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Lectures on the de Rham complex

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Plan

A. A brief historical survey (Lecture 1)

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B. New results around Deligne-Illusie
(after Drinfeld, Bhatt-Lurie, and Petrov) (Lectures 2, 3) (see [I1 22], [I2 22])

1. The Poincaré lemma

The exterior derivative

$U \subset \mathbb{R}^n$ open ; $f : U \to \mathbb{R}$ of class C^∞

differential of f at $x \in U$: the linear form $(df)(x) \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R})$ such that

$$f(x+h) = f(x) + (df)(x).h + o(h).$$

Example: $(dx_i)(x) = e_i^\vee : e_j \mapsto \delta_{ij}$

$$df : U \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}),$$

$$df = \sum_{1 \leqslant i \leqslant n} (\partial f / \partial x_i) dx_i .$$

$(\Rightarrow df \in C^\infty(U, \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R})))$

Define, for $i \in \mathsf{Z}$, the space of differential forms of degree i on U:

$$\Omega^i(U) := C^\infty(U, \Lambda^i \mathrm{Hom}_{\mathsf{R}}(\mathbb{R}^n, \mathbb{R}))$$

for $i \geqslant 0$ (and 0 for $i < 0$).

In particular, $\Omega^0(U) = C^\infty(U, \mathbb{R})$, and $\Omega^i(U) = 0$ for $i > n$.

Any $\omega \in \Omega^i(U)$ ($i \geqslant 1$) is uniquely written

$$\omega = \sum_{1 \leqslant j_1 < \cdots < j_i \leqslant n} a_{j_1 \cdots j_i} dx_{j_1} \wedge \cdots \wedge dx_{j_i},$$

with $a_{j_1 \cdots j_i} \in C^\infty(U, \mathbb{R})$.

Proposition. There exists a unique family of $\mathbb{R}$-linear operators

$$d : \Omega^i(U) \to \Omega^{i+1}(U)$$

such that:
(i) for $i = 0$, $df \in \Omega^1(U)$ is the exterior derivative of f;
(ii) $dd = 0$;
(iii) $d(a \wedge b) = da \wedge b + (-1)^p a \wedge db$ for $a \in \Omega^p(U)$, $b \in \Omega^q(U)$.

Definition. The complex

$$\Omega^\bullet(U) = (0 \to \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(U) \to 0)$$

is called the de Rham complex of U (Georges de Rham, 1903 - 1990)

Known to differential geometers of late 19th century: Bianchi, Poincaré, Ricci, Stokes, Volterra, ...

Its cohomology groups are called the de Rham cohomology groups of U:

$$H^i_{\mathrm{dR}}(U) := H^i\Omega^\bullet(U).$$

Theorem (Poincaré lemma). Assume U star-shaped, i.e., stable under $x \mapsto tx$, $t \in [0,1]$. Then the augmentation

$$\varepsilon : \mathbb{R} \to \Omega^\bullet(U),\ a \mapsto (x \mapsto a) \in \Omega^0(U) = C^\infty(U, \mathbb{R})$$

is a homotopy equivalence. In particular, $H^i_{\mathrm{dR}}(U) = 0$ for $i > 0$ and $H^0_{\mathrm{dR}} = \mathbb{R}$.

Proof. Let $h : [0,1] \times U \to U$, $h(t,x) := tx$. Define $k : \Omega^p(U) \to \Omega^{p-1}(U)$ by

$$k\omega = \int_0^1 i_{\partial t} h^*(\omega) dt,$$

where $i_{\partial t}$ is the interior product by ∂t applied to $h^*(\omega) \in \Omega^p([0,1] \times U)$. Then

$$\mathrm{Id} - \varepsilon \circ \pi = dk + kd : \Omega^\bullet(U) \to \Omega^\bullet(U),$$

where $\pi : \Omega^\bullet(U) \to \mathbb{R}$ is the projection given by $f \mapsto f(0)$.

Remarks.

- Proof by Volterra (1889); Poincaré: ? (cf. E. Cartan, de Rham)

- Avatars of Poincaré Lemma: analytic, crystalline (Berthelot-Grothendieck, 1970), ..., p-adic (Beilinson, 2012), ...

- If U not star shaped, the vanishing $H^i_{\mathrm{dR}}(U) = 0$ for $i > 0$ may not hold, e.g., for $n > 1$,

$$H^{n-1}_{\mathrm{dR}}(\mathbb{R}^n - \{0\}) = \mathbb{R}$$

(a consequence of the de Rham theorem).

2. The de Rham theorem

X: a C^∞-manifold of dimension n.

Using an atlas $X = \cup V_i$,
$\varphi_i : V_i \xrightarrow{\sim} \varphi_i(V_i) \subset \mathbb{R}^n, \varphi_{ij} : \varphi_i(V_i \cap V_j) \xrightarrow{\sim} \varphi_j(V_i \cap V_j)$,

for $U \subset X$ open, glue the $\Omega^\bullet(\varphi_i(U \cap V_i))$ into a complex $\Omega^\bullet_X(U)$.
For variable U, get a complex of sheaves of $\mathbb{R}$-vector spaces on X,

$$\Omega^\bullet_X : U \mapsto \Omega^\bullet_X(U)$$

called the de Rham complex of X,

$$\Omega^\bullet_X = (\mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_X \to 0).$$

$\mathcal{O}_X$: sheaf of real-valued C^∞ functions on X,

Ω^1_X: sheaf of C^∞ 1-forms on X, a rank n vector bundle on X, dual to the tangent bundle T_X

$\Omega^i_X = \Lambda^i \Omega^1_X$.

Poincaré lemma for star shaped open subsets of $\mathbb{R}^n$ implies:

Theorem (Poincaré lemma). The augmented complex

$$0 \to \mathbb{R}_X \to \mathcal{O}_X \to \Omega^1_X \to \cdots \to \Omega^n_X \to 0$$

is acyclic, in other words, the augmentation

$$\mathbb{R}_X \to \Omega^\bullet_X$$

is a quasi-isomorphism, hence induces isomorphisms[1]

$$R\Gamma(X, \mathbb{R}) \xrightarrow{\sim} R\Gamma(X, \Omega^\bullet_X),$$

$$H^i(X, \mathbb{R}) \xrightarrow{\sim} H^i_{\mathrm{dR}}(X) := H^i(X, \Omega^\bullet_X). \tag{1}$$

(first form of the de Rham theorem).

