Scaling Limits of Random Trees and Planar Maps

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Contents

1. Introduction

The main goal of these lectures is to present some of the recent progress in the asymptotics for large random planar maps. Recall that a planar map is simply a graph drawn on the two-dimensional sphere and viewed up to direct homeomorphisms of the sphere. The faces of the map are the connected components of the complement of edges, or in other words the regions of the sphere delimited by the graph. Special cases of planar maps are triangulations, respectively quadrangulations, respectively p-angulations, where each face is adjacent to exactly 3, respectively 4, respectively p, edges (see Section 4 for more precise definitions).

Planar maps play an important role in several areas of mathematics and physics. They have been studied extensively in combinatorics since the pioneering work of Tutte (see in particular [51]), which was motivated by the famous four-color theorem. Graphs drawn on surfaces also have important algebraic and geometric applications; see the book [27]. In theoretical physics, the enumeration of planar maps (and of maps on surfaces of higher genus) has strong connections with matrix models, as shown by the work of 't Hooft [24] and Brézin et al [10]. More recently, graphs on surfaces have been used in physics as discrete models of random geometry in the so-called two-dimensional quantum gravity; see in particular the book [3] (a different mathematical approach to quantum gravity using the Gaussian free field appears in the work of Duplantier and Sheffield [16]). A nice account of the

connections between planar maps and the statistical physics of random surfaces can be found in Bouttier's thesis [**7**]. From the probabilistic perspective, a planar map can be viewed as a discretization of a surface, and finding a continuous limit for large planar maps chosen at random in a suitable class should lead to an interesting model of a "Brownian surface". This is of course analogous to the well-known fact that Brownian motion appears as the scaling limit of long discrete random paths. In a way similar to the convergence of rescaled random walks to Brownian motion, one expects that the scaling limit of large random planar maps is universal in the sense that it should not depend on the details of the discrete model one is considering. These ideas appeared in the pioneering paper of Chassaing and Schaeffer [**12**] and in the subsequent work of Markert and Mokkadem [**37**] in the case of quadrangulations, and a little later in Schramm [**48**], who gave a precise form to the question of the existence of a scaling limit for large random triangulations of the sphere.

To formulate the latter question, consider a random planar map M_n which is uniformly distributed over a certain class of planar maps (for instance, triangulations, or quadrangulations) with n faces. Equip the vertex set $V(M_n)$ with the graph distance d_{gr}. It has been known for some time that the diameter of the resulting metric space is of order $n^{1/4}$ when n is large (see [**12**] for the case of quadrangulations). One then expects that the rescaled random metric spaces $(V(M_n), n^{-1/4}d_{gr})$ will converge in distribution as n tends to infinity towards a certain random metric space, which should be the same, up to trivial scaling factors, independently of the class of planar maps we started from. For the previous convergence to make sense, we need to say what it means for a sequence of metric spaces to converge. To this end we use the notion of the Gromov-Hausdorff distance, as it was suggested in [**48**]. Roughly speaking (see Section 2 for a more precise definition) a sequence (E_n) of compact metric spaces converges to a limiting space E_∞ if it is possible to embed isometrically all spaces E_n and E_∞ in the same "big" metric space E, in such a way that the Hausdorff distance between E_n and E_∞ tends to 0 as $n \to \infty$.

The preceding question of the existence of the scaling limit of large random planar maps is still open, but there has been significant progress in this direction, and our aim is to present some of the results that have been obtained in recent years.

Much of the recent progress in the understanding of asymptotic properties of large random planar maps was made possible by the use of bijections between different classes of planar maps and certain labeled trees. In the particular case of quadrangulations, such bijections were discovered by Cori and Vauquelin [**14**] and later popularized by Schaeffer [**47**] (see also Chassaing and Schaeffer [**12**]). The Cori-Vauquelin-Schaeffer bijection was extended to much more general planar maps by Bouttier, Di Francesco and Guitter [**9**]. In the case of bipartite planar maps, this extension takes a particularly simple form, which explains why some of the recent work [**36**, **31**, **32**] concentrates on the bipartite case. The reason why the bijections between maps and trees are interesting is the fact that properties of large (labeled) trees are often much easier to understand than those of large graphs. Indeed, it has been known for a long time and in particular since the work of Aldous [**1**, **2**] that one can often describe the asymptotic properties of large random trees in terms of "continuous trees" whose prototype is the so-called CRT or Brownian continuum random tree. In the case of trees with labels, the relevant scaling limit for most of
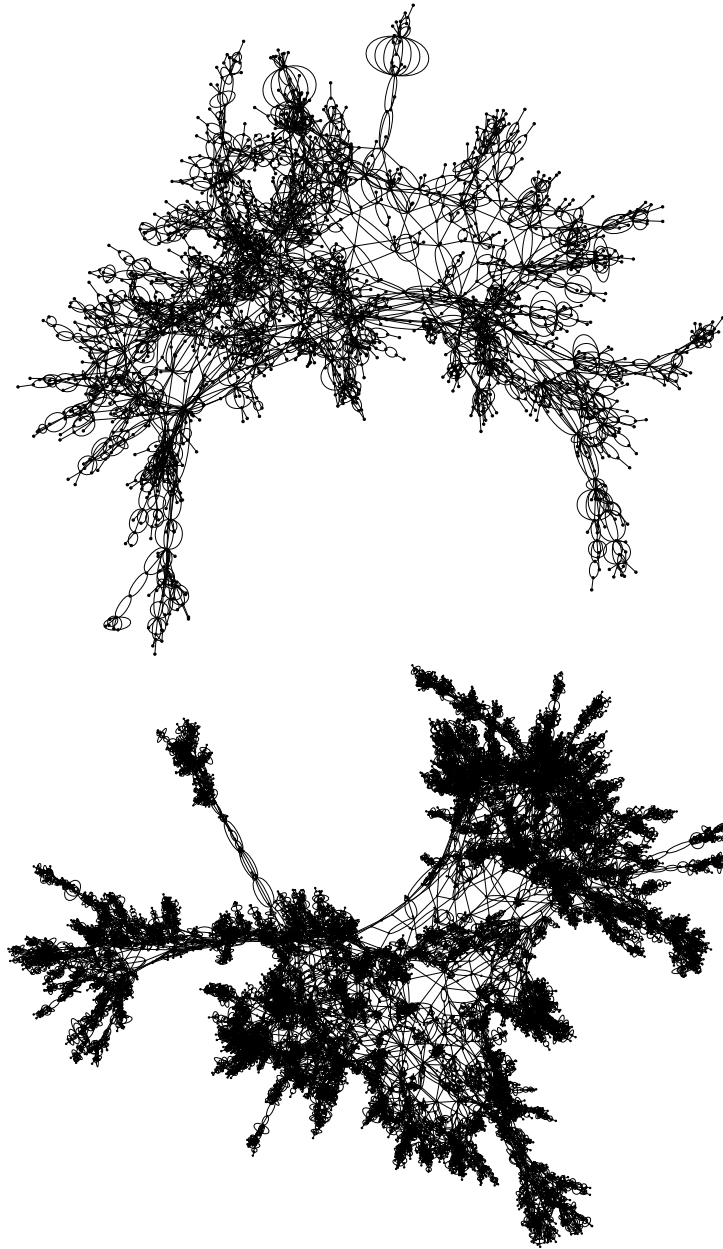
FIGURE 1. Two planar quadrangulations, with respectively 2500 and 20000 vertices. These pictures represent the quadrangulations as graphs, and do not take account of the embedding in the sphere. Simulations by J.-F. Marckert.

the discrete models of interest is the CRT equipped with Brownian labels, which can conveniently be constructed and studied via the path-valued process called the Brownian snake (see e.g. [**28**]).

A key feature of the bijections between planar maps and labeled trees is the fact that, up to an appropriate translation, labels on the tree correspond to distances in the map from a distinguished vertex that plays a special role. Therefore, the known results about scaling limits of labeled trees immediately give much information about asymptotics of distances from this distinguished vertex. This idea was exploited by Chassaing and Schaeffer [**12**] in the case of quadrangulations and then by Marckert and Miermont [**36**] (for bipartite planar maps) and Miermont [**38**] (for general planar maps). In view of deriving the Gromov-Hausdorff convergence of rescaled planar maps, it is however not sufficient to control distances from a distinguished vertex. Still, a simple argument gives an effective bound on the distance between two arbitrary vertices in terms of quantities depending only on the labels on the tree, or equivalently on the distances from the distinguished vertex (see Proposition 5.9(i) below). This bound was used in [**31**] to show via a compactness argument that the scaling limit of rescaled uniformly distributed $2p$-angulations with n faces exists along suitable subsequences. Furthermore, this scaling limit is a quotient space of the CRT for an equivalence relation defined in terms of Brownian labels on the CRT: Roughly speaking, two vertices of the CRT need to be identified if they have the same label and if, when travelling from one vertex to the other one along the contour of the CRT, one only encounters vertices with larger label. The results of [**31**] are not completely satisfactory, because they require the extraction of suitable subsequences. The reason why this is necessary is the fact that the distance on the limiting space (that is, on the quotient of the CRT we have just described) has not been fully identified, even though lower and upper bounds are available. Still we call Brownian map any random metric space that arises as the scaling limit of uniformly distributed $2p$-angulations with n faces. This terminology is borrowed from Marckert and Mokkadem [**37**], who studied a weaker form of the convergence of rescaled random quadrangulations. Although the distribution of the Brownian map has not been fully characterized, it is possible to derive many properties of this random object (these properties will be common to any of the limiting random metric spaces that can arise in the scaling limit). In particular, it has been shown that the Brownian map has dimension 4 [**31**] and that it is homeomorphic to the 2-sphere [**34, 39**]. The latter fact is maybe not surprising since we started from larger and larger graphs drawn on the sphere: Still it implies that large random planar maps will have no "bottlenecks", meaning cycles whose length is small in comparison with the diameter of the graph but such that both connected components of the complement of the cycle have a macroscopic size.

In the subsequent sections, we discuss most of the preceding results in detail. We restrict our attention to the case of quadrangulations, because the bijections with trees are simpler in that case: The labeled trees corresponding to quadrangulations are just plane trees (rooted ordered trees) equipped with integer labels, such that the label of the root is 0 and the label can change by at most 1 in absolute value along each edge of the tree.

The first three sections below are devoted to asymptotics for random (labeled) trees, in view of our applications to random planar maps. In Section 1, we discuss asymptotics for uniformly distributed plane trees with n edges. We give a detailed

proof of the fact that the suitably rescaled contour function of these discrete trees converges in distribution to the normalized Brownian excursion (this is a special case of the results of [**2**]). To this end, we first recall the basic facts of excursion theory that we need. In Section 2, we show that the convergence of rescaled contour functions can be restated as a convergence in the Gromov-Hausdorff sense of the trees viewed as random metric spaces for the graph distance. The limiting space is then the CRT, which we define precisely as the random real tree coded by a normalized Brownian excursion. Section 2 also contains basic facts about the Gromov-Hausdorff distance, and in particular its definition in terms of correspondences. In Section 3, we consider labeled trees and we give a detailed proof of the fact that rescaled labeled trees converge (in a suitable sense) towards the CRT equipped with Brownian labels.

