Logarithmic Kummer étale sites and Hodge degeneration

Luc Illusie (*)

These are notes of a talk given at the JAMI conference on *Hodge theory and logarithmic geometry* at Johns Hopkins, March 14, 2005, about some parts of the joint work [IKN]. I wish to thank C. Nakayama for his careful reading of a first draft and many helpful comments.

1. Log Hodge degeneration

The following theorem is due to Deligne [D].

Theorem 1.1. *Let Y be a scheme of characteristic zero, $f : X \to Y$ be a proper and smooth morphism. Then the Hodge to de Rham spectral sequence*

$$E_1^{pq} = R^q f_* \Omega^p_{X/Y} \Rightarrow R^{p+q} f_* \Omega^{\cdot}_{X/Y}$$

degenerates at E_1 and its initial term is locally free of finite type.

Let us briefly review the proof. By standard reduction arguments we may assume that $Y = \operatorname{Spec} A$, where A is an artinian local $\mathbb{C}$-algebra. Let $y = \operatorname{Spec} \mathbb{C} \in Y$ be the closed point. By the *relative Poincaré lemma* the augmentation

$$A_X \to \Omega^{\cdot,\mathrm{an}}_{X/Y} \tag{1}$$

is a quasi-isomorphism. Denoting by $\lg_A$ the length of an A-module, we have

$$\lg_A H^n(X, \Omega^{\cdot}_{X/Y}) = \lg_A H^n(X, \Omega^{\cdot,\mathrm{an}}_{X/Y})$$

(by GAGA), hence, by (1),

$$\lg_A H^n(X, \Omega^{\cdot}_{X/Y}) = \lg(A) \dim H^n(X_y, \mathbb{C}). \tag{2}$$

By classical Hodge degeneration for X_y,

$$\dim H^n(X_y, \mathbb{C}) = \sum_{p+q=n} \dim H^q(X_y, \Omega^p_{X_y}), \tag{3}$$

hence, by (2),

$$\lg_A H^n(X, \Omega^{\cdot}_{X/Y}) = \lg(A) \sum_{p+q=n} \dim H^q(X_y, \Omega^p_{X_y}). \tag{4}$$

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Therefore we have the following inequalities

$$\sum_{p+q=n} \lg_A H^q(X, \Omega^p_{X/Y}) \geq \lg_A H^n(X, \Omega^{\cdot}_{X/Y}) \geq \sum_{p+q=n} \lg_A H^q(X, \Omega^p_{X/Y}).$$

These have to be equalities. Therefore we get

$$\sum_{p+q=n} \lg_A H^q(X, \Omega^p_{X/Y}) = \lg_A H^n(X, \Omega^{\cdot}_{X/Y}), \tag{5}$$

which proves the degeneration at E_1, and (by (4))

$$\lg_A H^q(X, \Omega^p_{X/Y}) = \lg(A) \dim H^q(X_y, \Omega^p_{X_y}), \tag{6}$$

which proves that the initial term is free of finite type over A.

Remark 1.2. The degeneration result (3), and more generally 1.1 for Y smooth over a field of characteristic zero can be proved by "mod p^2" techniques ([DI], [I1]).

The main result I want to discuss in this talk is the following generalization of 1.1 :

Theorem 1.3 ([IKN 7.1]). *Let Y be an fs log scheme over $\mathbb{Q}$ and let $f : X \to Y$ be a proper, log smooth and exact morphism. Then the (log) Hodge to de Rham spectral sequence*

$$E_1^{pq} = R^q f_* \omega^p_{X/Y} \Rightarrow R^{p+q} f_* \omega^{\cdot}_{X/Y}$$

degenerates at E_1. Moreover, if for each point y of Y the monoid $M_y/\mathcal{O}^*_y$ is free, i. e. isomorphic to $\mathbb{N}^{r(y)}$ for some nonnegative integer $r(y)$, then its initial term is locally free of finite type.

