Probabilistic Approach to a Class of Semilinear Partial Differential Equations

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Dedicated to Professor Haïm Brézis.

Abstract. We discuss the recent progress about positive solutions of the semilinear equation $\Delta u = u^p$ in a domain, which has involved a combination of probabilistic and analytic methods. We emphasize the main ideas that have been used in the probabilistic approach. Special attention is given to the boundary trace problem, which consists in obtaining a one-to-one correspondence between the set of all solutions and a suitable set of admissible traces on the boundary. A few important open questions are also listed.

1. Introduction

It has been known for a long time that properties of random systems of branching particles are related to solutions of certain semilinear partial differential equations. In the last 15 years, these connections have given rise to fruitful developments in the setting of the theory of measure-valued branching processes, also called superprocesses. A major step was accomplished by Dynkin [**7**], who provided a simple probabilistic representation of the solution of the Dirichlet problem for the equation $\Delta u = u^p$, $1 < p \leq 2$ in a domain of $\mathbb{R}^d$, in terms of the so-called exit measure of the associated superprocess (see Theorem 2.1 below). A very interesting feature of this representation, in contrast to other probabilistic approaches, is its robustness: A formula that is a priori only valid for solutions with a given continuous boundary value can be generalized, by means of various limiting procedures, to yield similar representations for many other solutions whose behavior at the boundary can be very singular. In fact, as will be explained in Section 6 below, a generalized version of the probabilistic formula applies to *any* nonnegative solution of $\Delta u = u^p$, $1 < p \leq 2$.

Our goal in this work is to give an account of these developments, including the important recent contributions of Dynkin and Kuznetsov from the probabilistic side, and Marcus and Véron from the analytic side. We made no attempt at exhaustivity,

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and for instance we do not discuss parabolic equations which can also be handled by the same probabilistic tools. Rather, we try to explain as simply as possible the basic probabilistic ideas and the way these ideas can lead to analytic results. For this reason, we often concentrate on the particular case $p = 2$, where the random process called the Brownian snake can be used in the probabilistic representation of solutions. The Brownian snake was introduced in [**22**], and its connections with equation $\Delta u = u^2$ were first discussed in [**24**] (see also the monograph [**27**]). The Brownian snake is a simpler object than superprocesses and is sometimes more tractable for analytic applications, even though most of the analytic results that have been obtained for $p = 2$ via the Brownian snake could then be extended to the case $1 < p \leq 2$ using superprocesses. We do not discuss analytic methods here. The reader who is interested in the analytic approach to the problems discussed below should look at Laurent Véron's recent paper [**40**].

Above all, we tried to emphasize the nice interplay between analytic and probabilistic concepts. Already in [**7**], Dynkin used the characterization of removable singularities from Baras and Pierre [**2**] to solve the important problem of the description of polar sets for super-Brownian motion (see Theorem 4.1 below). In the reverse direction, we give examples of theorems that were first proved for $p = 2$ via the probabilistic approach, and then extended to arbitrary $p > 1$ by analytic methods. See Theorem 3.3 and its generalization Theorem 3.4 by Labutin, or Theorem 6.1 and its generalization Theorem 6.2 by Marcus and Véron. Obviously, the probabilistic approach, which does not apply to the case $p > 2$, does not replace analytic methods. Still we believe that in some particular cases the probabilistic intuition can help guessing or even proving new analytic results, which can then be generalized.

Section 2 below gives a brief presentation of the Brownian snake and states the key Theorem 2.1, from which the different probabilistic representation formulas can be deduced. This section should provide sufficient background to understand the probabilistic ideas that are explained in the remainder of the paper. Analytic questions, namely solutions with boundary blow-up, removable singularities, solutions with measure boundary data, and the trace problem are discussed in Sections 3 to 6. We have emphasized the boundary trace problem, which has given rise to recent major advances by Dynkin, Kuznetsov, Mselati, Marcus and Véron.

2. A probabilistic tool: The Brownian snake

In this section, we give a brief presentation of the Brownian snake, which will be our main tool in the probabilistic analysis of semilinear partial differential equations. We refer to the monograph [**27**] for a more detailed presentation. At an informal level, our aim is to construct a "tree of Brownian paths" originating from a given point $x \in \mathbb{R}^d$. More precisely, we will construct a collection (W_s) of random paths, indexed by a real parameter s varying in some interval. For each fixed value of the parameter s, $W_s = (W_s(t), 0 \leq t \leq \zeta_s)$ is thus a finite path in $\mathbb{R}^d$ starting from x, with lifetime denoted by ζ_s. If $s \neq s'$, the paths W_s and $W_{s'}$ coincide over an interval of the form $[0, m(s,s')]$, where $m(s,s') \leq \zeta_s \wedge \zeta_{s'}$. In this sense, the collection (W_s) forms a "tree" of paths. Assuming that x belongs to a domain D, a key role in our applications is played by the set $\mathcal{E}^D$ of all exit points from D of the paths W_s (more precisely of those paths W_s that do exit D), and by the exit measure from D, which is a finite measure supported on $\mathcal{E}^D$.

Let us turn to more rigorous definitions. The Brownian snake is a Markov process taking values in the set of finite paths in $\mathbb{R}^d$. By definition, a finite path in $\mathbb{R}^d$ is a continuous mapping $\mathrm{w} : [0, \zeta] \to \mathbb{R}^d$. The number $\zeta = \zeta_{(\mathrm{w})} \geq 0$ is called the lifetime of the path. We denote by $\mathcal{W}$ the set of all finite paths in $\mathbb{R}^d$. This set is equipped with the distance

$$d(\mathrm{w}, \mathrm{w}') = |\zeta_{(\mathrm{w})} - \zeta_{(\mathrm{w}')}| + \sup_{t\geq 0} |\mathrm{w}(t \wedge \zeta_{(\mathrm{w})}) - \mathrm{w}'(t \wedge \zeta_{(\mathrm{w}')})|.$$

Let us fix $x \in \mathbb{R}^d$ and denote by $\mathcal{W}_x$ the set of all finite paths with initial point $\mathrm{w}(0) = x$. The Brownian snake with initial point x is the continuous strong Markov process $W = (W_s, s \geq 0)$ in $\mathcal{W}_x$ whose law is characterized as follows.

1. If $\zeta_s = \zeta_{(W_s)}$ denotes the lifetime of W_s, the process $(\zeta_s, s \geq 0)$ is a reflecting Brownian motion in $\mathbb{R}_+$.

2. Conditionally on $(\zeta_s, s \geq 0)$, the process W is a (time-inhomogeneous) Markov process. Its conditional transition kernels are described by the following properties: For $s < s'$,
- $W_{s'}(t) = W_s(t)$ for every $t \leq m(s, s') := \inf_{[s,s']} \zeta_r$;
- $(W_{s'}(m(s, s') + t) - W_{s'}(m(s, s')), 0 \leq t \leq \zeta_{s'} - m(s, s'))$ is a standard Brownian motion in $\mathbb{R}^d$ independent of W_s.

