CALDERON - ZYGMUND ESTIMATE

<u>Thm</u>: $\Omega \subset \mathbb{R}^N$ bounded, open
$f \in L^p(\Omega)$, $1 < p < \infty$
let $u \in L^p(\Omega)$ s.t.

$$-\Delta u = f \quad \text{in } \mathcal{D}'(\Omega).$$

Then $u \in W^{2,p}_{loc}(\Omega)$ & $\|D^2 u\|_{L^p(\omega)} \le C\left[\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}\right]$

$u = v + w$, $w = G * f$
↳ fundamental sol
(f is extended by 0 outside Ω)

$\Delta v = 0$ in $\mathcal{D}'(\Omega) \Rightarrow v \in C^\infty(\Omega)$
$\Rightarrow v \in W^{2,p}_{loc}(\Omega)$

$\forall \omega \subset\subset \Omega$,

$$\|D^2 v\|_{L^p(\omega)} \le C\left[\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}\right]$$

It remains to prove that $w \in W^{2,p}_{loc}(\Omega)$ and

$$\|D^2 w\|_{L^p(\omega)} \le C\left[\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}\right]$$

<u>Lemma:</u> $w = G * f \in W^{1,p}(\Omega)$

<u>Proof:</u> * $\|w\|_{L^p(\Omega)} \leq \|G\|_{L^1(\Omega-\Omega)} \|f\|_{L^p(\Omega)} < \infty$

$G \in L^1_{loc}(\mathbb{R}^N)$

$\Rightarrow w \in L^p(\Omega)$

* $G \in W^{1,1}_{loc}(\mathbb{R}^N)$

$$G(x) = \frac{c_N}{|x|^{N-2}} \in L^1_{loc}(\mathbb{R}^N)$$

$(N \geq 3)$

$$\partial_i G(x) = c_N (2-N) \frac{x_i}{|x|^N}$$

$(x \neq 0)$

$$\Rightarrow |\partial_i G(x)| \leq c_N (N-2) \frac{1}{|x|^{N-1}} \in L^1_{loc}(\mathbb{R}^N)$$

This implies

$$\partial_i (G * f) \overset{?}{=} (\partial_i G) * f$$

$\forall \varphi \in C^\infty_c(\Omega)$

$$\int_\Omega (G * f)(x) \partial_i \varphi(x) dx = \int_\Omega \left(\int_{\mathbb{R}^N} G(x-y) f(y) dy \right) \partial_i \varphi(x) dx$$

$$\underset{\text{Fubini}}{=} \int_{\mathbb{R}^N} \left(\int_\Omega \underbrace{G(x-y)}_{\in W^{1,1}_{loc}} \partial_i \varphi(x) dx \right) f(y) dy$$

$$= - \int_{\mathbb{R}^N} \left(\int_\Omega \partial_i G(x-y) \varphi(x) dx \right) f(y) dy$$

$$\underset{\text{Fubini}}{=} -\int_{\Omega} \left(\int_{\mathbb{R}^N} \partial_i G(x-y) f(y)\,dy \right) \varphi(x)\,dx$$

$$= -\int_{\Omega} ((\partial_i G) * f)(x)\, \varphi(x)\,dx$$

$$\Rightarrow \partial_i (G * f) = (\partial_i G) * f$$

$f \in L^p(\Omega)$, $\partial_i G \in L^1_{loc}(\mathbb{R}^N)$

$$\Rightarrow \|\partial_i (G * f)\|_{L^p(\Omega)} \leq \|\partial_i G\|_{L^1(\Omega-\Omega)} \|f\|_{L^p(\Omega)} < \infty$$

$$\Rightarrow \partial_i (G * f) \in L^p(\Omega)$$

So $G * f \in W^{1,p}(\Omega)$

⚠ In general

$$\partial^2_{ij} w = \partial^2_{ij} (G * f) \neq (\partial^2_{ij} G) * f$$

$\Leftarrow \partial^2_{ij} G \notin L^1_{loc}(\mathbb{R}^N)$

$G(x) \sim \dfrac{1}{|x|^{N-2}}$, $\partial_i G(x) \sim \dfrac{1}{|x|^{N-1}}$

$(N \geq 3)$ $\quad \in L^1_{loc}$ $\quad \in L^1_{loc}$

$$\partial^2_{ij} G(x) \sim \frac{1}{|x|^N} \notin L^1_{loc}$$

Let $T : L^p(\Omega) \longrightarrow \mathcal{D}'(\Omega)$
$(1 < p < \infty)$

$$f \longmapsto T(f) = \partial^2_{ij}(G * f)$$

$$f \in L^p(\Omega) \Longrightarrow G * f \in L^1_{loc}(\Omega) \subset \mathcal{D}'(\Omega)$$
$$\Longrightarrow T(f) = \partial^2_{ij}(G * f) \in \mathcal{D}'(\Omega)$$

So T well defined and linear

We want to show that $T : L^p(\Omega) \longrightarrow L^p(\Omega)$ linear, continuous: $\exists\, C > 0$ s.t. $\forall f \in L^p(\Omega)$,

$$\|T(f)\|_{L^p(\Omega)} = \|\partial^2_{ij}(G * f)\|_{L^p(\Omega)} \le C \|f\|_{L^p(\Omega)}$$

Step ①: Prove the result for $p = 2$
("Easy")

Step ②: Prove a weaker result for $p = 1$

Instead of $L^1(\Omega) \longrightarrow L^{1,\infty}(\Omega)$
(Lorentz space)

$\exists\, C > 0$ s.t. for $f \in L^2(\Omega) \cap L^1(\Omega)$,

$$\underbrace{\|T(f)\|_{L^{1,\infty}(\Omega)}}_{\sup_{t>0} t\,|\{|T(f)| \ge t\}|} \le C \|f\|_{L^1(\Omega)}$$

Step ③ Interpolation (Marcinkiewicz)

$\Rightarrow \|T(f)\|_{L^p(\Omega)} \leq C \|f\|_{L^p(\Omega)}$

$(1 < p < 2)$

Step ④: $p > 2 \to$ Duality method.