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[1]where $\mathbb{R}$ is $\mathbb{R}_X$ by abuse

Remarks. 1. Assume X paracompact. Then $\mathcal{O}_X$ is a soft sheaf, the sheaves of $\mathcal{O}_X$-modules Ω^i_X are soft as well ([Go], II 3.4, 3.7), hence acyclic for $\Gamma(X, -)$ ([Go] II, 4.4). Therefore

$$\Gamma(X, \Omega^\bullet_X) \to R\Gamma(X, \Omega^\bullet_X)$$

is an isomorphism in $D(X, \mathbb{R})$, hence

$$H^i(\Gamma(X, \Omega^\bullet_X)) \xrightarrow{\sim} H^i_{\mathrm{dR}}(X),$$

and (1) can be rewritten

$$(1') \qquad H^i(X, \mathbb{R}) \xrightarrow{\sim} H^i(\Gamma(X, \Omega^\bullet_X))$$

2. For X compact, the $\mathbb{R}$-vector spaces $H^i(X, \mathbb{R})$ and $H^i_{\mathrm{dR}}(X)$ are finite dimensional[2]. Moreover, if X is orientable, the isomorphisms

$$H^i(X, \mathbb{R}) \xrightarrow{\sim} H^i_{\mathrm{dR}}(X).$$

are compatible with Poincaré duality and de Rham duality.

The original de Rham theorem[3] is a refinement of

$$H^i(X, \mathbb{R}) \xrightarrow{\sim} H^i(\Gamma(X, \Omega^\bullet_X)) \tag{1'}$$

using the description of $H^i(X, \mathbb{R})$ as the cohomology of the complex of singular cochains, integration of i-forms on singular i-cycles and Stokes formula (see the next Appendix).

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[2] By the Leray spectral sequence of an open cover and the existence of finite covers of X by open subsets U_i such that all finite intersections $U_{i_0} \cap \cdots \cap U_{i_j}$ are contractible ([Dem, 6.9]).

[3] Rigorously proved for the first time by Weil [W].

Appendix: the de Rham theorem and singular cohomology

Let

$$S_\bullet(X,\mathbb{R}) := (\cdots \to S_n(X,\mathbb{R}) \xrightarrow{d} S_{n-1}(X,\mathbb{R}) \to \cdots \to S_0(X,\mathbb{R}) \to 0)$$

be the complex of real C^∞ singular chains of X,

$$S_n(X,\mathbb{R}) = \mathbb{R}^{(\mathcal{C}^\infty(\Delta_n,X))},$$

(the real vector space of basis the singular n-chains),

$$d\gamma = \sum (-1)^i \partial_i \gamma,$$

$$S^\bullet(X,\mathbb{R}) := \mathrm{Hom}^\bullet(S_\bullet(X,\mathbb{R}),\mathbb{R}) = (0 \to S^0(X,\mathbb{R}) \to \cdots)$$

its dual, the complex of real C^∞ singular cochains, with $(da)(\gamma) = (-1)^{i+1} a(d\gamma)$ for $\gamma \in S_i$.

Let $\mathcal{S}_X^n$ be the sheaf associated to $U \mapsto S^n(U;\mathbb{R})$, hence a complex of sheaves $\mathcal{S}_X^\bullet$ on X. It is known (see [DM] for references) that:

- The augmentation $\mathbb{R} \to \mathcal{S}_X^\bullet$ is a quasi-isomorphism
- The sheaves $\mathcal{S}_X^n$ are soft, hence

$$\Gamma(X, \mathcal{S}_X^\bullet) \xrightarrow{\sim} R\Gamma(X, \mathbb{R})$$

- $S^\bullet(X, \mathbb{R}) \to \Gamma(X, \mathcal{S}_X^\bullet)$ is a quasi-isomorphism,

hence

- $S^\bullet(X, \mathbb{R}) \xrightarrow{\sim} R\Gamma(X, \mathbb{R})$ (in $D(\mathbb{R})$).

By Stokes formula

$$\int_\gamma d\omega = \int_{d\gamma} \omega,$$

the maps

$$\Omega^i_X(X) \to S^i(X, \mathbb{R}),\ \omega \mapsto (\gamma \mapsto (-1)^{i(i+1)/2} \int_\gamma \omega)$$

define a morphism of complexes

$$\Gamma(X, \Omega^\bullet_X) \to S^\bullet(X, \mathbb{R}). \tag{2}$$

This morphism corresponds to the pairing

$$\Gamma(X, \Omega^\bullet_X) \otimes S_\bullet(X, \mathbb{R}) \to \mathbb{R},\ \langle \omega, \gamma \rangle = (-1)^{i(i+1)/2} \int_\gamma \omega.$$

Theorem (G. de Rham, 1931 [dR]). The map (2) is a quasi-isomorphism, hence induces isomorphisms

$$H^i(X,\mathbb{R}) \xrightarrow{\sim} H^i_{\mathrm{dR}}(X).$$

Proof. (2) sheafifies to a morphism

$$\Omega^\bullet_X \to \mathcal{S}^\bullet_X \tag{2'}$$

and in the commutative square

$$\begin{array}{ccc} \Gamma(X,\Omega^\bullet_X) & \longrightarrow & S^\bullet(X,\mathbb{R}) \\ \downarrow & & \downarrow \\ R\Gamma(X,\Omega^\bullet_X) & \longrightarrow & R\Gamma(X,\mathcal{S}^\bullet_X) \end{array}$$

the vertical maps are isomorphisms. As (2') is a quasi-isomorphism (by the Poincaré lemma and acyclicity of open balls in $\mathbb{R}^n$), the bottow one is an isomorphism, and thus, the top one, too.

The numbers $\langle \omega, \gamma \rangle$ for a cocycle $\omega \in \Omega^i(X)$ (i.e., $d\omega = 0$) and a cycle $\gamma \in S_i(X)$ (i.e., $d\gamma = 0$) are called periods.

Example. $\langle (xdy - ydx)/(x^2 + y^2), \gamma : \theta \mapsto e^{i\theta}, \theta \in [0, 2\pi] \rangle = 2\pi$.

The de Rham theorem is equivalent to (a) + (b):

(a) (a cocycle ω is a boundary) $\Leftrightarrow$ (all periods of ω vanish);

(b) There exists a cocycle $\omega \in \Omega^i(X)$, unique up to a boundary, having prescribed periods on a set of cycles forming a basis of $H_i(X, \mathbb{R})$.

3. The analytic de Rham complex

X: a (paracompact) complex analytic manifold of (complex) dimension d.

$$\Omega_X^\bullet := (\mathcal{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^d \to 0)$$

the complex analytic de Rham complex.

$\mathcal{O}_X$: sheaf of holomorphic functions

Ω_X^1: sheaf of holomorphic 1-forms, dual of the tangent bundle T_X,

$\Omega_X^i := \Lambda^i \Omega_X^1$,

$d : \Omega_X^i \to \Omega_X^{i+1}$: the exterior differential (defined similarly to the real, C^∞-case).