Informally, one should think of W_s as a Brownian path in $\mathbb{R}^d$ with a random lifetime ζ_s evolving like (reflecting) linear Brownian motion. When ζ_s decreases, the path W_s is "erased" from its tip. When ζ_s increases, the path W_s is extended (independently of the past) by adding "small pieces" of Brownian motion at its tip. From this informal explanation, it should be clear that the evolution of the Brownian snake generates a "tree of Brownian paths" in the sense that was explained at the beginning of this section.

Denote by $\underline{x}$ the trivial path in $\mathcal{W}_x$ with lifetime 0. It is immediate that $\underline{x}$ is a regular recurrent point for the Markov process W. We denote by $\mathbb{N}_x$ the associated excursion measure. Under $\mathbb{N}_x$ the law of W is described by properties analogous to **1.** and **2.**, with the only difference that the law of reflecting Brownian motion in **1.** is replaced by the (infinite) Itô measure of positive excursions of linear Brownian motion (see [**27**]). In other words, the "lifetime process" $(\zeta_s, s \geq 0)$ is under $\mathbb{N}_x$ a positive Brownian excursion: It starts from 0, comes back to 0 at a finite time $\eta > 0$ called the duration of the excursion (and then stays at 0 over the time interval $[\eta, +\infty)$), whereas between times 0 and η it takes positive values and behaves like linear Brownian motion. Knowing the lifetime process $(\zeta_s, s \geq 0)$, the behavior of the Brownian snake under $\mathbb{N}_x$ is given by property **2.**, as was informally described above.

From our definitions, it is clear that $W_s = \underline{x}$ for all $s \geq \eta$, $\mathbb{N}_x$ a.e. Thus under $\mathbb{N}_x$, we will only be interested in the paths W_s for $0 \leq s \leq \eta$. We can normalize $\mathbb{N}_x$ so that, for every $\varepsilon > 0$, $\mathbb{N}_x(\sup_{s\geq 0} \zeta_s > \varepsilon) = (2\varepsilon)^{-1}$. Although $\mathbb{N}_x$ is an infinite measure, we have for every $\delta > 0$

$$\mathbb{N}_x\Big(\sup_{s\geq 0,\ 0\leq t\leq \zeta_s} |W_s(t) - x| \geq \delta\Big) = c_d\, \delta^{-2} < \infty, \tag{2.1}$$

where c_d is a positive constant (see [**27**], Proposition V.9).

For every fixed $s \geq 0$, conditionally on ζ_s, W_s is distributed under $\mathbb{N}_x$ as a d-dimensional Brownian path started at x and stopped at time ζ_s. If $0 < s < s' < \eta$, the paths W_s and $W_{s'}$ coincide up to time $m(s, s') > 0$, by Property **2.**, and this again corresponds to our concept of a tree of paths. In what follows, we always consider the Brownian snake under its excursion measure $\mathbb{N}_x$, for some $x \in \mathbb{R}^d$.

Next, consider a domain D such that $x \in D$. For any finite path $\mathrm{w} \in \mathcal{W}_x$, set

$$\tau(\mathrm{w}) = \inf\{t \in [0, \zeta_{(\mathrm{w})}] : \mathrm{w}(t) \notin D\}$$

with the usual convention $\inf \emptyset = \infty$. An important role in this work is played by the set of exit points

$$\mathcal{E}^D = \{W_s(\tau(W_s)) : s \geq 0, \tau(W_s) < \infty\}.$$

Notice that $\mathcal{E}^D$ is a random closed subset of ∂D. The exit measure $\mathcal{Z}^D$ from D is a random finite measure supported on the set $\mathcal{E}^D$. This measure can be defined by the following approximation ([**27**], Chapter V):

$$\langle \mathcal{Z}^D, \varphi \rangle = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^\eta ds\, \varphi(W_s(\tau(W_s)))\, 1_{\{\tau(W_s) < \zeta_s < \tau(W_s) + \varepsilon\}}$$

for any continuous function φ on ∂D, $\mathbb{N}_x$ a.e.

The following theorem is the basic ingredient needed for connections between the Brownian snake and semilinear partial differential equations. We let D be a bounded domain in $\mathbb{R}^d$. We say that D is Dirichlet regular if for every continuous function g on ∂D, the classical Dirichlet problem for the Laplace equation $\Delta u = 0$ in D with boundary value g has a (unique) solution. In probabilistic terms, this is equivalent to saying that for a Brownian motion $(B_t, t \geq 0)$ started at any point $y \in \partial D$, the first exit time $\inf\{t > 0 : B_t \notin D\}$ is zero a.s.

Theorem 2.1. *Let D be a bounded domain in $\mathbb{R}^d$. Assume that D is Dirichlet regular and let g be a nonnegative continuous function on ∂D. For every $x \in D$, set*

$$u(x) = \mathbb{N}_x(1 - \exp - \langle \mathcal{Z}^D, g \rangle).$$

Then the function u is the unique nonnegative solution of the boundary value problem

$$\begin{cases} \Delta u = 4\, u^2 & \textit{in } D\,, \\ u_{|\partial D} = g\,. \end{cases} \tag{2.2}$$

Remark. The boundary value $u_{|\partial D} = g$ should be understood in the pointwise sense

$$\lim_{x \to y, x \in D} u(x) = g(y)$$

for every $y \in \partial D$. The uniqueness of the nonnegative solution of (2.2) is an easy application of the maximum principle for elliptic equations. The factor 4 in (2.2) is an unimportant scaling constant due to our specific normalization.

Theorem 2.1 is a reformulation in terms of the Brownian snake of a more general result due to Dynkin ([**7**], Theorem 1.1). Dynkin's result, which is formulated in the setting of the theory of superprocesses, applies to the equation $\Delta u = u^p$ when $1 < p \leq 2$. Here we chose to concentrate on the case $p = 2$, where the description of the relevant probabilistic objects is easier to understand. See [**27**], Theorem V.6, for a proof of Theorem 2.1 in the present setting. For $1 < p \leq 2$, and equation $\Delta u = 4\, u^2$ replaced by $\Delta u = u^p$, the statement of Theorem 2.1 remains valid if we use the so-called p-stable Lévy snake instead of the Brownian snake (see Proposition 4.5.1 in [**6**]). On the other hand, it seems that no such probabilistic tool can be used to handle equation $\Delta u = u^p$ when $p > 2$.

In order to prove Theorem 2.1, one establishes the equivalent integral equation

$$u(x) + 2\int_D dy\, G_D(x,y)\, u^2(y) = \int_{\partial D} K_D(x,dz)\, g(z), \quad x \in D \tag{2.3}$$

where G_D is the usual Green function (for $\frac{1}{2}\Delta$) in D and $K_D(x,dz)$ is the harmonic measure on ∂D relative to the point x. In probabilistic terms, this equation can be rewritten in the form

$$u(x) + 2\, E_x\Big[\int_0^\tau ds\, u^2(B_s)\Big] = E_x[g(B_\tau)], \quad x \in D \tag{2.4}$$

where $(B_t, t \geq 0)$ is a d-dimensional Brownian motion that starts from x under the probability measure P_x, and $\tau = \inf\{t \geq 0 : B_t \notin D\}$. A computational way of proving (2.4) is to expand the exponential in the formula $u(x) = \mathbb{N}_x(1 - \exp -\langle \mathcal{Z}^D, g\rangle)$, and then to use recursion formulas for the moments of $\langle \mathcal{Z}^D, g\rangle$, which follow from the tree structure of the Brownian snake paths.