Step ①

Lemma: $f \in L^2(\Omega)$. Then

$$G * f \in H^2(\Omega)$$

$$-\Delta(G * f) = f \quad \text{a.e. in } \Omega$$

$$\|D^2(G * f)\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}$$

Proof: $w = G * f$

* $f \in C_c^\infty(\Omega)$ (Extended by 0 in $\mathbb{R}^n$)

$G * f \in C^\infty(\Omega)$, $-\Delta(G * f) = f$ in Ω

$\left(f \in C_c^\infty(\mathbb{R}^n) \text{ and } G * f \in C^\infty(\mathbb{R}^n)\right)$

Let $R > 0$ s.t. $\operatorname{Supp} f \subset B_{R/2}$

$$\int_\Omega |f|^2 = \int_{B_R} |f|^2 = \int_{B_R} |\Delta w|^2 = \sum_{i,j} \int_{B_R} (\partial_{ii} w)(\partial_{jj} w)$$

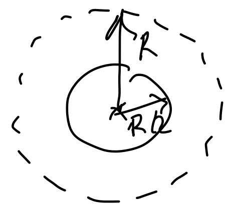
$$\underset{\text{Green}}{=} -\sum_{i,j} \int_{B_R} (\partial_i w)(\partial_{jji} w) + \sum_{i,j} \int_{\partial B_R} (\partial_i w)(\partial_{jj} w)\nu_j$$

$$\underset{\text{Green}}{=} \sum_{i,j} \int_{B_R} (\partial_{ij} w)(\partial_{ij} w) + \sum_{i,j} \int_{\partial B_R} \big[(\partial_i w)(\partial_{ij} w)\nu_j - (\partial_i w)(\partial_{jj} w)\nu_i \big]$$

$$= \int_{B_R} |D^2 w|^2 + \sum_{i,j} \int_{\partial B_R} [\longrightarrow]$$

$x \in \partial B_R$

$$w(x) = \int_{B_{R/2}} G(x-y) f(y)\,dy$$



$x \in \partial B_R,\ y \in B_{R/2} \Rightarrow |x-y| \geq \frac{R}{2} > 0$

$x \in \partial B_R,\ y \mapsto G(x-y)$ is $C^\infty(B_{R/2})$

$$\Rightarrow \partial_i w(x) = \int_{B_{R/2}} \partial_i G(x-y) f(y)\,dy$$

$$\Rightarrow \partial_{ij}^2 w(x) = \int_{B_{R/2}} \partial_{ij}^2 G(x-y) f(y)\,dy$$

$(N \geq 3)$ $$|G(x-y)| \leq \frac{C}{|x-y|^{N-2}} \leq \frac{C}{R^{N-2}}$$

$$|\partial_i G(x-y)| \leq \frac{C}{|x-y|^{N-1}} \leq \frac{C}{R^{N-1}}$$

$$|\partial^2_{ij} G(x-y)| \leq \frac{C}{|x-y|^{N}} \leq \frac{C}{R^{N}}$$

$$\Rightarrow |\partial_i w(x)| \leq \frac{C}{R^{N-1}}$$

$$|\partial^2_{ij} w(x)| \leq \frac{C}{R^{N}}$$

$$\int_{\partial B_R} [\text{———}] \leq C \cancel{R^{N-1}} \frac{1}{\cancel{R^{N-1}}} \frac{1}{R^N} \sim R^{-N} \xrightarrow[R \to \infty]{} 0$$

letting $R \to \infty$,

$$\int_\Omega |f|^2 = \int_{\mathbb{R}^N} |D^2 w|^2 \geq \int_\Omega |D^2 w|^2 \qquad (*)$$

$*$ Let $f \in L^2(\Omega)$, $\exists f_m \in C_c^\infty(\Omega)$
$f_m \to f$ in $L^2(\Omega)$

$w = G * f$

$w_m = G * f_m$

$$\|w - w_n\|_{L^2(\Omega)} \le \|G\|_{L^1(\Omega-\Omega)} \|f - f_n\|_{L^2(\Omega)} \longrightarrow 0$$

$$\|\partial_i w - \partial_i w_n\|_{L^2(\Omega)} \le \|\partial_i G\|_{L^1(\Omega-\Omega)} \|f - f_n\|_{L^2(\Omega)} \longrightarrow 0$$

$\Rightarrow w_n \longrightarrow w$ $H^1(\Omega)$

Apply (*) to $f_n - f_m$

$$\|D^2 w_n - D^2 w_m\|_{L^2(\Omega)} \le \|f_n - f_m\|_{L^2(\Omega)} \xrightarrow{n,m \to \infty} 0$$

