Duality in Lebesgue spaces and bounded Radon measures

1) Let Ω be an open subset of $\mathbb{R}^N$. Show that $f_n \rightharpoonup f$ weakly in $L^p(\Omega)$ for $1 < p < \infty$ (weakly* if $p = \infty$) if and only if

(i) there exists a constant $C > 0$ such that $\|f_n\|_p \leq C$ for all $n \in \mathbb{N}$;

(ii) $\displaystyle\int_\Omega f_n \varphi \, dx \to \int_\Omega f \varphi \, dx$ for all $\varphi \in \mathcal{C}_c^\infty(\Omega)$.

Why is it not true in general for $p = 1$?

2) Dirac mass. Let $\mathcal{C}_0(\mathbb{R}^N)$ be the space of all continuous functions vanishing at infinity endowed with the uniform norm

$$\|u\| := \max_{x \in \mathbb{R}^N} |u(x)|.$$

Let $\delta : \mathcal{C}_0(\mathbb{R}^N) \to \mathbb{R}$ be defined by

$$\delta(\varphi) = \varphi(0).$$

a) Show that δ is a bounded Radon measure, *i.e.* an element of the dual of $\mathcal{C}_0(\mathbb{R}^N)$.

b) Show that δ cannot be extended to an element of $(L^p(\mathbb{R}^N))'$, for $p \geqslant 1$.

3) Concentration-compactness.

(i) **Oscillation.** Let $u \in L^p(0,1)$ be such that $\int_0^1 u \, dx = 0$. We extended u to $\mathbb{R}$ by 1-periodicity, and we define the sequence

$$u_n(x) := u(nx) \text{ for a.e. } x \in \mathbb{R}.$$

Show that $u_n \rightharpoonup 0$ weakly in $L^p(0,1)$ (weakly* in $L^\infty(0,1)$) and that the convergence is not strong.

(ii) **Concentration.** We first assume that $1 < p < \infty$. Let $v \in \mathcal{C}_c^\infty(\mathbb{R})$ be such that $\int_{-\infty}^{+\infty} v \, dx = 1$, and define the sequence

$$v_n(x) := n^{1/p} v(nx) \text{ for all } x \in \mathbb{R}.$$

Show that $v_n \rightharpoonup 0$ weakly in $L^p(\mathbb{R})$ and that the convergence is not strong. Show that v_n do not converge weakly to 0 in $L^1(\mathbb{R})$ and that $v_n \rightharpoonup \delta_0$ weakly* in $\mathcal{M}(\mathbb{R})$.

(ii) **Evanescence.** Let v be as before and define the sequence

$$w_n(x) := v(n + x) \text{ for all } x \in \mathbb{R}.$$

Show that $w_n \rightharpoonup 0$ weakly in $L^p(\mathbb{R})$ (for $1 < p < \infty$), and weakly* in $L^\infty(\mathbb{R})$ (for $p = \infty$), and that the convergence is not strong. Show that w_n do not converge weakly to 0 in $L^1(\mathbb{R})$ and that $w_n \rightharpoonup 0$ weakly* in $\mathcal{M}(\mathbb{R})$.

4) Vitali-Hahn-Saks Theorem. *Let $(X, \mathfrak{M}, \mu)$ be a finite measure space ($\mu(X) < \infty$), and $(f_n)_{n \geq 1} \subset L^1(X, \mu)$ be a sequence such that the limit*

$$\lim_{n \to \infty} \int_E f_n \, d\mu$$

exists for every $E \in \mathfrak{M}$. Then for each $\varepsilon > 0$, there exists $\delta > 0$ such that for every $E \in \mathfrak{M}$ with $\mu(E) < \delta$, then

