Université Paris-Saclay 2025-2026

Linear and nonlinear elliptic partial differential equations
Final exam

In the sequel, we assume that $\Omega \subset \mathbb{R}^N$ is a bounded open set with $N \geq 2$.

Exercise 1. Let $A = (a_{ij})_{1\leq i,j\leq N}$ be a matrix whose coefficients satisfy $a_{ij} \in L^\infty(\Omega)$ for all $1 \leq i, j \leq N$ as well as the ellipticity condition : there exists $\lambda > 0$ such that

$$A(x)\xi \cdot \xi \geq \lambda|\xi|^2 \quad \text{for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^N.$$

Let us also consider $h = (h_1, \dots, h_N)$ with $h_i \in L^2(\Omega)$ for all $1 \leq i \leq N$.

1. Show that there exists a unique $u \in H^1_0(\Omega)$ satisfying

$$\int_\Omega A\nabla u \cdot \nabla v \, dx = \int_\Omega h \cdot \nabla v \, dx \quad \text{for all } v \in H^1_0(\Omega). \tag{1}$$

2. Show that (1) is equivalent to

$$\int_\Omega A\nabla u \cdot \nabla v \, dx = \int_\Omega h \cdot \nabla v \, dx \quad \text{for all } v \in \mathcal{C}^\infty_c(\Omega). \tag{2}$$

3. We next intend to show that, if $h_i \in L^q(\Omega)$ for all $1 \leq i \leq N$ with $q > N$, then $u \in L^\infty(\Omega)$, and

$$\|u\|_{L^\infty(\Omega)} \leq C\|h\|_{L^q(\Omega)},$$

where $C > 0$ only depends on N, q and Ω.

(a) Let $k \geq 0$. We already know that $(u-k)_+ = \max(u-k, 0) \in H^1(\Omega)$ and $\nabla(u-k)_+ = \mathbf{1}_{\{u\geq k\}}\nabla u$. Show that, actually, $(u-k)_+ \in H^1_0(\Omega)$.

(b) Show that

$$\|\nabla(u-k)_+\|_{L^2(\Omega)} \leq \frac{|\Omega_k|^{\frac{1}{2}-\frac{1}{q}}}{\lambda}\|h\|_{L^q(\Omega)},$$

where $\Omega_k = \{x \in \Omega : u(x) \geq k\}$.

(c) We recall a version of *Sobolev's imbedding* which states that if $N \geq 2$, then there exists a constant $C_* > 0$ such that

$$\|w\|_{L^r(\Omega)} \leq C_*\|\nabla w\|_{L^2(\Omega)} \quad \text{for all } w \in H^1_0(\Omega),$$

where $r = 2N/(N-2)$ if $N > 2$, while $r \in [2, +\infty[$ is arbitrary if $N = 2$. Setting $1/p+1/q = 1$, show that

$$\|(u-k)_+\|_{L^1(\Omega)} \leq \frac{C_*|\Omega_k|^{\frac{1}{2}-\frac{1}{r}+\frac{1}{p}}}{\lambda}\|h\|_{L^q(\Omega)}.$$

(d) Using Fubini's Theorem, show that

$$\|(u-k)_+\|_{L^1(\Omega)} = \int_k^\infty |\Omega_s| \, ds.$$

and deduce that

$$H(k) \leq \frac{C_*\|h\|_{L^q(\Omega)}}{\lambda}[-H'(k)]^{\frac{1}{2}-\frac{1}{r}+\frac{1}{p}}, \tag{3}$$

where $H(k) := \int_k^\infty |\Omega_s| \, ds$.

(e) Let $k_0 = \sup\{k \geq 0 : H(k) > 0\}$. Show that

$$k_0 \leq \frac{\beta^\gamma H(0)^{1-\gamma}}{1-\gamma},$$

where $\beta = C_*\|h\|_{L^q(\Omega)}/\lambda$ and $1/\gamma = 1/2 + 1/p - 1/r$.

(f) Deduce that $k_0 \leq C\|h\|_{L^q(\Omega)}$ and show that

$$u \leq C\|h\|_{L^q(\Omega)} \quad \text{a.e. in } \Omega$$

where $C = C(N, p, \Omega) > 0$.

