Hilbert analysis

1) Let $(e_n)_{n\geq 1}$ be an Hilbertian basis in a separable Hilbert space H.

a) Show that $e_n \rightharpoonup 0$ as $n \to \infty$.

Let $(a_n)_{n\geq 1}$ be a bounded sequence of real numbers. We set

$$u_n = \frac{1}{n}\sum_{i=1}^{n} a_i e_i.$$

b) Show that $\|u_n\| \to 0$.

c) Show that $\sqrt{n}\, u_n \rightharpoonup 0$ weakly in H.

2) Let H be a real Hilbert space, and $K \subset H$ be a convex closed cone with vertex 0 (*i.e.* if $x \in K$ and $\lambda \geq 0$, then $\lambda x \in K$). Show that if $f \in H$, then its projection $u = P_K(f)$ is characterized by the properties

- $u \in K$
- $(f - u, u) = 0$
- for all $v \in K$, $(f - u, v) \leq 0$.

3) Let Ω be an open subset of $\mathbb{R}^N$ with finite (Lebesgue) measure, and $\varphi : \Omega \to \mathbb{R}$ be a nonnegative measurable function. We define

$$K = \left\{ u \in L^2(\Omega)\, ;\ |u(x)| \leq \varphi(x) \text{ for a.e. } x \in \Omega \right\}.$$

Show that K is a non empty convex and closed subset of $L^2(\Omega)$. Show that the orthonormal projection $P_K(u)$ of any element $u \in L^2(\Omega)$ is given by

$$P_K(u) = u\chi_{\{|u|\leq\varphi\}} + \varphi\chi_{\{u>\varphi\}} - \varphi\chi_{\{u<-\varphi\}}.$$

4) Let H be a real Hilbert space, and M be a non zero closed linear subspace of H. Show that for all $f \in H \setminus M^{\perp}$, the infimum

$$\mu = \inf_{\substack{u\in M\\ \|u\|=1}} (f, u)$$

is reached at a unique point. (*Hint : consider a minimizing sequence*)

5) All functions of this exercise are real valued. Let us consider the equation

$$\begin{cases} u''(t) = f(t) \text{ for all } t \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{1}$$

a) Show that if $f \in \mathcal{C}([0,1])$, then the solution of (1) is given by

$$u(t) = \int_0^1 K(t,s)\, f(s)\, ds \quad \forall t \in [0,1],$$

for some $K : [0,1] \times [0,1] \to \mathbb{R}$ to be determined.

b) Show that the operator $T : f \mapsto Tf$ defined by

$$(Tf)(t) = \int_0^1 K(t,s)\, f(s)\, ds$$

is well defined from $L^2(0,1)$ to $L^2(0,1)$, that it is linear, continuous, symmetric and compact.

c) Compute explicitly the eigenvalues of T.

d) For all $n \in \mathbb{N}^*$ and $t \in [0,1]$, let $e_n(t) = \sin(n\pi t)$. Deduce from b) and c) that the family

$$\left\{ \frac{e_n}{\|e_n\|} : n \in \mathbb{N}^* \right\}$$

is a Hilbertian basis of $L^2(0,1)$.

6) Let X and Y be two normed linear spaces. Let $\mathcal{K}(X,Y)$ be the space of all compact operators from X to Y.

1. Show that $\mathcal{K}(X,Y)$ is a linear subspace of $\mathcal{L}(X,Y)$.
2. Show that any operator $T \in \mathcal{L}(X,Y)$ with finite range (*i.e.* $\dim(T(X)) < \infty$) is compact.
3. Assume that Y is complete. Show that $\mathcal{K}(X,Y)$ is closed in $\mathcal{L}(X,Y)$.
4. Assume that Y is a Hilbert space. Show that any compact operator $T \in \mathcal{L}(X,Y)$ is the limit of a sequence of operators with finite range.

Hint : We recall that a metric space is compact if and only if it is complete and totally bounded.

7) Hilbert-Schmidt operators. Let $(X, \mathfrak{M}, \mu)$ be a finite measure space, and let $(X \times X, \mathfrak{M} \otimes \mathfrak{M}, \mu \otimes \mu)$ be the product measure space. Consider a Hilbert basis $(\phi_n)_{n \geq 1}$ of $L^2(X,\mu)$.

a) For every m, $n \geq 1$, and every $(x,y) \in X \times X$, define the function $\psi_{mn}(x,y) := \phi_m(x)\phi_n(y)$. Show that $(\psi_{mn})_{m,n\geq 1}$ is a Hilbert basis of $L^2(X \times X, \mu \otimes \mu)$. (*Hint : use Fubini's Theorem*).

b) Let $K \in L^2(X \times X, \mu \otimes \mu)$, and define the operator $T_K : L^2(X,\mu) \to L^2(X,\mu)$ by

$$T_K f(x) := \int_X K(x,y) f(y)\, d\mu(y) \qquad \text{for all } f \in L^2(X,\mu).$$

Show that $T_K(x)$ is well defined for a.e. $x \in X$, and that T_K is a linear continuous mapping from $L^2(X,\mu)$ into itself.

c) Show that T_K is compact (*Hint : one can show that it is the limit of a sequence of operators with finite range*).