Analytic Poincaré lemma: The augmentation

$$\mathbb{C} \to \Omega_X^\bullet$$

is a quasi-isomorphism (of complexes of sheaves of $\mathbb{C}$-vector spaces), hence induces isomorphisms

$$R\Gamma(X, \mathbb{C}) \xrightarrow{\sim} R\Gamma(X, \Omega_X^\bullet),$$

$$H^n(X, \mathbb{C}) \xrightarrow{\sim} H^n_{dR}(X)(:= H^n(X, \Omega_X^\bullet)).$$

Same proof. But contrary to the C^∞-case, the sheaves Ω_X^i are not in general acyclic for $\Gamma(X, -)$.

Relation with the C^∞-de Rham complex.

$X_{\mathbb{R}}$: underlying real, C^∞ manifold (of dimension $2d$).

$$\Omega^n_{X_{\mathbb{R}}} \otimes_{\mathbb{R}} \mathbb{C} = \oplus_{i+j=n} \Omega^{i,j}_X,$$

where $\Omega^{i,j}_X$: sheaf of C^∞-forms of type (i,j), with $\overline{\Omega^{i,j}_X} = \Omega^{j,i}_X$. Then

$$\Omega^\bullet_{X_{\mathbb{R}}} \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Tot}(\Omega^{\bullet,\bullet}_X, d', d''),$$

where $(\Omega^{\bullet,\bullet}_X, d', d'')$ = Dolbeault bi-complex. Recall Dolbeault's quasi-isomorphisms

$$\Omega^i_X \to (\Omega^{i,\bullet}_X, d''),$$

hence a quasi-isomorphism

$$\Omega^\bullet_X \to \Omega^\bullet_{X_{\mathbb{R}}} \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Tot}(\Omega^{\bullet,\bullet}_X).$$

and, as $\Omega^{i,j}_X$ is soft, isomorphisms

$$\Omega^{i,\bullet}(X) \xrightarrow{\sim} R\Gamma(X, \Omega^i_X),$$

$$\mathrm{Tot}(\Omega^{\bullet,\bullet}(X)) \xrightarrow{\sim} R\Gamma(X, \Omega^\bullet_X) (\xrightarrow{\sim} R\Gamma(X, \mathbb{C})).$$

The compact Kähler case.

Assume X compact, Kähler.

Let h be a Kähler metric on X: a hermitian form such that $d(\mathrm{Im}(h)) = 0$, where $\mathrm{Im}(h) \in \Omega^{1,1}(X)$ is the imaginary part of h.

Let d^*, d'^*, d''^* be the adjoints of the operators d, d', d'' for the Riemannian metric $g = \mathrm{Re}(h)$, and

$$\Delta = dd^* + d^*d,\ \Delta' = d'd'^* + d'^*d', \Delta'' = d''d''^* + d''^*d''$$

the corresponding Laplacian operators on $\Omega^{\bullet,\bullet}(X)$, so that

$$\Delta = 2\Delta' = 2\Delta''.$$

Let

$$H^{i,j}(X) := \{\omega \in \Omega^{i,j}(X) | \Delta\omega = 0\}$$

$$= \mathrm{Ker}(d) \cap \mathrm{Ker}(d^*) = \mathrm{Ker}(d'') \cap \mathrm{Ker}(d''^*) \subset \Omega^{i,j}(X)$$

be the space of harmonic forms of type (i, j).

Theorem (Hodge). The inclusions

$$H^{i,j}(X) \subset \Omega^{i,j}(X)$$

induce isomorphisms

$$H^{i,j}(X) \xrightarrow{\sim} H^j(X, \Omega^i_X)$$

and a decomposition (the Hodge decomposition)

$$\oplus_{i+j=n} H^{i,j}(X) \xrightarrow{\sim} H^n(X, \Omega^\bullet_X) \, (\xrightarrow{\sim} H^n(X, \mathbb{C})),$$

with

$$\overline{H^{i,j}(X)} = H^{j,i}(X).$$

Hodge filtration and the Hodge to de Rham spectral sequence

Let

$$\Omega_X^{\geqslant i} := (0 \to 0 \cdots \to 0 \to \Omega_X^i \to \Omega_X^{i+1} \to \cdots \to \Omega_X^d \to 0)$$

be the naive filtration of the de Rham complex. By the Dolbeault isomorphisms, the associated spectral sequence coincides with the first spectral sequence of the bicomplex $\Omega^{\bullet,\bullet}(X)$, and reads

$$(*) \qquad E_1^{i,j} = H^j(X, \Omega_X^i) \Rightarrow H_{\mathrm{dR}}^{i+j}(X).$$

By the Hodge theorem, (*) degenerates at E_1, the map

$$H^n(X, \Omega_X^{\geqslant i}) \to H_{\mathrm{dR}}^n(X)$$

is injective for all i and its image is the abutment filtration $F^i H_{\mathrm{dR}}^n(X) \subset H_{\mathrm{dR}}(X)$, called the Hodge filtration.

For $i+j=n$, the inclusions $H^{i,j}(X) \subset H^n_{\mathrm{dR}}(X)$ induce isomorphisms

$$H^{i,j}(X) \xrightarrow{\sim} (F^i \cap \overline{F^j})H^n_{\mathrm{dR}}(X) \xrightarrow{\sim} \mathrm{gr}^i_F H^n_{\mathrm{dR}}(X) = H^j(X, \Omega^i_X),$$

and the Hodge decomposition can be rewritten

$$H^n_{\mathrm{dR}}(X) = \oplus_{i+j=n} H^j(X, \Omega^i_X).$$

In particular the spaces $H^n_{\mathrm{dR}}(X)$, $H^j(X, \Omega^i_X)$ are finite dimensional, and if

$$h^n(X) := \dim_{\mathbb{C}} H^n_{\mathrm{dR}}(X) (= \dim_{\mathbb{C}} H^n(X, \mathbb{C})),$$
$$h^{i,j}(X) := \dim_{\mathbb{C}} H^j(X, \Omega^i_X),$$

we have, for all n,

$$\sum_{i+j=n} h^{i,j}(X) = h^n(X),$$

and for all i, j, the Hodge symmetry

$$h^{i,j}(X) = h^{j,i}(X).$$

4. Algebraic de Rham complexes

Let $f : X \to S$ be a morphism of schemes.

A construction of Grothendieck functorially associates to f a quasi-coherent sheaf of $\mathcal{O}_X$-modules, $\Omega^1_{X/S}$, called the module of differential forms of degree 1 of X/S and an S-derivation $d : \mathcal{O}_X \to \Omega^1_{X/S}$ defined as follows:

$$\Omega^1_{X/S} := \mathcal{I}/\mathcal{I}^2,$$

where $\mathcal{I} \subset i^{-1}(\mathcal{O}_{X \times_S X})$ is the ideal of the (locally closed) diagonal immersion $i : X \hookrightarrow X \times_S X$, and $\mathcal{I}/\mathcal{I}^2$ is viewed as an $\mathcal{O}_X$-module via $\mathcal{O}_X = i^{-1}(\mathcal{O}_{X \times_S X})/\mathcal{I}$.

