# Topology

### Topological spaces

**1)** Let $(X, \mathcal{T})$ be a topological space. Show that $\mathcal{T} = \mathcal{P}(X)$ (we say that $\mathcal{T}$ is the *discrete topology*) if and only if every point is an open set.

**2)** Let $(X, \mathcal{T})$ be a topological space, and $A$ be a non empty subset of $X$. We define the *relative topology* on $A$ by

$$\mathcal{T}_A = \{U \cap A : U \in \mathcal{T}\}.$$

a) Prove that $\mathcal{T}_A$ is a topology on $A$.

b) Prove that if $A \in \mathcal{T}$, then $\mathcal{T}_A = \{U \in \mathcal{T} : U \subset A\}$. Show that it is not the case in general.

c) We denote by $\mathcal{C} = \{C : X \setminus C \in \mathcal{T}\}$ the family of all closed sets in $X$. Prove that the family of all closed subsets of $A$ is $\mathcal{C}_A = \{C \cap A : C \in \mathcal{C}\}$.

d) Prove that if $A \in \mathcal{C}$, then $\mathcal{C}_A = \{C \in \mathcal{C} : C \subset A\}$. Show that it is not the case in general.

**3)** Let $X$ be a set, and let $\mathcal{T}_1$, $\mathcal{T}_2$ be two topologies on $X$ such that for each $x \in X$, every neighborhood of $x$ for $\mathcal{T}_1$ is a neighborhood of $x$ for $\mathcal{T}_2$, and conversely. Show that $\mathcal{T}_1 = \mathcal{T}_2$.

**4)** Let $X$ be a topological space, $(x_n)_{n\in\mathbb{N}} \subset X$, and $x \in X$. We assume that $x$ is an accumulation point of every subsequence of $(x_n)_{n\in\mathbb{N}}$. Show that the full sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x$.

### Separated topological spaces

**5)** Show that in every separated topological space (also called *Hausdorff space*) every point is a closed set.

**6)** Let $(X, \mathcal{T})$ be a separated topological space, and let $A \subset X$. Show that $(A, \mathcal{T}_A)$ is a separated topological space.

**7)** Let $(X, \mathcal{T})$ be a topological space, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $X$. We assume that a subsequence $(x_n)_{n\in\mathbb{N}}$ converges to $x \in X$. Show that $x$ is an accumulation point of $(x_n)_{n\in\mathbb{N}}$.

### Compact (separated) topological spaces

**8)** A topological space $(X, \mathcal{T})$ is *discrete* if $\mathcal{T} = \mathcal{P}(X)$, *i.e.*, if all subsets of $X$ are open. Show that in a discrete topological space, all compact subsets are finite.

**9)** A family $(F_i)_{i\in I}$ in a topological space is said to have the *finite intersection property* if and only if all finite subfamily has a nonempty intersection. Show that a topological space is compact if and only if all family of closed sets $(F_i)_{i\in I}$ having the finite intersection property admits a common point (*i.e.*, $\bigcap F_i \neq \emptyset$).

**10)** Show that a compact set in a separated topological space is closed.

**11)** Let $(X, \mathcal{T})$ be a separated and compact topological space.

a) Show that every closed set is compact.

b) Let $x \in X$ and $C \subset X$ be a closed set such that $x \notin C$. Show that there exist two disjoint open sets $V_1$ and $V_2$ such that $x \in V_1$ et $C \subset V_2$.

c) Let $C_1$ and $C_2$ be two disjoint closed sets. Show that there exist two disjoint open sets $U_1$ and $U_2$ satisfying $C_i \subset U_i\,(i = 1, 2)$.

### Baire spaces

**12)** We recall that a topological space is a Baire space if all countable intersection of dense open sets is dense. Show that a topological space is a Baire space if and only if all countable union of closed sets with empty interior has empty interior.

### Metric spaces

**13)** Let $X$ be a set. We define the mapping $d : X \times X \to \mathbb{R}$ by $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ if $x \neq y$.

a) Show that $d$ is a distance.

b) Show that the topology $\mathcal{T}$ induced by $d$ is the discrete topology, *i.e.*, $\mathcal{T} = \mathcal{P}(X)$.