Here "fs" means "fine (= integral and of finite type) and saturated", i. e. the log scheme in question has local (étale) charts by fine and saturated monoids ; "exact" means that for all $x \in X$, $y = f(x)$ the map $M_y/\mathcal{O}^*_y \to M_x/\mathcal{O}^*_x$ is exact, where a homomorphism of fs monoids $h : P \to Q$ is called exact if $h^{-1}(Q) = P$ in the group envelope P^{gp} ; "log smooth" means that X is locally smooth over the pull-back by a strict morphism $g : Y \to \operatorname{Spec} \mathbb{Q}[P]$ ("strict" meaning that g is a chart) of a morphism $\operatorname{Spec} \mathbb{Q}[Q] \to \operatorname{Spec} \mathbb{Q}[P]$ given by an *injective* homomorphism $P \to Q$ of fs monoids. A typical example of a log smooth and exact morphism is a morphism of the form $\mathbb{A}^n_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$, $(x_1, \cdots, x_n) \mapsto x_1^{m_1} \cdots x_r^{m_r}$ (for some $r \leq n$ and nonnegative integers m_i) ("generalized semistable reduction"). Finally, $\omega^{\cdot}_{X/Y}$ denotes the (log) de Rham complex of X/Y.

Remarks 1.4. (a) If one drops the assumption of freeness on the stalks of $M_Y/\mathcal{O}^*_Y$, it may happen that the initial term is not locally free. For example, if $X = \mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$, $Y = \operatorname{Spec}[x^2, xy, y^2]$ and $f : X \to Y$ is the morphism given by the inclusion of $\mathbb{C}[x^2, xy, y^2]$ into $\mathbb{C}[x, y]$(corresponding to taking the quotient of X by the involution $a \to -a$), then f is log étale and exact, $\omega^{\cdot}_{X/Y} = \mathcal{O}_X$ and $f_*\mathcal{O}_X$ is not locally free.

(b) Steenbrink [St] treated the case of semistable reduction over a curve. Since then several other cases of 1.3 were obtained by Cailotto [C], Fujisawa [F], Illusie [I1], Kawamata-Namikawa [KwN]. Variants involving certain coefficients are considered by Kato-Matsubara-Nakayama in [KMN].

(c) Note that in 1.3 f is not assumed to be "vertical" : if X and Y are log smooth over a field k, "vertical" means that the inverse image of the open subset of triviality of the log structure of Y, which may be bigger than that of X, is equal to it ; in other words, in this case, in 1.3 "horizontal divisors" are permitted.

2. Kummer étale sites and log degeneration

In this section, for simplicity, we work over $\mathbb{Q}$, all schemes (resp. log schemes) are of characteristic zero.

2.1. A morphism $f : X \to Y$ of fs log schemes is said to be *Kummer étale* (*ket* for short) if it is log étale and exact. It is equivalent to asking that locally X is (classicaly) étale over the pull-back by a strict map $g : Y \to \operatorname{Spec} \mathbb{Q}[P]$ ("strict" meaning that g is a chart) of a map $\operatorname{Spec} \mathbb{Q}[Q] \to \operatorname{Spec} \mathbb{Q}[P]$ given by an injective morphism of fs monoids $P \to Q$ such that there exists an integer $n \geq 1$ such that $nQ \subset P$. A typical example is $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ given by $t \to t^n$ $(n \geq 1)$, where $\mathbb{A}^1_{\mathbb{C}}$ is endowed with the log structure given by the origin, i. e. given by $\mathbb{N} \to \mathbb{C}[t], 1 \mapsto t$.

The *Kummer étale site* X_{ket} of an fs log scheme X consists of the category of ket maps $U \to X$ equipped with the topology given by surjective families. We have a natural morphism of ringed toposes

$$\varepsilon : X_{\text{ket}} \to X_{\text{et}}.$$

If $f : X \to Y$ is a morphism of fs log schemes, we define

$$\omega^p_{X/Y,\text{ket}} := \varepsilon^* \omega^p_{X/Y},$$

i. e. $\omega^p_{X/Y,\text{ket}}$ is the sheaf associating to U Kummer étale over X $\Gamma(U, \omega^p_{U/Y})$. The ket site is functorial, so f defines a morphism $f_{\text{ket}} : X_{\text{ket}} \to Y_{\text{ket}}$.

We shall deduce 1.3 from the following ket variant :

Theorem 2.2. *Let $f : X \to Y$ be a proper, log smooth and exact morphism between fs log schemes of characteristic zero. Then the Hodge to de Rham spectral sequence*

$$E_1^{pq} = R^q f_* \omega^p_{X/Y,\text{ket}} \Rightarrow R^{p+q} f_* \omega^{\cdot}_{X/Y,\text{ket}}$$

degenerates at E_1 and E_1^{pq} is locally free of finite type for all p, q.