Let us summarize the contents of this section. Under the measure $\mathbb{N}_x$, the paths $(W_s, 0 \leq s \leq \eta)$ form a tree of Brownian paths started from x, each individual path W_s having a finite lifetime ζ_s. The set $\mathcal{E}^D$ consists of all exit points from D of the paths W_s (for those that do exit D), and the exit measure $\mathcal{Z}^D$ is in a sense uniformly spread over $\mathcal{E}^D$. We will also use the range $\mathcal{R}$, which is defined by

$$\mathcal{R} := \{y = W_s(t); 0 \leq s \leq \eta, 0 \leq t \leq \zeta_s\}.$$

This is simply the union of the Brownian snake paths.

3. Solutions with boundary blow-up

According to Keller [**18**] and Osserman [**38**], if D is a bounded smooth domain and ψ is a function that satisfies an appropriate integral condition, there exists a nonnegative solution of equation $\Delta u = \psi(u)$ in D that blows up everywhere at the boundary. This holds in particular if $\psi(u) = u^p$ for some $p > 1$. This raises the following two questions:

(a) For which non-smooth domains does there exist a solution that blows up everywhere at the boundary ?

(b) Assuming that there exists a solution with boundary blow-up, is it unique ?

The probabilistic approach turns out to be rather efficient in providing answers to these questions. Let us start by reformulating in terms of the Brownian snake two key theorems again due to Dynkin [**7**].

Theorem 3.1. *Let D be a bounded domain. Assume that D is Dirichlet regular. Then $u_1(x) = \mathbb{N}_x(\mathcal{Z}^D \neq 0)$, $x \in D$ is the minimal nonnegative solution of the problem*

$$\begin{cases} \Delta u = 4u^2 & \text{in } D \\ u_{|\partial D} = +\infty. \end{cases} \tag{3.1}$$

The proof of this theorem is easy from Theorem 2.1. Simply consider for every $n \geq 1$ the function $v_n(x) = \mathbb{N}_x(1 - \exp -n\langle \mathcal{Z}^D, 1\rangle)$ that solves (2.2) with $g = n$. Clearly, $v_n \uparrow u_1$ as $n \uparrow \infty$, and it follows that u_1 also solves $\Delta u = 4u^2$ in D. Since $u_1 \geq v_n$ for every n, we have $u_{1|\partial D} = +\infty$. Finally, if u is any (nonnegative) solution of (3.1), the maximum principle implies that $u \geq v_n$ for every n and so $u \geq u_1$.

To state the second theorem, recall our notation $\mathcal{R}$ for the range of the Brownian snake.

THEOREM 3.2. *Let D be any open set in $\mathbb{R}^d$ and $u_2(x) = \mathbb{N}_x(\mathcal{R} \cap D^c \neq \emptyset)$ for $x \in D$. Then u_2 is the maximal nonnegative solution of $\Delta u = 4u^2$ in D (in the sense that $u \le u_2$ for any other nonnegative function u of class C^2 in D such that $\Delta u = 4u^2$ in D).*

Remark. By combining Theorem 3.2 and (2.1), we recover the classical a priori bound

$$u(x) \le u_2(x) \le c_d \operatorname{dist}(x, \partial D)^{-2}\ , \quad x \in D$$

which holds for any nonnegative solution of $\Delta u = 4\,u^2$ in D.

Again the proof of Theorem 3.2 is relatively easy from the preceding theorem. One can argue separately on each connected component of D, and thus assume that D is connected (notice that by construction the range $\mathcal{R}$ is also connected). It is then easy to construct an increasing sequence $(D_n)_{n\ge1}$ of bounded Dirichlet regular subdomains of D such that $\overline{D}_n \subset D_{n+1}$ for every n, and $D = \cup D_n$. By Theorem 3.1, $u_1^n(x) = \mathbb{N}_x(\mathcal{Z}^{D_n} \neq 0)$ is a solution in D_n with infinite boundary conditions. On the other hand, from the probabilistic formulas of Theorems 3.1 and 3.2, one can check that $u_1^n(x) \downarrow u_2(x)$ as $n \uparrow \infty$, for every $x \in D$. It follows that u_2 is also a solution of $\Delta u = 4u^2$ in D. Moreover any other nonnegative solution u is bounded above by u_1^n in D_n (by the maximum principle in D_n) and therefore is bounded above by u_2.

The previous theorems already shed some light on questions (a) and (b). From Theorem 3.1, a solution with boundary blow-up exists as soon as D is Dirichlet regular. Under this assumption, question (b) reduces to giving conditions ensuring that $u_1 = u_2$. In the case when D is not Dirichlet regular, it is easy to construct examples where $u_1 \neq u_2$. Let $B(x,r)$ denote the open ball with radius r centered at x. Then if $d = 2$ or 3 and if $D = B(0,1)\backslash\{0\}$ is the punctured unit ball (which is *not* Dirichlet regular), the function u_2 blows up near the origin, as a consequence of Theorem 3.3 below, whereas the function u_1 stays bounded near the origin, because the exit measure "does not see" the origin.

To state conditions ensuring that $u_1 = u_2$, assume that $d \ge 2$ (the case $d = 1$ is trivial) and denote by $\mathcal{C}_{d-2}(K)$ the Newtonian capacity (or the logarithmic capacity if $d = 2$) of a compact subset K of $\mathbb{R}^d$. According to Theorem IV.9 of [**27**], the answer to question (b) is positive under the following assumption: For every $y \in \partial D$, there exists a positive constant $c(y)$ such that the inequality

$$\mathcal{C}_{d-2}(D^c \cap \bar{B}(y, 2^{-n})) \ge c(y)\, \mathcal{C}_{d-2}(\bar{B}(y, 2^{-n})) \tag{3.2}$$

holds for all n belonging to a sequence of positive density in $\mathbb{N}$ (here $\bar{B}(x,r)$ is the closed ball of radius r centered at x). See also Marcus and Véron [**29**] for related results obtained by analytic methods for the more general equation $\Delta u = u^p$.

It is interesting to compare (3.2) with the classical Wiener test, which gives a necessary and sufficient condition for the Dirichlet regularity: The bounded domain D is Dirichlet regular if and only if for every $y \in \partial D$,

$$\sum_{n=1}^{\infty} \frac{\mathcal{C}_{d-2}(D^c \cap \bar{B}(y, 2^{-n}))}{\mathcal{C}_{d-2}(\bar{B}(y, 2^{-n}))} = \infty. \tag{3.3}$$

Clearly, assumption (3.2) is stronger than (3.3). However, it is very plausible that (3.2) is not the best possible assumption, and this leads to the following question.