The last four sections are devoted to planar maps and their scaling limits. Section 4 presents the combinatorial facts about planar maps that we need. In particular, we describe the Cori-Vauquelin-Schaeffer bijection between (rooted and pointed) quadrangulations and labeled trees. We also explain how labels on the tree give access to distances from the distinguished vertex in the map, and provide useful upper and lower bounds for other distances. In Section 5, we give the compactness argument that makes it possible to get sequential limits for rescaled uniformly distributed quadrangulations with n faces, in the Gromov-Hausdorff sense. The identification of the limit (or Brownian map) as a quotient space of the CRT for the equivalence relation described above is explained in Section 6. In that section, we are not able to give the full details of the proofs, but we try to present the main ideas. As a simple consequence of some of the estimates needed in the identification of the Brownian map, we also compute its Hausdorff dimension. Finally, Section 7 is devoted to the homeomorphism theorem. We follow the approach of [**39**], which consists in establishing the absence of "bottlenecks" in the Brownian map before proving via a theorem of Whyburn that this space is homeomorphic to the sphere.

To conclude this introduction, let us mention that, even though the key problem of the uniqueness of the Brownian map remains unsolved, many properties of this space have been investigated successfully. Often these results give insight into the properties of large planar maps. This is in particular the case for the results of [**32**], which give a complete description of all geodesics connecting an arbitrary point of the Brownian map to the distinguished point. Related results have been obtained in the paper [**40**], which deals with maps on surfaces of arbitrary genus. Very recently, the homeomorphism theorem of [**34**] has been extended by Bettinelli [**5**] to higher genus. As a final remark, one expects that the Brownian map should be the scaling limit for all random planar maps subject to some bound on the maximal degree of faces. One may ask what happens for random planar maps such that the distribution of the degree of a typical face has a heavy tail: This problem is discussed in [**33**], where it is shown that this case leads to different scaling limits.

2. Discrete trees and convergence towards the Brownian excursion

2.1. Plane trees. We will be interested in (finite) rooted ordered trees, which are called plane trees in combinatorics (see e.g. [**50**]). We set $\mathbb{N} = \{1, 2, \ldots\}$ and

by convention $\mathbb{N}^0 = \{\varnothing\}$. We introduce the set

$$\mathcal{U} = \bigcup_{n=0}^{\infty} \mathbb{N}^n.$$

An element of $\mathcal{U}$ is thus a sequence $u = (u^1, \ldots, u^n)$ of elements of $\mathbb{N}$, and we set $|u| = n$, so that $|u|$ represents the "generation" of u. If $u = (u^1, \ldots, u^k)$ and $v = (v^1, \ldots, v^{\ell})$ belong to $\mathcal{U}$, we write $uv = (u^1, \ldots, u^k, v^1, \ldots, v^{\ell})$ for the concatenation of u and v. In particular $u\varnothing = \varnothing u = u$.

The mapping $\pi : \mathcal{U} \backslash \{\varnothing\} \longrightarrow \mathcal{U}$ is defined by $\pi((u^1, \ldots, u^n)) = (u^1, \ldots, u^{n-1})$ ($\pi(u)$ is the "parent" of u).

A plane tree τ is a finite subset of $\mathcal{U}$ such that:

(i) $\varnothing \in \tau$.
(ii) $u \in \tau \backslash \{\varnothing\} \Rightarrow \pi(u) \in \tau$.
(iii) For every $u \in \tau$, there exists an integer $k_u(\tau) \geq 0$ such that, for every $j \in \mathbb{N}$, $uj \in \tau$ if and only if $1 \leq j \leq k_u(\tau)$

The number $k_u(\tau)$ is interpreted as the "number of children" of u in τ.

We denote by $\mathbf{A}$ the set of all plane trees. In what follows, we see each vertex of the tree τ as an individual of a population whose τ is the family tree. By definition, the size $|\tau|$ of τ is the number of edges of τ, $|\tau| = \#\tau - 1$. For every integer $k \geq 0$, we put

$$\mathbf{A}_k = \{\tau \in \mathbf{A} : |\tau| = k\}.$$

EXERCISE 2.1. *Verify that the cardinality of $\mathbf{A}_k$ is the k-th Catalan number*

$$\#\mathbf{A}_k = \mathrm{Cat}_k := \frac{1}{k+1}\binom{2k}{k}.$$

A plane tree can be coded by its Dyck path or **contour function**. Suppose that the tree is embedded in the half-plane in such a way that edges have length one. Informally, we imagine the motion of a particle that starts at time $t = 0$ from the root of the tree and then explores the tree from the left to the right, moving continuously along the edges at unit speed (in the way explained by the arrows of Fig.2), until all edges have been explored and the particle has come back to the root. Since it is clear that each edge will be crossed twice in this evolution, the total time needed to explore the tree is $2|\tau|$. The value $C(s)$ of the contour function at time $s \in [0, 2|\tau|]$ is the distance (on the tree) between the position of the particle at time s and the root. By convention $C(s) = 0$ if $s \geq 2|\tau|$. Fig.2 explains the construction of the contour function better than a formal definition.

Let $k \geq 0$ be an integer. A Dyck path of length $2k$ is a sequence $(x_0, x_1, x_2, \ldots, x_{2k})$ of nonnegative integers such that $x_0 = x_{2k} = 0$, and $|x_i - x_{i-1}| = 1$ for every $i = 1, \ldots, 2k$. Clearly, if τ is a plane tree of size k, and $(C(s))_{s \geq 0}$ is its contour function, the sequence $(C(0), C(1), \ldots, C(2k))$ is a Dyck path of length $2k$. More precisely, we have the following easy result.

PROPOSITION 2.2. *The mapping $\tau \mapsto (C(0), C(1), \ldots, C(2k))$ is a bijection from $\mathbf{A}_k$ onto the set of all Dyck paths of length $2k$.*
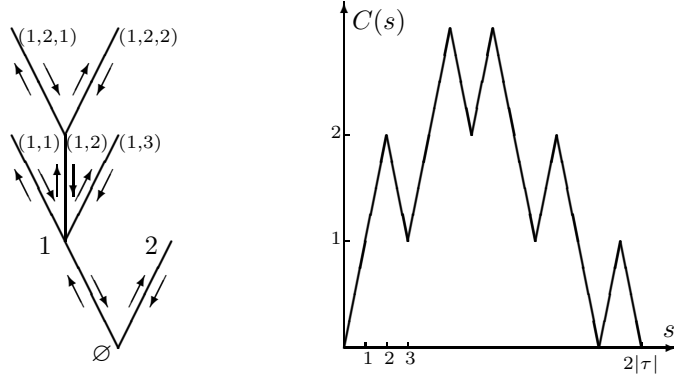
FIGURE 2. A tree and its contour function

2.2. Galton-Watson trees. Let μ be a critical or subcritical offspring distribution. This means that μ is a probability measure on $\mathbb{Z}_+$ such that

$$\sum_{k=0}^{\infty} k\mu(k) \leq 1.$$

We exclude the trivial case where $\mu(1) = 1$.

To define Galton-Watson trees, we let $(K_u, u \in \mathcal{U})$ be a collection of independent random variables with law μ, indexed by the set $\mathcal{U}$. Denote by θ the random subset of $\mathcal{U}$ defined by

$$\theta = \{u = (u^1, \ldots, u^n) \in \mathcal{U} : u^j \leq K_{(u^1, \ldots, u^{j-1})} \text{ for every } 1 \leq j \leq n\}.$$

PROPOSITION 2.3. θ *is a.s. a tree. Moreover, if*

$$Z_n = \#\{u \in \theta : |u| = n\},$$

$(Z_n, n \geq 0)$ *is a Galton-Watson process with offspring distribution μ and initial value $Z_0 = 1$.*

REMARK 2.4. *Clearly $k_u(\theta) = K_u$ for every $u \in \theta$.*

The tree θ, or any random tree with the same distribution, will be called a Galton-Watson tree with offspring distribution μ, or in short a μ-Galton-Watson tree. We also write Π_μ for the distribution of θ on the space $\mathbf{A}$.

We leave the easy proof of the proposition to the reader. The finiteness of the tree θ comes from the fact that the Galton-Watson process with offspring distribution μ becomes extinct a.s., so that $Z_n = 0$ for n large.

If τ is a tree and $1 \leq j \leq k_\varnothing(\tau)$, we write $T_j\tau$ for the tree τ shifted at j:

$$T_j\tau = \{u \in \mathcal{U} : ju \in \tau\}.$$

Note that $T_j\tau$ is a tree.

Then Π_μ may be characterized by the following two properties (see e.g. [44] for more general statements):

 (i) $\Pi_\mu(k_\varnothing = j) = \mu(j), \quad j \in \mathbb{Z}_+$.
 (ii) For every $j \geq 1$ with $\mu(j) > 0$, the shifted trees $T_1\tau, \ldots, T_j\tau$ are independent under the conditional probability $\Pi_\mu(d\tau \mid k_\varnothing = j)$ and their conditional distribution is Π_μ.

Property (ii) is often called the branching property of the Galton-Watson tree. We now give an explicit formula for Π_μ.

PROPOSITION 2.5. *For every $\tau \in \mathbf{A}$,*

$$\Pi_\mu(\tau) = \prod_{u \in \tau} \mu(k_u(\tau)).$$

PROOF. We can easily check that

$$\{\theta = \tau\} = \bigcap_{u \in \tau} \{K_u = k_u(\tau)\},$$

so that

$$\Pi_\mu(\tau) = P(\theta = \tau) = \prod_{u \in \tau} P(K_u = k_u(\tau)) = \prod_{u \in \tau} \mu(k_u(\tau)).$$

$\square$

We will be interested in the particular case when $\mu = \mu_0$ is the (critical) geometric offspring distribution, $\mu_0(k) = 2^{-k-1}$ for every $k \in \mathbb{Z}_+$. In that case, the proposition gives

$$\Pi_{\mu_0}(\tau) = 2^{-2|\tau|-1}$$

(note that $\sum_{u \in \tau} k_u(\tau) = |\tau|$ for every $\tau \in \mathbf{A}$).

In particular $\Pi_{\mu_0}(\tau)$ only depends on $|\tau|$. As a consequence, for every integer $k \geq 0$, the conditional probability distribution $\Pi_{\mu_0}(\cdot \mid |\tau| = k)$ is just the uniform probability measure on $\mathbf{A}_k$. This fact will be important later.

2.3. The contour function in the geometric case. In general, the Dyck path of a Galton-Watson tree does not have a "nice" probabilistic structure (see however Section 1 of [**29**]). In this section we restrict our attention to the case when $\mu = \mu_0$ is the critical geometric offspring distribution.

First recall that $(S_n)_{n \geq 0}$ is a simple random walk on $\mathbb{Z}$ (started from 0) if it can be written as

$$S_n = X_1 + X_2 + \cdots + X_n$$

where $X_1, X_2, \ldots$ are i.i.d. random variables with distribution $P(X_n = 1) = P(X_n = -1) = \frac{1}{2}$.