To deduce 1.3 from 2.2 one needs the following easy lemma ([IKN, 3.7] for (a) (b) and an algebraic variant of 6.4.1 for (c)) :

Lemma 2.3. *(a) For all sheaves F of $\mathbb{Q}$-vector spaces on X_{ket}, one has $\varepsilon_* F = R\varepsilon_* F$.*

(b) The natural map $\mathcal{O}_{X_{\text{et}}} \to R\varepsilon_* \mathcal{O}_{X_{\text{ket}}}$ is an isomorphism.

(c) Assume that for all $y \in Y$, $M_y/\mathcal{O}_y^*$ is free, then for any locally free $\mathcal{O}_{Y_{\text{ket}}}$-module of finite type, $\varepsilon_* F$ is locally free of finite type.

Let us show that 2.2 implies 1.3. By 2.3 (a) and (b) one has

$$\omega_{X/Y}^{[a,b]} = \varepsilon_* \omega_{X/Y,\mathrm{ket}}^{[a,b]}$$

for all intervals $[a, b]$, where $\omega^{[a,b]}$ denote the naive truncation in degrees $[a, b]$, and therefore

$$(*) \qquad Rf_* \omega_{X/Y}^{[a,b]} = \varepsilon_* Rf_* \omega_{X/Y,\mathrm{ket}}^{[a,b]}.$$

By the ket degeneration 2.2, the short exact sequences

$$0 \to \omega_{X/Y,\mathrm{ket}}^{[b,c]} \to \omega_{X/Y,\mathrm{ket}}^{[a,c]} \to \omega_{X/Y,\mathrm{ket}}^{[a,b-1]} \to 0$$

then give short exact sequences

$$0 \to R^n f_* \omega_{X/Y,\mathrm{ket}}^{[b,c]} \to R^n f_* \omega_{X/Y,\mathrm{ket}}^{[a,c]} \to R^n f_* \omega_{X/Y,\mathrm{ket}}^{[a,b-1]} \to 0.$$

The degeneration in 1.3 follows, using 2.3 (a) and (*). By (*) again, the last asssertion of 1.3 is a consequence of 2.2 and 2.3 (c).

3. Kato-Nakayama spaces and relative log Poincaré lemma

3.1. To any fs log analytic complex space X Kato and Nakayama associate a topological space $X^{\log}$ together with a proper map $\tau : X^{\log} \to X$, which somehow "displays" the log structure of X (cf. [KN], [I2]). The points of $X^{\log}$ are the pairs (x, h) where x is a point of X and $h : M_x^{\mathrm{gp}} \to S^1$ is a homomorphism, which we will call the *angle map*, extending the usual angle map $a \to a(x)/|a(x)|$ for $a \in \mathcal{O}_x^*$, and τ is the first projection : $\tau(x, h) = x$. The fiber of τ at x is homeomorphic to $(S^1)^{r(x)}$ where $r(x)$ is the rank of the log structure at x, i. e. $r(x) = \mathrm{rk} M_x^{\mathrm{gp}}/\mathcal{O}_x^*$. If X is the log analytic space associated to $\operatorname{Spec} \mathbb{C}[P]$, for an fs monoid P, $X^{\log}$ is the set of monoid homomorphisms from P to the (multiplicative) monoid $\mathbb{R}_{\geq 0} \times S^1$, with the topology given by that of $\mathbb{R}_{\geq 0} \times S^1$, and τ is given by the "polar map" $\mathbb{R}_{\geq 0} \times S^1 \to \mathbb{C}, (r, z) \mapsto rz$. For example, if $P = \mathbb{N}$, then X is the affine line $\mathbb{A}^1_{\mathbb{C}}$ with the standard log structure given by the origin, $X^{\log}$ is the real blow-up $\mathbb{R}_{\geq 0} \times S^1$ of the origin, and τ the polar map. If X is log smooth and $j : U \to X$ is the open subset of triviality of its log structure, then $U = U^{\log}$, $X^{\log}$ is a topological manifold with boundary [KjN] and $j^{\log} : U^{\log} \to X^{\log}$ is the inclusion of the complement of the boundary.