Open problem. *Is the solution with boundary blow-up unique in the case of a general Dirichlet regular domain ?*

Let us discuss question (a). Here Theorem 3.2 immediately tells us that for a general open set D in $\mathbb{R}^d$, the existence of a nonnegative solution of $\Delta u = u^2$ in D that blows up everywhere at the boundary of D is equivalent to the property

$$\lim_{x\to y, x\in D} \mathbb{N}_x(\mathcal{R}\cap D^c \neq \emptyset) = +\infty \tag{3.4}$$

for every $y \in \partial D$. It is not hard to see that this condition holds if and only if for every $y \in \partial D$,

$$\mathbb{N}_y(W_s(t) \notin D \text{ for some } s \geq 0 \text{ and } t \in (0, \zeta_s]) = +\infty.$$

In this form, (3.2) is quite similar to the probabilistic version of the characterization of the Dirichlet regularity, with the difference that a single Brownian path started from y is replaced by a tree of Brownian paths started from the same point.

In order to state the next result, we need to introduce some notation. If $a \geq 0$ and K is a compact subset of $\mathbb{R}^d$, we define the capacity $\mathcal{C}_a(K)$ by setting

$$\mathcal{C}_a(K) = \Big(\inf_{\nu\in\mathcal{M}_1(K)} \int\!\!\int \nu(dy)\nu(dz) f_a(|y-z|)\Big)^{-1}$$

wehere $\mathcal{M}_1(K)$ is the set of all probability measures on K, and

$$f_a(r) = \begin{cases} 1 + \log^+ \frac{1}{r} & \text{if } a = 0\ , \\ r^a & \text{if } a > 0\ . \end{cases}$$

Theorem 3.3. [**4**] *Let D be a domain in $\mathbb{R}^d$. Then the following two properties are equivalent.*

(i) *The problem*

$$\begin{cases} \Delta u = u^2 & \text{in } D \\ u_{|\partial D} = +\infty \end{cases}$$

has a nonnegative solution.

(ii) *Either $d \leq 3$, or $d \geq 4$ and for every $y \in \partial D$,*

$$\sum_{n=1}^{\infty} 2^{n(d-2)}\, \mathcal{C}_{d-4}(D^c \cap \bar{B}(y, 2^{-n})) = +\infty.$$

This theorem, which was proved in [**4**] by probabilistic methods involving the Brownian snake, thus gives a complete answer to question (a) above.

Theorem 3.3 was generalized a few years later by Labutin [**21**] using purely analytic methods. To state Labutin's result, and in view of further statements, we introduce Bessel capacities in $\mathbb{R}^d$. For every $\gamma > 0$, consider the classical Bessel kernel

$$G_\gamma^{(d)}(x) = a_\gamma \int_0^\infty t^{\frac{\gamma-d}{2}} \exp\Big(-\frac{\pi|x|^2}{2} - \frac{t}{4\pi}\Big)\,\frac{dt}{t} \tag{3.5}$$

where $a_\gamma = (4\pi)^{-\gamma/2}\Gamma(\gamma/2)^{-1}$. For any compact subset K of $\mathbb{R}^d$, and every $p > 1$, we then set

$$C_{\gamma,p}(K) = \sup_{\mu\in\mathcal{M}_1(K)} \Big(\int_{\mathbb{R}^d} dx \Big(\int \mu(dy)\, G_\gamma^{(d)}(y-x)\Big)^{p'}\Big)^{-p/p'} \tag{3.6}$$

where $\frac{1}{p}+\frac{1}{p'}=1$ as usual. The capacity $C_{\gamma,p}$ can also be viewed as the capacity associated with the Sobolov space $W^{\gamma,p}$: See Theorem 2.2.7 in [**1**].

THEOREM 3.4. [**21**] *Let D be a bounded domain in $\mathbb{R}^d$, $d\geq 3$ and $p>1$. Let p' be defined by $\frac{1}{p}+\frac{1}{p'}=1$. Then the following two properties are equivalent.*

(i) *The problem*

$$\begin{cases} \Delta u = u^p & \text{in } D \\ u_{|\partial D} = +\infty \end{cases}$$

has a nonnegative solution.

(ii) *Either $p<\frac{d}{d-2}$, or $p\geq\frac{d}{d-2}$ and for every $y\in\partial D$,*

$$\sum_{n=1}^{\infty} 2^{n(d-2)}\, C_{2,p'}(D^c\cap\bar{B}(y,2^{-n})) = +\infty.$$

In the case $p=2$, we recover the preceding result. Indeed, a few lines of calculations show that, if $d\geq 4$, there exist two positive constants a_1 and a_2 such that, for every compact subset K of the unit ball,

$$a_1\, C_{2,2}(K) \leq \mathcal{C}_{d-4}(K) \leq a_2\, C_{2,2}(K). \tag{3.7}$$

4. Removable singularities

Let K be a compact subset of $\mathbb{R}^d$. We say that K is an interior removable singularity for $\Delta u = u^2$ if the only nonnegative solution of $\Delta u = u^2$ in $\mathbb{R}^d\backslash K$ is the function identically equal to 0. This turns out to be equivalent to saying that for any open set O containing K, any nonnegative solution on $O\backslash K$ can be extended to a solution on O.

From the probabilistic point of view, interior removable singularities correspond to (interior) polar sets. The compact set K is said to be polar if for every $x\in\mathbb{R}^d\backslash K$,

$$\mathbb{N}_x(\mathcal{R}\cap K\neq\emptyset)=0.$$

In other words, the compact set K will never be hit by the tree of Brownian paths which is the range of the Brownian snake.

THEOREM 4.1. *Let $d\geq 4$ and let K be a compact subset of $\mathbb{R}^d$. The following are equivalent.*

(i) *K is an interior removable singularity for $\Delta u = u^2$.*
(ii) *K is polar.*
(iii) *$\mathcal{C}_{d-4}(K)=0$.*

In dimension $d\leq 3$, the equivalence (i)$\Leftrightarrow$(ii) also holds trivially, since (i) or (ii) can only be true if K is empty.

The equivalence (i)$\Leftrightarrow$(ii) is an immediate consequence of Theorem 3.2 above applied with $D=K^c$. The equivalence (i)$\Leftrightarrow$(iii) was obtained by Baras and Pierre [**2**]: More generally, Baras and Pierre have shown that K is a removable singularity for $\Delta u = u^p$ if and only if $C_{2,p'}(K)=0$ (see also [**3**] for an earlier discussion of removable singularities for semilinear equations).