**14)** Let $(X, d)$ be a metric space, and let $A$ be a compact subset of $X$.

a) Show that $A$ is bounded and closed.

b) Show on a counterexample that the converse is wrong.

**15)** Let $(X, d)$ be a complete metric space, and $A \subset X$. Show that

$$(A, d) \text{ is complete if and only if } A \text{ is closed in } X.$$

**16)** Let $(X, d)$ be a metric space, and $A \subset X$ a compact set. Show that

$$B \subset A \text{ is compact if and only if } B \text{ is closed in } X.$$

**17)** Let $(X, d)$ be a metric space. For all $a \in X$ and $r \geq 0$, we define

$$B(a, r) := \{x \in X : d(x, a) < r\} \quad \text{(open ball of center } a \text{ and radius } r\text{)},$$

$$\overline{B}(a, r) := \{x \in X : d(x, a) \leq r\} \quad \text{(closed ball of center } a \text{ and radius } r\text{)},$$

Show that $\overline{B(a, r)} \subset \overline{B}(a, r)$ and that $B(a, r) \subset int(\overline{B}(a, r))$. Give an example of metric space where these inclusions are strict.

**18)** (*Hausdorff distance*) Let $(X, d)$ be a metric space, $x \in X$ and $A \subset X$ non empty. We define the distance between $x$ and $A$ by :

$$dist(x, A) := \inf_{a \in A} d(x, a).$$

a) Show that $x \mapsto dist(x, A)$ is 1-Lipschitz ;

b) Assume that $A$ is closed. Show that $x \in A$ if and only if $dist(x, A) = 0$.

c) Show that if $A$ is compact, there exists $a \in A$ such that $dist(x, A) = d(x, a)$.

d) Assume now that $X$ is compact, and let $A$ and $B$ be two closed subset of $X$. We define the Hausdorff distance between $A$ and $B$ by

$$d_{\mathcal{H}}(A, B) := \max \left\{ \sup_{x \in A} dist(x, B), \sup_{y \in B} dist(y, A) \right\}.$$

Show that $d_{\mathcal{H}}$ is a distance on the family of all closed subsets of $X$.

e) Show that $d_{\mathcal{H}}(A_n, A) \to 0$ if and only if the following properties hold :

- each $x \in A$ is the limit of a sequence $(x_n)_{n \in \mathbb{N}}$, with $x_n \in A_n$ for all $n \in \mathbb{N}$ ;
- if $x_n \in A_n$, any limit point of $(x_n)_{n \in \mathbb{N}}$ belongs to $A$.

### ~~Baire~~ Banach spaces

**19)** Let $E$ be a Banach space (*i.e.* a complete normed vector space) with infinite dimension. Show that the dimension of $E$ is uncountable. Deduce that there is no norm on the space of real polynomials $\mathbb{R}[X]$ which makes it into a Banach space.

**20)** Let $E$ be a Banach space, and $f : E \to E$ be a continuous linear map such that for all $x \in E$ there exists $n_x \in \mathbb{N}$ such that $f^{(n_x)}(x) = 0$. Show that $f$ is nilpotent, *i.e.*, there exists $N \in \mathbb{N}$ such that $f^{(N)} = 0$.

**21) Riesz Theorem.** *Let $E$ be a Banach space. Then the closed unit ball $B_1 := \{x \in E : \|x\|_E \leq 1\}$ is compact if and only if $E$ has finite dimension.*

a) Show that if $F$ is a closed subspace of $E$, then for any $\varepsilon > 0$ there exists $x_\varepsilon \in E$ such that

$$\|x_\varepsilon\|_E = 1, \qquad \text{and } \|x_\varepsilon - y\|_E \geq 1 \text{ for all } y \in F.$$

*Hint : Fix $x_0 \in E \setminus F$, and consider $d := dist(x_0, F)$. Then choose an element $x_1 \in F$ whose distance to $x_0$ is almost $d$.*

b) Show that if $B_1$ is compact, there exists a finite dimension space $F$ such that $B_1 \subset F \subset E$.

c) Conclude.

**Separability.**

**22)** Show that $\ell^\infty$ is not separable