The projections $\mathrm{pr}_1, \mathrm{pr}_2 : X \times_S X \to X$ (which retract i) induce ring homomorphisms $p_1^*, p_2^* : \mathcal{O}_X \to i^{-1}(\mathcal{O}_{X\times_S X})/\mathcal{I}^2$, and an S-derivation[4]

$$d := p_2^* - p_1^* : \mathcal{O}_X \to \mathcal{I}/\mathcal{I}^2 \subset i^{-1}(\mathcal{O}_{X\times_S X})/\mathcal{I}^2.$$

For X, S affine, $X = \mathrm{Spec}(B)$, $S = \mathrm{Spec}(A)$, f given by a homomorphism of rings $A \to B$, then $\Omega^1_{X/S}$ is the quasi-coherent sheaf associated to

$$\Omega^1_{B/A} := I/I^2,$$

where $I = \mathrm{Ker}(B \otimes_A B \to B,\ b_1 \otimes b_2 \mapsto b_1 b_2)$, and $d : B \to \Omega^1_{B/A}$ is defined by $da = 1 \otimes a - a \otimes 1$ modulo I^2.

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[4] As Grothendieck observed, the definition of the sheaf of 1-forms Ω^1 as $\mathcal{I}/\mathcal{I}^2$ works in other contexts as well: complex analytic, real anaytic, and even, more surprisingly, C^∞: for a real analytic manifold X, with associated C^∞-manifold X_∞, by the division theorem of Malgrange the sheaf $\mathcal{O}_{X_\infty}$ is flat over $\mathcal{O}_X$ ([Tou], VI 1.3).

The B-module $\Omega^1_{B/A}$ is the module of Kähler differentials of B/A (Kähler, 1953). The pair $(\Omega^1_{B/A}, d)$ is universal among A-derivations of B into B-modules.

Example. For $B = A[t_1, \cdots, t_n]$,

$$\Omega^1_{B/A} = \oplus_{1 \leqslant i \leqslant n} B dt_i,\ db = \sum (\partial b / \partial t_i) dt_i.$$

The image of the derivation

$$d : \mathcal{O}_X \to \Omega^1_{X/S}$$

$\mathcal{O}_X$-linearly generates $\Omega^1_{X/S}$, and d can be uniquely extended to a complex

$$\Omega^\bullet_{X/S} = (\mathcal{O}_X \xrightarrow{d} \Omega^1_{X/S} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^i_{X/S} \xrightarrow{d} \Omega^{i+1}_{X/S} \xrightarrow{d} \cdots)$$

where $\Omega^i_{X/S} = \Lambda^i_{\mathcal{O}_X}\Omega^1_{X/S}$, in such a way that $d(ab) = da \wedge b + (-1)^i a \wedge db$ for a of degree i.

This complex is called the de Rham complex of X/S. The $\mathcal{O}_X$-module $\Omega^1_{X/S}$ and the complex $\Omega^\bullet_{X/S}$ have nice functorial properties. In particular, $\Omega^\bullet_{X/S}$ commutes with base change: for X'/S' pulled-back by $g : S' \to S$ from X/S,

$$g^*\Omega^\bullet_{X/S} \xrightarrow{\sim} \Omega^\bullet_{X'/S'}.$$

Cotangent and derived de Rham complexes

For X/S smooth, $\Omega^1_{X/S}$ is locally free of finite type (with basis $(dx_1, \cdots, dx_n)$ if $x = (x_1, \cdots, x_n) : X \to \mathbb{A}^n_S$ is étale), and for a first order thickening $S \hookrightarrow S'$ of ideal I, the groups

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\Omega^1_{X/S}, I \otimes \mathcal{O}_X) = H^i(X, T_{X/S} \otimes I), \ \ T_{X/S} := \mathcal{H}om(\Omega^1_{X/S}, \mathcal{O}_X)$$

for $i = 0, 1, 2$ control flat (hence smooth) deformations of X over S'.

No longer the case if X/S is only assumed to be flat. Need to replace $\Omega^1_{X/S}$ by the cotangent complex [I 71]

$$L_{X/S} \in D^{\leqslant 0}(X, \mathcal{O}_X),$$

more often denoted $L\Omega^1_{X/S}$ today.

For X/S corresponding to an A-algebra B, $L\Omega^1_{X/S}$ is the complex of quasi-coherent sheaves on $X = \mathrm{Spec}(B)$ associated to the cotangent complex $L\Omega^1_{B/A}$ ($= L_{B/A}$) defined (independently) by André and Quillen (around 1968):

$$L\Omega^1_{B/A} := \Omega^1_{P_\bullet/A} \otimes_{P_\bullet} B \in D(B)$$

for a resolution (quasi-isomorphism) $P_\bullet \to B$ by a simplicial A-algebra which is polynomial in each degree.

Definition extends to simplicial A-algebras $B_\bullet$.

In modern language, $B_\bullet \mapsto L\Omega^1_{B_\bullet/A}$ is the left Kan extension

$$\begin{array}{ccc} \mathrm{Poly}_A & \xrightarrow{\Omega^1_{-/A}} & D(\mathrm{Mod}(A-\mathrm{Alg})) \\ \downarrow & \nearrow_{L\Omega^1_{-/A}} & \\ D(A-\mathrm{alg}) & & \end{array}$$

of the functor $\Omega^1_{-/A}$ from the category Poly_A of finitely generated polynomial A-algebras to the ∞-category of animated A-algebras $D(A-\mathrm{alg})$. Here $D(\mathrm{Mod}(A-\mathrm{Alg}))$ is the ∞-category of animated pairs (B, M), B an A-algebra, M a B-module.

This is the unique extension commuting with sifted colimits (filtering colimits, and simplicial realizations).

$L\Omega^1_{X/S}$ is recovered from the $L\Omega^1_{B/A}$'s (for $(\mathrm{Spec}(B) \subset X) \to (\mathrm{Spec}(A) \subset S)$) by Zariski sheafification (works in the ∞-categorical context).

By left Kan extension one defines similarly

$$L\Omega^i_{B/A} = L\Lambda^i L\Omega^1_{B/A}$$

and the derived de Rham complex

$$L\Omega^\bullet_{B/A},$$

and its Zariski sheafification

$$L\Omega^\bullet_{X/S}.$$

Explicitly,

$$L\Omega^\bullet_{B/A} = \mathrm{Tot}(\Omega^\bullet_{P_\bullet/A})$$

for a simplicial resolution $P_\bullet \to B$ by polynomial algebras, and $\mathrm{Tot}^n = \oplus_{i+j=n}$.