Set $T = \inf\{n \geq 0 : S_n = -1\} < \infty$ a.s. The random finite path

$$(S_0, S_1, \ldots, S_{T-1})$$

(or any random path with the same distribution) is called an excursion of simple random walk. Obviously this random path is a random Dyck path of length $T - 1$.

PROPOSITION 2.6. *Let θ be a μ_0-Galton-Watson tree. Then the Dyck path of θ is an excursion of simple random walk.*

PROOF. Since plane trees are in one-to-one correspondence with Dyck paths (Proposition 2.2), the statement of the proposition is equivalent to saying that the random plane tree θ coded by an excursion of simple random walk is a μ_0-Galton-Watson tree. To see this, introduce the upcrossing times of the random walk S from 0 to 1:

$$U_1 = \inf\{n \geq 0 : S_n = 1\} \ , \ V_1 = \inf\{n \geq U_1 : S_n = 0\}$$

and by induction, for every $j \geq 1$,

$$U_{j+1} = \inf\{n \geq V_j : S_n = 1\} \ , \ V_{j+1} = \inf\{n \geq U_{j+1} : S_n = 0\}.$$

Let $K = \sup\{j : U_j \leq T\}$ ($\sup \varnothing = 0$). From the relation between a plane tree and its associated Dyck path, one easily sees that $k_\varnothing(\theta) = K$, and that for every $i = 1, \ldots, K$, the Dyck path associated with the subtree $T_i\theta$ is the path ω_i, with

$$\omega_i(n) := S_{(U_i+n)\wedge(V_i-1)} - 1 \quad, \ 0 \leq n \leq V_i - U_i - 1.$$

A simple application of the Markov property now shows that K is distributed according to μ_0 and that conditionally on $K = k$, the paths $\omega_1, \ldots, \omega_k$ are k independent excursions of simple random walk. The characterization of Π_{μ_0} by properties (i) and (ii) listed before Proposition 2.5 now shows that θ is a μ_0-Galton-Watson-tree. $\square$

2.4. Brownian excursions. Our goal is to prove that the (suitably rescaled) contour function of a tree uniformly distributed over $\mathbf{A}_k$ converges in distribution as $k \to \infty$ towards a normalized Brownian excursion. We first need to recall some basic facts about Brownian excursions.

We consider a standard linear Brownian motion $B = (B_t)_{t\geq 0}$ starting from the origin. The process $\beta_t = |B_t|$ is called reflected Brownian motion. We denote by $(L_t^0)_{t\geq 0}$ the local time process of B (or of β) at level 0, which can be defined by the approximation

$$L_t^0 = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_0^t ds\, \mathbf{1}_{[-\varepsilon,\varepsilon]}(B_s) = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_0^t ds\, \mathbf{1}_{[0,\varepsilon]}(\beta_s),$$

for every $t \geq 0$, a.s.

Then $(L_t^0)_{t\geq 0}$ is a continuous increasing process, and the set of increase points of the function $t \to L_t^0$ coincides with the set

$$\mathcal{Z} = \{t \geq 0 : \beta_t = 0\}$$

of all zeros of β. Consequently, if we introduce the right-continuous inverse of the local time process,

$$\sigma_\ell := \inf\{t \geq 0 : L_t^0 > \ell\}\ , \quad \text{for every } \ell \geq 0,$$

we have

$$\mathcal{Z} = \{\sigma_\ell : \ell \geq 0\} \cup \{\sigma_{\ell-} : \ell \in D\}$$

where D denotes the countable set of all discontinuity times of the mapping $\ell \to \sigma_\ell$.

The connected components of the open set $\mathbb{R}_+ \backslash \mathcal{Z}$ are called the *excursion intervals* of β away from 0. The preceding discussion shows that, with probability one, the excursion intervals of β away from 0 are exactly the intervals $(\sigma_{\ell-}, \sigma_\ell)$ for $\ell \in D$. Then, for every $\ell \in D$, we define the excursion $e_\ell = (e_\ell(t))_{t\geq 0}$ associated with the interval $(\sigma_{\ell-}, \sigma_\ell)$ by setting

$$e_\ell(t) = \begin{cases} \beta_{\sigma_{\ell-}+t} & \text{if } 0 \leq t \leq \sigma_\ell - \sigma_{\ell-}, \\ 0 & \text{if } t > \sigma_\ell - \sigma_{\ell-}. \end{cases}$$

We view e_ℓ as an element of the excursion space E, which is defined by

$$E = \{e \in C(\mathbb{R}_+, \mathbb{R}_+) : e(0) = 0 \text{ and } \zeta(e) := \sup\{s > 0 : e(s) > 0\} \in (0, \infty)\},$$

where $\sup \varnothing = 0$ by convention. Note that we require $\zeta(e) > 0$, so that the zero function does not belong to E. The space E is equipped with the metric d defined by

$$d(e, e') = \sup_{t\geq 0} |e(t) - e'(t)| + |\zeta(e) - \zeta(e')|$$

and with the associated Borel σ-field. Notice that $\zeta(e_\ell) = \sigma_\ell - \sigma_{\ell-}$ for every $\ell \in D$. The following theorem is the basic result of excursion theory in our particular setting.

THEOREM 2.7. *The point measure*

$$\sum_{\ell \in D} \delta_{(\ell, e_\ell)}(ds\, de)$$

is a Poisson measure on $\mathbb{R}_+ \times E$, with intensity

$$2ds \otimes \mathbf{n}(de)$$

where $\mathbf{n}(de)$ is a σ-finite measure on E.

The measure $\mathbf{n}(de)$ is called the Itô measure of positive excursions of linear Brownian motion, or simply the Itô excursion measure (our measure $\mathbf{n}$ corresponds to the measure n_+ in Chapter XII of [**46**]). The next corollary follows from standard properties of Poisson measures.

COROLLARY 2.8. *Let A be a measurable subset of E such that $0 < \mathbf{n}(A) < \infty$, and let $T_A = \inf\{\ell \in D : e_\ell \in A\}$. Then, T_A is exponentially distributed with parameter $\mathbf{n}(A)$, and the distribution of e_{T_A} is the conditional measure*

$$\mathbf{n}(\cdot \,|\, A) = \frac{\mathbf{n}(\cdot \cap A)}{\mathbf{n}(A)}.$$

Moreover, T_A and e_{T_A} are independent.

This corollary can be used to calculate various distributions under the Itô excursion measure. The distribution of the height and the length of the excursion are given as follows: For every $\varepsilon > 0$,

$$\mathbf{n}\Big(\max_{t \geq 0} e(t) > \varepsilon \Big) = \frac{1}{2\varepsilon}$$

and

$$\mathbf{n}(\zeta(e) > \varepsilon) = \frac{1}{\sqrt{2\pi\varepsilon}}.$$

The Itô excursion measure enjoys the following scaling property. For every $\lambda > 0$, define a mapping $\Phi_\lambda : E \longrightarrow E$ by setting $\Phi_\lambda(e)(t) = \sqrt{\lambda}\, e(t/\lambda)$, for every $e \in E$ and $t \geq 0$. Then we have $\Phi_\lambda(\mathbf{n}) = \sqrt{\lambda}\, \mathbf{n}$.

This scaling property is useful when defining conditional versions of the Itô excursion measure. We discuss the conditioning of $\mathbf{n}(de)$ with respect to the length $\zeta(e)$. There exists a unique collection $(\mathbf{n}_{(s)}, s > 0)$ of probability measures on E such that the following properties hold:

(i) For every $s > 0$, $\mathbf{n}_{(s)}(\zeta = s) = 1$.
(ii) For every $\lambda > 0$ and $s > 0$, we have $\Phi_\lambda(\mathbf{n}_{(s)}) = \mathbf{n}_{(\lambda s)}$.
(iii) For every measurable subset A of E,

$$\mathbf{n}(A) = \int_0^\infty \mathbf{n}_{(s)}(A)\, \frac{ds}{2\sqrt{2\pi s^3}}.$$

We may and will write $\mathbf{n}_{(s)} = \mathbf{n}(\cdot \,|\, \zeta = s)$. The measure $\mathbf{n}_{(1)} = \mathbf{n}(\cdot \,|\, \zeta = 1)$ is called the law of the normalized Brownian excursion.

There are many different descriptions of the Itô excursion measure: See in particular [**46**, Chapter XII]. We state the following proposition, which emphasizes the Markovian properties of **n**. For every $t > 0$ and $x > 0$, we set

$$q_t(x) = \frac{x}{\sqrt{2\pi t^3}} \exp(-\frac{x^2}{2t}).$$

Note that the function $t \mapsto q_t(x)$ is the density of the first hitting time of x by B. For $t > 0$ and $x, y \in \mathbb{R}$, we also let

$$p_t(x, y) = \frac{1}{\sqrt{2\pi t}} \exp(-\frac{(y-x)^2}{2t})$$

be the usual Brownian transition density.

PROPOSITION 2.9. *The Itô excursion measure* **n** *is the only σ-finite measure on E that satisfies the following two properties:*

  (i) *For every $t > 0$, and every $f \in C(\mathbb{R}_+, \mathbb{R}_+)$,*

$$\mathbf{n}(f(e(t))\,\mathbf{1}_{\{\zeta > t\}}) = \int_0^\infty f(x)\,q_t(x)\,dx.$$

  (ii) *Let $t > 0$. Under the conditional probability measure $\mathbf{n}(\cdot \mid \zeta > t)$, the process $(e(t+r))_{r \geq 0}$ is Markov with the transition kernels of Brownian motion stopped upon hitting 0.*

This proposition can be used to establish absolute continuity properties of the conditional measures $\mathbf{n}_{(s)}$ with respect to **n**. For every $t \geq 0$, let $\mathcal{F}_t$ denote the σ-field on E generated by the mappings $r \mapsto e(r)$, for $0 \leq r \leq t$. Then, if $0 < t < 1$, the measure $\mathbf{n}_{(1)}$ is absolutely continuous with respect to **n** on the σ-field $\mathcal{F}_t$, with Radon-Nikodým density

$$\frac{d\mathbf{n}_{(1)}}{d\mathbf{n}}\Big|_{\mathcal{F}_t}(e) = 2\sqrt{2\pi}\,q_{1-t}(e(t)).$$

This formula provides a simple derivation of the finite-dimensional marginals under $\mathbf{n}_{(1)}$, noting that the finite-dimensional marginals under **n** are easily obtained from Proposition 2.9. More precisely, for every integer $p \geq 1$, and every choice of $0 < t_1 < t_2 < \cdots < t_p < 1$, we get that the distribution of $(e(t_1), \ldots, e(t_p))$ under $\mathbf{n}_{(1)}(de)$ has density

$$(1) \qquad 2\sqrt{2\pi}\,q_{t_1}(x_1)\,p^*_{t_2-t_1}(x_1, x_2)\,p^*_{t_3-t_2}(x_2, x_3) \cdots p^*_{t_p - t_1}(x_{p-1}, x_p)\,q_{1-t_p}(x_p)$$

where

$$p_t^*(x, y) = p_t(x, y) - p_t(x, -y)\ ,\ t > 0\ ,\ x, y > 0$$

is the transition density of Brownian motion killed when it hits 0. As a side remark, formula (1) shows that the law of $(e(t))_{0 \leq t \leq 1}$ under $\mathbf{n}_{(1)}$ is invariant under time-reversal.

