UPMC
Master 1, MM05E

Basic functional analysis
2011-2012

Spaces of continuous functions

1) Let E and F be two metric spaces with E complete, and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions from E to F such that $f_n(y)$ converges to $f(y)$ for each $y \in E$.

a) Show that f is continuous on a dense G_δ set. We can consider the sets

$$E_{n,p} := \{y \in E : \ \forall\, q \geq p,\ d(f_p(y), f_q(y)) \leq 1/n\}$$

and

$$O_n := \bigcup_{p\in\mathbb{N}} \mathring{E}_{n,p}.$$

b) Deduce that if $f : \mathbb{R} \to \mathbb{R}$ is derivable, then it is of class $\mathcal{C}^1$ on a dense G_δ.

2) Let $\mathcal{C}([0,1])$, endowed with the uniform norm. Show that the subsets $A \subset \mathcal{C}([0,1])$ of all continuous functions nowhere derivable is a dense G_δ subset of $\mathcal{C}([0,1])$. We can consider the sets

$$A_n := \left\{ f \in \mathcal{C}([0,1]) : \ \exists\, y \in [0,1] \text{ with } \sup_{x\in[0,1]} \frac{|f(y)-f(x)|}{|y-x|} \leq n \right\}.$$

3) Urysohn Lemma. *Let Ω be an open subset of $\mathbb{R}^N$, K be a compact set, and V be an open set such that $K \subset V \subset \overline{V} \subset \Omega$. Then there exists a continuous function $f \in \mathcal{C}(\Omega)$ such that $0 \leq f \leq 1$ in Ω, $f = 1$ on K and $f = 0$ on $\Omega \setminus \overline{V}$.*

4) Partition of unity. *Let Ω be an open subset of $\mathbb{R}^N$, K be a compact set, and $V_1, \ldots, V_k$ be open sets satisfying $\overline{V_i} \subset \Omega$ for all $i = 1, \ldots, k$, and $K \subset \bigcup_{i=1}^k V_i$. Then there exist continuous functions $f_i \in \mathcal{C}(\Omega)$ such that $0 \leq f_i \leq 1$ on Ω, $f_i = 0$ on $\Omega \setminus \overline{V_i}$ for all $i = 1, \ldots, k$, and $\sum_{i=1}^k f_i = 1$ on K.*

5) Tietze extension theorem. *Let f be a bounded and continuous function from a closed set $C \subset \mathbb{R}^N$ into $\mathbb{R}$. Then there exists a continuous real valued function f^* defined on the whole $\mathbb{R}^N$ such that $f = f^*$ on C. Moreover, if*

$$|f(x)| \leq M \quad \text{for all } x \in C \text{ for some } M > 0,$$

then

$$|f^*(x)| \leq M \quad \text{for all } x \in \mathbb{R}^N.$$

a) Assume first that f is bounded on C by 1. Using Urysohn Lemma, show the existence of a continuous function $g_1 : \mathbb{R}^N \to [-1/3, 1/3]$ such that

$$g_1 = \frac{1}{3} \text{ on } \left\{ f \geq \frac{1}{3} \right\} \text{ and } g_1 = -\frac{1}{3} \text{ on } \left\{ f \leq -\frac{1}{3} \right\}.$$

b) By iteration, show the existence of a sequence of continuous functions $(g_n)_{n\geq 1}$ on $\mathbb{R}^N$ such that for all $n \geq 1$,

$$|g_n(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} \quad \text{for all } x \in \mathbb{R}^N,$$

and

$$\left| f(x) - \sum_{i=1}^n g_i(x) \right| \leq \left(\frac{2}{3}\right)^n \quad \text{for all } x \in C.$$

c) Conclude for the case where f is bounded on C by 1.

d) Deduce the case where f is bounded on C by some constant M.