The derived de Rham complex comes equipped with the Hodge filtration (deduced from the naive filtration of $\Omega^\bullet$)

$$\mathrm{Fil}^i_{\mathrm{Hdge}} \Omega^\bullet_{X/S} := L\Omega^{\geqslant i}_{X/S}$$

with associated graded

$$\mathrm{gr}^i = L\Omega^i_{X/S}[-i].$$

Applications of cotangent complex and derived de Rham complex theory

• first order deformation theory: schemes, group schemes, etc. (Grothendieck, I., ...)

• relation with crystalline cohomology in mixed characteristic (I., Bhatt, Beilinson, ...)

• use in p-adic comparison theorems of p-adic Hodge theory (Bhatt, Beilinson, ...)

• use in perfectoid geometry (Bhatt-Morrow-Scholze, Cesnavicius, Mathew, ...) (starting point: $L\Omega^1_{B/\mathbb{F}_p} = 0$ if B is perfect)

• relation with Hochschild homology ($B \otimes^L_{(B \otimes^L_A B)} B$), cyclic homology, syntomic cohomology, and K-theory (Bhatt-Morrow-Scholze, Mathew, ...)

• use in prismatic cohomology theory (Bhatt-Lurie, Drinfeld, Mathew, ...).

5. The case of smooth, complex, algebraic varieties

(A brief review of theorems of Serre (GAGA), Grothendieck, and Deligne).

X: a smooth $\mathbb{C}$-scheme, separated and of finite type, $\dim(X) = d$.

Then: $\Omega^1_X := \Omega^1_{X/\mathbb{C}}$ is locally free of rank d (hence Ω^i_X locally free $\forall i$).

Poincaré lemma fails for $\Omega^\bullet_X$: $\mathcal{H}^i(\Omega^\bullet_X) \neq 0$ for $i > 0$ (deep relations with algebraic cycles (Bloch-Ogus)).

But, let

$$X_{\mathrm{an}} = X(\mathbb{C})$$

the associated complex analytic variety, and

$$\varepsilon : X_{\mathrm{an}} \to X$$

the canonical morphism (of ringed spaces). By Serre $\mathcal{O}_{X_{\mathrm{an}}}$ is flat over $\mathcal{O}_X$, and

$$\Omega^i_{X_{\mathrm{an}}} = \varepsilon^* \Omega^i_X := \mathcal{O}_{X_{\mathrm{an}}} \otimes_{\mathcal{O}_X} \Omega^i_X,$$

hence a canonical morphism of complexes

$$(*) \qquad \Omega^\bullet_X \to \varepsilon_* \Omega^\bullet_{X_{\mathrm{an}}}.$$

Remark For $U \subset X$ open affine, U^{an} is Stein (as closed in some $(\mathbb{A}^n_{\mathbb{C}})^{\mathrm{an}}$), hence $H^j(U^{\mathrm{an}}, \Omega^i) = 0$ for all $j > 0$, hence $\varepsilon_* \Omega^\bullet_{X_{\mathrm{an}}} \xrightarrow{\sim} R\varepsilon_* \Omega^\bullet_{X_{\mathrm{an}}}$.

Theorem (Serre, Grothendieck, Deligne).

(∗) $$\Omega_X^\bullet \to \varepsilon_* \Omega_{X_{\mathrm{an}}}^\bullet.$$

is a quasi-isomorphism, hence induces an isomorphism (in $D(\mathbb{C})$)

(∗∗) $$R\Gamma(X, \Omega_X^\bullet) \xrightarrow{\sim} R\Gamma(X_{\mathrm{an}}, \Omega_{X_{\mathrm{an}}}^\bullet).$$

Combining with the (analytic) Poincaré lemma, we get:

(∗ ∗ ∗) $$R\Gamma(X, \Omega_X^\bullet) \xrightarrow{\sim} R\Gamma(X_{\mathrm{an}}, \mathbb{C}).$$

hence, for all n,

$$H^n_{\mathrm{dR}}(X) \xrightarrow{\sim} H^n(X_{\mathrm{an}}, \mathbb{C}).$$

When X is affine, $H^j(X, \Omega^i_X) = 0$ for all $j > 0$ by Serre, so in this case

$$H^n_{\mathrm{dR}}(X) = H^n(\Gamma(X, \Omega^\bullet_X)),$$

and the theorem is equivalent to its special case (the most difficult one!):

Theorem'. For X affine, the canonical map

$$\Omega^\bullet_X(X) \to \Omega^\bullet_{X^{\mathrm{an}}}(X^{\mathrm{an}})$$

is a quasi-isomorphism.

Glimpses on proof.

(a) The proper case. Assume $X/\mathbb{C}$ proper. The morphism (**) induces a morphism of Hodge to de Rham spectral sequences

$$(E_1^{i,j}(X) = H^j(X, \Omega^i_X) \Rightarrow H^{i+j}_{\mathrm{dR}}(X))$$

$$\to (E_1^{i,j}(X^{\mathrm{an}}) = H^j(X^{\mathrm{an}}, \Omega^i_{X^{\mathrm{an}}}) \Rightarrow H^{i+j}_{\mathrm{dR}}(X^{\mathrm{an}}).$$

By Serre's GAGA, this is an isomorphism on the E_1 terms, hence an isomorphism.

(b) The general case. By Nagata's compactification theorem and Hironaka's resolution of singularities there exists a dense open immersion

$$j : X \hookrightarrow \overline{X},$$

with $\overline{X}/\mathbb{C}$ proper and smooth and $D := \overline{X} - X$ the support of a strictly normal crossings divisor. Then the proof uses de Rham complexes with logarithmic poles, both in the algebraic and analytic contexts, whose local study near D provides isomorphisms

$$R\Gamma(\overline{X}, \Omega^\bullet_{\overline{X}}(\log D)) \xrightarrow{\sim} R\Gamma(X, \Omega^\bullet_X),$$

$$R\Gamma(\overline{X}_{\mathrm{an}}, \Omega^\bullet_{\overline{X}_{\mathrm{an}}}(\log D_{\mathrm{an}}) \xrightarrow{\sim} R\Gamma(X_{\mathrm{an}}, \Omega^\bullet_{X_{\mathrm{an}}})$$

with left hand sides isomorphic by GAGA.

Application to Hodge theory

(1) The proper smooth case. Let $X/\mathbb{C}$ be smooth and projective. Hence X^{an} is Kähler. Then degeneration and decomposition results for X^{an} imply, by GAGA,

(a) The (algebraic) Hodge to de Rham spectral sequence

$$E_1^{i,j} = H^j(X, \Omega^i) \Rightarrow H^{i+j}_{\mathrm{dR}}(X/\mathbb{C})$$

degenerates at E_1.