2.5. Convergence of contour functions to the Brownian excursion. The following theorem can be viewed as a special case of the results in Aldous [**2**]. The space of all continuous functions from $[0, 1]$ into $\mathbb{R}_+$ is denoted by $C([0, 1], \mathbb{R}_+)$, and is equipped with the topology of uniform convergence.

THEOREM 2.10. *For every integer $k \geq 1$, let θ_k be a random tree that is uniformly distributed over $\mathbf{A}_k$, and let $(C_k(t))_{t \geq 0}$ be its contour function. Then*

$$\left(\frac{1}{\sqrt{2k}} C_k(2k\,t) \right)_{0 \leq t \leq 1} \xrightarrow[k \to \infty]{\text{(d)}} (\mathbb{e}_t)_{0 \leq t \leq 1}$$

where $\mathbb{e}$ is distributed according to $\mathbf{n}_{(1)}$ (i.e. $\mathbb{e}$ is a normalized Brownian excursion) and the convergence holds in the sense of weak convergence of the laws on the space $C([0,1], \mathbb{R}_+)$.

PROOF. We already noticed that $\Pi_{\mu_0}(\cdot \mid |\tau| = k)$ coincides with the uniform distribution over $\mathbf{A}_k$. By combining this with Proposition 2.6, we get that $(C_k(0), C_k(1), \ldots, C_k(2k))$ is distributed as an excursion of simple random walk conditioned to have length $2k$. Recall our notation $(S_n)_{n \geq 0}$ for simple random walk on $\mathbb{Z}$ starting from 0, and $T = \inf\{n \geq 0 : S_n = -1\}$. To get the desired result, we need to verify that the law of

$$\left(\frac{1}{\sqrt{2k}} S_{\lfloor 2kt \rfloor} \right)_{0 \leq t \leq 1}$$

under $P(\cdot \mid T = 2k+1)$ converges to $\mathbf{n}_{(1)}$ as $k \to \infty$. This result can be seen as a conditional version of Donsker's theorem (see Kaigh [26] for similar statements). We will provide a detailed proof, because this result plays a major role in what follows, and because some of the ingredients of the proof will be needed again in Section 3 below. As usual, the proof is divided into two parts: We first check the convergence of finite-dimensional marginals, and then establish the tightness of the sequence of laws.

Finite-dimensional marginals. We first consider one-dimensional marginals. So we fix $t \in (0,1)$, and we will verify that

$$(2) \qquad \lim_{k \to \infty} \sqrt{2k}\, P\Big(S_{\lfloor 2kt \rfloor} = \lfloor x\sqrt{2k} \rfloor \text{ or } \lfloor x\sqrt{2k} \rfloor + 1 \,\Big|\, T = 2k+1 \Big)$$
$$= 4\sqrt{2\pi}\, q_t(x)\, q_{1-t}(x),$$

uniformly when x varies over a compact subset of $(0, \infty)$. Comparing with the case $p = 1$ of formula (1), we see that the law of $(2k)^{-1/2} S_{\lfloor 2kt \rfloor}$ under $P(\cdot \mid T = 2k+1)$ converges to the law of $e(t)$ under $n_{(1)}(de)$ (we even get a local version of this convergence).

In order to prove (2), we will use two lemmas. The first one is a very special case of classical local limit theorems (see e.g. Chapter 2 of Spitzer [49]).

LEMMA 2.11. *For every $\varepsilon > 0$,*

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}} \sup_{s \geq \varepsilon} \left| \sqrt{n} P\Big(S_{\lfloor ns \rfloor} = \lfloor x\sqrt{n} \rfloor \text{ or } \lfloor x\sqrt{n} \rfloor + 1 \Big) - 2\, p_s(0, x) \right| = 0.$$

In our special situation, the result of the lemma is easily obtained by direct calculations using the explicit form of the law of S_n and Stirling's formula.

The next lemma is (a special case of) a famous formula of Kemperman (see e.g. [45] Chapter 6). For every integer $\ell \in \mathbb{Z}$, we use P_ℓ for a probability measure under which the simple random walk S starts from ℓ.

LEMMA 2.12. *For every $\ell \in \mathbb{Z}_+$ and every integer $n \geq 1$,*

$$P_\ell(T = n) = \frac{\ell+1}{n}\, P_\ell(S_n = -1).$$

PROOF. It is easy to see that

$$P_\ell(T = n) = \frac{1}{2} P_\ell(S_{n-1} = 0, T > n - 1).$$

On the other hand,

$$
\begin{aligned}
P_\ell(S_{n-1} = 0, T > n - 1) &= P_\ell(S_{n-1} = 0) - P_\ell(S_{n-1} = 0, T \le n - 1) \\
&= P_\ell(S_{n-1} = 0) - P_\ell(S_{n-1} = -2, T \le n - 1) \\
&= P_\ell(S_{n-1} = 0) - P_\ell(S_{n-1} = -2),
\end{aligned}
$$

where the second equality is a simple application of the reflection principle. So we have

$$P_\ell(T = n) = \frac{1}{2}\Big(P_\ell(S_{n-1} = 0) - P_\ell(S_{n-1} = -2) \Big)$$

and an elementary calculation shows that this is equivalent to the statement of the lemma. $\square$

Let us turn to the proof of (2). We first write for $i \in \{1, \ldots, 2k\}$ and $\ell \in \mathbb{Z}_+$,

$$P(S_i = \ell \mid T = 2k + 1) = \frac{P(\{S_i = \ell\} \cap \{T = 2k + 1\})}{P(T = 2k + 1)}.$$

By an application of the Markov property of S,

$$P(\{S_i = \ell\} \cap \{T = 2k + 1\}) = P(S_i = \ell, T > i)\, P_\ell(T = 2k + 1 - i).$$

Furthermore, a simple time-reversal argument (we leave the details to the reader) shows that

$$P(S_i = \ell, T > i) = 2\, P_\ell(T = i + 1).$$

Summarizing, we have obtained

$$
\begin{aligned}
(3) \qquad P(S_i = \ell \mid T = 2k + 1) &= \frac{2 P_\ell(T = i + 1) P_\ell(T = 2k + 1 - i)}{P(T = 2k + 1)} \\
&= \frac{2(2k+1)(\ell + 1)^2}{(i+1)(2k+1-i)} \frac{P_\ell(S_{i+1} = -1) P_\ell(S_{2k+1-i} = -1)}{P(S_{2k+1} = -1)}
\end{aligned}
$$

using Lemma 2.12 in the second equality.

We apply this identity with $i = \lfloor 2kt \rfloor$ and $\ell = \lfloor x\sqrt{2k} \rfloor$ or $\ell = \lfloor x\sqrt{2k} \rfloor + 1$. Using Lemma 2.11, we have first

$$\frac{2(2k+1)(\lfloor x\sqrt{2k} \rfloor + 1)^2}{(\lfloor 2kt \rfloor + 1)(2k + 1 - \lfloor 2kt \rfloor)} \times \frac{1}{P(S_{2k+1} = -1)} \approx 2\sqrt{2\pi}\,(k/2)^{1/2}\,\frac{x^2}{t(1-t)}$$

and, using Lemma 2.11 once again,

$$
\begin{aligned}
&P_{\lfloor x\sqrt{2k} \rfloor}(S_{\lfloor 2kt \rfloor + 1} = -1) P_{\lfloor x\sqrt{2k} \rfloor}(S_{2k+1-\lfloor 2kt \rfloor} = -1) \\
&\quad + \ P_{\lfloor x\sqrt{2k} \rfloor + 1}(S_{\lfloor 2kt \rfloor + 1} = -1) P_{\lfloor x\sqrt{2k} \rfloor + 1}(S_{2k+1-\lfloor 2kt \rfloor} = -1) \\
&\quad \approx 2\, k^{-1}\, p_t(0, x) p_{1-t}(0, x).
\end{aligned}
$$

Putting these estimates together, and noting that $q_t(x) = (x/t) p_t(0, x)$, we arrive at (2).

Higher order marginals can be treated in a similar way. Let us sketch the argument in the case of two-dimensional marginals. We observe that, if $0 < i < j < 2k$ and if $\ell, m \in \mathbb{Z}_+$, we have, by the same arguments as above,

$$
\begin{aligned}
&P(S_i = \ell, S_j = m, T = 2k + 1) \\
&\quad = \ 2\, P_\ell(T = i + 1)\, P_\ell(S_{j-i} = m, T > j - i)\, P_m(T = k + 1 - j).
\end{aligned}
$$

Only the middle term $P_\ell(S_{j-i} = m, T > j-i)$ requires a different treatment than in the case of one-dimensional marginals. However, by an application of the reflection principle, one has

$$P_\ell(S_{j-i} = m, T > j - i) = P_\ell(S_{j-i} = m) - P_\ell(S_{j-i} = -m - 2).$$

Hence, using Lemma 2.11, we easily obtain that for $x, y > 0$ and $0 < s < t < 1$,

$$P_{\lfloor x\sqrt{2k}\rfloor}(S_{\lfloor 2kt\rfloor - \lfloor 2ks\rfloor} = \lfloor y\sqrt{2k}\rfloor) + P_{\lfloor x\sqrt{2k}\rfloor+1}(S_{\lfloor 2kt\rfloor - \lfloor 2ks\rfloor} = \lfloor y\sqrt{2k}\rfloor)$$
$$\approx (2k)^{-1/2}\, p^*_{t-s}(x, y),$$

and the result for two-dimensional marginals follows in a straightforward way.

Tightness. We start with some combinatorial considerations. We fix $k \geq 1$. Let $(x_0, x_1, \ldots, x_{2k})$ be a Dyck path with length $2k$, and let $i \in \{0, 1, \ldots, 2k - 1\}$. We set, for every $j \in \{0, 1, \ldots, 2k\}$,

$$x_j^{(i)} = x_i + x_{i\oplus j} - 2 \min_{i \wedge (i\oplus j) \leq n \leq i \vee (i\oplus j)} x_n$$

with the notation $i \oplus j = i + j$ if $i + j \leq 2k$, and $i \oplus j = i + j - 2k$ if $i + j > 2k$. It is elementary to see that $(x_0^{(i)}, x_1^{(i)}, \ldots, x_{2k}^{(i)})$ is again a Dyck path with length $2k$. Moreover, the mapping $\Phi_i : (x_0, x_1, \ldots, x_{2k}) \longrightarrow (x_0^{(i)}, x_1^{(i)}, \ldots, x_{2k}^{(i)})$ is a bijection from the set of all Dyck paths with length $2k$ onto itself. To see this, one may check that the composition $\Phi_{2k-i} \circ \Phi_i$ is the identity mapping. This property is easily verified by viewing Φ_i as a mapping defined on plane trees with $2k$ edges (using Proposition 2.2): The plane tree corresponding to the image under Φ_i of the Dyck path associated with a tree τ is the "same" tree τ re-rooted at the corner corresponding to the i-th step of the contour exploration of τ. From this observation it is obvious that the composition $\Phi_{2k-i} \circ \Phi_i$ leads us back to the original plane tree.