$$\sup_{n\geq 1} \int_E |f_n|\, d\mu < \varepsilon.$$

1. Show that $A := \{\chi_E : E \in \mathfrak{M}\}$ is a closed subset of $L^1(X,\mu)$. Deduce that A is a complete metric space.
2. Show that the sets
$$A_k := \left\{ \chi_E \in A : \sup_{n,l\geq k} \left| \int_X (f_n - f_l)\chi_E\, d\mu \right| \leq \frac{\varepsilon}{8} \right\}$$
are closed in A, and that $A = \bigcup_{k\geq 1} A_k$.
3. Applying Baire's Theorem, show that there exist $k_0 \geq 1$, $\delta' > 0$ and $\chi_{E_0} \in A_{k_0}$ such that if $\chi_E \in A$ satisfies
$$\int_X |\chi_E - \chi_{E_0}|\, d\mu < \delta',$$
then $\chi_E \in A_{k_0}$.
4. Show that there exists $0 < \delta \leq \delta'$ such that if $E \in \mathfrak{M}$ is such that $\mu(E) \leq \delta$, then
$$\sup_{1\leq n\leq k_0} \int_E |f_n|\, d\mu < \frac{\varepsilon}{4}.$$
5. Show that if $E \in \mathfrak{M}$ is such that $\mu(E) \leq \delta$, then $\chi_{E\cup E_0}$ and $\chi_{E_0\setminus E} \in A_{k_0}$.
6. Deduce that for all $n \geq k_0$,
$$\left| \int_E f_n\, d\mu \right| \leq \frac{\varepsilon}{2}.$$
7. Deduce that for all $n \geq 1$,
$$\int_E |f_n|\, d\mu \leq \varepsilon.$$

5) Dunford-Pettis Theorem. *Let $(X, \mathfrak{M}, \mu)$ be a finite measure space ($\mu(X) < \infty$), and $(f_n)_{n\geq 1} \subset L^1(X,\mu)$ be a sequence such that*

$$\sup_{n\geq 1} \|f_n\|_1 < +\infty.$$

(i) If $f_n \rightharpoonup f$ weakly in $L^1(X,\mu)$ for some $f \in L^1(X,\mu)$, then the sequence (f_n) is equi-integrable.
(ii) If (f_n) is equi-integrable, then there exist a subsequence $(f_{n_j}) \subset (f_n)$ and $f \in L^1(X,\mu)$ such that $f_{n_j} \rightharpoonup f$ weakly in $L^1(X,\mu)$.

1. Using the Vitali-Hahn-Saks Theorem, show that (i) holds.
2. The rest of the exercise consists in showing (ii).
   (a) Show that there is no loss of generality to assume that $f_n \geq 0$ for all $n \geq 1$.
   (b) Let $g_n^k := f_n \chi_{\{f_n \leq k\}}$ for all n, $k \geq 1$. Show that
   $$\sup_{n\geq 1} \|g_n^k - f_n\|_1 \to 0$$
   as $k \to \infty$.
   (c) Show that there exists a subsequence $(g_{n_j}^k)_{j\geq 1} \subset (g_n^k)_{n\geq 1}$ and $g^k \in L^1(X,\mu)$ such that $g_{n_j}^k \rightharpoonup g^k$ weakly in $L^1(X,\mu)$ for all $k \geq 1$.

(d) Show that $g^k \to f$ strongly in $L^1(X,\mu)$ for some $f \in L^1(X,\mu)$.
(e) Conclude that $f_{n_j} \rightharpoonup f$ weakly in $L^1(X,\mu)$.

6) Dual of $L^p(X,\mu)$ for $0 < p < 1$.
Let $(X, \mathfrak{M}, \mu)$ be a measure space.

1. For any $0 < p < 1$, let
$$d(f,g) = \int_X |f-g|^p \, d\mu.$$
Show that d is a distance on $L^p(X,\mu)$ and that $L^p(X,\mu)$ is a complete metric space.
2. Assume now that $X = [0,1]$, $\mathfrak{M} = \mathcal{L}([0,1])$ and $\mu = \mathcal{L}^1$. Let $V \subset L^p([0,1])$ be an open convex set containing 0, $r > 0$ such that $B_r(0) \subset V$, and $f \in L^p([0,1])$. Show that there exists $n \in \mathbb{N}$ such that $n^{p-1} d(f,0) < r$.
3. Show that there exist points $0 = x_0 < x_1 < \ldots < x_n = 1$ such that
$$\int_{x_{i-1}}^{x_i} |f(t)|^p \, dt = n^{-1} d(f,0).$$
4. Let $g_i := n f \chi_{(x_{i-1}, x_i]}$ for all $i \in \{1, \ldots, n\}$. Show that $g_i \in V$ and that
$$d(g_i, 0) = n^{p-1} d(f,0) < r.$$
5. Show that $f = \frac{1}{n} \sum_{i=1}^n g_i \in V$.
6. Deduce that $V = L^p([0,1])$.
7. Deduce that $(L^p([0,1]))' = \{0\}$.