From the probabilistic viewpoint, it is worthwile to look for a direct proof of the equivalence (ii)$\Leftrightarrow$(iii). A simple argument gives the implication (ii)$\Rightarrow$(iii) (this implication was first obtained, independently of [**2**], by Perkins [**39**], and later the connection with [**2**] was made by Dynkin [**7**]). Indeed, suppose that $\mathcal{C}_{d-4}(K) > 0$, and so that there is a probability measure ν supported on K such that

$$\iint \nu(dy)\nu(dz) f_{d-4}(|y-z|) < \infty \tag{4.1}$$

where $f_{d-4}(r)$ is as above in the definition of $\mathcal{C}_a(K)$. Let h be a radial nonnegative continuous function on $\mathbb{R}^d$ with compact support contained in the unit ball, and for every $\varepsilon \in (0,1]$, set $h_\varepsilon(x) = \varepsilon^{-d} h(x/\varepsilon)$. Finally let $\mathcal{I}$ be the "total occupation" measure of the Brownian snake defined by

$$\langle \mathcal{I}, g \rangle = \int_0^\eta ds\, g(\widehat{W}_s)$$

where $\widehat{W}_s = W_s(\zeta_s)$ is the terminal point of the finite path W_s, and we recall that η is the duration of the excursion under $\mathbb{N}_x$. Notice that by construction $\mathcal{I}$ is supported on $\mathcal{R}$. If $x \in \mathbb{R}^d \backslash K$ is fixed, explicit moment calculations using (4.1) give the bounds

$$\mathbb{N}_x(\langle \mathcal{I}, h_\varepsilon * \nu \rangle) \geq c_1 > 0 \quad , \quad \mathbb{N}_x(\langle \mathcal{I}, h_\varepsilon * \nu \rangle^2) \leq c_2 < \infty \, ,$$

where the constants c_1 and c_2 do not depend on $\varepsilon \in (0,1]$ (see Chapter VI in [**27**] for details). Let K_ε denote the closed tubular neighborhood of radius ε of the set K. From an application of the Cauchy-Schwarz inequality, it follows that

$$\mathbb{N}_x(\mathcal{R} \cap K_\varepsilon \neq \emptyset) \geq \mathbb{N}_x(\langle \mathcal{I}, h_\varepsilon * \nu \rangle > 0) \geq \frac{(\mathbb{N}_x(\langle \mathcal{I}, h_\varepsilon * \nu \rangle))^2}{\mathbb{N}_x(\langle \mathcal{I}, h_\varepsilon * \nu \rangle^2)} \geq \frac{c_1^2}{c_2}.$$

By letting ε go to 0, it follows that $\mathbb{N}_x(\mathcal{R} \cap K \neq \emptyset) > 0$, and thus K is not polar.

In view of the simplicity of the preceding argument, one would expect that similar probabilistic proof should also give the converse implication (iii)$\Rightarrow$(ii). Surprisingly this is not the case, and the only known way to obtain this implication is via Baras and Pierre's result (i)$\Leftrightarrow$(iii).

Open problem. *Give a direct probabilistic proof of the implication* (iii)$\Rightarrow$(ii) *in Theorem 4.1.*

Finding such a proof would be of interest for other related problems where the analogues of the results of [**2**] are not always available. An example of such problems is provided by the notion of boundary removable singularity.

From now on, consider a bounded domain D in $\mathbb{R}^d$, with a smooth (C^∞) boundary ∂D. A compact subset K of ∂D is called boundary removable for $\Delta u = u^p$ (in D) if the only nonnegative function u of class C^2 in D such that $\Delta u = u^p$ and u tends to 0 pointwise at every point of $\partial D \backslash K$ is the function identically equal to 0. Boundary singularities were studied first by Gmira and Véron [**17**], who proved in particular that singletons are removable if $p \geq \frac{d+1}{d-1}$.

To introduce the corresponding probabilistic notion, recall that $\mathcal{E}^D$ is the set of all exit points from D of the Brownian snake paths. The compact set $K \subset \partial D$ is said to be boundary polar if

$$\mathbb{N}_x(\mathcal{E}^D \cap K \neq \emptyset) = 0$$

for every $x \in D$. The following analogue of Theorem 4.1 was obtained in [**25**], confirming a conjecture of Dynkin [**8**].

THEOREM 4.2. *Suppose that $d \geq 3$ and let K be a compact subset of $\mathbb{R}^d$. Then the following are equivalent.*

(i) *K is a boundary removable singularity for $\Delta u = u^2$ in D.*
(ii) *K is boundary polar.*
(iii) *$\mathcal{C}_{d-3}(K) = 0$.*

If $d < 3$, (i) and (ii) only hold if $K = \emptyset$. The equivalence (i)$\Leftrightarrow$(ii) is an immediate consequence of the following lemma (Proposition VII.1 in [**27**]), which is analogous to Theorem 3.2.

LEMMA 4.3. *If K is a compact subset of D, the function*

$$u_K(x) = \mathbb{N}_x(\mathcal{E}^D \cap K \neq \emptyset), \qquad x \in D$$

is the maximal nonnegative solution of the problem

$$\begin{cases} \Delta u = 4u^2 & \textit{in } D \\ u_{|\partial D \setminus K} = 0. \end{cases} \tag{4.2}$$

Lemma 4.3 is essentially a consequence of Theorem 2.1 above. Roughly speaking, one can find a sequence a sequence (g_n) of continuous functions on ∂D, such that $\langle \mathcal{Z}^D, g_n \rangle$ converges to $+\infty$ on the event $\{\mathcal{E}^D \cap K \neq \emptyset\}$, and to 0 on the complementary event. It follows that u_K solves $\Delta u = 4u^2$ and it is also not hard to see that u_K vanishes on $\partial D \setminus K$. A suitable application of the maximum principle gives the maximality property stated in the lemma.

Coming back to Theorem 4.2, the implication (ii)$\Rightarrow$(iii) can be established in a way very similar to the probabilistic proof of (ii)$\Rightarrow$(iii) in Theorem 4.1 that was described above (compute the first and second moments of $\langle \mathcal{Z}^D, g \rangle$ for suitable functions g that vanish outside a small neighborhood of K). The implication (iii)$\Rightarrow$(i) was obtained in [**25**] by Fourier analytic methods, using some ideas from [**2**].

The analytic part of Theorem 4.2, that is the equivalence (i)$\Leftrightarrow$(iii), can in fact be extended to equation $\Delta u = u^p$. This extension again involves the Bessel capacities that were introduced above, but now considered for subsets of the boundary ∂D. If K is a compact subset of ∂D, we set

$$C^{\partial D}_{\gamma,p}(K) = \sup_{\mu \in \mathcal{M}_1(K)} \Big(\int_{\partial D} \sigma(dx) \Big(\int \mu(dy)\, G^{(d-1)}_\gamma(y-x) \Big)^{p'} \Big)^{-p/p'} \tag{4.3}$$

where $\sigma(dx)$ stands for Lebesgue measure on ∂D, and the Bessel kernels $G^{(d)}$ were defined in (3.5). This is of course analogous to (3.6), but $\mathbb{R}^d$ is replaced by the $(d-1)$-dimensional manifold ∂D, and consequently $G^{(d)}$ is replaced by $G^{(d-1)}$.

THEOREM 4.4. *Let K be a compact subset of ∂D. Then the following are equivalent.*

(i) *K is a boundary removable singularity for $\Delta u = u^p$ in D.*
(ii) *$C^{\partial D}_{2/p,p'}(K) = 0$.*

In the case $p = 2$, we recover the preceding theorem, since an easy calculation shows that $C^{\partial D}_{1,2}(K) = 0$ if and only if $\mathcal{C}_{d-3}(K) = 0$. Theorem 4.4 was proved in the case $1 < p \leq 2$ by Dynkin and Kuznetsov [**13**] using a combination of probabilistic and analytic techniques (in the case $1 < p \leq 2$, boundary removable singularities still have a probabilistic interpretation in terms of superprocesses with a p-stable branching mechanism). The case $p > 2$ of Theorem 4.4 was obtained by Marcus and Véron [**31**]. Rather surprisingly, the analytic techniques of [**31**] did not apply to the case $p < 2$ treated in [**13**]. In a subsequent paper [**32**], Marcus and Véron developed a different approach that allowed them to give a unified treatment of all cases of Theorem 4.4.