To simplify notation, we set for every $i, j \in \{0, 1, \ldots, 2k\}$,

$$\check{C}_k^{i,j} = \min_{i \wedge j \leq n \leq i \vee j} C_k(n).$$

The preceding discussion then gives the identity in distribution

$$(4) \qquad \left(C_k(i) + C_k(i \oplus j) - 2\check{C}_k^{i,i\oplus j}\right)_{0 \leq j \leq 2k} \overset{(d)}{=} (C_k(j))_{0 \leq j \leq 2k}.$$

LEMMA 2.13. *For every integer $p \geq 1$, there exists a constant K_p such that, for every $k \geq 1$ and every $i \in \{0, 1, \ldots, 2k\}$,*

$$E[C_k(i)^{2p}] \leq K_p\, i^p.$$

Assuming that the lemma holds, the proof of tightness is easily completed. Using the identity (4), we get for $0 \leq i < j \leq 2k$,

$$\begin{aligned} E[(C_k(j) - C_k(i))^{2p}] &\leq E[(C_k(i) + C_k(j) - 2\check{C}_k^{i,j})^{2p}] \\ &= E[C_k(j - i)^{2p}] \\ &\leq K_p(j - i)^p. \end{aligned}$$

It readily follows that the bound

$$E\left[\left(\frac{C_k(2kt) - C_k(2ks)}{\sqrt{2k}}\right)^{2p}\right] \leq K_p\,(t - s)^p.$$

holds at least if s and t are of the form $s = i/2k$, $t = j/2k$, with $0 \le i < j \le 2k$. Since the function C_k is 1-Lipschitz, a simple argument shows that the same bound holds (possibly with a different constant K_p) whenever $0 \le s < t \le 1$. This gives the desired tightness, but we still have to prove the lemma.

Proof of Lemma 2.13. Clearly, we may restrict our attention to the case $1 \le i \le k$ (note that $(C_k(2k - i))_{0 \le i \le 2k}$ has the same distribution as $(C_k(i))_{0 \le i \le 2k}$). Recall that $C_k(i)$ has the same distribution as S_i under $P(\cdot \mid T = 2k + 1)$. By formula (3), we have thus, for every integer $\ell \ge 0$,

$$P(C_k(i) = \ell) = \frac{2(2k+1)(\ell+1)^2}{(i+1)(2k+1-i)} \frac{P_\ell(S_{i+1} = -1)P_\ell(S_{2k+1-i} = -1)}{P(S_{2k+1} = -1)}.$$

From Lemma 2.11 (and our assumption $i \le k$), we can find two positive constants c_0 and c_1 such that

$$P(S_{2k+1} = -1) \ge c_0(2k)^{-1/2} , \quad P_\ell(S_{2k+1-i} = -1) \le c_1(2k)^{-1/2}.$$

It then follows that

$$
\begin{aligned}
P(C_k(i) = \ell) &\le 4c_1(c_0)^{-1} \frac{(\ell+1)^2}{i+1} P_\ell(S_{i+1} = -1) \\
&= 4c_1(c_0)^{-1} \frac{(\ell+1)^2}{i+1} P(S_{i+1} = \ell+1).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
E[C_k(i)^{2p}] &= \sum_{\ell=0}^{\infty} \ell^{2p} P(C_k(i) = \ell) \\
&\le \frac{4c_1(c_0)^{-1}}{i+1} \sum_{\ell=0}^{\infty} \ell^{2p}(\ell+1)^2 P(S_{i+1} = \ell+1) \\
&\le \frac{4c_1(c_0)^{-1}}{i+1} E[(S_{i+1})^{2p+2}].
\end{aligned}
$$

However, it is well known and easy to prove that $E[(S_{i+1})^{2p+2}] \le K'_p(i+1)^{p+1}$, with some constant K'_p independent of i. This completes the proof of the lemma and of Theorem 2.10. $\square$

Extensions and variants of Theorem 2.10 can be found in [**2**], [**17**] and [**18**]. To illustrate the power of this theorem, let us give a typical application. The height $H(\tau)$ of a plane tree τ is the maximal generation of a vertex of τ.

COROLLARY 2.14. *Let θ_k be uniformly distributed over $\mathbf{A}_k$. Then*

$$\frac{1}{\sqrt{2k}} H(\theta_k) \xrightarrow[k \to \infty]{(d)} \max_{0 \le t \le 1} \mathbb{e}_t.$$

Since

$$\frac{1}{\sqrt{2k}} H(\theta_k) = \max_{0 \le t \le 1} \left(\frac{1}{\sqrt{2k}} C_k(2k\,t) \right)$$

the result of the corollary is immediate from Theorem 2.10.

The limiting distribution in Corollary 2.14 is known in the form of a series: For every $x > 0$,

$$P\left(\max_{0 \le t \le 1} \mathbb{e}_t > x \right) = 2 \sum_{k=1}^{\infty} (4k^2x^2 - 1) \exp(-2k^2x^2).$$

See Chung [**13**].

3. Real trees and the Gromov-Hausdorff convergence

Our main goal in this section is to interpret the convergence of contour functions in Theorem 2.10 as a convergence of discrete random trees towards a "continuous random tree" which is coded by the Brownian excursion in the same sense as a plane tree is coded by its contour function. We need to introduce a suitable notion of a continuous tree, and then to explain in which sense the convergence takes place.

3.1. Real trees. We start with a formal definition. In these notes, we consider only *compact* real trees, and so we include this compactness property in the definition.

DEFINITION 3.1. *A compact metric space* $(\mathcal{T}, d)$ *is a real tree if the following two properties hold for every* $a, b \in \mathcal{T}$.
  (i) *There is a unique isometric map* $f_{a,b}$ *from* $[0, d(a, b)]$ *into* $\mathcal{T}$ *such that* $f_{a,b}(0) = a$ *and* $f_{a,b}(d(a, b)) = b$.
  (ii) *If* q *is a continuous injective map from* $[0, 1]$ *into* $\mathcal{T}$, *such that* $q(0) = a$ *and* $q(1) = b$, *we have*

$$q([0, 1]) = f_{a,b}([0, d(a, b)]).$$

A rooted real tree is a real tree $(\mathcal{T}, d)$ *with a distinguished vertex* $\rho = \rho(\mathcal{T})$ *called the root. In what follows, real trees will always be rooted, even if this is not mentioned explicitly.*

Informally, one should think of a (compact) real tree as a connected union of line segments in the plane with no loops. Asssume for simplicity that there are finitely many segments in the union. Then, for any two points a and b in the tree, there is a unique path going from a to b in the tree, which is the concatenation of finitely many line segments. The distance between a and b is then the length of this path.

Let us consider a rooted real tree $(\mathcal{T}, d)$. The range of the mapping $f_{a,b}$ in (i) is denoted by $[\![a, b]\!]$ (this is the "line segment" between a and b in the tree). In particular, $[\![\rho, a]\!]$ is the path going from the root to a, which we will interpret as the ancestral line of vertex a. More precisely, we can define a partial order on the tree by setting $a \preccurlyeq b$ (a is an ancestor of b) if and only if $a \in [\![\rho, b]\!]$.

If $a, b \in \mathcal{T}$, there is a unique $c \in \mathcal{T}$ such that $[\![\rho, a]\!] \cap [\![\rho, b]\!] = [\![\rho, c]\!]$. We write $c = a \wedge b$ and call c the most recent common ancestor to a and b.

By definition, the multiplicity of a vertex $a \in \mathcal{T}$ is the number of connected components of $\mathcal{T} \backslash \{a\}$. Vertices of $\mathcal{T}$ which have multiplicity 1 are called leaves.

3.2. Coding real trees. In this subsection, we describe a method for constructing real trees, which is well-suited to our forthcoming applications to random trees. This method is nothing but a continuous analog of the coding of discrete trees by contour functions.

We consider a (deterministic) continuous function $g : [0, 1] \longrightarrow [0, \infty)$ such that $g(0) = g(1) = 0$. To avoid trivialities, we will also assume that g is not identically zero. For every $s, t \in [0, 1]$, we set
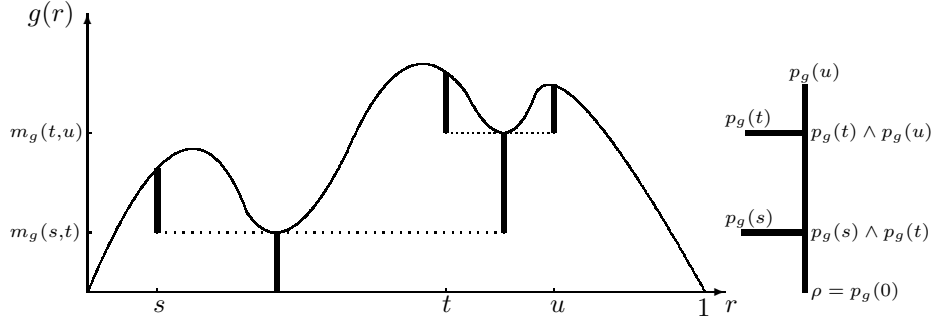
$$m_g(s, t) = \inf_{r \in [s \wedge t, s \vee t]} g(r),$$

FIGURE 3. Coding a tree by a continuous function

and
$$d_g(s,t) = g(s) + g(t) - 2m_g(s,t).$$
Clearly $d_g(s,t) = d_g(t,s)$ and it is also easy to verify the triangle inequality
$$d_g(s,u) \leq d_g(s,t) + d_g(t,u)$$
for every $s, t, u \in [0,1]$. We then introduce the equivalence relation $s \sim t$ iff $d_g(s,t) = 0$ (or equivalently iff $g(s) = g(t) = m_g(s,t)$). Let $\mathcal{T}_g$ be the quotient space
$$\mathcal{T}_g = [0,1]/\sim .$$
Obviously the function d_g induces a distance on $\mathcal{T}_g$, and we keep the notation d_g for this distance. We denote by $p_g : [0,1] \longrightarrow \mathcal{T}_g$ the canonical projection. Clearly p_g is continuous (when $[0,1]$ is equipped with the Euclidean metric and $\mathcal{T}_g$ with the metric d_g), and the metric space $(\mathcal{T}_g, d_g)$ is thus compact.

