Integration and measure theory.

Riesz Representation Theorem. *Let Ω be an open subset of $\mathbb{R}^N$, and $\mathcal{C}_c(\Omega)$ be the space of all continuous functions having compact support in Ω. Let $L : \mathcal{C}_c(\Omega) \to \mathbb{R}$ be a positive linear functional, i.e.,*

$$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g) \quad \text{for all } f,\, g \in \mathcal{C}_c(\Omega) \text{ and all } \alpha,\, \beta \in \mathbb{R},$$
$$L(f) \geq 0 \quad \text{for all } f \in \mathcal{C}_c(\Omega) \text{ with } f \geq 0.$$

Then there exist a σ-algebra $\mathfrak{M}$ (containing the Borel σ-algebra $\mathcal{B}(\Omega)$) and a unique measure μ on $\mathfrak{M}$ such that

$$L(f) = \int_\Omega f\, d\mu, \tag{1}$$

for every $f \in \mathcal{C}_c(\Omega)$. Moreover, for each compact set $K \subset \Omega$, $\mu(K) < \infty$, and for every $E \in \mathfrak{M}$,

$$\mu(E) = \inf\{\mu(V) : E \subset V,\, V \text{ open}\}, \tag{2}$$

and every open set E, and every $E \in \mathfrak{M}$ with $\mu(E) < \infty$,

$$\mu(E) = \sup\{\mu(K) : K \subset E,\, K \text{ compact}\}. \tag{3}$$

1) Show the uniqueness : if μ_1 and μ_2 are two measures satisfying the conclusion of the theorem, show that $\mu_1 = \mu_2$ (*hint : use Urysohn's theorem*).

2) For every open set $V \subset \Omega$, we define

$$\mu(V) := \sup\{L(f) : f \in \mathcal{C}_c(\Omega),\ \|f\|_\infty \leq 1,\ \mathrm{supp}(f) \subset V\}. \tag{4}$$

Show that if $V_1 \subset V_2$, then $\mu(V_1) \leq \mu(V_2)$.

3) We extend μ to any arbitrary $E \subset \Omega$ by setting

$$\mu(E) := \inf\{\mu(V) : E \subset V,\, V \text{ open}\}. \tag{5}$$

Show that both definitions (4) and (5) are coincide on open sets, and that property (2) is satisfied.

4) Show that μ is an increasing set function : if $E_1 \subset E_2$, then $\mu(E_1) \leq \mu(E_2)$.

5) Let $\mathfrak{M}_F$ be the family of all sets $E \subset \Omega$ such that $\mu(E) < \infty$ and

$$\mu(E) = \sup\{\mu(K) : K \subset E,\, K \text{ compact}\}.$$

Finally, let $\mathfrak{M}$ be the class of all $E \subset \Omega$ such that $E \cap K \in \mathfrak{M}_F$ for any compact K. Show that if K is compact, then $K \in \mathfrak{M}_F$, and

$$\mu(K) = \inf\{L(f) : f \in \mathcal{C}_c(\Omega; [0,1]),\ f = 1 \text{ on } K\}.$$

6) Show that any open set V satisfies (3), and that if $\mu(V) < \infty$, then $V \in \mathfrak{M}_F$.

7) Show that μ is finitely subadditive on open sets : if V_1 and V_2 are open sets, $\mu(V_1 \cup V_2) \leq \mu(V_1) + \mu(V_2)$.

8) Show that μ is countably subadditive : for every sets $E_n \subset \Omega$ for $n \in \mathbb{N}^*$, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n).$$

9) Show that μ is finitely additive on compact sets : if K_1 and K_2 are disjoint compact sets, $\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2)$.

10) Show that if $E = \bigcup_{n=1}^{\infty} E_n$, where E_n are pairwise disjoint elements of $\mathfrak{M}_F$ for all $n \in \mathbb{N}^*$, then,

$$\mu(E) = \sum_{n=1}^{\infty} \mu(E_n).$$

Prove that if $\mu(E) < \infty$, then $E \in \mathfrak{M}_F$.

11) Show that if $E \in \mathfrak{M}_F$ and $\varepsilon > 0$, there is a compact set K and an open set V such that $K \subset E \subset V$ and $\mu(V \setminus K) < \varepsilon$.

12) Deduce that if A and $B \in \mathfrak{M}_F$, then $A \setminus B$, $A \cup B$ and $A \cap B \in \mathfrak{M}_F$.

13) Show that $\mathfrak{M}$ is a σ-algebra which contains all Borel sets and all sets E such that $\mu(E) = 0$. Deduce that $\mathfrak{M}_F = \{E \subset \mathfrak{M} : \mu(E) < \infty\}$.

14) Show that μ is a measure on $\mathfrak{M}$ satisfying (2) and (3).

15) Prove that in order to show the representation property (1) it is enough to check the inequality $L(f) \leq \int_\Omega f\, d\mu$ for any $f \in \mathcal{C}_c(\Omega)$.

16) Let $f \in \mathcal{C}_c(\Omega)$, $K := \operatorname{supp}(f)$ and $[a, b]$ be an interval which contains the image of f. For $\varepsilon > 0$, let $y_0, \ldots, y_n \in \mathbb{R}$ be such that $y_0 < a < y_1 < \ldots < y_n = b$, and $\max_{1 \leq i \leq n}(y_i - y_{i-1}) < \varepsilon$. Define

$$E_i := \{x \in \Omega : y_{i-1} < f(x) \leq y_i\} \cap K.$$

Show that E_i are disjoint Borel sets whose union is K.