5. Solutions with measure boundary data

In this section, as well as in the next one, we keep assuming that D is a bounded domain in $\mathbb{R}^d$ with a smooth boundary ∂D. Many of the subsequent results hold under weaker regularity assumptions on D, but for the sake of simplicity we will omit the precise minimal assumptions.

We are now interested in the problem

$$\begin{cases} \Delta u = u^p & \text{in } D\ , \\ u_{|\partial D} = \nu\ , \end{cases} \tag{5.1}$$

where ν is a finite (positive) measure on ∂D. Similarly as for (2.2), the boundary condition $u_{|\partial D} = \nu$ may be interpreted via the integral equation

$$u(x) + \frac{1}{2}\int_D dy\, G_D(x,y)\, u^p(y) = \int_{\partial D} \nu(dz)\, P_D(x,z)\ , \quad x \in D\ , \tag{5.2}$$

where G_D is as in (2.3) the Green function of D, and P_D is the Poisson kernel of D (in the notation of (2.3), $K_D(x,dz) = P_D(x,z)\sigma(dz)$). (5.2) makes it obvious that u is bounded above by the harmonic function $P_D\nu$. Conversely, any nonnegative solution of $\Delta u = u^p$ in D which is bounded above by a harmonic function solves a problem of the type (5.2), for some finite measure ν on the boundary: See e.g. Proposition 4.1 in [**25**], for an argument in the case $p = 2$ which is easily extended. A nonnegative solution that is bounded above by a harmonic function will be called *moderate*.

Gmira and Véron [**17**] considered the problem (5.1) (in fact for more general nonlinearities). They proved in particular that (5.1) has a unique solution for any finite measure μ on the boundary if $p < \frac{d+1}{d-1}$. Notice that this condition corresponds to the case when singletons are not boundary polar.

We fix $p > 1$ and to simplify terminology, we call *boundary polar* any compact subset K of ∂D that satisfies the equivalent conditions of Theorem 4.4. This is of course consistent with our preceding terminology for $p = 2$.

Theorem 5.1. *Suppose that $p \geq \frac{d+1}{d-1}$ and let μ be a finite measure on ∂D. The following two conditions are equivalent:*

(i) *The problem (5.1), or equivalently the integral equation (5.2), has a unique nonnegative solution.*

(ii) *The measure μ does not charge boundary polar sets.*

Consequently, there is a one-to-one correspondence between the set of all moderate solutions of $\Delta u = u^p$ in D and the class of all finite measures on ∂D that do not charge boundary polar sets.

In the case $p = 2$, this theorem was proved in [**25**] (again confirming a conjecture of Dynkin [**8**]) using both analytic and probabilistic arguments. In that case, there is a probabilistic representation of the solution in terms of the Brownian snake: This is analogous to Theorem 2.1 with the difference that the quantity $\langle \mathcal{Z}^D, g\rangle$ should be replaced by a suitable additive functional of the Brownian snake.

Similarly as for Theorem 4.4, the general form of Theorem 5.1 was obtained by Dynkin and Kuznetsov (see [**13**] and [**14**]) when $1 < p < 2$ and by Marcus and Véron [**31**] when $p > 2$. A unified treatment was provided in [**32**].

6. The boundary trace problem

The classical Poisson representation states that nonnegative harmonic functions h in D are in one-to-one correspondence with finite measures ν on ∂D, and this correspondence is made explicit by the formula $h = P_D\nu$, where P_D is as above the Poisson kernel of ν. We may say that the measure ν is the trace of the harmonic function h on the boundary.

Our goal in this section is to discuss a similar trace representation for nonnegative solutions of $\Delta u = u^p$ in D. We will deal separately with the subcritical case $p < \frac{d+1}{d-1}$ (where there are no nonempty boundary polar sets) and the supercritical case $p \geq \frac{d+1}{d-1}$.

6.1. The subcritical case. We first consider $p = 2$, so that the Brownian snake approach is available. Then the subcritical case holds if and only if $d \leq 2$. Since the case $d = 1$ is trivial, we concentrate on $d = 2$, where we have the following theorem ([**23**], [**26**]). Recall that $\sigma(dz)$ denotes Lebesgue measure on ∂D.

Theorem 6.1. *Assume that $d = 2$. There is a one-to-one correspondence between nonnegative solutions of $\Delta u = 4\,u^2$ in D and pairs (K,ν), where K is a (possibly empty) compact subset of ∂D, and ν is a Radon measure on $\partial D\backslash K$.*

If a solution u is given, the associated pair (K,ν) is determined as follows. For every $z \in \partial D$, denote by N_z the inward-pointing normal unit vector to ∂D at z, then:

(i) *A point $y \in \partial D$ belongs to K if and only if, for every neighborhood U of y in ∂D,*
$$\lim_{r\downarrow 0} \int_U \sigma(dz)\, u(z + rN_z) = +\infty.$$

(ii) *For every continuous function g with compact support on $\partial D\backslash K$,*
$$\lim_{r\downarrow 0} \int_{\partial D\backslash K} \sigma(dz)\, u(z + rN_z)\, g(z) = \int \nu(dz)\, g(z).$$

Conversely, if the pair (K,ν) is given, the solution u can be obtained by the formula

$$u(x) = \mathbb{N}_x\Big(1 - \mathbf{1}_{\{\mathcal{E}^D\cap K=\emptyset\}} e^{-\langle \nu, Z^D\rangle}\Big)\;, \quad x \in D \tag{6.1}$$

where $(Z^D(z), z \in \partial D)$ is the continuous density of the exit measure $\mathcal{Z}^D$ with respect to Lebesgue measure $\sigma(dz)$ on ∂D.

The pair (K,ν) will be called the trace of u on the boundary. Informally, K is a set of singular points on the boundary (this is the set of points where u blows up as the square of the inverse of the distance from the boundary) and ν is a measure corresponding to the boundary value of u on $\partial D\backslash K$. The formula (6.1) contains as special cases the other probabilistic representations that have appeared previously. The formula of Theorem 2.1 corresponds to $K = \emptyset$, $\nu(dz) = g(z)\sigma(dz)$. The function u_2 of Theorem 3.2 (which here coincides with u_1 of Theorem 3.1) is obtained by taking $K = \partial D$. More generally the functions u_K in Lemma 4.3 correspond to the case $\nu = 0$. Finally, the moderate solutions of Theorem 5.1 are obtained when $K = \emptyset$.