3.2. The space $X^{\log}$ is endowed with a sheaf of rings $\mathcal{O}_X^{\log}$, whose definition we briefly recall.

One first defines a sheaf $\mathcal{L}$ of *logarithms* of sections of $\tau^{-1}(M_X^{\mathrm{gp}})$. This sheaf $\mathcal{L}$ fits in an exact sequence

$$(3.2.1) \qquad 0 \to \mathbb{Z}(1) \to \mathcal{L} \to \tau^{-1}(M_X^{\mathrm{gp}}) \to 0$$

obtained by pulling back the exact sequence of abelian sheaves on $X^{\log}$

$$0 \to \mathbb{Z}(1) \to \mathcal{C}ont(-, i\mathbb{R}) \to \mathcal{C}ont(-, S^1) \to 0$$

by the angle map $h : \tau^{-1}(M_X^{\mathrm{gp}}) \to \mathcal{C}ont(-, S^1)$. Here $\mathcal{C}ont$ means a sheaf of continuous functions, and the second map is the exponential. The sequence (3.2.1) extends the usual exponential sequence (on $X^{\log}$)

$$0 \to \mathbb{Z}(1) \to \tau^{-1}(\mathcal{O}_X) \to \tau^{-1}(\mathcal{O}_X^*) \to 0.$$

In other words, sections of $\mathcal{L}$ are pairs $(i\theta, a)$ where θ is a real continuous function on $X^{\log}$, and a a section of $\tau^{-1}(M^{\mathrm{gp}})$ such that $e^{i\theta} = h(a)$. We will denote again by exp the map $\mathcal{L} \to \tau^{-1}(M_X^{\mathrm{gp}})$ in (3.2.1). The sheaf $\mathcal{L}$ contains $\tau^{-1}(\mathcal{O}_X)$ as an abelian subsheaf. One defines a sheaf $\mathcal{O}_X^{\log}$ of $\tau^{-1}(\mathcal{O}_X)$-algebras, containing $\mathcal{L}$, by

$$\mathcal{O}_X^{\log} = (\tau^{-1}(\mathcal{O}_X) \otimes_{\mathbb{Z}} \mathrm{Sym}_{\mathbb{Z}}(\mathcal{L}))/I,$$

where I is the ideal generated by sections of the form $f \otimes 1 - 1 \otimes f$. The stalk of $\mathcal{O}_X^{\log}$ at each point (x, h) of $X^{\log}$ is a polynomial algebra over $\mathcal{O}_{X,x}$ on $r(x)$ generators T_i, sent to logarithms of elements of M_x^{gp} forming a basis modulo $\mathcal{O}_x^*$.

3.3. One defines the ket site X_{ket} of X as in the case of schemes. As a ket map $U \to V$ induces a local homeomorphism $U^{log} \to V^{\log}$, the map τ from X^{log} to X factors through X_{ket} via a map still denoted $\tau : X^{\log} \to X_{\mathrm{ket}}$. One defines the sheaf of rings $\mathcal{O}_X^{\mathrm{klog}}$ on $X^{\log}$ by

$$\mathcal{O}_X^{\mathrm{klog}} = \tau^{-1}(\mathcal{O}_{X_{\mathrm{ket}}}) \otimes_{\tau^{-1}(\mathcal{O}_X)} \mathcal{O}_X^{\log}.$$

Finally, one defines the de Rham complex

$$\omega_{X/Y}^{\cdot,\mathrm{klog}}$$

where

$$\omega_{X/Y}^{p,\mathrm{klog}} = \mathcal{O}_X^{\mathrm{klog}} \otimes_{\mathcal{O}_{X_{\mathrm{ket}}}} \omega_{X/Y}^{p,\mathrm{ket}},$$

and the differential $d : \mathcal{O}^{\mathrm{klog}} \to \omega^{1,\mathrm{klog}}$ is defined on $\mathcal{L}$ by $da = \mathrm{dlog}(\exp(a))$.

The main tool in the proof of 2.2 is the following *relative log Poincaré lemma* :

Theorem 3.4. *Let $f : X \to Y$ be an exact and log smooth map of fs complex analytic spaces. Then the natural augmentation*

$$(f^{\log})^{-1}(\mathcal{O}_Y^{\mathrm{klog}}) \to \omega_{X/Y}^{\cdot,\mathrm{klog}}$$

is a quasi-isomorphism.