17) Show that $L(f) \leq \int_\Omega f\, d\mu$ for any $f \in \mathcal{C}_c(\Omega)$.

Regularity of Radon measures

1) With the notations of the Riesz Representation Theorem, show that for every measurable set $E \in \mathfrak{M}$ and every $\varepsilon > 0$, there exist a closed set C and an open set V such that $C \subset E \subset V$ and $\mu(V \setminus C) < \varepsilon$.

2) Let λ be a Radon measure in an open set $\Omega \subset \mathbb{R}^N$.

a) Show that

$$\lambda(E) = \inf\{\lambda(V) : E \subset V,\ V \text{ open}\} \text{ for every Borel set } E \subset \Omega,$$

and

$$\lambda(E) = \sup\{\lambda(K) : K \subset E,\ K \text{ compact}\} \text{ for every Borel set } E \subset \Omega \text{ with } \lambda(E) < \infty.$$

b) Show that for every Borel set $E \subset \Omega$ and any $\varepsilon > 0$, there exist a closed set C and an open set V such that $C \subset E \subset V$ and

$$\lambda(V \setminus C) < \varepsilon.$$

Hint : Consider the positive linear form $L(\varphi) := \int_\Omega \varphi\, d\lambda$ *for all* $\varphi \in \mathcal{C}_c(\Omega)$.

Existence of the Lebesgue measure. *There exists a σ-algebra $\mathcal{L}(\mathbb{R}^N)$ (containing the Borel σ-algebra $\mathcal{B}(\mathbb{R}^N)$) and a unique measure $\mathcal{L}^N$ on $\mathcal{L}(\mathbb{R}^N)$ such that*

1. $\mathcal{L}^N([0,1]^N) = 1$ *;*

2. For any $x \in \mathbb{R}^N$ *and any* $E \in \mathcal{L}(\mathbb{R}^N)$*,* $\mathcal{L}^N(x+E) = \mathcal{L}^N(E)$*.*

The measure $\mathcal{L}^N$ *is called the Lebesgue measure, and* $\mathcal{L}(\mathbb{R}^N)$ *is the* σ*-algebra of all Lebesgue measurable sets.*

1) Show that there exist a σ-algebra $\mathcal{L}(\mathbb{R}^N)$ containing $\mathcal{B}(\mathbb{R}^N)$, and a measure $\mathcal{L}^N$ on $\mathcal{L}(\mathbb{R}^N)$ satisfying (2), (3) and such that

$$\int_{\mathbb{R}^N} f(x)\,dx = \int_{\mathbb{R}^N} f\,d\mathcal{L}^N,$$

where the first integral is the Riemann integral of f, and the second one is the Lebesgue integral of f with respect to the measure $\mathcal{L}^N$.

2) Show that $E \in \mathcal{L}(\mathbb{R}^N)$ if and only if there exist an F_σ set A (a countable union of closed sets) and a G_δ set B (a countable intersection of open sets) such that $A \subset E \subset B$ and $\mathcal{L}^N(B \setminus A) = 0$. Deduce that $\mathcal{L}(\mathbb{R}^N) = \{E \cup Z, \text{ with } E \in \mathcal{B}(\mathbb{R}^N) \text{ and } \mathcal{L}^N(Z) = 0\}$.

3) Let $a < b$ be real numbers. Construct a sequence of functions $\varphi_n \in \mathcal{C}_c(\mathbb{R})$ such that $\varphi_n = 1$ in $(a + 1/n, b - 1/n)$ and $\varphi_n = 0$ in $\mathbb{R} \setminus [a,b]$.

4) Show that if $a_i < b_i$ for all $i \in \{1,\dots,N\}$, then

$$\mathcal{L}^N\left(\prod_{i=1}^N (a_i,b_i)\right) = \prod_{i=1}^N (b_i - a_i) \le \mathcal{L}^N\left(\prod_{i=1}^N [a_i,b_i]\right).$$

5) Show that for any $i \in \{1,\dots,N\}$ and any $a \in \mathbb{R}$, $\mathcal{L}^N(\{x_i = a\}) = 0$. Deduce that

$$\mathcal{L}^N\left(\prod_{i=1}^N [a_i,b_i] \setminus \prod_{i=1}^N (a_i,b_i)\right) = 0$$

and that

$$\mathcal{L}^N\left(\prod_{i=1}^N (a_i,b_i)\right) = \prod_{i=1}^N (b_i - a_i) = \mathcal{L}^N\left(\prod_{i=1}^N [a_i,b_i]\right).$$

6) Show that for any $x \in \mathbb{R}^N$ and any open set $V \subset \mathbb{R}^N$, $\mathcal{L}^N(x+V) = \mathcal{L}^N(V)$.

7) Show that the Borel σ-algebra $\mathcal{B}(\mathbb{R}^N)$ is stable by translation : if $E \in \mathcal{B}(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$, then $x + E \in \mathcal{B}(\mathbb{R}^N)$. Deduce that $\mathcal{L}^N(x+E) = \mathcal{L}^N(E)$ for any $E \in \mathcal{B}(\mathbb{R}^N)$.

8) Using question 3, deduce that for any $x \in \mathbb{R}^N$ and any $E \in \mathcal{L}(\mathbb{R}^N)$, then $\mathcal{L}^N(x+E) = \mathcal{L}^N(E)$.

9) Show that any open set can be covered by countably many closed cubes with pairwise disjoint interior.

10) Show that if μ is a Radon measure invariant by translation, then there exists a constant $c > 0$ such that $\mu = c\mathcal{L}^N$. Deduce that if $\mu([0,1]^N) = 1$, then $\mu = \mathcal{L}^N$.