THEOREM 3.1. *The metric space $(\mathcal{T}_g, d_g)$ is a real tree. We will view $(\mathcal{T}_g, d_g)$ as a rooted tree with root $\rho = p_g(0) = p_g(1)$.*

REMARK 3.2. *It is also possible to prove that any (rooted) real tree can be represented in the form $\mathcal{T}_g$. We will leave this as an exercise for the reader.*

To get an intuitive understanding of Theorem 3.1, the reader should have a look at Fig.3. This figure shows how to construct a simple subtree of $\mathcal{T}_g$, namely the "reduced tree" consisting of the union of the ancestral lines in $\mathcal{T}_g$ of three vertices $p_g(s), p_g(t), p_g(u)$ corresponding to three (given) times $s, t, u \in [0,1]$. This reduced tree is the union of the five bold line segments that are constructed from the graph of g in the way explained on the left part of the figure. Notice that the lengths of the horizontal dotted lines play no role in the construction, and that the reduced tree should be viewed as pictured on the right part of Fig.3. The ancestral line of $p_g(s)$ (resp. $p_g(t), p_g(u)$) is a line segment of length $g(s)$ (resp. $g(t), g(u)$). The ancestral lines of $p_g(s)$ and $p_g(t)$ share a common part, which has length $m_g(s,t)$ (the line segment at the bottom in the left or the right part of Fig.3), and of course a similar property holds for the ancestral lines of $p_g(s)$ and $p_g(u)$, or of $p_g(t)$ and $p_g(u)$.

The following re-rooting lemma, which is of independent interest, is a useful ingredient of the proof of Theorem 3.1 (a discrete version of this lemma already appeared at the beginning of the proof of tightness in Theorem 2.10).

LEMMA 3.3. *Let $s_0 \in [0, 1)$. For any real $r \geq 0$, denote the fractional part of r by $\overline{r} = r - \lfloor r \rfloor$. Set*

$$g'(s) = g(s_0) + g(\overline{s_0 + s}) - 2m_g(s_0, \overline{s_0 + s}),$$

for every $s \in [0, 1]$. Then, the function g' is continuous and satisfies $g'(0) = g'(1) = 0$, so that we can define $\mathcal{T}_{g'}$. Furthermore, for every $s, t \in [0, 1]$, we have

$$(5) \qquad\qquad d_{g'}(s, t) = d_g(\overline{s_0 + s}, \overline{s_0 + t})$$

and there exists a unique isometry R from $\mathcal{T}_{g'}$ onto $\mathcal{T}_g$ such that, for every $s \in [0, 1]$,

$$(6) \qquad\qquad R(p_{g'}(s)) = p_g(\overline{s_0 + s}).$$

Assuming that Theorem 3.1 is proved, we see that $\mathcal{T}_{g'}$ coincides with the real tree $\mathcal{T}_g$ re-rooted at $p_g(s_0)$. Thus the lemma tells us which function codes the tree $\mathcal{T}_g$ re-rooted at an arbitrary vertex.

Proof. It is immediately checked that g' satisfies the same assumptions as g, so that we can make sense of $\mathcal{T}_{g'}$. Then the key step is to verify the relation (5). Consider first the case where $s, t \in [0, 1 - s_0)$. Then two possibilities may occur.

If $m_g(s_0 + s, s_0 + t) \geq m_g(s_0, s_0 + s)$, then $m_g(s_0, s_0 + r) = m_g(s_0, s_0 + s) = m_g(s_0, s_0 + t)$ for every $r \in [s, t]$, and so

$$m_{g'}(s, t) = g(s_0) + m_g(s_0 + s, s_0 + t) - 2m_g(s_0, s_0 + s).$$

It follows that

$$\begin{aligned}
d_{g'}(s, t) &= g'(s) + g'(t) - 2m_{g'}(s, t) \\
&= g(s_0 + s) - 2m_g(s_0, s_0 + s) + g(s_0 + t) \\
&\quad - 2m_g(s_0, s_0 + t) - 2(m_g(s_0 + s, s_0 + t) - 2m_g(s_0, s_0 + s)) \\
&= g(s_0 + s) + g(s_0 + t) - 2m_g(s_0 + s, s_0 + t) \\
&= d_g(s_0 + s, s_0 + t).
\end{aligned}$$

If $m_g(s_0 + s, s_0 + t) < m_g(s_0, s_0 + s)$, then the minimum in the definition of $m_{g'}(s, t)$ is attained at r_1 defined as the first $r \in [s, t]$ such that $g(s_0 + r) = m_g(s_0, s_0 + s)$ (because for $r \in [r_1, t]$ we will have $g(s_0 + r) - 2m_g(s_0, s_0 + r) \geq -m_g(s_0, s_0 + r) \geq -m_g(s_0, s_0 + r_1)$). Therefore,

$$m_{g'}(s, t) = g(s_0) - m_g(s_0, s_0 + s),$$

and

$$\begin{aligned}
d_{g'}(s, t) &= g(s_0 + s) - 2m_g(s_0, s_0 + s) + g(s_0 + t) \\
&\quad - 2m_g(s_0, s_0 + t) + 2m_g(s_0, s_0 + s) \\
&= d_g(s_0 + s, s_0 + t).
\end{aligned}$$

The other cases are treated in a similar way and are left to the reader.

By (5), if $s, t \in [0, 1]$ are such that $d_{g'}(s, t) = 0$, then $d_g(\overline{s_0 + s}, \overline{s_0 + t}) = 0$ so that $p_g(\overline{s_0 + s}) = p_g(\overline{s_0 + t})$. Noting that $\mathcal{T}_{g'} = p_{g'}([0, 1])$, we can define R in a unique way by the relation (6). From (5), R is an isometry, and it is also immediate that R takes $\mathcal{T}_{g'}$ onto $\mathcal{T}_g$. $\square$

Thanks to the lemma, the fact that $\mathcal{T}_g$ verifies property (i) in the definition of a real tree is obtained from the particular case when $a = \rho$ and $b = p_g(s)$ for some

$s \in [0, 1]$. In that case however, the isometric mapping $f_{\rho,b}$ is easily constructed by setting

$$f_{\rho,b}(t) = p_g(\sup\{r \leq s : g(r) = t\}) , \quad \text{for every } 0 \leq t \leq g(s) = d_g(\rho, b).$$

The remaining part of the argument is straightforward: See Section 2 in [**19**].

REMARK 3.4. *A short proof of Theorem 3.1 using the characterization of real trees via the so-called four-point condition can be found in* [**20**].

The following simple observation will be useful in Section 7: If $s, t \in [0, 1]$, the line segment $[\![p_g(s), p_g(t)]\!]$ in the tree $\mathcal{T}_g$ coincides with the collection of the vertices $p_g(r)$, for all $r \in [0, 1]$ such that either $g(r) = m_g(r, s) \geq m_g(s, t)$ or $g(r) = m_g(r, t) \geq m_g(s, t)$. This easily follows from the construction of the distance d_g.

3.3. The Gromov-Hausdorff convergence. In order to make sense of the convergence of discrete trees towards real trees, we will use the Gromov-Hausdorff distance between compact metric spaces, which has been introduced by Gromov (see e.g. [**22**]) in view of geometric applications.

If (E, δ) is a metric space, the notation $\delta_{Haus}(K, K')$ stands for the usual Hausdorff metric between compact subsets of E :

$$\delta_{Haus}(K, K') = \inf\{\varepsilon > 0 : K \subset U_\varepsilon(K') \text{ and } K' \subset U_\varepsilon(K)\},$$

where $U_\varepsilon(K) := \{x \in E : \delta(x, K) \leq \varepsilon\}$.

A pointed metric space is just a pair consisting of a metric space E and a distinguished point ρ of E. We often write E instead of (E, ρ) to simplify notation.

Then, if (E_1, ρ_1) and (E_2, ρ_2) are two pointed compact metric spaces, we define the distance $d_{GH}(E_1, E_2)$ by

$$d_{GH}(E_1, E_2) = \inf\{\delta_{Haus}(\varphi_1(E_1), \varphi_2(E_2)) \vee \delta(\varphi_1(\rho_1), \varphi_2(\rho_2))\}$$

where the infimum is over all possible choices of the metric space (E, δ) and the isometric embeddings $\varphi_1 : E_1 \longrightarrow E$ and $\varphi_2 : E_2 \longrightarrow E$ of E_1 and E_2 into E.

Two pointed compact metric spaces E_1 and E_2 are called equivalent if there is an isometry that maps E_1 onto E_2 and preserves the distinguished points. Obviously $d_{GH}(E_1, E_2)$ only depends on the equivalence classes of E_1 and E_2. We denote by $\mathbb{K}$ the space of all equivalence classes of pointed compact metric spaces.

THEOREM 3.5. d_{GH} *defines a metric on the set* $\mathbb{K}$. *Furthermore the metric space* $(\mathbb{K}, d_{GH})$ *is separable and complete.*

A proof of the fact that d_{GH} is a metric on the set $\mathbb{K}$ can be found in [**11**, Theorem 7.3.30]. This proof is in fact concerned with the non-pointed case, but the argument is easily adapted to our setting. The separability of the space $(\mathbb{K}, d_{GH})$ follows from the fact that finite metric spaces are dense in $\mathbb{K}$. Finally the completeness of $(\mathbb{K}, d_{GH})$ can be obtained as a consequence of the compactness theorem in [**11**, Theorem 7.4.15].

In our applications, it will be important to have the following alternative definition of d_{GH}. First recall that if (E_1, d_1) and (E_2, d_2) are two compact metric spaces, a correspondence between E_1 and E_2 is a subset $\mathcal{R}$ of $E_1 \times E_2$ such that for every $x_1 \in E_1$ there exists at least one $x_2 \in E_2$ such that $(x_1, x_2) \in \mathcal{R}$ and conversely for every $y_2 \in E_2$ there exists at least one $y_1 \in E_1$ such that $(y_1, y_2) \in \mathcal{R}$.

The distortion of the correspondence $\mathcal{R}$ is defined by

$$\text{dis}(\mathcal{R}) = \sup\{|d_1(x_1, y_1) - d_2(x_2, y_2)| : (x_1, x_2), (y_1, y_2) \in \mathcal{R}\}.$$

PROPOSITION 3.6. *Let (E_1, ρ_1) and (E_2, ρ_2) be two pointed compact metric spaces. Then,*

(7) $$d_{GH}(E_1, E_2) = \frac{1}{2} \inf_{\mathcal{R} \in \mathcal{C}(E_1, E_2),\, (\rho_1, \rho_2) \in \mathcal{R}} \text{dis}(\mathcal{R}),$$

where $\mathcal{C}(E_1, E_2)$ denotes the set of all correspondences between E_1 and E_2.

See [**11**, Theorem 7.3.25] for a proof of this proposition in the non-pointed case, which is easily adapted.

The following consequence of Proposition 3.6 will be very useful. Notice that a rooted real tree can be viewed as a pointed compact metric space, whose distinguished point is the root.

COROLLARY 3.7. *Let g and g' be two continuous functions from $[0, 1]$ into $\mathbb{R}_+$, such that $g(0) = g(1) = g'(0) = g'(1) = 0$. Then,*

$$d_{GH}(\mathcal{T}_g, \mathcal{T}_{g'}) \leq 2\|g - g'\|,$$

where $\|g - g'\| = \sup_{t \in [0,1]} |g(t) - g'(t)|$ is the supremum norm of $g - g'$.

