*W_n\mathcal{O}_{Z_0} \to W_n\mathcal{O}_{X_0}.$$

This is a new construction, similar to the previous one, but using the *de Rham-Witt complex* (of Langer-Zink) of P_0/X_0.

- *Step 3*

Comparison of trace morphisms and proof of the key formula

$$(\tau_{f_0})_n(f_0)_n^* = r.\mathrm{Id}, \tag{8.4.2}$$

where $(f_0)_n^* : W_n\mathcal{O}_{X_0} \to R(f_0)_*W_n\mathcal{O}_{Z_0}$ is the adjunction map. (This formula implies (8.3.4) because $Z_k \subset Z_0$, $X_k \subset X_0$ are nilpotent immersions, and (by a result of [BBE]) the restriction maps $H^q(X_0, W\mathcal{O})\otimes\mathbf{Q} \to H^q(X_k, W\mathcal{O})\otimes\mathbf{Q}$, $H^q(Z_0, W\mathcal{O}) \otimes \mathbf{Q} \to H^q(Z_k, W\mathcal{O}) \otimes \mathbf{Q}$ are isomorphisms.)

This is the most ingenious part of the proof of 8.2.4. The basic tool is the unique factorization of the n-th phantom map

$$w_n = F^n : W_{n+1}(\mathcal{O}_{X_{n-1}}) \to \mathcal{O}_{X_{n-1}},$$

$$w_n(b_0, \cdots, b_n) = b_0^{p^n} + \cdots + p^{n-1}b_{n-1}^p + p^nb_n = b_0^{p^n} + \cdots + p^{n-1}b_{n-1},$$

into

$$\begin{tikzcd} W_{n+1}(\mathcal{O}_{X_{n-1}}) \arrow[r, "F^n"] \arrow[d] & \mathcal{O}_{X_{n-1}} . \\ W_n(\mathcal{O}_{X_0}) \arrow[ur, "\tilde{F}^n"] & \end{tikzcd}$$

Comparing cohomology classes of a regular immersion in both theories, one shows the commutativity of the diagram

$$\begin{tikzcd} f_{0*}W_n(\mathcal{O}_{Z_0}) \arrow[r, "H^0(\tau)"] \arrow[d] & W_n(\mathcal{O}_{Z_0}) \arrow[d] \\ f_{n-1*}\mathcal{O}_{Z_{n-1}} \arrow[r, "H^0(\mathrm{Tr})"] & \mathcal{O}_{X_{n-1}} \end{tikzcd} ,$$

where the vertical maps are given by $\tilde{F}^n$. It follows that $(\tau_{f_0})_n(f_0)^*_n$ is the multiplication by a class $c_n \in H^0(X_0, W_n(\mathcal{O}_{X_0}))$ such that $c := \text{proj.lim} c_n \in H^0(X_0, W\mathcal{O}_{X_0})$ has the following two properties :

(i) $Fc = c$,

(ii) $\tilde{F}^n(c - r) = 0$ for all $n \geq 1$.

One shows that this implies that $c - r = 0$, hence $c_n = r$. One shows more generally that $\mathrm{Ker}(F-1) \cap \bigcap_{n\geq 1} \mathrm{Ker}(\tilde{F}^n : W\mathcal{O}_{X_0} \to \mathcal{O}_{X_{n-1}}) = 0$.

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