Continuous linear maps

1) Let $(E, \|\cdot\|_E)$ be a normed linear space of dimension N. Show that any weakly converging sequence in E is also strongly converging.

2) Let $(X, \|\cdot\|)$ be a real normed linear space

a) Show that for all $x \in X$, there exists $f \in X'$ such that $f(x) = \|x\|$ and $\|f\|_{X'} = 1$. *Hint : Use the Hahn-Banach Theorem.*

b) Show that for all $x \in X$,

$$\|x\| = \max_{f \in X', \|f\| \le 1} f(x) = \max_{f \in X', f \neq 0} \frac{f(x)}{\|f\|_{X'}}.$$

c) Deduce that X is isometrically isomorphic to a subspace of its bidual $X'' := (X')'$ (*i.e.* there exists a subspace $\widetilde{X} \subset X''$ and a one to one linear continuous map $J : X \to \widetilde{X}$ such that $\|J(x)\|_{X''} = \|x\|$ for all $x \in X$). When is $\widetilde{X}$ closed ?

<u>Remark</u> : If $\widetilde{X} = X''$, we say that X is *reflexible.* In particular, uniformly convex Banach spaces or Hilbert spaces are reflexible.

3) Let $(X, \|\cdot\|)$ be a normed linear space.

a) show that if $x_n \to x$, then $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$.

b) Show that if $x_n \rightharpoonup x$, then

$$\|x\| \le \liminf_{n \to \infty} \|x_n\|$$

and that there exists a constant $C > 0$ such that $\|x_n\| \le C$ for all $n \in \mathbb{N}$.

c) Show that if X is uniformly convex, then

$$x_n \rightharpoonup x \text{ and } \|x_n\| \to \|x\| \Rightarrow x_n \to x.$$

4) Let $(X, \|\cdot\|)$ be a uniformly convex Banach space, and let $A \subset X$ be non empty closed and convex set.

a) Show that for all $x \in X$, there exists a unique $a_x \in A$ such that

$$\|x - a_x\| = \mathrm{dist}(x, A) := \inf_{a \in A} \|x - a\|.$$

b) Show that the map $x \mapsto a_x$ is continuous.

5) Open mapping Theorem. *Let E and F be two Banach spaces, and $\ell \in \mathcal{L}_c(E, F)$ be a surjective continuous map. Then ℓ is an open mapping, i.e., for every open set $U \subset E$, then $\ell(U)$ is <u>open</u> in F.*

a) We denote by B_E and B_F the open unit balls in E and F respectively. Show that $\overline{\ell(B_E)}$ has non empty interior in F (we can consider $X_n := n\,\overline{\ell(B_E)}$).

b) Deduce that there exists $r > 0$, such that $2rB_F \subset \overline{\ell(B_E)}$.

c) Let $y \in rB_F$, show that there exists $x_1 \in \frac{1}{2}B_E$ such that $y_1 := y - \ell(x_1) \in \frac{r}{2}B_F$. Construct then two sequences (x_n), (y_n) such that $x_n \in 2^{-n}B_E$ and $y_n = y_{n-1} - \ell(x_n) \in 2^{-n}rB_F$. Deduce that $y \in \ell(2B_E)$.

d) Show that

$$\exists r > 0,\ \forall y \in F, \|y\|_F < r, \exists x \in E, \|x\| < 2 \text{ and } \ell(x) = y. \qquad (\star)$$

Deduce that for each open set $U \subset E$, then $\ell(U)$ is open in F.

6) Banach Theorem. *Let E and F be two Banach spaces and $\ell \in \mathcal{L}_c(E, F)$ be a linear one to one continuous map. Show that $\ell^{-1} \in \mathcal{L}_c(F, E)$, and that $\|\ell^{-1}\|_{\mathcal{L}_c(F,E)} \geq 1/\|\ell\|_{\mathcal{L}_c(E,F)}$.*

7) Closed graph Theorem. *Let E and F be two Banach spaces, and T be a linear map from E to F. We suppose that the graph of T, $G(T) := \{(x, Tx) : x \in E\}$ is a closed subset of $E \times F$. Then T is continuous.*

8) Let F be a closed linear subspace of $\mathcal{C}([0,1])$ which is contained in $\mathcal{C}^1([0,1])$.

a) Show that the derivation map $D : f \in F \to f' \in \mathcal{C}([0,1])$ is continous.

b) Deduce that F has finite dimension.

9) Grothendieck Theorem. *Let $(X, \mathfrak{M})$ be a measure space and μ be a probability measure on $\mathfrak{M}$. Let S be a closed subspace of $L^p(X, \mu)$ $(p > 0)$, contained in $L^\infty(X, \mu)$. Then S has finite dimension.*

a) Show that there exists a constant $K < \infty$ such that for all $f \in S$, then $\|f\|_\infty \leq K\|f\|_p$.

b) Deduce that there exists a constant $M < \infty$ such that for all $f \in S$, then $\|f\|_\infty \leq M\|f\|_2$.

c) Show that for $c := (c_1, \ldots, c_n) \in \mathbb{R}^n$ with $\|c\|_2 \leq 1$ and for $\phi_1, \ldots, \phi_n$ orthonormal in S, then the function $f_c := \sum_{i=1}^n c_i \phi_i$ satisfies $\|f_c\|_\infty \leq M$.

d) Deduce that there exists $X' \subset X$ with $\mu(X') = 1$ such that for all $c := (c_1, \ldots, c_n) \in \mathbb{R}^n$ with $\|c\|_2 \leq 1$ and for all $x \in X'$, one has $|f_c(x)| \leq M$.

e) Conclude.