$(D^2 w_n)$ Cauchy in $L^2(\Omega)$

$\Rightarrow D^2 w_n \longrightarrow D^2 w$ in $L^2(\Omega)$

$\Rightarrow w \in H^2(\Omega)$

$$(*) \Rightarrow \begin{array}{ccc} \|D^2 w_n\|_{L^2(\Omega)} & \le & \|f_n\|_{L^2(\Omega)} \\ \downarrow & & \downarrow \\ \|D^2 w\|_{L^2(\Omega)} & \le & \|f\|_{L^2(\Omega)} \end{array}$$

$$\begin{array}{ccc} -\Delta w_n & = & f_n \text{ in } \Omega \\ \downarrow L^2 & & \downarrow L^2 \\ -\Delta w & = & f \text{ a.e. in } \Omega \end{array}$$

$\#$

Step ②

Def: (Lorentz spaces)

$1 \leq p < \infty$

$$L^{p,\infty}(\Omega) = \{ f : \Omega \to \mathbb{R} \text{ meas. s.t. } \sup_{t>0} t^p |\{|f| > t\}| < \infty \}$$

$$\|f\|_{L^{p,\infty}(\Omega)} = \left[\sup_{t>0} t^p |\{|f| > t\}| \right]^{1/p}$$

NOT A NORM

Quasi-norm: $\|f+g\|_{L^{p,\infty}(\Omega)} \leq 2(\|f\|_{L^{p,\infty}} + \|g\|_{L^{p,\infty}})$

Rk: * $L^p(\Omega) \subset L^{p,\infty}(\Omega)$

Chebychev's inequality: $f \in L^p(\Omega)$

$$|\{|f| > t\}| \leq \frac{1}{t^p} \int_\Omega |f|^p$$

$$\Rightarrow \|f\|_{L^{p,\infty}} \leq \|f\|_{L^p}$$

* $L^{1,\infty}(\Omega) \not\subset L^1(\Omega)$

Take $f(x) = \frac{1}{|x|^N} \in L^{1,\infty}(\Omega) \setminus L^1(\Omega)$

$$\partial^2_{ij} G(x) \sim \frac{1}{|x|^N} \in L^{1,\infty}(\Omega) \setminus L^1(\Omega).$$

Lemma: $\exists\, C \geq 0$ (only depending on N) s.t.

$\forall f \in L^1(\Omega) \cap L^2(\Omega)$ $(= L^2(\Omega))$, then

$$\|T(f)\|_{L^{1,\infty}(\Omega)} \leq C \|f\|_{L^1(\Omega)}$$

$$\|\partial^2_{ij}(G * f)\|_{L^{1,\infty}(\Omega)} \leq C\|f\|_{L^1(\Omega)}$$

Thm (Calderon - Zygmund decomposition)

$f \in L^1(\mathbb{R}^N)$, $t > 0$.

Then $\exists \{Q_j\}_{j \in \mathbb{N}}$ where Q_j are cubes with pairwise disjoint interiors

$$(\mathring{Q}_i \cap \mathring{Q}_j = \phi \quad \forall i \neq j)$$

s.t.

(i) $|f(x)| \leq t$ a.e. in $\mathbb{R}^N \setminus \bigcup_j Q_j$.

(ii) $\forall j \in \mathbb{N}$,

$$t < \frac{1}{|Q_j|} \int_{Q_j} |f| \leq 2^N t$$

(iii) $\sum_j |Q_j| \leq \frac{1}{t} \int_{\mathbb{R}^N} |f|$

Moreover, we can decompose

$$f = b + g \qquad b = \text{bad}$$
$$g = \text{good}$$

where $g(x) = \begin{cases} f(x), & x \in \mathbb{R}^N \setminus (\cup Q_j) \\ \frac{1}{|Q_j|}\int_{Q_j} f, & x \in Q_j \end{cases}$

$$b = f - g = \begin{cases} 0, & x \in \mathbb{R}^N \setminus \cup Q_j \\ f - \frac{1}{|Q_j|}\int_{Q_j} f, & x \in Q_j \end{cases}$$

① $|g| \leq 2^N t$ a.e. in $\mathbb{R}^N$ → in ${}^c(\cup Q_j)$: $|g| = |f| \leq t \leq 2^N t$; → in Q_j: $|g| \leq \frac{1}{|Q_j|}\int_{Q_j}|f| \leq 2^N t$

② $b = 0$ in ${}^c(\cup Q_j)$ OK

③ $\int_{Q_j} b = 0$

$$\int_{Q_j} b = \int_{Q_j} \left[f - \frac{1}{|Q_j|}\int_{Q_j} f \right] = \int_{Q_j} f - \cancel{|Q_j|} \frac{1}{\cancel{|Q_j|}} \int_{Q_j} f = 0$$