Let us outline the proof of the probabilistic representation formula (6.1), assuming for simplicity that D is the unit disk of the plane. Fix a sequence r_n of real numbers in $(0,1)$ such that $r_n \uparrow 1$ as $n \uparrow \infty$. For every $n \geq 1$ and $x \in \bar{D}$, set $u_n(x) = r_n^2 u(r_n x)$, so that we have $\Delta u_n = 4u_n^2$ in D. Since u_n obviously has a continuous boundary value on ∂D, we may use Theorem 2.1 to write, for every $x \in D$,

$$u_n(x) = \mathbb{N}_x(1 - \exp -\langle \mathcal{Z}^D, u_n\rangle) = \mathbb{N}_x(1 - \exp - \int \sigma(dz)\, Z^D(z) u_n(z)) \tag{6.2}$$

using the fact that the exit measure $\mathcal{Z}^D$ has a continuous density Z^D with respect to σ (this property only holds when $d = 2$). Note that we can identify ∂D with $\mathbb{R}/\mathbb{Z}$. Using a compactness argument and replacing (r_n) by a subsequence if necessary, we may assume that for every open subinterval I of ∂D with rational ends, we have

$$\lim_{n\to\infty} \int_I \sigma(dz)\, u_n(z) = a(I)$$

where $a(I) \in [0, +\infty]$. We then set

$$K = \{y \in \partial D : a(I) = +\infty \text{ if } y \in I\}.$$

Replacing again (r_n) by a subsequence, we may also assume that the sequence of measures $\mathbf{1}_{\partial D\setminus K}(z) u_n(z)\sigma(dz)$ converges to a limiting measure $\nu(dz)$, in the sense of vague convergence of Radon measures on $\partial D \setminus K$. From the definition of K and ν, it can then be proved that, for every $x \in D$,

$$\lim_{n\to\infty} \int_{\partial D} \sigma(dz)\, u_n(z)\, Z^D(z) = +\infty \quad , \ \mathbb{N}_x \text{ a.e. on } \{\mathcal{E}^D \cap K \neq \emptyset\} \tag{6.3}$$

and

$$\lim_{n\to\infty} \int_{\partial D} \sigma(dz)\, u_n(z)\, Z^D(z) = \langle \nu, Z^D\rangle \quad , \ \mathbb{N}_x \text{ a.e. on } \{\mathcal{E}^D \cap K = \emptyset\}. \tag{6.4}$$

Indeed, (6.4) is easy if we observe that the support of Z^D is contained in $\partial D \setminus K$, on the event $\{\mathcal{E}^D \cap K = \emptyset\}$. The proof of (6.3) reduces to checking that on the event $\{\mathcal{E}^D \cap K \neq \emptyset\}$ there is a (random) point $z \in \partial D$ such that $Z^D(z) > 0$.

Using (6.3) and (6.4), we can pass to the limit $n \to \infty$ in the right-hand side of (6.2), and we arrive at the representation formula (6.1). The other assertions of Theorem 6.1 then follow rather easily.

Let us come back to the general case of equation $\Delta u = u^p$. Marcus and Véron ([**30**], Theorem 1) proved that, for any $p > 1$ and in any dimension $d \geq 2$, the trace (K, ν) of a nonnegative solution u of $\Delta u = u^p$ in D can be defined by properties (i) and (ii) of Theorem 6.1. Independently, Dynkin and Kuznetsov [**15**] gave a slightly different but equivalent definition of the trace.

The one-to-one correspondence between solutions and their traces can in fact be extended to the general subcritical case. The following theorem was proved by Marcus and Véron [**30**].

Theorem 6.2. *Assume that $d < \frac{p+1}{p-1}$. Then the mapping $u \longrightarrow (K,\nu)$ associating with u its trace (K,ν) (defined by* (i) *and* (ii) *of Theorem 6.1) gives a one-to-one correspondence between the set of all nonnegative solutions of $\Delta u = u^p$ in D and the set of all pairs (K,ν), where K is a (possibly empty) compact subset of ∂D, and ν is a Radon measure on $\partial D \setminus K$.*

When $1 < p \leq 2$, the probabilistic representation formula (6.1) can be extended to this more general setting: See Theorem 1.3 in [**28**] (which elaborates on preceding results of Dynkin and Kuznetsov [**15**], [**16**]).

6.2. The supercritical case. The supercritical case $p \geq \frac{1+d}{1-d}$ is more complicated and in a sense more interesting. As was mentioned above, properties (i) and (ii) of Theorem 6.1 can still be used to define the trace of any nonnegative solution of $\Delta u = u^p$ in D.

However, the fact that there are nontrivial boundary polar sets now suggests that all pairs (K, ν) cannot occur as possible traces. More precisely, Theorem 4.4 indicates that the pair $(K, 0)$ cannot be a possible trace if K is boundary polar, and similarly, Theorem 5.1 suggests that ν should not charge boundary polar sets in order for $(\emptyset, \nu)$ to be a possible trace. The characterization of possible traces was obtained independently by Marcus and Véron [**31**] and Dynkin and Kuznetsov [**15**] (the latter in the case $p \leq 2$).

THEOREM 6.3. *Let K be a compact subset of ∂D, and let ν be a Radon measure on $\partial D \backslash K$. Then the pair (K, ν) is the trace of a nonnegative solution of $\Delta u = u^p$ in D if and only if:*

(i) *The measure ν does not charge boundary polar sets.*
(ii) *The set K is the union of the two sets*

$$K_p^* = \{y \in K : K \cap U \text{ is not boundary polar for every neighborhood } U \text{ of } y\}$$

and

$$\partial_\nu K = \{y \in K : \nu(K \cap U) = \infty \text{ for every neighborhood } U \text{ of } y\}.$$

Another problem in the supercritical case is the lack of uniqueness of the solution corresponding to a given (admissible) trace. To give an example of this phenomenon, consider the case $p = 2$, $d \geq 3$. Let (y_n) be a dense sequence in ∂D and, for every n, let $(r_n^k, k = 1, 2, \ldots)$ be a decreasing sequence of positive numbers. For every $k \geq 1$, set

$$H_k = \bigcup_{n=1}^{\infty} \{y \in \partial D : |y - y_n| < r_n^k\} ,$$

and

$$u_k(x) = \mathbb{N}_x(\mathcal{E}^D \cap H_k \neq \emptyset) , \qquad x \in D.$$

Then it is easy to see that, for every $k \geq 1$, u_k is a solution with trace $(\partial D, 0)$. On the other hand, the fact that singletons are boundary polar implies that $u_k \downarrow 0$ as $k \uparrow \infty$, provided that the sequences $(r_n^k, k = 1, 2, \ldots)$ decrease sufficiently fast. Therefore infinitely many of the functions u_k must be different.

In view of this nonuniqueness problem, Dynkin and Kuznetsov [**19**], [**16**] have proposed to use a finer definition of the trace, where the set K is no longer closed with respect to the Euclidean topology. We will explain this definition in the general case of equation $\Delta u = u^p$. We first need to introduce the analogue of the singular part for the fine trace of a solution u.