(g) Conclude that

$$\|u\|_{L^\infty(\Omega)} \leq C\|h\|_{L^q(\Omega)}.$$

Exercise 2. Let $A = (a_{ij})_{1\leq i,j\leq N}$ be a matrix whose coefficients satisfy $a_{ij} \in L^\infty(\Omega)$ for all $1 \leq i, j \leq N$ as well as the ellipticity condition : there exists $\lambda > 0$ such that

$$A(x)\xi \cdot \xi \geq \lambda|\xi|^2 \quad \text{for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^N.$$

The goal of this exercise is to show the existence of weak solutions to the PDE

$$\begin{cases} -\mathrm{div}(A\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

when the source term f only belongs to $L^1(\Omega)$.

1. Assume first that $f \in L^2(\Omega)$ and let $u \in H_0^1(\Omega)$ be such that

$$\int_\Omega A\nabla u \cdot \nabla\varphi \, dx = \int_\Omega f\varphi \, dx \quad \text{for all } \varphi \in H_0^1(\Omega).$$

Let $1 \leq p < N/(N-1)$ and $q = p/(p-1) > N$ its conjugate exponent. For $h = (h_1, \dots, h_N)$ with $h_i \in L^q(\Omega)$ for all $1 \leq i \leq N$, let $v_h \in H_0^1(\Omega)$ be the unique solution of

$$\int_\Omega A^T\nabla v_h \cdot \nabla\varphi \, dx = \int_\Omega h \cdot \nabla\varphi \, dx \quad \text{for all } \varphi \in H_0^1(\Omega),$$

where A^T is the transpose of the matrix A.

(a) Show that

$$\int_\Omega A\nabla u \cdot \nabla v_h \, dx = \int_\Omega f v_h \, dx = \int_\Omega h \cdot \nabla u \, dx.$$

(b) Using the conclusion of exercise 1, deduce that

$$\int_\Omega h \cdot \nabla u \, dx \leq C\|f\|_{L^1(\Omega)}\|h\|_{L^q(\Omega)},$$

where $C > 0$ only depends on Ω, N and p.

(c) Deduce that

$$\|\nabla u\|_{L^p(\Omega)} \leq C\|f\|_{L^1(\Omega)}.$$

2. We now discuss the existence of solutions $u \in W_0^{1,p}(\Omega)$, with $1 \leq p < N/(N-1)$, to

$$-\operatorname{div}(A\nabla u) = f \quad \text{in } \mathcal{D}'(\Omega). \tag{4}$$

(a) For $f \in L^1(\Omega)$, let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}_c^\infty(\Omega)$ such that $f_n \to f$ in $L^1(\Omega)$. For each $n \in \mathbb{N}$, let us $u_n \in H_0^1(\Omega)$ be the unique weak solution of

$$-\operatorname{div}(A\nabla u_n) = f_n \text{ in } \mathcal{D}'(\Omega).$$

Show that for all $n \in \mathbb{N}$, then $u_n \in W_0^{1,p}(\Omega)$ for any $p < N/(N-1)$, and that if n and $m \in \mathbb{N}$, then

$$\|u_n - u_m\|_{W_0^{1,p}(\Omega)} \leq C\|f_n - f_m\|_{L^1(\Omega)}.$$

(b) Deduce that $u_n \to u$ in $W^{1,p}(\Omega)$, where $u \in W_0^{1,p}(\Omega)$ satisfies (4).

Exercise 3. (Quasi-linear equation) The goal of this problem is to show the existence of weak solutions to the following quasi-linear PDE

$$\begin{cases} -\operatorname{div} a(x, \nabla u) = f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

We will do the following assumptions :

(H_1) Ω is a bounded open set of $\mathbb{R}^N$;

(H_2) $f : \Omega \times (\mathbb{R} \times \mathbb{R}^N) \to \mathbb{R}$ is a Carathéodory function which satisfies

$$|f(x, s, \xi)| \leq c(1 + |s|^\beta + |\xi|^\beta) \quad \text{for a.e. } x \in \Omega \text{ and all } (s, \xi) \in \mathbb{R} \times \mathbb{R}^N,$$

where $c > 0$, $0 < \beta < p - 1$ and $1 < p < \infty$.