Proof. We rely on formula (7). We can construct a correspondence between $\mathcal{T}_g$ and $\mathcal{T}_{g'}$ by setting

$$\mathcal{R} = \{(a, a') : \exists t \in [0, 1] \text{ such that } a = p_g(t) \text{ and } a' = p_{g'}(t)\}.$$

Note that $(\rho, \rho') \in \mathcal{R}$, if $\rho = p_g(0)$, resp. $\rho' = p_{g'}(0)$, is the root of $\mathcal{T}_g$, resp. the root of $\mathcal{T}_{g'}$. In order to bound the distortion of $\mathcal{R}$, let $(a, a') \in \mathcal{R}$ and $(b, b') \in \mathcal{R}$. By the definition of $\mathcal{R}$ we can find $s, t \geq 0$ such that $p_g(s) = a$, $p_{g'}(s) = a'$ and $p_g(t) = b$, $p_{g'}(t) = b'$. Now recall that

$$d_g(a, b) = g(s) + g(t) - 2m_g(s, t),$$
$$d_{g'}(a', b') = g'(s) + g'(t) - 2m_{g'}(s, t),$$

so that

$$|d_g(a, b) - d_{g'}(a', b')| \leq 4\|g - g'\|.$$

Thus we have $\text{dis}(\mathcal{R}) \leq 4\|g - g'\|$ and the desired result follows from (7). $\square$

3.4. Convergence towards the CRT. As in subsection 2.5, we use the notation $\mathbb{e}$ for a normalized Brownian excursion. We view $\mathbb{e} = (\mathbb{e}_t)_{0 \leq t \leq 1}$ as a (random) continuous function over the interval $[0, 1]$, which satisfies the same assumptions as the function g in subsection 3.2.

DEFINITION 3.2. *The Brownian continuum random tree, also called the CRT, is the random real tree $\mathcal{T}_{\mathbb{e}}$ coded by the normalized Brownian excursion.*

The CRT $\mathcal{T}_{\mathbb{e}}$ is thus a random variable taking values in the set $\mathbb{K}$. Note that the measurability of this random variable follows from Corollary 3.7.

REMARK 3.8. *Aldous [**1**],[**2**] uses a different method to define the CRT. The preceding definition then corresponds to Corollary 22 in [**2**]. Note that our normalization differs by an unimportant scaling factor 2 from the one in Aldous' papers: The CRT there is the tree $\mathcal{T}_{2\mathbb{e}}$ instead of $\mathcal{T}_{\mathbb{e}}$.*

We will now restate Theorem 2.10 as a convergence in distribution of discrete random trees towards the CRT in the space $(\mathbb{K}, d_{GH})$.

THEOREM 3.9. *For every $k \geq 1$, let θ_k be uniformly distributed over $\mathbf{A}_k$, and equip θ_k with the usual graph distance d_{gr}. Then*

$$(\theta_k, (2k)^{-1/2} d_{gr}) \xrightarrow[k \to \infty]{(\mathrm{d})} (\mathcal{T}_\mathbb{e}, d_\mathbb{e})$$

in the sense of convergence in distribution for random variables with values in $(\mathbb{K}, d_{GH})$.

PROOF. As in Theorem 2.10, let C_k be the contour function of θ_k, and define a rescaled version of C_k by setting

$$\widetilde{C}_k(t) = (2k)^{-1/2} C_k(2k\,t)$$

for every $t \in [0,1]$. Note that the function $\widetilde{C}_k$ is continuous and nonnegative over $[0,1]$ and vanishes at 0 and at 1. Therefore we can define the real tree $\mathcal{T}_{\widetilde{C}_k}$.

Now observe that this real tree is very closely related to the (rescaled) discrete tree θ_k. Indeed $\mathcal{T}_{\widetilde{C}_k}$ is (isometric to) a finite union of line segments of length $(2k)^{-1/2}$ in the plane, with genealogical structure prescribed by θ_k, in the way suggested in the left part of Fig.2. From this observation, and the definition of the Gromov-Hausdorff distance, we easily get

(8) $$d_{GH}\Big((\theta_k, (2k)^{-1/2} d_{gr}), (\mathcal{T}_{\widetilde{C}_k}, d_{\widetilde{C}_k})\Big) \leq (2k)^{-1/2}.$$

On the other hand, by combining Theorem 2.10 and Corollary 3.7, we have

$$(\mathcal{T}_{\widetilde{C}_k}, d_{\widetilde{C}_k}) \xrightarrow[k \to \infty]{(\mathrm{d})} (\mathcal{T}_\mathbb{e}, d_\mathbb{e}).$$

The statement of Theorem 3.9 now follows from the latter convergence and (8). $\square$

REMARK 3.10. *Theorem 3.9 contains in fact less information than Theorem 2.10, because the lexicographical ordering that is inherent to the notion of a plane tree (and also to the coding of real trees by functions) disappears when we look at a plane tree as a metric space. Still, Theorem 3.9 is important from the conceptual viewpoint: It is crucial to think of the CRT as a continuous limit of rescaled discrete random trees.*

There are analogs of Theorem 3.9 for other classes of combinatorial trees. For instance, if τ_n is distributed uniformly among all rooted Cayley trees with n vertices, then $(\tau_n, (4n)^{-1/2} d_{gr})$ converges in distribution to the CRT $\mathcal{T}_\mathbb{e}$, in the space $\mathbb{K}$. Similarly, discrete random trees that are uniformly distributed over binary trees with $2k$ edges converge in distribution (modulo a suitable rescaling) towards the CRT. All these results can be derived from a general statement of convergence of conditioned Galton-Watson trees due to Aldous [2] (see also [29]). A recent work of Haas and Miermont [23] provides further extensions of Theorem 3.9 to Pólya trees (unordered rooted trees).

4. Labeled trees and the Brownian snake

4.1. Labeled trees. In view of forthcoming applications to random planar maps, we now introduce labeled trees. A labeled tree is a pair $(\tau, (\ell(v))_{v \in \tau})$ that consists of a plane tree τ (see subsection 2.1) and a collection $(\ell(v))_{v \in \tau}$ of integer

labels assigned to the vertices of τ – in our formalism for plane trees, the tree τ coincides with the set of all its vertices. We assume that labels satisfy the following three properties:

(i) for every $v \in \tau$, $\ell(v) \in \mathbb{Z}$;
(ii) $\ell(\varnothing) = 0$;
(iii) for every $v \in \tau \backslash \{\varnothing\}$, $\ell(v) - \ell(\pi(v)) = 1, 0,$ or -1,

where we recall that $\pi(v)$ denotes the parent of v. Condition (iii) just means that when crossing an edge of τ the label can change by at most 1 in absolute value.

The motivation for introducing labeled trees comes from the fact that (rooted and pointed) planar quadrangulations can be coded by such trees (see Section 4 below). Our goal in the present section is to derive asymptotics for large labeled trees chosen uniformly at random, in the same way as Theorem 2.10, or Theorem 3.9, provides asymptotics for large plane trees. For every integer $k \geq 0$, we denote by $\mathbf{T}_k$ the set of all labeled trees with k edges. It is immediate that

$$\#\mathbf{T}_k = 3^k \#\mathbf{A}_k = \frac{3^k}{k+1} \binom{2k}{k}$$

simply because for each edge of the tree there are three possible choices for the label increment along this edge.

Let $(\tau, (\ell(v))_{v \in \tau})$ be a labeled tree with k edges. As we saw in subsection 2.1, the plane tree τ is coded by its contour function $(C_t)_{t \geq 0}$. We can similarly encode the labels by another function $(V_t)_{t \geq 0}$, which is defined as follows. If we explore the tree τ by following its contour, in the way suggested by the arrows of Fig.2, we visit successively all vertices of τ (vertices that are not leaves are visited more than once). Write $v_0 = \varnothing, v_1, v_2, \ldots, v_{2k} = \varnothing$ for the successive vertices visited in this exploration. For instance, in the particular example of Fig.1 we have

$$v_0 = \varnothing, v_1 = 1, v_2 = (1,1), v_3 = 1, v_4 = (1,2), v_5 = (1,2,1), v_6 = (1,2), \ldots$$

The finite sequence $v_0, v_1, v_2, \ldots, v_{2k}$ will be called the contour exploration of the vertices of τ.

Notice that $C_i = |v_i|$, for every $i = 0, 1, \ldots, 2k$, by the definition of the contour function. We similarly set

$$V_i = \ell(v_i) \quad \text{for every } i = 0, 1, \ldots, 2k.$$

To complete this definition, we set $V_t = 0$ for $t \geq 2k$ and, for every $i = 1, \ldots, 2k$, we define V_t for $t \in (i-1, i)$ by using linear interpolation. We will call $(V_t)_{t \geq 0}$ the "label contour function" of the labeled tree $(\tau, (\ell(v))_{v \in \tau})$ Clearly $(\tau, (\ell(v))_{v \in \tau})$ is determined by the pair $(C_t, V_t)_{t \geq 0}$.

Our goal is now to describe the scaling limit of this pair when the labeled tree $(\tau, (\ell(v))_{v \in \tau})$ is chosen uniformly at random in $\mathbf{T}_k$ and $k \to \infty$. As an immediate consequence of Theorem 2.10 (and the fact that the number of possible labelings is the same for every plane tree with k edges), the scaling limit of $(C_t)_{t \geq 0}$ is the normalized Brownian excursion. To describe the scaling limit of $(V_t)_{t \geq 0}$ we need to introduce the Brownian snake.

4.2. The snake driven by a deterministic function. Consider a continuous function $g : [0, 1] \longrightarrow \mathbb{R}_+$ such that $g(0) = g(1) = 0$ (as in subsection 3.2). We also assume that g is Hölder continuous: There exist two positive constants K and

γ such that, for every $s, t \in [0, 1]$,

$$|g(s) - g(t)| \leq K \, |s - t|^{\gamma}.$$

As in subsection 3.2, we also set, for every $s, t \in [0, 1]$,

$$m_g(s, t) = \min_{r \in [s \wedge t, s \vee t]} g(r).$$

LEMMA 4.1. *The function* $(m_g(s, t))_{s, t \in [0,1]}$ *is nonnegative definite in the sense that, for every integer* $n \geq 1$, *for every* $s_1, \ldots, s_n \in [0, 1]$ *and every* $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, *we have*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j \, m_g(s_i, s_j) \geq 0.$$

PROOF. Fix $s_1, \ldots, s_n \in [0, 1]$, and let $t \geq 0$. For $i, j \in \{1, \ldots, n\}$, put $i \approx j$ if $m_g(s_i, s_j) \geq t$. Then $\approx$ is an equivalence relation on $\{i : g(s_i) \geq t\} \subset \{1, \ldots, n\}$. By summing over the different classes of this equivalence relation, we get that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j \mathbf{1}_{\{t \leq m_g(s_i, s_j)\}} = \sum_{\mathcal{C} \text{ class of } \approx} \left(\sum_{i \in \mathcal{C}} \lambda_i \right)^2 \geq 0.$$

Now integrate with respect to dt to get the desired result. $\qquad\square$

By Lemma 4.1 and a standard application of the Kolmogorov extension theorem, there exists a centered Gaussian process $(Z_s^g)_{s \in [0,1]}$ whose covariance is

$$E[Z_s^g Z_t^g] = m_g(s, t)$$

for every $s, t \in [0, 1]$. Consequently we have

$$
\begin{aligned}
E[(Z_s^g - Z_t^g)^2] &= E[(Z_s^g)^2] + E[(Z_t^g)^2] - 2E[Z_s^g Z_t^g] \\
&= g(s) + g(t) - 2m_g(s, t) \\
&\leq 2K \, |s - t|^{\gamma},
\end{aligned}
$$

where the last bound follows from our Hölder continuity assumption on g (this calculation also shows that $E[(Z_s^g - Z_t^g)^2] = d_g(s, t)$, in the notation of subsection 3.2). From the previous bound and an application of the Kolmogorov continuity criterion, the process $(Z_s^g)_{s \in [0,1]}$ has a modification with continuous sample paths. This leads us to the following definition.