(b) The Hodge decomposition of $H^n_{\mathrm{dR}}(X^{\mathrm{an}}/\mathbb{C})$ induces by GAGA a decomposition

$$H^n_{\mathrm{dR}}(X/\mathbb{C}) = \oplus_{i+j=n} H^{i,j},\ \overline{H}^{i,j} = H^{j,i},$$

also called the Hodge decomposition, where

$$H^{i,j} = F^i \cap \overline{F}^j,$$

$F^i := H^n(X, \tau^{\geqslant i}\Omega^\bullet_X) \subset H^n_{\mathrm{dR}}(X/\mathbb{C})$ denoting the Hodge filtration.

Deligne generalized (a) and (b) to X proper (not necessarily projective), while X^{an} may fail to be Kähler.

Statement (a) is purely algebraic (equivalent to

$$h^n_{\mathrm{dR}} = \sum_{i+j=n} h^{i,j},$$

with $h^{i,j} := \dim_{\mathbb{C}} H^j(X, \Omega^i)$. It generalizes to any proper, smooth X/k, k a field of characteristic zero. A purely algebraic proof was given in [DI].

In contrast, statement (b), which involves complex conjugation, is of analytic nature.

(2) The general case. For $X/\mathbb{C}$ proper smooth, the data of the lattice $H^n(X^{\mathrm{an}}, \mathbb{Z})$ and the decomposition

$$H^n(X^{\mathrm{an}}, \mathbb{Z}) \otimes \mathbb{C} \xrightarrow{\sim} \oplus_{i+j=n} H^{i,j},\ \overline{H}^{i,j} = H^{j,i}$$

is called a pure Hodge structure of weight n. (Classical) Hodge theory is the study of such structures.

In a series of remarkable papers (Hodge I, Hodge II, Hodge III) Deligne constructed an extension of this theory to arbitrary $X/\mathbb{C}$ (separated and of finite type), called mixed Hodge theory.

In the past 50 years Hodge theory of complex algebraic varieties has become a central topic in algebraic geometry, with deep connections with number theory and representation theory.

6. De Rham complexes in positive characteristic

In positive characteristic the Poincaré lemma is outrageously false. But this failure is the source of a miraculous isomorphism, the Cartier isomorphism, which turns out to control the whole differential calculus in positive and mixed characteristic.

Let S be an $\mathbb{F}_p$-scheme, and X a smooth S-scheme. We have a commutative square

$$\begin{array}{ccccc} X & \longleftarrow & X^{(1)} & \xleftarrow{F} & X \\ \downarrow & & \downarrow & \swarrow & \\ S & \xleftarrow{F_S} & S, & & \end{array}$$

where F_S (resp. the upper composite) is the absolute Frobenius of S (resp. X), F the relative Frobenius of X/S, and the square is cartesian. The complex $F_*\Omega^\bullet_{X/S}$ is $\mathcal{O}_{X^{(1)}}$-linear. In particular, $\oplus_i H^i(F_*\Omega^\bullet_{X/S})$ is a graded commutative $\mathcal{O}_{X^{(1)}}$-algebra.

The following result (due to Katz in its formulation) is classical ([K]):

Theorem (Cartier). The homomorphism $\mathcal{O}_{X^{(1)}} \to \mathrm{Ker}(d) \subset F_* \mathcal{O}_X$ extends uniquely to a homomorphism of graded algebras

$$C^{-1} : \oplus_i \Omega^i_{X^{(1)}/S} \to \oplus_i H^i(F_* \Omega^\bullet_{X/S})$$

such that, for any local section f of $\mathcal{O}_X$, $C^{-1}(d(1 \otimes f)) =$ class of $f^{p-1} df$ in $H^1 F_* \Omega^\bullet_{X/S}$. And C^{-1} is an isomorphism.

Proof. Existence and uniqueness easy:
(∗)

$$(f+g)^{p-1}(df+dg)-f^{p-1}df-g^{p-1}dg = d(\sum_{0<i<p}(1/p)\binom{p}{i}f^{p-i}g^{i}).$$

Proof that C^{-1} is an isomorphism by reduction to $S = \mathrm{Spec}(\mathbb{F}_p)$, $X = \mathrm{Spec}(\mathbb{F}_p[t_1, \cdots, t_n])$ (and finally, $n = 1$). In this case, C^{-1} is an isomorphism

$$\Omega^* := \oplus\Omega^i_{X/S} \xrightarrow{\sim} \oplus H^i(\Omega^\bullet_{X/S})$$

($H^*(\Omega^\bullet)$ as big as Ω^*!).

Remark. (*) has tight links with: δ-structures, Witt vectors, liftings mod p^2.

Derived Cartier isomorphism

For $S/\mathbb{F}_p$, and X/S, the canonical filtration $\tau_{\leqslant i}$ on $F_*\Omega^\bullet$ defines an increasing filtration

$$\mathrm{Fil}_i^{\mathrm{conj}} F_* L\Omega^\bullet_{X/S}$$

on $F_* L\Omega^\bullet_{X/S}$, called the conjugate filtration, with associated graded calculated by a derived Cartier isomorphism

$$C^{-1} : L\Omega^i_{X^{(1)}/S}[-i] \xrightarrow{\sim} \mathrm{gr}_i F_* L\Omega^\bullet_{X/S},$$

where $X^{(1)}$ is the derived pull-back of X/S by F_S, and $F : X \to X^{(1)}$ the relative Frobenius.

This filtration, defined and studied by Bhatt [Bh], plays an important role in p-adic Hodge theory.

7. Crystalline cohomology

Let k be a perfect field of characteristic p and $W(k)$ its ring of Witt vectors (e.g., $k = \mathbb{F}_p$, $W(k) = \mathbb{Z}_p$).

Let Y/k be proper smooth, and suppose $X_1/W(k)$, $X_2/W(k)$ are proper, smooth liftings of Y. Analogy with work of Monsky-Washnitzer (in the affine case) and an algebraic construction (due to him and, independently, Katz-Oda) of the Gauss-Manin connection on relative de Rham cohomology led Grothendieck to conjecture (in [GCJ])

(1) There should exist a canonical isomorphism

$$\chi(X_1, X_2) : H^*_{\mathrm{dR}}(X_1/W(k)) \xrightarrow{\sim} H^*_{\mathrm{dR}}(X_2/W(k)),$$

(satisfying the natural transitivity condition for X_1, X_2, X_3);

(2) A new cohomology theory, crystalline cohomology, $Y/k \mapsto H^*(Y/W(k))$ defined for all Y/k proper and smooth (not necessarily liftable), of which he proposed a definition, should give rise, for any (proper smooth) lifting $X/W(k)$, to a canonical isomorphism

$$\chi(X) : H^*(Y/W(k)) \xrightarrow{\sim} H^*_{\mathrm{dR}}(X/W(k),$$

with $\chi(X_1, X_2)\chi(X_1) = \chi(X_2)$ for any X_1, X_2 lifting Y.

This theory (with a slightly modified definition), and in a much greater generality, was worked out by Berthelot [B].