Let b be a nonnegative continuously differentiable function on D. We can then consider the Poisson kernel $(P_D^b(x, y), x \in D, y \in \partial D)$ associated with the operator $\Delta u - bu$ in D (see Section 11.1.2 in Dynkin [**9**] for a detailed construction of P_D^b). A point y of the boundary ∂D is called singular for b if $P_D^b(x, y) = 0$ for some, or equivalently for every, $x \in D$. Informally, this corresponds to points of rapid growth of b. A simple equivalent probabilistic definition can be given as follows. If

$(B_t, 0 \le t \le \tau)$ is under $P_{x\to y}$ a Brownian motion started from x and conditioned to exit D at y (in the sense of [**5**]), the point y is singular for b if and only if

$$\int_0^\tau dt\, b(B_t) = +\infty \ , \quad P_{x\to y} \text{ a.s.}$$

Consider now a nonnegative solution u of $\Delta u = u^p$. The *singular set* of u, which is denoted by $\mathrm{SG}(u)$ is the set of all boundary points that are singular for u^{p-1}. Note that $\mathrm{SG}(u)$ is a Borel subset of ∂D, but needs not be closed in general.

We denote by $\mathcal{N}$ the set of all finite measures on the boundary that do not charge boundary polar sets. For every $\nu \in \mathcal{N}$, we denote by u_ν the unique solution of the problem (5.1), or equivalently the solution associated with ν via the correspondence of Theorem 5.1.

Definition 6.4. Let u be a nonnegative solution of $\Delta u = u^p$ in D. The fine trace of u is the pair (Γ, μ) that is defined as follows:

(i) $\Gamma = \mathrm{SG}(u)$.
(ii) μ is the σ-finite measure on $\partial D\backslash\Gamma$ such that, for every Borel subset A of $\partial D\backslash\Gamma$,

$$\mu(A) = \sup\{\nu(A) : \nu \in \mathcal{N},\ u_\nu \le u\}. \tag{6.5}$$

Remark. It is not obvious that formula (6.5) defines a *measure*. See Theorem 1.3 in [**16**]. It is clear from (ii) that ν does not charge boundary polar sets.

It can be checked that in the subcritical case this definition is equivalent to the one given by (i) and (ii) of Theorem 6.1. The interest of this definition comes from the following theorem (Theorem 1.4 in [**16**]).

Theorem 6.5. [**16**] *Let us call σ-moderate any nonnegative solution of $\Delta u = u^p$ that is the increasing limit of a sequence of moderate solutions. Then σ-moderate solution are characterized by their fine traces.*

This theorem shows that the lack of uniqueness mentioned above disappears if one considers the fine trace instead of the (rough) trace discussed in the previous subsection. Dynkin and Kuznetsov [**16**] also give a description of those pairs (Γ, ν) that can occur as fine traces of solutions. Provided one considers only σ-moderate solutions, the fine trace thus yields a one-to-one correspondence between solutions and admissible pairs (Γ, ν). The obvious question, which was stated in the epilogue of [**9**] is thus:

Are all nonnegative solutions σ-moderate ?

This question was answered positively first in the case $p = 2$ in Mselati's thesis [**36**], [**37**]. In addition, Mselati's work gives a probabilistic representation of solutions, which is analogous to Theorem 6.1. To state this representation, we need to introduce some additional notation. Let $\nu \in \mathcal{N}$, and let h_ν be the harmonic function in D associated with ν ($h_\nu = P_D\nu$ in our previous notation). Then, if (D_n) is an increasing sequence of smooth subdomains of D such that $\bar{D}_n \subset D_{n+1}$ and $D = \cup D_n$, we can define

$$Z_\nu := \lim_{n\uparrow\infty} \langle \mathcal{Z}_{D_n}, h_\nu\rangle \ , \quad \mathbb{N}_x \text{ a.e.}$$

and the resulting variable Z_ν does not depend on the choice of the sequence (D_n). Note that the existence of the limit defining Z_ν is easy because $\langle \mathcal{Z}_{D_n}, h_\nu\rangle$ is a nonnegative martingale. Then, if ν is a σ-finite measure on ∂D that does not charge boundary polar sets, we can find an increasing sequence (ν_k) in $\mathcal{N}$ such that $\nu = \lim \uparrow \nu_k$, and we set $Z_\nu = \lim \uparrow Z_{\nu_k}$ (again this does not depend on the choice of the sequence (ν_k)).

THEOREM 6.6. [**37**] *All nonnegative solutions of $\Delta u = 4\,u^2$ are σ-moderate. Moreover, if u is a solution and (Γ, ν) is its fine trace, we have for every $x \in D$,*

$$u(x) = \mathbb{N}_x\Big(1 - \mathbf{1}_{\{\mathcal{E}^D \cap \Gamma = \emptyset\}} \exp(-Z_\nu)\Big). \tag{6.6}$$

A major step in the proof of Theorem 6.6 was to prove that the solution u_K defined in Lemma 4.3 is σ-moderate, for any compact subset K of ∂D. The proof depends on delicate upper bounds on u_K near the boundary, and analogous lower bounds for certain σ-moderate solutions, which are obtained via probabilistic methods. Motivated by Mselati's work, Marcus and Véron [**33**], [**34**] were able to obtain very precise capacitary estimates in the general case $p > 1$. Part of their results is summarized in the following theorem.

THEOREM 6.7. [**34**] *Consider the general case $p > 1$. Let K be a compact subset of ∂D and assume that K is not boundary polar. Let u_K be the maximal nonnegative solution of $\Delta u = u^p$ in D that vanishes on $\partial D \backslash K$. Then u_K is σ-moderate. Moreover, for every $x \in D$,*

$$u_K(x) \le c_1\,\rho(x)\,\rho_K(x)^{-1-2/(p-1)}\,C^{\partial D}_{2/p,p'}(K/\rho_K(x)) \tag{6.7}$$

where c_1 is a positive constant, $\rho(x) = \mathrm{dist}(x, \partial D)$ and $\rho_K(x) = \mathrm{dist}(x, K)$.

In addition to the upper bound (6.7), the main result of [**34**] also gives a sharp lower bound, which we omit here.

Recently, Dynkin [**10**], [**11**], [**12**] was able to extend to extend Mselati's result to all values $p \in (1, 2]$.

THEOREM 6.8. [**12**] *If $1 < p \le 2$, all solutions of $\Delta u = u^p$ are σ-moderate. Moreover, if u is a solution with fine trace (Γ, ν), we have*

$$u = u_\Gamma \oplus u_\nu$$

where:

- *u_Γ is the supremum of the functions u_K for all compact subsets K of Γ.*
- *u_ν is the supremum of the functions u_μ for all measures $\mu \in \mathcal{N}$ such that $\mu \le \nu$.*
- *The notation $u_\Gamma \oplus u_\nu$ stands for the maximal solution dominated by $u_\Gamma + u_\nu$.*

See the monograph [**12**] for a detailed proof. In addition to some ideas taken from [**37**], an important role is played by an upper bound similar to (6.7), which was obtained by Kuznetsov [**20**] independently of [**34**]. A probabilistic representation formula analogous to (6.6) also holds in the setting of Theorem 6.8.

Open problem. *Extend Theorem 6.8 to the case $p > 2$.*

The very recent paper [**35**] by Marcus and Véron contains important progress towards the solution of this open problem.

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