The proof is a local calculation imitating the construction of a homotopy operator in the classical (non log) case.

Remarks 3.5. (a) The analogue of 3.4 for the non ket case is true and was proven by F. Kato [FK].

(b) As F. Kato observed in (*loc. cit.*), the exactness assumption on f is essential, as the example of a log blow-up shows.

Corollary 3.6. *Under the assumptions of 2.2, the sheaves* $R^n f_* \omega^{\cdot,\mathrm{ket}}_{X/Y}$ *are locally free of finite type for all* n.

To deduce 3.6 from 3.4 one uses the following two facts :

(i) The natural map

$$\omega^{\cdot,\mathrm{ket}}_{X/Y} \to R\tau_* \omega^{\cdot,\mathrm{klog}}_{X/Y}$$

is an isomorphism.

(ii) For all n, the sheaf $R^n f^{\log}_* \mathbb{C}_{X^{\log}}$ is a locally constant sheaf of $\mathbb{C}$-vector spaces on $Y^{\log}$; moreover, it is *quasi-unipotent*, which means that, locally over Y, it admits a finite filtration whose successive quotients are inverse images of locally constant sheaves (of $\mathbb{C}$-vector spaces) on Y_{ket}.

The first one is easy ([IKN, 3.7 (4)]). The second one ([IKN, 6.1]) relies on a delicate result of Kajiwara-Nakayama [KjN].

To prove 3.6, consider the commutative square

$$\begin{array}{ccc} X^{\log} & \xrightarrow{\tau} & X^{\mathrm{ket}} \\ \downarrow{\scriptstyle f^{\log}} & & \downarrow{\scriptstyle f^{\mathrm{ket}}} \\ Y^{\log} & \xrightarrow{\tau} & Y^{\mathrm{ket}} \end{array} .$$

Using (i) one has

$$Rf^{\mathrm{ket}}_* \omega^{\cdot,\mathrm{ket}}_{X/Y} = R\tau_* Rf^{\log}_* \omega^{\cdot,\mathrm{klog}}_{X/Y},$$

hence, by 3.4 and the projection formula,

$$R^n f^{\mathrm{ket}}_* \omega^{\cdot,\mathrm{ket}}_{X/Y} = \tau_*(\mathcal{O}^{\mathrm{klog}}_Y \otimes R^n f^{\log}_* \mathbb{C}).$$

Now, using (ii) (plus (i)) one finds that $R^n f^{\mathrm{ket}}_* \omega^{\cdot,\mathrm{ket}}_{X/Y}$ is locally free of finite type on Y^{ket} for all n.

4. Outline of the proof of 2.2

4.1. As the question is ket local on Y, by a result of Nakayama-Tsuji (cf. [IKN A 4.3]) one may (and will) assume that f is *saturated*, which means that f has reduced fibers, and implies that the underlying scheme of any fs pull-back of X by a map of fs log schemes $g : Y' \to Y$ is the usual schematic pull-back of X by g. By 3.6 the ket de Rham cohomology sheaves

$$H^n_{dR,\mathrm{ket}} = R^n f^{\mathrm{ket}}_* \omega^{\cdot,\mathrm{ket}}_{X/Y}$$

are locally free on Y_{ket}. Therefore, by ket localization on Y, we may assume that they are of the form $H^n_{dR,\mathrm{ket}} = \varepsilon^* M^n$, where M^n is a free module of finite type on Y_{et}. Then, by 2.3,

$$R^n f_* \omega^{\cdot}_{X/Y} = \varepsilon_* H^n_{dR,\mathrm{ket}} = M^n$$

is free of finite type on Y_{et} for all n. Therefore it suffices to show the following :

Proposition 4.2. *Let $f : X \to Y$ be a proper, log smooth and saturated morphism of fs log schemes of characteristic zero. Assume that the sheaves $R^n f_* \omega^{\cdot}_{X/Y}$ are locally free of finite type on Y_{et}. Then :*

(a) The (étale) spectral sequence

$$E_1^{pq} = R^q f_* \omega^p_{X/Y} \Rightarrow R^{p+q} f_* \omega^{\cdot}_{X/Y} \tag{1}$$

degenerates at E_1 and has locally free initial terms.

