Tempered distributions and Sobolev spaces

1) Topology of the Schwartz space. We recall that a function $f \in \mathcal{S}(\mathbb{R}^N)$ if for any multi-indexes α and $\beta \in \mathbb{N}^N$,

$$\sup_{x\in\mathbb{R}^N} |x^\alpha D^\beta f(x)| < +\infty.$$

For each $n \in \mathbb{N}$ and $f \in \mathcal{S}(\mathbb{R}^N)$, define

$$p_n(f) := \sup_{\alpha,\beta\in\mathbb{N}^N, |\alpha|\le n, |\beta|\le n} \sup_{x\in\mathbb{R}^N} |x^\alpha D^\beta f(x)|.$$

For all f and $g \in \mathcal{S}(\mathbb{R}^N)$, let

$$d(f,g) := \sum_{n\in\mathbb{N}} \frac{1}{2^n} \frac{p_n(f-g)}{1+p_n(f-g)}.$$

Show that d is a distance on $\mathcal{S}(\mathbb{R}^N)$ whose induced topology defines a complete metric space.

2) Compute the Fourier transform of the following tempered distributions :

a) δ_0.

b) 1.

c) δ_0', δ_0'', $\ldots$, $\delta_0^{(k)}$.

d) δ_a.

e) $x \mapsto e^{-2i\pi a x}$.

3) Let $\mu \in \mathcal{M}(\mathbb{R}^N)$ be a bounded Radon measure in $\mathbb{R}^N$. Show that its Fourier transform is given by the function

$$\mathcal{F}\mu(\xi) = \int_{\mathbb{R}^N} e^{-2i\pi\xi\cdot x}\, d\mu(x) \quad \text{for all } \xi \in \mathbb{R}^N.$$

4) Cauchy principal value. For all $x \in \mathbb{R}^*$, define $f(x) := \ln|x|$.

a) Show that $f \in L^1_{\mathrm{loc}}(\mathbb{R})$.

b) Show that the mapping

$$T : \varphi \in \mathcal{S}(\mathbb{R}) \mapsto \int_{\mathbb{R}} \varphi(x) \ln|x|\, dx \in \mathbb{R}$$

defines a tempered distribution.

c) Show that the derivative of T is the *principal value* of $1/x$, which is defined by the tempered distribution

$$\mathrm{pv}\left(\frac{1}{x}\right) : \varphi \in \mathcal{S}(\mathbb{R}) \mapsto \lim_{\varepsilon\to 0} \int_{\{|x|>\varepsilon\}} \frac{\varphi(x)}{x}\, dx.$$

5) Hilbert transform. For all $u \in \mathcal{S}(\mathbb{R})$, we denote by $Hu \in \mathcal{S}(\mathbb{R})$ the *Hilbert transform* of u defined by

$$Hu := \frac{1}{\pi} u * \mathrm{pv}\left(\frac{1}{x}\right).$$

a) Show that for every $x \in \mathbb{R}$,

$$(Hu)(x) = \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{\{|x-y|>\varepsilon\}} \frac{u(y)}{x-y}\, dy.$$

b) Show that its Fourier transform is given by

$$\widehat{Hu}(\xi) = -isign(\xi)\hat{u}(\xi) \quad \text{for all } \xi \in \mathbb{R}.$$

Deduce that for every $u \in \mathcal{S}(\mathbb{R})$,

$$\|Hu\|_{L^2(\mathbb{R})} = \|u\|_{L^2(\mathbb{R})}$$

c) Deduce that H extends to an isometry from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$, and that $H^2 = -I$, where I is the identity mapping over $L^2(\mathbb{R})$.

6) Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ be such that $\chi = 1$ in a neighborhood of 0.

a) Show that any function $\varphi \in \mathcal{S}(\mathbb{R})$ admits the following decomposition

$$\varphi(x) = \varphi(0)\chi(x) + x\psi(x) \quad \forall x \in \mathbb{R},$$

for some $\psi \in \mathcal{S}(\mathbb{R})$.

b) Solve $xT = 0$ in $\mathcal{S}'(\mathbb{R})$.

c) Solve $xT = 1$ in $\mathcal{S}'(\mathbb{R})$.

7) We recall that a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^N)$ belongs to the *Sobolev space* $H^s(\mathbb{R}^N)$ (with $s \in \mathbb{R}$) if and only if $\hat{u} \in L^2_{\text{loc}}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi < \infty.$$

Show that $\delta_0 \in H^s(\mathbb{R}^N)$ if and only if $s < -N/2$.

8) The aim of this exercise is to show that for each $s \geq 0$, $(H^s(\mathbb{R}^N))'$ can be identified to $H^{-s}(\mathbb{R}^N)$. More precisely, for each $T \in (H^s(\mathbb{R}^N))'$ there exists a unique $u \in H^{-s}(\mathbb{R}^N)$ such that

$$\langle T, v \rangle_{(H^s(\mathbb{R}^N))', H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \hat{u}\hat{v} \, d\xi \quad \text{for every } v \in H^s(\mathbb{R}^N),$$

and

$$\|T\|_{(H^s(\mathbb{R}^N))'} = \|u\|_{H^{-s}(\mathbb{R}^N)}.$$

- Show that if $u \in H^{-s}(\mathbb{R}^N)$ and $v \in H^s(\mathbb{R}^N)$, then $\hat{u}\hat{v} \in L^1(\mathbb{R}^N)$.
- Deduce that the mapping $v \mapsto \int_{\mathbb{R}^N} \hat{u}\hat{v} \, d\xi$ defines an element of $(H^s(\mathbb{R}^N))'$.
- Show that the Fourier transform $\mathcal{F}$ is an isometrical isomorphism from $H^s(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N, \mu)$, where μ is the absolutely continuous Radon measure $(1 + |\xi|^2)^s \mathcal{L}^N$.
- Let $T \in (H^s(\mathbb{R}^N))'$, and define $\tilde{T} := T \circ \mathcal{F}^{-1}$. Deduce that $\tilde{T}$ is a linear continuous map on $L^2(\mathbb{R}^N, \mu)$.
- Conclude.