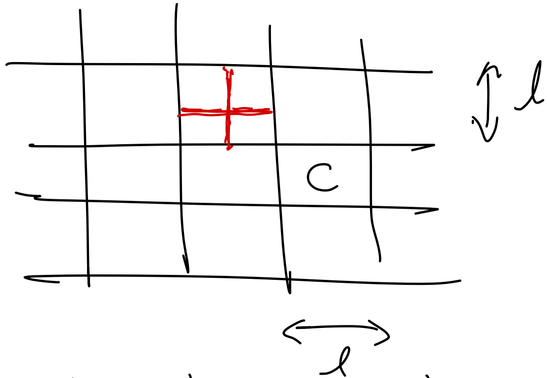
Proof of the CZ decomposition

$\int_{\mathbb{R}^N} |f| < \infty$ because $f \in L^1(\mathbb{R}^N)$, $t > 0$

Let $\ell \in \mathbb{N}$ large enough so that

$$\frac{1}{\ell^N}\int_{\mathbb{R}^N} |f| \leq t \qquad (1)$$

We subdivide $\mathbb{R}^N$ into cubes of side length l (which have pairwise disjoint interior)



$(1) \Rightarrow \forall$ such cube C, $\frac{1}{l^N}\int_C |f| \le t$

$|C| = l^N$

$$\Rightarrow \frac{1}{|C|}\int_C |f| \le t$$

Then subdivide each C into 2^N subcubes with side length $l/2$. Among these 2^N subcubes, there is at most one, say Q, s.t.

$$\frac{1}{|Q|}\int_Q |f| \le t$$

Otherwise all cubes satisfy

$$\frac{1}{|Q|}\int_Q |f| > t \qquad |Q| = \frac{|C|}{2^N} = \left(\frac{l}{2}\right)^N$$

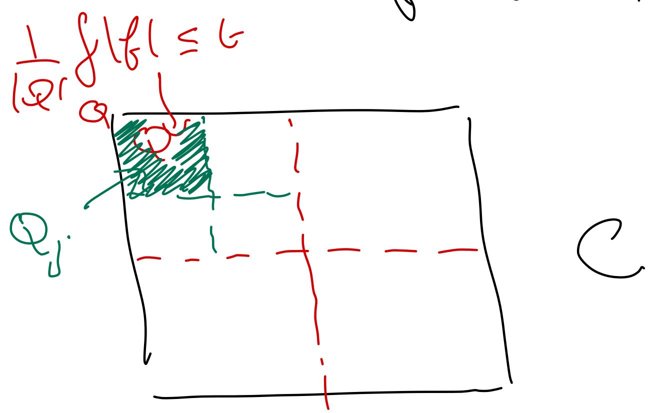
$$\Rightarrow \frac{2^N}{|C|}\int_Q |f| > t \qquad \forall\, Q \subset C$$

Sum wrt all $Q \subset C$,

$$\frac{\cancel{2^N}}{|C|} \underbrace{\sum_{Q \subset C} \int_Q |f|}_{\int_C |f|} > t \, \#\{Q_C\} = t \, \cancel{2^N}$$

$$= \frac{1}{|C|} \int_C |f| > t \quad \text{impossible.}$$

We repeat this procedure: we decompose each cubes Q s.t. $\frac{1}{|Q|}\int_Q |f| \le t$ into 2^N subcubes with half side length



The cubes $\{Q_j\}$ correspond to the cubes that have not be decomposed:

$$\frac{1}{|Q_j|}\int_{Q_j} |f| > t \qquad \forall j \in \mathbb{N}$$

By construction $\mathring{Q}_j \cap \mathring{Q}_{j'} = \phi \quad \forall i \neq j$.

$$t < \frac{1}{|Q_j|} \int_{Q_j} |f|$$

$\exists\, !\, C$ cube that has been decomposed s.t.

$$\frac{1}{|C|} \int_C |f| \leq t \quad \text{and} \quad |C| = \frac{|Q_j|}{2^N}$$

$$\Longrightarrow \int_{Q_j} |f| \leq \int_C |f| \leq t\,|C| = 2^N t\,|Q_j|$$

$$(Q_j \subset C)$$

$$\Longrightarrow \frac{1}{|Q_j|} \int_{Q_j} |f| \leq 2^N t \qquad (ii)$$

$\forall j \in \mathbb{N}$

$$\frac{1}{|Q_j|} \int_{Q_j} |f| > t$$

$$\Longrightarrow t\,|Q_j| < \int_{Q_j} |f|$$

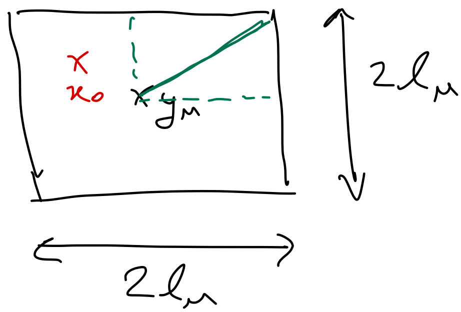
$$\Longrightarrow t \sum_j |Q_j| \leq \sum \int_{Q_j} |f| \leq \int_{\mathbb{R}^N} |f| \qquad (iii)$$

Proof (i): $x_0 \in \mathbb{R}^N \setminus (\bigcup Q_j)$

$\exists \{C_n\}_{n\in\mathbb{N}}$ of good cubes $\frac{1}{|C_n|}\int_{C_n} |f| \le t$.
with $\operatorname{diam}(C_n) \longrightarrow 0$
s.t. $x_0 \in C_n \ \forall n \in \mathbb{N}$

$$C_n = y_n + [-l_n, l_n]^N$$

C_n



$\forall z \in C_n$, $|z - y_n| \le \sqrt{N}\, l_n$

$$\Rightarrow |z - x_0| \le |z - y_n| + |x_0 - y_n| \le 2\sqrt{N}\, l_n = r_n$$

So $C_n \subset B_{r_n}(x_0)$ $\quad r_n = 2\sqrt{N}\, l_n \xrightarrow[n\to\infty]{} 0$