(b) The (ket) spectral sequence

$$E_1^{pq} = R^q f_*^{\mathrm{ket}} \omega^{p,\mathrm{ket}}_{X/Y} \Rightarrow R^{p+q} f_*^{\mathrm{ket}} \omega^{\cdot,\mathrm{ket}}_{X/Y} \tag{2}$$

degenerates at E_1 and has locally free initial terms.

(c) For any interval $[a,b]$ and any integer n the natural map

$$\varepsilon^* R^n f_* \omega^{[a,b]}_{X/Y} \to R^n f_*^{\mathrm{ket}} \omega^{[a,b],\mathrm{ket}}_{X/Y}$$

is an isomorphim, in other words, the spectral sequence (2) is the inverse image by ε of the spectral sequence (1).

4.3. The proof of 4.2 is in several steps.

(1) One first shows that (a) implies (c), hence (b). This is easy.

(2) One then treats the particular case of (a) where $Y = \operatorname{Spec} \mathbb{C}$. One does this using standard spreading out arguments and mod p^2 techniques as in [DI]. The saturatedness property of f ensures "Cartier type" and one can apply the degeneration result of Kato [K, 4.12].

(3) By standard reductions (spreading out, limit arguments, Lefschetz principle) one reduces the problem to the case where Y is the spectrum of a local artinian $\mathbb{C}$-algebra A. Then, using the hypothesis that $H^n(X, \omega^{\cdot}_{X/Y})$ is free of finite type over A for all n, the same calculation of lengths as in the proof of Deligne's theorem 1.1, with $\Omega^{\cdot}_{X/Y}$ replaced by $\omega^{\cdot}_{X/Y}$, proves the degeneration at E_1 and the freeness of the initial terms.

5. Complements and questions

In 1.1, Deligne also showed that E_1^{pq} and E_1^{qp} have the same rank (*Hodge symmetry*). In the situation of 2.2 we also get a similar result, under additional hypotheses.

Theorem 5.1. *Under the hypotheses of 2.2, let us assume moreover that f is* projective *and* vertical. *Then E_1^{pq} and E_1^{qp} have the same rank for all p, q.*

Here "vertical" means that for all $x \in X$, $y = f(x)$, the homomorphism $M_y/\mathcal{O}_y^* \to M_x/\mathcal{O}_x^*$ is vertical. A homomorphism of monoids $h : P \to Q$ is called *vertical* if for all $q \in Q$ there exists $p \in P$ and $q' \in Q$ such that $h(p) = q + q'$. For example, the first projection $\operatorname{Spec} \mathbb{C}[\mathbb{N}^2] \to \operatorname{Spec} \mathbb{C}[\mathbb{N}]$ corresponding to the inclusion $\mathbb{N} \to \mathbb{N}^2$ of the first factor is not vertical (presence of the *horizontal* divisor $y = 0$) while the semistable reduction map $(x, y) \to xy$ (corresponding to the diagonal inclusion) is. As recalled in 1.4

(c), if X and Y are log smooth over some field k, f is vertical if and only if the inverse image by f of the open subset of triviality of Y is that of X.

Here is a brief sketch the proof. By standard reductions, we may assume that f is saturated and that $Y = \operatorname{Spec}\mathbb{C}$, with $M_Y/\mathcal{O}_Y^* = P$. Choosing a local homomorphism $u : P \to \mathbb{N}$ (i. e. such that $u^{-1}(0) = 0$), we may assume that $P = \mathbb{N}$, i. e. Y is a standard log point. By 4.2 we have to show that

$$\dim H^q(X, \omega^p_{X/Y}) = \dim H^p(X, \omega^q_{X/Y})$$

for all p, q. By a result of Vidal [V] (a variant of a result of Yoshioka, see also [Sa]), up to further ket localization on Y, there is a log blow-up $h : X' \to X$ such that $f' : X' \to Y$ is strictly semistable. Using that $\mathcal{O}_X = Rh_*\mathcal{O}_{X'}$, and hence $\omega^{\cdot}_{X/Y} = Rh_*\omega^{\cdot}_{X'/Y}$, one is then reduced to the case where f is strictly semistable. In this case, the result is due to Y. Nakkajima [N], using a duality theorem of Tsuji and the (classical) hard Lefschetz theorem.