6) Topology of the space of continuous functions. Let E be a subset of $\mathbb{R}^N$.

a) Define for all f and $g \in \mathcal{C}(E)$,

$$d(f,g) := \sup_{x\in E} |f(x) - g(x)|.$$

Show that if E is compact, then d defines a distance on $\mathcal{C}(E)$ which generates a topology of complete metric space.

b) Assume from now on that E is open. Construct a sequence of compact sets $(K_n)_{n\geq 1}$ such that $E = \bigcup_{n\geq 1} K_n$.

c) For every f and $g \in \mathcal{C}(E)$,

$$d_n(f,g) := \sup_{x\in K_n} |f(x) - g(x)|.$$

Show that d_n is not a distance on $\mathcal{C}(E)$ (d_n is called a semi-norm).

d) Show that

$$d(f,g) := \sum_{n\geq 1} \frac{1}{2^n} \frac{d_n(f,g)}{1 + d_n(f,g)}$$

defines a distance on $\mathcal{C}(E)$.

e) Show that $d(f_n, f) \to 0$ if and only if $f_n \to f$ uniformly on any compact subset of E.

f) Show that $\mathcal{C}(E)$ endowed with this distance is a complete metric space.

7) Ascoli Theorem. *Let (E, d_E) be a compact metric space, (F, d_F) be a complete metric space and $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $\mathcal{C}(E; F)$. Assume that*

– *$(f_n)_{n\in\mathbb{N}}$ is equi-continuous ;*

– *for all $x \in E$, the set $\overline{\{f_n(x) : n \in \mathbb{N}\}}$ is compact in F.*

Then $(f_n)_{n\in\mathbb{N}}$ admits a subsequence uniformly converging in E.

a) Show that E is separable, *i.e.*, there exists a countable dense subset $X := \{x_p\}_{p\in\mathbb{N}}$ in E.

b) Show the existence of a subsequence $(f_{\psi(n)})_{n\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ (where $\psi : \mathbb{N} \to \mathbb{N}$ is increasing) such that $f_{\psi(n)}(x_p) \to f(x_p)$ for all $p \in \mathbb{N}$, for some $f(x_p) \in F$. (*Hint : use Cantor's diagonalization procedure*).

c) Show that $f : X \to F$ is uniformly continuous and that it can be extended to a uniformly continuous function (still denoted f) from E to F.

d) Show that for all $x \in E$, $f_{\psi(n)}(x) \to f(x)$ in F.

e) Conclude.

8) Compactness of the Hausdorff distance. Let (X, d) be a compact metric space. For all closed subsets A and B of X, we recall that the Hausdorff distance between A and B is defined by

$$d_{\mathcal{H}}(A,B) := \max\left\{\sup_{x\in A} dist(x,B), \sup_{y\in B} dist(y,A)\right\}.$$

Show that if $(A_n)_{n\in\mathbb{N}}$ is a sequence of closed subsets of X, then there exists a subsequence converging to some closed set in the Hausdorff metric.

9) Show that the following sets are compact or not in $(\mathcal{C}([0,1];\mathbb{R}), d_{[0,1]})$:

a) $A = \{f : [0,1] \to \mathbb{R} : f$ continuous on $[0,1]$ and $\sup_{x\in[0,1]} |f(x)| \leq 1\}$.

b) $A = \{f : [0,1] \to \mathbb{R} : f$ polynom and $\sup_{x\in[0,1]} |f(x)| \leq 1\}$.

c) $A_N = \{f : [0,1] \to \mathbb{R} : f$ polynom of degree less than or equal to N and $\sup_{x\in[0,1]} |f(x)| \leq 1\}$.

d) $\overline{A}$, where $A = \{f : [0,1] \to \mathbb{R} : f$ derivable and $|f'(x)| \leq 1\ \forall x \in [0,1]\}$.

e) $\overline{A}$, where $A = \{f : [0,1] \to \mathbb{R} : f(1) = 2,\ f$ derivable and $|f'(x)| \leq 1\ \forall x \in [0,1]\}$.

f) $\overline{A}$, where $A = \left\{f : [0,1] \to \mathbb{R} : |f(0)| \leq 3 \text{ and } \dfrac{|f(x) - f(y)|}{|x-y|^{1/3}} \leq 5\ \forall x, y \in [0,1], x \neq y\right\}$.

g) $A = \left\{f : [0,1] \to \mathbb{R} : \dfrac{|f(x) - f(y)|}{|x-y|^2} \leq 4\ \forall x, y \in [0,1], x \neq y\right\}$.

h) $A = \left\{f : [0,1] \to \mathbb{R} : |f(1/2)| + \frac{|f(x)-f(y)|}{|x-y|^{3/2}} \leq 4\ \forall x, y \in [0,1], x \neq y\right\}$.