By the Lebesgue point Thm.

$$\frac{1}{C_n}\int_{C_n} |f - f(x_0)| = \frac{|B_{r_n}(x_0)|}{|C_n|}\frac{1}{|B_{r_n}(x_0)|}\int_{C_n} |f - f(x_0)|$$

$$|B_{r_n}(x_0)| = \omega_N r_n^N = \omega_N (2\sqrt{N})^N \ell_n^N$$

$$|C_n| = (2\ell_n)^N$$

$$\text{So } \frac{|B_{r_n}(x_0)|}{|C_n|} = \omega_N N^{N/2}$$

$$\Rightarrow \frac{1}{C_n} \int_{C_n} |f - f(x_0)| \leq \omega_N N^{N/2} \frac{1}{|B_{r_n}(x_0)|} \int_{B_{r_n}(x_0)} |f - f(x_0)|$$

$$\xrightarrow[r_n \to 0]{} 0 \quad \text{(by the Lebesgue pt Thm)}$$

a.e. $x_0 \in \mathbb{R}^N$

$$\Rightarrow |f(x_0)| = \lim_{n \to \infty} \frac{1}{|C_n|} \int_{C_n} |f| \leq t$$

for a.e $x_0 \in \mathbb{R}^N \setminus (\cup Q_j)$

$$\Rightarrow |f| \leq t \quad \text{a.e. in } \mathbb{R}^N \setminus (\cup Q_j) \qquad (i)$$

Proof of the lemma: $f \in L^2(\Omega) \subset L^1(\Omega)$

extended by zero to $\mathbb{R}^N$, $f \in L^1(\mathbb{R}^N)$.

CZ decomposition. Given $t \geq 0$,

$$f = g + b$$

$g, b \in L^1(\mathbb{R}^N)$

$$|g| \leq \begin{cases} |f| & \text{on } {}^c(\cup Q_j) \\ \dfrac{1}{|Q_j|}\displaystyle\int_{Q_j} |f| & \text{on } Q_j \end{cases}$$

$$\Rightarrow \int_{\mathbb{R}^N} |g| \leq \int_{{}^c(\cup Q_j)} |f| + \sum_j \cancel{|Q_j|} \frac{1}{\cancel{|Q_j|}} \int_{Q_j} |f|$$

$$= \int_{\mathbb{R}^N} |f|$$

$$\int_{\mathbb{R}^N} |b| = \int_{\mathbb{R}^N} |f - g| \leq \int_{\mathbb{R}^N} |f| + |g| \leq 2 \int_{\mathbb{R}^N} |f|$$

Since $f \in L^2(\mathbb{R}^N) \Rightarrow g, b \in L^2(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} |g|^2 \leq \int_{\mathbb{R}^N} |f|^2 \quad \text{(Jensen or Cauchy-Schwarz)}$$

$$f = g + b$$

$T : L^2(\Omega) \to \mathcal{D}'(\Omega)$ linear

$$\Rightarrow \quad T(f) = T(g) + T(b)$$

$$\{|T(f)| > t\} \subset \{|T(g)| > \tfrac{t}{2}\} \cup \{|T(b)| > \tfrac{t}{2}\}$$

$$\Rightarrow \quad |\{|T(f)| > t\}| \leq |\{|T(g)| > \tfrac{t}{2}\}| + |\{|T(b)| > \tfrac{t}{2}\}| \qquad (0)$$

Estimate of the good part:

Chebychev's inequality:

$$|\{|T(g)| > \tfrac{t}{2}\}| \leq \left(\frac{2}{t}\right)^2 \int_\Omega |T(g)|^2$$

$$\underset{\substack{(p=2)\\(\text{step }1)}}{\leq} \left(\frac{2}{t}\right)^2 \int_\Omega |g|^2$$

$$\leq \frac{2^{2+N}}{t^{\cancel{2}}} \cancel{t} \int_\Omega |g|$$

$$\leq \frac{C_N}{t} \int_\Omega |g|$$

$$\sup_{t \geq 0} t\, |\{|T(g)| > \tfrac{t}{2}\}| \leq C_N \int_\Omega |f| \qquad (1)$$

Estimate of the bad part:

$b = 0$ outside $\bigcup Q_j$ so

$$b = \sum_{k\in\mathbb{N}} b_k \quad (*) \qquad \text{where } b_k = b\,\mathbb{1}_{Q_k}$$

$f \in L^2(\mathbb{R}^N) \Longrightarrow b \in L^2(\mathbb{R}^N)$

$\Longrightarrow$ the series $(*)$ is converging in $L^2(\mathbb{R}^N)$ (by dominated convergence).