Remark 5.2. If Y is a standard log point, the conclusion may fail if f is not assumed to be projective, as an example of Nakkajima shows. If Y is log smooth over $\mathbb{C}$, it is not known, however if one can remove the hypothesis that f is projective.

5.3. In [IKN, 6.3] the result in 3.6 is extended to certain *quasi-unipotent* coefficients. In particular, a Riemann-Hilbert type equivalence

$$L_{\text{qunip}}(X) \to V_{\text{qnilp}}(X) \tag{5.3.1}$$

is defined for any log smooth fs log analytic complex space X, and shown to be functorial with respect to maps $f : X \to Y$ which are *proper* and *log smooth* (no exactness assumption is needed in this case). Let $L(X)$ be the category of locally constant sheaves of finite dimensional $\mathbb{C}$-vector spaces on $X^{\log}$. By restriction to the open subset U of triviality of the log structure, this category is equivalent to the category of locally constant sheaves of finite dimensional $\mathbb{C}$-vector spaces on U. In (5.3.1) $L_{\text{qunip}}(X)$ denotes the full subcategory of $L(X)$ consisting of those local systems L which have quasi-unipotent local monodromies (this means that at any point $y \in X^{\log}$ with $\tau(x) = y$, $\pi_1(\tau^{-1}(x), y)$ acts quasi-unipotently on L_y, or, equivalently, that, locally over X, L admits a finite filtration whose successive quotients are inverse images by τ of locally constant sheaves (of finite dimensional $\mathbb{C}$-vector spaces) on X_{ket}). The correspondence (5.3.1) associates to L a vector bundle V on X_{ket}, defined by

$$V = \tau_*(\mathcal{O}_X^{\text{klog}} \otimes_{\mathbb{C}} L),$$

equipped with an integrable connection $\nabla : V \to \omega^{1,ket}_{X/Y} \otimes V$, which is *quasi-nilpotent*, which means that, locally on X_{ket}, V admits a finite filtration (V_i) by vector bundles stable under ∇ such that the successive quotients are vector bundles on which the induced connection has no pole, i. e. $\nabla(V_i/V_{i-1}) \subset \operatorname{Im}(\Omega^1_{X/\mathbb{C}} \otimes (V_i/V_{i-1}) \to \omega^{1,\text{ket}}_{X/\mathbb{C}} \otimes (V_i/V_{i-1}))$. The functoriality indicated above says, in particular, that for V in $V_{\text{qnilp}}(X)$, and $f : X \to Y$ proper and log smooth, then, for all n, $R^n f_*^{\text{ket}} \omega^{\cdot,\text{ket}}_{X/Y}(V)$ is locally free of finite type on Y_{ket}.

Let us finally mention that Kato-Matsubara-Nakayama [KMN] have proved variants of the main degeneration result 2.2 for certain *variations of log Hodge structures*. In particular, they show the following (see (*loc. cit*) for the relevant definitions) :

Theorem 5.4. *Let Y be a log smooth fs log scheme over $\mathbb{C}$ and $f : X \to Y$ a projective, log smooth and* vertical *morphism. Let $(\mathcal{H}_{\mathbb{Z}}, \mathcal{V}, (-,-))$ be a variation of polarized log Hodge structures of pure weight w on X. Then, for all n, $R^n f_*^{\log}\mathcal{H}_{\mathbb{Z}}$ underlies a variation of polarized log Hodge structures of pure weight $w+n$ on Y. In particular, the Hodge to de Rham ket spectral sequence*

$$E_1^{pq} = R^{p+q} f_*^{\text{ket}} \text{gr}^p(\omega_{X/Y}^{\cdot,ket}(\mathcal{V})) \Rightarrow R^{p+q} f_*^{\text{ket}} \omega_{X/Y}^{\cdot,\text{ket}}(\mathcal{V})$$

degenerates at E_1 and its E_1-term is locally free of finite type on Y_{ket}.

5.5. *Questions.* (a) In 2.2, is the conclusion still valid without the hypothesis "exact", but assuming that Y is log smooth over $\mathbb{C}$? (Note that, as mentioned in 5.3, $R^n f_*^{\text{ket}} \omega_{X/Y}^{\cdot,\text{ket}}$ is locally free of finite type in this case.)

(b) Can one find a common generalization of 2.2 and 5.4 ?

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