Fourier series

1) Let $f : \mathbb{R} \to \mathbb{C}$ be a T-periodic function such that $f \in L^2(0, T)$.

a) Show that

$$\hat{f}(k) := \frac{1}{T} \int_0^T f(t)\, e^{-i2\pi kt/T}\, dt = \frac{1}{T} \int_a^{a+T} f(t)\, e^{-i2\pi kt/T}\, dt \quad \forall a \in \mathbb{R}, \forall k \in \mathbb{Z}.$$

b) Show that

$$f(t) = \sum_{k\in\mathbb{Z}} \hat{f}(k)\, e^{i2\pi kt/T} \quad \text{for a.e. } t \in \mathbb{R}.$$

c) Show that

$$\frac{1}{T} \int_0^T |f(t)|^2\, dt = \sum_{k\in\mathbb{Z}} |\hat{f}(k)|^2.$$

2) Let f be a T-periodic function such that $f \in L^2(0, T)$. Let

$$\hat{f}_T(k) := \frac{1}{T} \int_0^T f(t)\, e^{-i2\pi kt/T}\, dt \quad \text{and} \quad \hat{f}_{2T}(k) := \frac{1}{2T} \int_0^{2T} f(t)\, e^{-i\pi kt/T}\, dt$$

for all $k \in \mathbb{Z}$. Show that $\hat{f}_{2T}(2k) = \hat{f}_T(k)$, and $\hat{f}_{2T}(2k + 1) = 0$ for all $k \in \mathbb{Z}$.

3) Let g be the 2-periodic function whose restriction to $[0, 2)$ is $g(x) = (1 - x)\chi_{[0,1]}(x)$.

a) Expand g in Fourier series.

b) Find a 2-periodic function such that $f' + f = g$ in $(0, 2)$.

4) Let us define the functions $P_m : \mathbb{R} \to \mathbb{R}$ for all $m \in \mathbb{N}^*$ in the following way :

- P_1 is the 2π-periodic function such that $P_1(x) = \pi - x$ if $x \in]0, 2\pi]$.
- $\forall m \geq 1$, we define P_{m+1} in terms of P_m by

$$P_{m+1}(x) = C_{m+1} + \int_0^x P_m(t)\, dt,$$

where the constant C_{m+1} is such that the average of P_{m+1} over $(0, 2\pi)$ is zero.

a) Show that P_m is 2π-periodic and that $P_m \in L^2(0, 2\pi)$ for all $m \in \mathbb{N}^*$.

b) Expand P_m in Fourier series.

c) Deduce that

$$\sum_{n\in\mathbb{N}^*} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n\in\mathbb{N}^*} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

5) Let $\mathcal{C}_{\text{per}}([0, 2\pi])$ be the space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$, which are 2π-periodic. We want to show that for all $x \in [0, 2\pi)$, the space of all functions $f \in \mathcal{C}_{\text{per}}([0, 2\pi])$ whose Fourier series diverges at x, defines a G_δ dense subspace.

1. Prove the following refinement of the Banach-Steinhaus Theorem : Let $(T_i)_{i\in I}$ be an arbitrary family of continuous linear maps from a Banach space E to a normed linear space F. Then we have the following alternative :
   (a) $\sup_{i\in I} \|T_i\|_{\mathcal{L}(E,F)} < +\infty$,
   (b) For all y in a G_δ dense subset of E, $\sup_{i\in I} \|T_i(y)\|_F = +\infty$.

   Let us fix $x \in [0, 2\pi)$. For $f \in \mathcal{C}_{\text{per}}([0, 2\pi])$, we introduce the Fourier coefficients $c_n(f)$, and the partial Fourier sum :
   $$S_N(f)(x) := \sum_{-N\leqslant n\leqslant N} c_n e^{inx} = \frac{1}{2\pi}\int_0^{2\pi} f(t)D_N(x-t)dt,$$
   with
   $$D_N(x) := \sum_{-N\leqslant n\leqslant N} e^{inx}.$$
2. Show that $\|D_N\|_{L^1(0,2\pi)} \to +\infty$ when $N \to +\infty$.
3. Show that
   $$\|S_N(\cdot)(x)\|_{\mathcal{L}(\mathcal{C}_{\text{per}}([0,2\pi]),\mathbb{R})} = \frac{1}{2\pi}\|D_N\|_{L^1(0,2\pi)}.$$
4. Conclude. Show that on a G_δ dense subspace of $\mathcal{C}_{\text{per}}([0, 2\pi])$ the Fourier series diverges.
5. Show that the map $f \in L^1(0, 2\pi) \to (c_n(f))_{n\in\mathbb{Z}} \in c_0$, where c_0 is the space of all sequences converging to zero, is not surjective.