Since (Step ①) $T : L^2(\Omega) \to L^2(\Omega)$ is continuous linear,

then $T(b) = \sum_{k\in\mathbb{N}} T(b_k)$

$$T(b_k) = \partial^2_{ij}(G * b_k)$$

$$G * b_k(x) = \int_{Q_k} G(x-y)\, b_k(y)\,dy$$

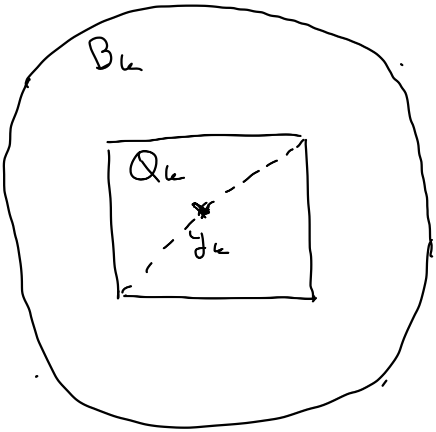
$Q_k = y_k + (-l_k, l_k)^N$, y_k is the center of the cube Q_k

$$\delta_k = \operatorname{diam}(Q_k) = \sqrt{N}\,(2l_k)$$

$$B_k = B_{\delta_k}(y_k) \supset Q_k$$

$x \notin B_k$, $y \in Q_k$

$|x-y| \geq \frac{\delta_k}{2} > 0$



$\Rightarrow$ If $x \notin B_k$, $y \in Q_k \longmapsto G(x-y) \in C^\infty(Q_k)$

$$\Rightarrow T(b_k)(x) = \partial^2_{ij}(G * b_k)(x)$$

$$= \int_{Q_k} \partial^2_{ij} G(x-y)\, b_k(y)\, dy \quad , x \notin B_k$$

$0 = \int_{Q_k} b = \int_{Q_k} b_k$ (by the CZ decomposition)

$\forall x \notin B_k$

$$T(b_k)(x) = \int_{Q_k} \left[\partial^2_{ij} G(x-y) - \partial^2_{ij} G(x-y_k) \right] b_k(y)\, dy$$

By the mean value thm, $\exists\ \hat{y}_k \in [y, y_k]$ s.t.

$$|\partial^2_{ij} G(x-y) - \partial^2_{ij} G(x-y_k)|$$

$$\leq |\nabla \partial^2_{ij} G(x-\hat{y}_k)|\,|y-y_k| \leq \frac{C_N}{|x-\hat{y}_k|^{N+1}} |y-y_k|$$

because:

$$G \sim \frac{1}{|x|^{N-2}}, \quad \partial^2_{ij} G \sim \frac{1}{|x|^N}, \quad |\nabla \partial^2_{ij} G| \sim \frac{1}{|x|^{N+1}}$$

Q_k convex, $y, y_k \in Q_k \Rightarrow \hat{y}_k \in [y, y_k] \subset Q_k$

$$\Rightarrow \operatorname{dist}(x, Q_k) \leq |x - \hat{y}_k|$$

So

$$|T(b_k)(x)| \leq C_N \int_{Q_k} \frac{\overbrace{|y-y_k|}^{\leq \delta_k}}{\operatorname{dist}(x, Q_k)^{N+1}} |b_k(y)|\,dy$$

$\forall x \notin B_k$

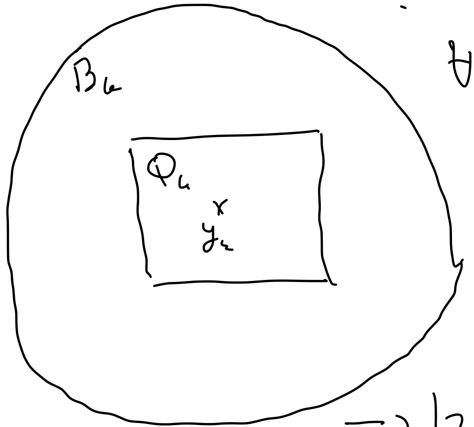
$$\leq C_N \delta_k \int_{Q_k} \frac{|b_k(y)|}{\operatorname{dist}(x, Q_k)^{N+1}}\,dy$$

$$\Rightarrow \int_{{}^cB_k} |T(b_k)|\,dx \leq C_N \delta_k \int_{{}^cB_k} \left(\int_{Q_k} \frac{|b_k(y)|}{\operatorname{dist}(x, Q_k)^{N+1}}\,dy \right) dx$$

$$\underset{\text{Fubini}}{=} C_N \delta_k \int_{Q_k} \left[\int_{{}^cB_k} \frac{dx}{\operatorname{dist}(x, Q_k)^{N+1}} \right] |b_k(y)|\,dy$$

$B_k = B_{\delta_k}(y_k)$

? $\mathrm{dist}(x, Q_k) \geq \frac{\delta_k}{2}$?



$\forall z \in Q_k$,

$|x-z| \geq |x-y_k| - |y_k - z|$

$x \notin B_k \Rightarrow |x-y_k| \geq \delta_k$

$z \in Q_k \Rightarrow |z-y_k| \leq \frac{\delta_k}{2}$

$\Rightarrow |z-y_k| \leq \frac{|x-y_k|}{2}$

So $|x-z| \geq \frac{|x-y_k|}{2} \quad \forall z \in Q_k$

$\Rightarrow \mathrm{dist}(x, Q_k) \geq \frac{|x-y_k|}{2}$

$$\int_{{}^cB_k} \frac{dx}{\mathrm{dist}(x,Q_k)^{N+1}} \leq 2^{N+1} \int_{{}^cB_k} \frac{dx}{|x-y_k|^{N+1}} =$$