Berthelot proved that, for Y/k proper and smooth, $H^*(Y/W(k))$ is finitely generated over $W(k)$, and $Y \mapsto H^*(Y/W(k)) \otimes \mathbb{Q}_p$ is a Weil cohomology theory.

The definition of $H^*(Y/W(k))$ uses the crystalline site $(Y/W_n(k))_{\mathrm{crys}}$, whose objects (for any Y/k) are $W_n(k)$-thickenings $U \hookrightarrow V$ of open subschemes of Y, endowed with a divided power structure on the ideal of the thickening, with coverings defined by the Zariski topology.

There is a natural sheaf of rings $\mathcal{O}_{Y/W_n(k)}$, $(U \hookrightarrow V) \mapsto \mathcal{O}_V$, and

$$R\Gamma(Y/W_n(k)) := R\Gamma((Y/W_n(k))_{\mathrm{crys}}, \mathcal{O}_{Y/W_n(k)}),$$

and (for Y/k proper, smooth)

$$H^*(Y/W(k)) := \varprojlim H^*(Y/W_n(k)).$$

For any $X/W(k)$ proper smooth lifting Y, there exists a canonical inverse system of isomorphisms

$$\chi_n(X) : R\Gamma(Y/W_n(k)) \xrightarrow{\sim} R\Gamma_{dR}(X_n/W_n(k))$$

where $X_n = X \otimes W_n(k)$, giving the above $\chi(X)$.

A different construction of $H^*(Y/W_n(k)$ was later provided by the de Rham-Witt complex, an inverse system of (strictly) graded commutative differential algebras on Y

$$W_n\Omega^\bullet_Y = (W_n\mathcal{O}_Y \xrightarrow{d} W_n\Omega^1_Y \xrightarrow{d} \cdots \xrightarrow{d} W_n\Omega^i_Y \xrightarrow{d} \cdots),$$

with

$$W_1\Omega^\bullet_Y = \Omega^\bullet_{Y/k},$$

operators $F : W_n\Omega^i_Y \to W_{n-1}\Omega^i_Y$, $V : W_n\Omega^i_Y \to W_{n+1}\Omega^i_Y$ extending F and V on $W\mathcal{O}_Y$, and satisfying a number of relations (such as $FV = VF = p$, $FdV = d$).

Moreover, for Y/k smooth, an inverse system of isomorphisms

$$R\Gamma(Y/W_n(k)) \xrightarrow{\sim} R\Gamma(Y, W_n\Omega^\bullet_Y)$$

(functorial in Y, compatible with products, and, for Y/k proper, smooth, identifying the Frobenius morphism on $R\Gamma(Y/W(k))$ with the endomorphism of $R\Gamma(Y, W\Omega^\bullet_Y)$ defined by p^iF on $W\Omega^i_Y$.

First constructed by Bloch (under some restrictions), then by I. in general, following suggestions by Deligne. Further generalizations by Langer-Zink and Hesselholt-Madsen.

New, simplified approach by Bhatt-Lurie-Mathew [BLM], giving reasonable results for certain singular Y/k (saturated de Rham-Witt complexes).

8. p-adic Hodge theory

Let k, $W(k)$ as before, $K := \mathrm{Frac}(W(k))$, $\overline{K}$ an algebraic closure of K, $G_K = \mathrm{Gal}(\overline{K}/K)$.

Let $X/W(k)$ be proper and smooth, and let $Y = X \otimes k$, $X_{\overline{K}} = X \otimes \overline{K}$. Associated with X are two kinds of cohomological objects:

(a) de Rham cohomology $H^*_{\mathrm{dR}}(X/W(k))$

(b) p-adic étale cohomology $H^*(X_{\overline{K}}, \mathbb{Z}_p)$.

For all n, $H^n_{\mathrm{dR}}(X/W)$ is finitely generated over $W(k)$, in particular

$$\dim_K(H^n_{\mathrm{dR}}(X_K/K)) < +\infty.$$

Similarly, $H^n(X_{\overline{K}}, \mathbb{Z}_p)$ is finitely generated over $\mathbb{Z}_p$, in particular

$$\dim_{\mathbb{Q}_p}(H^n(X_{\overline{K}}, \mathbb{Q}_p)) < +\infty.$$

It follows from the comparison theorems between algebraic and analytic de Rham cohomology on one hand, and between Betti cohomology and p-adic étale cohomology (over complex algebraic varieties) (Artin) on the other hand that

$$(*) \qquad \dim_K(H^n_{\mathrm{dR}}(X_K/K)) = \dim_{\mathbb{Q}_p}(H^n(X_{\overline{K}}, \mathbb{Q}_p)).$$

Natural to ask whether (*) could be refined into an isomorphism

after a suitable extension of scalars. But de Rham cohomology and p-adic étale cohomology are quite different in nature:

- $H^n(X_{\overline{K}}, \mathbb{Q}_p)$ has a continuous Galois action (of G_K);

- $H^n_{\mathrm{dR}}(X_K/K)$ has no Galois action. As a K-vector spaces, it depends only on the special fiber Y:

$$H^n_{\mathrm{dR}}(X_K/K) \xrightarrow{\sim} H^n(Y/W(k)) \otimes K.$$

But $H^n_{\mathrm{dR}}(X_K/K)$ has other pieces of structure:

(i) the Hodge filtration

$$F^i H^n_{\mathrm{dR}}(X_K/K) = H^n(X_K, \Omega^{\geqslant i}_{X_K/K}) \subset H^n_{\mathrm{dR}}(X_K/K).$$

(ii) the σ-linear Frobenius automorphism

$$\varphi : H^n_{\mathrm{dR}}(X_K/K) \xrightarrow{\sim} H^n_{\mathrm{dR}}(X_K/K),$$

deduced from the Frobenius isogeny φ on crystalline cohomology and the isomorphism $H^n(Y/W(k)) \xrightarrow{\sim} H^n_{\mathrm{dR}}(X/W(k))$.

Example. Suppose $X/W(k)$ is an abelian scheme of dimension g. Then

$$H^1_{\mathrm{dR}}(X/W(k)) = H^1(Y/W(k))$$

is free of rank $2g$ over W(k), and with its natural operators F, V satisfying $FV = VF = p$, is the Dieudonné module of the p-divisible group $Y[p^\infty]/k$. One has $\varphi = F$, and the Hodge filtration is given by

$$F^1 H^1_{\mathrm{dR}}(X/W(k)) = H^0(X, \Omega^1_{X/W(k)}) = \mathrm{Lie}(X)^\vee.$$

On the other hand, $H^1(X_{\overline{K}}, \mathbb{Q}_p)$ is the Tate module of $X^\vee$, with its natural Galois action

$$H^1(X_{\overline{K}}, \mathbb{Q}_p) = T_p(X^\vee_{\overline{K}}) \otimes \mathbb{Q}_p = (\varprojlim X^\vee_{\overline{K}}[p^n]) \otimes \mathbb{Q}_p.$$

By results of Serre-Tate, Tate and Grothendieck, both the filtered Dieudonné module $(H^1(Y/W(k)) \otimes K, F^1)$ and the Galois representation $T_p(X^{\vee}_{\overline{K}}) \otimes \mathbb{Q}_p$ characterize X_K.

