Lebesgue spaces

In the sequel, $(X, \mathfrak{M}, \mu)$ stands for a measure space.

1) Interpolation inequality. Let $1 \leq p \leq q \leq +\infty$.

a) Show that if $f \in L^p(X, \mu) \cap L^q(X, \mu)$, then $f \in L^r(X, \mu)$ for all $r \in [p, q]$, and that

$$\|f\|_r \leq \|f\|_p^\alpha \|f\|_q^{1-\alpha},$$

where $\alpha \in [0,1]$ is defined by $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$.

b) Show that if $\mu(X) < +\infty$ and $f \in L^p(X, \mu)$, then $f \in L^r(X, \mu)$ for all $r \in [1, p]$, and that there exists a constant $C > 0$ (independent of f) such that

$$\|f\|_r \leq C\|f\|_p.$$

2) Generalized Hölder inequality. Let $f_1 \in L^{p_1}(X, \mu), \dots, f_k \in L^{p_k}(X, \mu)$ be such that

$$\frac{1}{p_1} + \cdots + \frac{1}{p_k} =: \frac{1}{r} \leq 1.$$

Show that the product $\prod_{i=1}^k f_i$ belongs to $L^r(X, \mu)$ and

$$\left\| \prod_{i=1}^k f_i \right\|_r \leq \prod_{i=1}^k \|f_i\|_{p_i}.$$

3) Let $1 \leq p_0 < +\infty$.

a) Show that if $f \in L^{p_0}(X, \mu) \cap L^\infty(X, \mu)$, then

$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty.$$

b) Let $f \in L^p(X, \mu)$ for all $p \in [p_0, +\infty)$ such that $\|f\|_p \to \infty$ as $p \to \infty$. Show that $f \notin L^\infty(X, \mu)$.

c) Let $f \in L^p(X, \mu)$ for all $p \in [p_0, +\infty)$ such that $f \notin L^\infty(X, \mu)$. Show that $\|f\|_p \to \infty$ as $p \to \infty$.

4) Continuity of the translation in $L^p(\mathbb{R}^N)$. Let $X = \mathbb{R}^N$, $\mathfrak{M} = \mathcal{L}(\mathbb{R}^N)$ be the σ-algebra of all Lebesgue measurable subsets of $\mathbb{R}^N$, and $\mu = \mathcal{L}^N$ be the Lebesgue measure. Let $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}^N)$. For each $h \in \mathbb{R}^N$, we define the translation of f by

$$\tau_h f(x) := f(x - h) \quad \forall x \in \mathbb{R}^N.$$

Show that

$$\lim_{|h| \to 0} \|\tau_h f - f\|_p = 0.$$

5) Assume that μ is a probability measure, *i.e.*, $\mu(X) = 1$. Let $f : X \to [0, +\infty)$ be a function in $L^1(X, \mu)$.

a) Using Hölder's inequality, show that if $\mu(\{f > 0\}) < 1$, then $\|f\|_p \to 0$ as $p \to 0$.

b) Show that
$$\lim_{p\to 0} \int_X f^p \, d\mu = \mu(\{f > 0\}).$$

c) Show that for all $p \in (0, 1)$, and all $y \in (0, +\infty)$, then
$$\frac{|y^p - 1|}{p} \leq y + |\log y|.$$

d) From now on, we assume that $f > 0$ on X, and that $\log f \in L^1(X, \mu)$. Show that
$$\lim_{p\to 0} \int_X \frac{f^p - 1}{p} \, d\mu = \int_X \log f \, d\mu.$$

e) Show that
$$\lim_{p\to 0} \|f\|_p = \exp\left(\int_X \log f \, d\mu\right).$$

6) Jensen's inequality. Assume that μ is a probability measure, *i.e.*, $\mu(X) = 1$. Let $\varphi : (a, b) \to \mathbb{R}$ be a convex function (with $-\infty \leq a < b \leq +\infty$).

a) Show that
$$\frac{\varphi(t) - \varphi(s)}{t - s} \leq \frac{\varphi(u) - \varphi(t)}{u - t}$$
whenever $a < s < t < u < b$.

b) Deduce that φ is continuous, and that for each $s \in (a, b)$, there exists $\beta_s \in \mathbb{R}$ such that
$$\varphi(t) \geq \varphi(s) + \beta_s(t - s)$$
for every $t \in (a, b)$.

c) Let $f : X \to (a, b)$ such that $f \in L^1(X, \mu)$. Show that $\varphi \circ f$ is measurable and that
$$\varphi\left(\int_X f \, d\mu\right) \leq \int_X \varphi \circ f \, d\mu.$$

7) Let $1 \leq p < \infty$ and $p' = p/(p - 1)$. Show that for every $u \in L^p(X, \mu)$, then
$$\|u\|_p = \sup\left\{\int_X |uv| \, d\mu : v \in L^{p'}(X, \mu), \ \|v\|_{p'} \leq 1\right\}.$$

If $p = \infty$ and X is σ-finite, show that for every $u \in L^\infty(X, \mu)$, then
$$\|u\|_\infty = \sup\left\{\int_X |uv| \, d\mu : v \in L^1(X, \mu), \ \|v\|_1 \leq 1\right\}.$$

We recall that X is σ-finite is there exists an increasing sequence of measurable sets $(X_n)_{n\in\mathbb{N}} \subset \mathfrak{M}$ such that $X = \bigcup_{n\in\mathbb{N}}$ and $\mu(X_n) < \infty$ for each $n \in \mathbb{N}$.