UPMC | Basic functional analysis
Master 1, MM05E | 2011-2012

Fourier transform of integrable and square integrable functions

1) Let $f \in L^1(\mathbb{R})$. The Fourier transform of f is defined on $\mathbb{R}$ by $\hat{f}(\xi) = \int_{-\infty}^{+\infty} f(x) e^{-2i\pi\xi x}\, dx$.

1. Let $f(x) = e^{-\pi x^2}$ for all $x \in \mathbb{R}$. Compute $\frac{d}{d\xi}\hat{f}(\xi)$. Deduce $\hat{f}(\xi)$.
2. Let $g(x) = e^{-|x|}$ for all $x \in \mathbb{R}$. Compute $\hat{g}$.
3. Deduce the Fourier transform of $\mathbb{R} \ni x \mapsto h(x) = \frac{1}{1+x^2}$.

2) a) Let χ be the characteristic function of the interval $[-\frac{1}{2}, \frac{1}{2}]$ and $\phi = \chi * \chi$. Compute ϕ, $\widehat{\phi}$ and $\widehat{\chi}$.
b) Find two functions f and $g \in L^2(\mathbb{R})$, non identically zero and such that $f * g = 0$.
c) Show that there is no function $u \in L^1(\mathbb{R})$ such that $f * u = f$, $\forall f \in L^1$.

3) Let $A \subset \mathbb{R}^n$ be a (Lebesgue) measurable set such that $0 < \mathcal{L}^n(A) < \infty$, and let χ_A be its characteristic function. Show that $\widehat{\chi}_A \in L^2(\mathbb{R}^n)$ but $\widehat{\chi}_A \notin L^1(\mathbb{R}^n)$.

4) We consider the differential equation

$$u'(x) + \lambda x u(x) = 0 \ \forall x \in \mathbb{R},$$

where $\lambda > 0$.

a) Find u such that $u(0) = 1$.
b) Transform the previous equation to get an equation solved by $\hat{u}$.
c) Deduce $\hat{u}$.

5) Compute the Fourier transform of the following functions :

1. $f_1(x) = (1 - |x|)\chi_{[-1,1]}(x)$.
2. $f_2(x) = x^n e^{-x} \chi_{[0,+\infty)}(x)$, $n \in \mathbb{N}$.
3. $f_3(x) = \frac{\sin(\pi x)}{\pi x}$.
4. $f_4(x) = \left(\frac{\sin(\pi x)}{\pi x}\right)^2$.
5. $f_5(x) = \frac{1}{(1+2i\pi x)^{n+1}}$, $n \in \mathbb{N}$.