$$= 2^{N+1} \left(\int_{\delta_k}^{+\infty} \frac{r^{N-1}\,dr}{r^{N+1}} \right) N\omega_N$$

polar coordinates

$$= C_N \int_{\delta_k}^{\infty} \frac{dr}{r^2} = \frac{C_N}{\delta_k}$$

So $\displaystyle\int_{{}^cB_k} |T(b_k)| \leq C_N \int_{Q_k} |b_k|$

$$\mathcal{U} = \bigcup_k B_k \quad , \quad F = \Omega \setminus \mathcal{U}$$

$$\int_F |T(b)| = \int_F \Big| \sum_k T(b_k) \Big| \leq \int_F \sum_k |T(b_k)|$$

$$\leq \sum_k \int_F |T(b_k)| \leq \sum_k \int_{{}^cB_k} |T(b_k)|$$

$$\leq C_N \sum_k \int_{Q_k} |b_k| = C_N \int_{\mathbb{R}^N} |b|$$

$$\leq C_N \int_\Omega |f|$$

$$|\{|T(b)| > \tfrac{t}{2}\}| = |\{|T(b)| > \tfrac{t}{2}\} \cap \mathcal{U}|$$

$$+ |\{|T(b)| > \tfrac{t}{2}\} \cap F|$$

$$\leq |\mathcal{U}| + \frac{2}{t} \int_F |T(b)|$$

Chebyshev

$$\leq \sum_k |B_k| + \frac{C_N}{t} \int_\Omega |f|$$

$$|B_k| = |B_{\delta_k}(y_k)| = \omega_N \delta_k^N$$

Set $\delta_k = \mathrm{diam}(Q_k)$, $Q_k = y_k + [-l_k, l_k]^N$

$= \sqrt{N}(2 l_k)$

$$|B_k| = \omega_N \delta_k^N = \omega_N N^{N/2} (2 l_k)^N$$

$$= C_N |Q_k|$$

$$\sum_k |B_k| \le C_N \sum_k |Q_k| \underset{\substack{(iii)\\ \text{of CZ})}}{\le} \frac{C_N}{t} \int_\Omega |f|$$

So we get that

$$\left| \left\{ |T(b)| > \frac{t}{2} \right\} \right| \le \frac{C_N}{t} \int_\Omega |f|$$

$$\Longrightarrow \sup_{t>0} t \left| \left\{ |T(b)| > \frac{t}{2} \right\} \right| \le C_N \|f\|_{L^1} \quad (2)$$

Put together (0), (1) and (2),

$$\underbrace{\sup_{t>0} t \left| \{ |T(f)| \ge t \} \right|}_{\|T(f)\|_{L^{1,\infty}(\Omega)}} \le C_N \|f\|_{L^1(\Omega)}$$

$\#$

Step ③ (Interpolation for $1 < p < 2$)

Thm (Marcinkiewicz interpolation)

$\Omega \subset \mathbb{R}^N$ open

$1 \leq q < r < \infty$

$T : L^q(\Omega) \cap L^r(\Omega) \longrightarrow L^q(\Omega) \cap L^r(\Omega)$ linear

Assume that $\exists K_1, K_2 > 0$

$$\|T(f)\|_{L^{q,\infty}(\Omega)} \leq K_1 \|f\|_{L^q(\Omega)}$$

$$\|T(f)\|_{L^{r,\infty}(\Omega)} \leq K_2 \|f\|_{L^r(\Omega)}$$

$$\forall f \in L^q(\Omega) \cap L^r(\Omega)$$

Then T extends to a continuous and linear map from $L^p(\Omega) \to L^p(\Omega)$

$$\forall \; q < p < r.$$

Application of this result:

$\Omega \subset \mathbb{R}^N$ bdd, open.

$q = 1$, $p = 2$ $\quad L^1(\Omega) \cap L^2(\Omega) = L^2(\Omega)$.

$T(f) = \partial_{ij}^2 (G * f)$, $\underset{(\text{Step } ①)}{T} : L^2(\Omega) \longrightarrow L^2(\Omega)$ linear

Step ③ : $\|T(f)\|_{L^{1,\infty}(\Omega)} \leq C_N \|f\|_{L^1(\Omega)}$

Step ① $\|T(f)\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}$

Chebychev :

$$|\{|T(f)|>t\}| \leq \frac{1}{t^2}\int_\Omega |T(f)|^2 \leq \frac{1}{t^2}\int_\Omega |f|^2$$

$$\Rightarrow \sup_{t>0} t^2 |\{|T(f)|>t\}| \leq \int_\Omega |f|^2$$

$\forall f \in L^2(\Omega)$

$$\Rightarrow \|T(f)\|_{L^{2,\infty}(\Omega)} \leq \|f\|_{L^2(\Omega)}$$

Marcinkiewicz : $\forall\, 1<p<2$,

$T : L^p(\Omega) \to L^p(\Omega)$ linear continuous

$\exists\, C>0$ s.t. $\forall f \in L^p(\Omega)$

$$\|\partial^2_{ij}(G*f)\|_{L^p(\Omega)} = \|T(f)\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}$$

Step ④. (Duality $2 < p < \infty$)