Around 1970 Grothendieck asked whether one could find an algebraic machinery enabling to recover the filtered Dieudonné module from the p-adic Galois representation and vice-versa. A special case of his problem of the mysterious functor.

On the other hand, let $C := \widehat{\overline{K}}$. The action of G_K on $\overline{K}$ extends to a continuous action on C. Let $C(1)$ be the rank one G_K-module over C deduced from the cyclotomic character $G_K \to \mathbb{Z}_p^*$. His results on abelian varieties and p-divisible groups led Tate to conjecture (around 1968) the existence of canonical, G_K-equivariant decomposition for any proper smooth Z/K (not necessarily of the form X_K as above)

$$(HT) \qquad \oplus_{i+j=n} H^i(Z, \Omega^j_{Z/K}) \otimes C(-j) \xrightarrow{\sim} H^n(Z_{\overline{K}}, \mathrm{Q}_p) \otimes C,$$

later called Hodge-Tate decomposition.

Fontaine's period rings and the birth of p-adic Hodge theory

The sought for algebraic machinery was patiently built by Fontaine in the 1970's and early 1980's. He constructed rings denoted B (for Barsotti), equipped with filtrations, φ and Galois actions, called rings of p-adic periods, and conjectured the existence of canonical period isomorphisms of the form

$$B \otimes H^*_{\mathrm{dR}} \xrightarrow{\sim} B \otimes H^*(-, \mathbb{Q}_p),$$

compatible with the induced Galois and φ-actions and filtrations, and in such a way that H^*_{dR} (resp. $H^*(-, \mathbb{Q}_p)$) could be recovered from $B \otimes H^*(-, \mathbb{Q}_p)$ (resp. $B \otimes H^*_{\mathrm{dR}}$) by simple operations. There were 3 rings,

$$B_{\mathrm{cris}} \subset B_{\mathrm{st}} \subset B_{\mathrm{dR}},$$

and corresponding comparison conjectures denoted C_{cris}, C_{st}, C_{dR}.

The simplest one: B_{dR}

B_{dR} is a complete discrete valuation field with residue field C, equipped with a filtration (from the valuation) and a continuous action of G_K. Technically:

$$B_{\mathrm{dR}}^{+} := \varprojlim_{n} (A_{\mathrm{inf}} \otimes K/J_K^n), \; B_{\mathrm{dR}} := \mathrm{Frac}(B_{\mathrm{dR}}^{+}),$$

where

$$A_{\mathrm{inf}} := \varprojlim_{n} W(\mathcal{O}_C^{\flat})/((\xi) + (p))^n$$

is the perfect prism associated with the perfectoid ring $\mathcal{O}_C$ ($\mathcal{O}_C^{\flat} := \varprojlim_F \mathcal{O}_C/p$),

$$\theta : W(\mathcal{O}_C^{\flat}) \to \mathcal{O}_C$$

the (surjective) Fontaine map, with kernel (ξ), and

$$J_K := \mathrm{Ker}(\theta : A_{\mathrm{inf}} \otimes K \to \mathcal{O}_C).$$

Construction works more generally for any finite, totally ramified extension K of $\mathrm{Frac}(W(k))$. See [Ber] for a nice exposition. One has

$$B_{\mathrm{dR}}^{G_K} = K,$$

$$\mathrm{gr} B_{\mathrm{dR}} = \oplus_i C(i).$$

Fontaine's C_{dR} conjecture was the existence, for Z proper and smooth over K, of a functorial isomorphism

$$(C_{\mathrm{dR}}) \qquad B_{\mathrm{dR}} \otimes_K H^*_{\mathrm{dR}}(Z/K) \xrightarrow{\sim} B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} H^*(Z_{\overline{K}}, \mathbb{Q}_p),$$

compatible with filtrations and Galois actions. Implies the Hodge-Tate decomposition (HT), and $H^*_{\mathrm{dR}}(Z/K)$, with its Hodge filtration, is recovered as

$$H^*_{\mathrm{dR}}(Z/K) = (B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} H^*(Z_{\overline{K}}, \mathbb{Q}_p))^{G_K}.$$

Fontaine's C_{dR} conjecture, as well as the companion conjectures C_{cris}, C_{st}, was eventually proven by various authors, using different methods:

Tsuji (for C_{st}, plus de Jong to get C_{dR}) (1999), Faltings (2002), Niziol (2008), Beilinson (2012). See [Ber] for a historical survey.

Integral p-adic Hodge theory, prismatic cohomology

Let $X/W(k)$ be proper, smooth as above, and $Y = X \otimes W(k)$. For any n,

$$H^n_{\mathrm{dR}}(X/W(k))(\stackrel{\sim}{\to} H^n(Y/W(k))) \text{ and } H^n(X_{\overline{K}}, \mathbb{Z}_p)$$

are finitely generated modules over $W(k)$ and $\mathbb{Z}_p$ respectively, of the same rank.

In the late 1960's Grothendieck asked:

Question. Compare the torsion subgroups

$$H^n_{\mathrm{dR}}(X/W(k))_{\mathrm{tors}} \text{ and } H^n(X_{\overline{K}}, \mathbb{Z}_p)_{\mathrm{tors}}.$$

This question was out of reach of Fontaine et al.'s comparison theorems, which all neglect torsion.

Answer recently given by Bhatt-Morrow-Scholze:

Theorem [BMS1, Th. 1.1 (ii)]. We have, for all n, and all $m \geqslant 1$,

$$\mathrm{lgth}_{W(k)}(H^n_{\mathrm{dR}}(X/W(k))/p^m) \geqslant \mathrm{lgth}_{\mathbb{Z}_p}(H^n(X_{\overline{K}}, \mathbb{Z}_p)/p^m),$$

in particular,

$$\dim_k H^n_{\mathrm{dR}}(Y/k) \geqslant \dim_{\mathbb{F}_p} H^n(X_{\overline{K}}, \mathbb{F}_p).$$

Remark. Inequality of lengths can be strict, and in case of equality, structures of elementary divisors can be different.

The proof relies on a new theory, the A_{inf}-cohomology theory, enabling, in the case of good reduction (over $W(k)$ or ramified rings over $W(k)$), to compare crystalline cohomology and p-adic étale cohomology integrally. Which theory turned out to be a special case of a more general and flexible one, prismatic cohomology, due to Bhatt-Scholze, Bhatt-Lurie, Drinfeld ([BL], [BL1], [Dr]), of which we will discuss a few aspects in the second part of these lectures.

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