(H_3) $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a (vector valued) Carathéodory function which satisfies

(a) (coercivity) there exists $\lambda > 0$ such that

$$a(x, \xi) \cdot \xi \geq \lambda|\xi|^p \quad \text{for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^N ;$$

(b) (growth) there exists $\Lambda > 0$ such that

$$|a(x, \xi)| \leq \Lambda(1 + |\xi|^{p-1}) \quad \text{for a.e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^N ;$$

(c) (strong monotonicity) there exists $\alpha > 0$ such that

$$[a(x, \xi) - a(x, \xi')] \cdot (\xi - \xi') \geq \alpha|\xi - \xi'|^p \quad \text{for a.e. } x \in \Omega \text{ and all } \xi, \xi' \in \mathbb{R}^N.$$

1. Show that if $u \in W_0^{1,p}(\Omega)$, then $f(\cdot, u, \nabla u) \in L^q(\Omega)$ and $a(\cdot, \nabla u) \in L^q(\Omega; \mathbb{R}^N)$ where $\frac{1}{p} + \frac{1}{q} = 1$.
2. Show that if $(u_n)_{n\in\mathbb{N}}$ is a sequence in $W_0^{1,p}(\Omega)$ such that $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$, then $f(\cdot, u_n, \nabla u_n) \to f(\cdot, u, \nabla u)$ strongly in $L^q(\Omega)$ and $a(\cdot, \nabla u_n) \to a(\cdot, \nabla u)$ strongly in $L^q(\Omega; \mathbb{R}^N)$.
3. Let $T : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ be defined by $T(u) = v$, where $v \in W_0^{1,p}(\Omega)$ is the solution of

$$\int_\Omega a(x, \nabla v) \cdot \nabla w \, dx = \int_\Omega f(x, u, \nabla u) w \, dx \quad \text{for all } w \in W_0^{1,p}(\Omega).$$

Which results of course justifies the well-definiteness of the mapping T ?

4. Show that for all $u \in W_0^{1,p}(\Omega)$, we have

$$\|v\|_{W^{1,p}(\Omega)} \leq K\left(1 + \|u\|_{W^{1,p}(\Omega)}^{\frac{\beta}{p-1}}\right),$$

for some constant $K = K(\lambda, p, c, \Omega, \beta) > 0$.

5. Deduce the existence of $R = R(\lambda, p, c, \Omega, \beta) > 0$ such that $T(B_R) \subset B_R$, where B_R is the close ball of $W_0^{1,p}(\Omega)$ of center 0 and radius R.

6. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $W_0^{1,p}(\Omega)$ such that $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$, and let us denote by $v_n := T(u_n)$.

   (a) Show that there exist a subsequence $(v_{\sigma(n)})_{n\in\mathbb{N}}$, $v \in W_0^{1,p}(\Omega)$ and $\xi \in L^q(\Omega;\mathbb{R}^N)$ such that

$$\begin{cases} v_{\sigma(n)} \rightharpoonup v & \text{weakly in } W_0^{1,p}(\Omega), \\ a(\cdot, \nabla v_{\sigma(n)}) \rightharpoonup \xi & \text{weakly in } L^q(\Omega;\mathbb{R}^N) \end{cases}$$

   and

$$\int_\Omega \xi \cdot \nabla w \, dx = \int_\Omega f(x, u, \nabla u) w \, dx \quad \text{for all } w \in W_0^{1,p}(\Omega).$$

   (b) Show that $v_{\sigma(n)} \to v$ strongly in $W_0^{1,p}(\Omega)$.

   (c) Deduce that T is continuous.

7. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in B_R, and let us denote by $v_n := T(u_n)$.

   (a) Show that there exist subsequences $(u_{\sigma(n)})_{n\in\mathbb{N}}$, $(v_{\sigma(n)})_{n\in\mathbb{N}}$, u, $v \in W_0^{1,p}(\Omega)$, $\bar{f} \in L^q(\Omega)$ and $\xi \in L^q(\Omega;\mathbb{R}^N)$ such that

$$\begin{cases} u_{\sigma(n)} \rightharpoonup u & \text{weakly in } W_0^{1,p}(\Omega), \\ v_{\sigma(n)} \rightharpoonup v & \text{weakly in } W_0^{1,p}(\Omega), \\ f(\cdot, u_{\sigma(n)}, \nabla u_{\sigma(n)}) \rightharpoonup \bar{f} & \text{weakly in } L^q(\Omega), \\ a(\cdot, \nabla v_{\sigma(n)}) \rightharpoonup \xi & \text{weakly in } L^q(\Omega;\mathbb{R}^N) \end{cases}$$

   and

$$\int_\Omega \xi \cdot \nabla w \, dx = \int_\Omega \bar{f} w \, dx \quad \text{for all } w \in W_0^{1,p}(\Omega).$$

   (b) Show that $v_{\sigma(n)} \to v$ strongly in $W_0^{1,p}(\Omega)$.

   (c) Deduce that $T(B_R)$ is relatively compact in $W_0^{1,p}(\Omega)$.

8. Conclude.