$f, g \in C_c^\infty(\Omega)$

$2 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ $q \in]1, 2[$

$$\int_\Omega T(f)\, g\, dx = \int_\Omega \partial^2_{ij} (G * f)(x)\, g(x)\, dx$$

$$= \int_\Omega (G * f)(x)\, \partial^2_{ij} g(x)\, dx$$

$$= \int_\Omega f(x) \left(G * \partial^2_{ij} g \right)(x)\, dx$$

$$= \int_\Omega f\, \partial^2_{ij} (G * g)\, dx = \int_\Omega f\, T(g)\, dx$$

Hölder:

$$\left| \int_\Omega T(f)\, g \right| = \left| \int_\Omega f\, T(g) \right|$$

$$\leq \|f\|_{L^p(\Omega)} \|T(g)\|_{L^q(\Omega)}$$

$$\underset{\text{Step 3}}{\leq} C_N \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}$$

$$\underbrace{\sup_{\substack{g \in \cancel{L^q} C_c^\infty \\ g \neq 0}} \frac{\left|\int_\Omega T(f) g\right|}{\|g\|_{L^q(\Omega)}}}_{\substack{\| \\ \|T(f)\|_{L^p(\Omega)}}} \leq C_N \|f\|_{L^p(\Omega)}$$

$$\|T(f)\|_{L^p(\Omega)} \leq C_N \|f\|_{L^p(\Omega)}$$

$T : C_c^\infty(\Omega) \to L^p(\Omega)$ linear continuous and $C_c^\infty(\Omega)$ dense in $L^p(\Omega)$

$\Rightarrow T \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$: $\exists\, C > 0$ s.t.

$$\|T(f)\|_{L^p(\Omega)} \leq C \|f\|_{L^p(\Omega)}$$

$$\|$$

$$\|\partial^2_{ij}(\Gamma * f)\|_{L^p(\Omega)}$$

Proof of Marcinkiewicz Thm:

Lemma: $f \in L^p(\Omega)$

$$\int_\Omega |f|^p \, dx = p \int_0^\infty t^{p-1} |\{|f| > t\}| \, dt$$

$1 \le q < r < \infty$, $t > 0$ $\quad (q < p < r)$

$f \in L^q \cap L^r$, $f = f_1 + f_2$

$$f_1(x) = \begin{cases} f(x) & \text{if } |f(x)| > t \\ 0 & \text{if } |f(x)| \le t \end{cases} \in L^q \cap L^r$$

$$f_2(x) = \begin{cases} 0 & \text{if } |f(x)| > t \\ f(x) & \text{if } |f(x)| \le t \end{cases} \in L^q \cap L^r.$$

$T : L^q \cap L^r \to L^q \cap L^r$ linear

$T(f) = T(f_1) + T(f_2)$

$\Rightarrow |T(f)| \le |T(f_1)| + |T(f_2)|$

$\{|T(f)| > t\} \subset \{|T(f_1)| > \frac{t}{2}\} \cup \{|T(f_2)| > \frac{t}{2}\}$

$\Rightarrow |\{|T(f)| > t\}| \le |\{|T(f_1)| > \frac{t}{2}\}| + |\{|T(f_2)| > \frac{t}{2}\}|$

$$T : L^q \longrightarrow L^{q,\infty}$$
$$T : L^r \longrightarrow L^{r,\infty}$$

$$|\{|T(f_1)| > \tfrac{t}{2}\}| \leq \left(\frac{2}{t}\right)^q K_1^q \, \|f_1\|_{L^q}^q$$
$$|\{|T(f_2)| \geq \tfrac{t}{2}\}| \leq \left(\frac{2}{t}\right)^r K_2^r \, \|f_2\|_{L^r}^r$$

$$\int_\Omega |T(f)|^p \, dx = p \int_0^\infty t^{p-1} \, |\{|T(f)| \geq t\}| \, dt$$
$$\leq p(2K_1)^q \int_0^\infty t^{p-1-q} \left(\int_{|f| \geq t} |f(x)|^q dx \right) dt$$
$$+ p(2K_2)^r \int_0^\infty t^{p-1-r} \left(\int_{|f| \leq t} |f(x)|^r dx \right) dt$$

Fubini

$$= p(2K_1)^q \int_\Omega |f(x)|^q \left(\int_0^{|f(x)|} t^{p-1-q} \, dt \right) dx \qquad = \frac{1}{p-q} |f(x)|^{p-q}$$
$$+ p(2K_2)^r \int_\Omega |f(x)|^r \left(\int_{|f(x)|}^\infty t^{p-1-r} \, dt \right) dx \qquad = \frac{1}{p-r} \left(-|f(x)|^{p-r} \right)$$

$$= \underbrace{\frac{p(2K_1)^q}{p-q}}_{>0} \int_\Omega |f(x)|^p dx + \underbrace{\frac{p(2K_2)^r}{r-p}}_{>0} \int_\Omega |f(x)|^p dx$$

So $$\int_\Omega |T(f)|^p \le \underbrace{\left[\frac{p(2K_1)^q}{p-q} + \frac{p(2K_2)^r}{r-p}\right]}_{C=C(p,q,r,K_1,K_2)} \int_\Omega |f|^p$$

$T: L^q \cap L^r \longrightarrow L^q \cap L^r$ is linear and continuous in L^p. Since $L^q \cap L^r$ is dense in L^p, T extends by continuity to

$$T \in \mathcal{L}(L^p, L^p).$$

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