DEFINITION 4.1. *The snake driven by the function* g *is the centered Gaussian process* $(Z_s^g)_{s \in [0,1]}$ *with continuous sample paths and covariance*

$$E[Z_s^g Z_t^g] = m_g(s, t) \ , \quad s, t \in [0, 1].$$

Notice that we have in particular $Z_0^g = Z_1^g = 0$. More generally, for every $t \in [0, 1]$, Z_t^g is normal with mean 0 and variance $g(t)$.

REMARK 4.2. *Recall from subsection 3.2 the definition of the equivalence relation* $\sim$ *associated with* g: $s \sim t$ *iff* $d_g(s, t) = 0$. *Since we have* $E[(Z_s^g - Z_t^g)^2] = d_g(s, t)$, *a simple argument shows that almost surely for every* $s, t \in [0, 1]$, *the condition* $s \sim t$ *implies that* $Z_s^g = Z_t^g$. *In other words we may view* Z^g *as a process indexed by the quotient* $[0, 1] / \sim$, *that is by the tree* $\mathcal{T}_g$. *Indeed, it is then very natural to interpret* Z^g *as Brownian motion indexed by the tree* $\mathcal{T}_g$: *In the particular case when* $\mathcal{T}_g$ *is a finite union of segments (which holds if* g *is piecewise monotone),* Z^g *can be constructed by running independent Brownian motions along the branches of*

$\mathcal{T}_g$. It is however more convenient to view Z^g as a process indexed by $[0,1]$ because later the function g (and thus the tree $\mathcal{T}_g$) will be random and we avoid considering a random process indexed by a random set.

4.3. Convergence towards the Brownian snake. Let e be as previously a normalized Brownian excursion. By standard properties of Brownian paths, the function $t \mapsto \mathrm{e}_t$ is a.s. Hölder continuous (with exponent $\frac{1}{2} - \varepsilon$ for any $\varepsilon > 0$), and so we can apply the construction of the previous subsection to (almost) every realization of e.

In other words, we can construct a pair $(\mathrm{e}_t, Z_t)_{t \in [0,1]}$ of continuous random processes, whose distribution is characterized by the following two properties:

(i) e is a normalized Brownian excursion;
(ii) conditionally given e, Z is distributed as the snake driven by e.

The process Z will be called the Brownian snake (driven by e). This terminology is a little different from the usual one: Usually, the Brownian snake is viewed as a path-valued process (see e.g. [**28**]) and Z_t would correspond only to the terminal point of the value at time t of this path-valued process.

We can now answer the question raised at the end of subsection 4.1. The following theorem is due to Chassaing and Schaeffer [**12**]. More general results can be found in [**25**].

THEOREM 4.3. *For every integer $k \geq 1$, let $(\theta_k, (\ell^k(v))_{v \in \theta_k})$ be distributed uniformly over the set $\mathbf{T}_k$ of all labeled trees with k edges. Let $(C_k(t))_{t \geq 0}$ and $(V_k(t))_{t \geq 0}$ be respectively the contour function and the label contour function of the labeled tree $(\theta_k, (\ell^k(v))_{v \in \theta_k})$. Then,*

$$\left(\frac{1}{\sqrt{2k}} C_k(2k\,t), \left(\frac{9}{8k}\right)^{1/4} V_k(2k\,t) \right)_{t \in [0,1]} \xrightarrow[k \to \infty]{(d)} (\mathrm{e}_t, Z_t)_{t \in [0,1]}$$

where the convergence holds in the sense of weak convergence of the laws on the space $C([0,1], \mathbb{R}_+^2)$.

PROOF. From Theorem 2.10 and the Skorokhod representation theorem, we may assume without loss of generality that

$$(9) \qquad \sup_{0 \leq t \leq 1} |(2k)^{-1/2} C_k(2kt) - \mathrm{e}_t| \xrightarrow[k \to \infty]{\text{a.s.}} 0.$$

We first discuss the convergence of finite-dimensional marginals: We prove that for every choice of $0 \leq t_1 < t_2 < \cdots < t_p \leq 1$, we have

$$(10) \qquad \left(\frac{1}{\sqrt{2k}} C_k(2k\,t_i), \left(\frac{9}{8k}\right)^{1/4} V_k(2k\,t_i) \right)_{1 \leq i \leq p} \xrightarrow[k \to \infty]{(d)} (\mathrm{e}_{t_i}, Z_{t_i})_{1 \leq i \leq p}.$$

Since for every $i \in \{1, \ldots, n\}$,

$$|C_k(2kt_i) - C_k(\lfloor 2kt_i \rfloor)| \leq 1 , \quad |V_k(2kt_i) - V_k(\lfloor 2kt_i \rfloor)| \leq 1$$

we may replace $2kt_i$ by its integer part $\lfloor 2kt_i \rfloor$ in (10).

Consider the case $p = 1$. We may assume that $0 < t_1 < 1$, because otherwise the result is trivial. It is immediate that conditionally on θ_k, the label increments $\ell^k(v) - \ell^k(\pi(v))$, $v \in \theta_k \backslash \{\emptyset\}$, are independent and uniformly distributed over $\{-1, 0, 1\}$. Consequently, we may write

$$(C_k(\lfloor 2kt_1 \rfloor), V_k(\lfloor 2kt_1 \rfloor)) \overset{(d)}{=} \left(C_k(\lfloor 2kt_1 \rfloor), \sum_{i=1}^{C_k(\lfloor 2kt_1 \rfloor)} \eta_i \right)$$

where the variables $\eta_1, \eta_2, \ldots$ are independent and uniformly distributed over $\{-1, 0, 1\}$, and are also independent of the trees θ_k. By the central limit theorem,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \eta_i \xrightarrow[n\to\infty]{\text{(d)}} \left(\frac{2}{3}\right)^{1/2} N$$

where N is a standard normal variable. Thus if we set for $\lambda \in \mathbb{R}$,

$$\Phi(n, \lambda) = E\Big[\exp\Big(i\frac{\lambda}{\sqrt{n}} \sum_{i=1}^{n} \eta_i \Big) \Big]$$

we have $\Phi(n, \lambda) \longrightarrow \exp(-\lambda^2/3)$ as $n \to \infty$.

Then, for every $\lambda, \lambda' \in \mathbb{R}$, we get by conditioning on θ_k

$$E\Big[\exp\Big(i\frac{\lambda}{\sqrt{2k}} C_k(\lfloor 2kt_1 \rfloor) + i\frac{\lambda'}{\sqrt{C_k(\lfloor 2kt_1 \rfloor)}} \sum_{i=1}^{C_k(\lfloor 2kt_1 \rfloor)} \eta_i \Big) \Big]$$

$$= E\Big[\exp\Big(i\frac{\lambda}{\sqrt{2k}} C_k(\lfloor 2kt_1 \rfloor) \Big) \times \Phi(C_k(\lfloor 2kt_1 \rfloor), \lambda') \Big]$$

$$\xrightarrow[k\to\infty]{} E[\exp(i\lambda e_{t_1})] \times \exp(-\lambda'^2/3)$$

using the (almost sure) convergence of $(2k)^{-1/2} C_k(\lfloor 2kt_1 \rfloor)$ towards $e_{t_1} > 0$. In other words we have obtained the joint convergence in distribution

$$(11) \qquad \Big(\frac{C_k(\lfloor 2kt_1 \rfloor)}{\sqrt{2k}}, \frac{1}{\sqrt{C_k(\lfloor 2kt_1 \rfloor)}} \sum_{i=1}^{C_k(\lfloor 2kt_1 \rfloor)} \eta_i \Big) \xrightarrow[k\to\infty]{\text{(d)}} (e_{t_1}, (2/3)^{1/2} N),$$

where the normal variable N is independent of e.

From the preceding observations, we have

$$\Big(\frac{C_k(\lfloor 2kt_1 \rfloor)}{\sqrt{2k}}, \Big(\frac{9}{8k}\Big)^{1/4} V_k(\lfloor 2kt_1 \rfloor) \Big)$$

$$\overset{\text{(d)}}{=} \Big(\frac{C_k(\lfloor 2kt_1 \rfloor)}{\sqrt{2k}}, \Big(\frac{3}{2}\Big)^{1/2} \Big(\frac{C_k(\lfloor 2kt_1 \rfloor)}{\sqrt{2k}}\Big)^{1/2} \frac{1}{\sqrt{C_k(\lfloor 2kt_1 \rfloor)}} \sum_{i=1}^{C_k(\lfloor 2kt_1 \rfloor)} \eta_i \Big)$$

and from (11) we get

$$\Big(\frac{C_k(\lfloor 2kt_1 \rfloor)}{\sqrt{2k}}, \Big(\frac{9}{8k}\Big)^{1/4} V_k(\lfloor 2kt_1 \rfloor) \Big) \xrightarrow[k\to\infty]{\text{(d)}} (e_{t_1}, \sqrt{e_{t_1}} N).$$

This gives (10) in the case $p = 1$, since by construction it holds that $(e_{t_1}, Z_{t_1}) \overset{\text{(d)}}{=} (e_{t_1}, \sqrt{e_{t_1}} N)$.

Let us discuss the case $p = 2$ of (10). We fix t_1 and t_2 with $0 < t_1 < t_2 < 1$. Recall the notation

$$\check{C}_k^{i,j} = \min_{i \wedge j \le n \le i \vee j} C_k(n), \qquad i, j \in \{0, 1, \ldots, 2k\}$$

introduced in Section 1. Write $v_0^k = \varnothing, v_1^k, \ldots, v_{2k}^k = \varnothing$ for the contour exploration of vertices of θ_k (see the end of subsection 4.1). Then we know that

$$C_k(\lfloor 2kt_1 \rfloor) = |v_{\lfloor 2kt_1 \rfloor}^k|, \ C_k(\lfloor 2kt_2 \rfloor) = |v_{\lfloor 2kt_2 \rfloor}^k|,$$
$$V_k(\lfloor 2kt_1 \rfloor) = \ell^k(v_{\lfloor 2kt_1 \rfloor}^k), \ V_k(\lfloor 2kt_2 \rfloor) = \ell^k(v_{\lfloor 2kt_2 \rfloor}^k),$$

and furthermore $\check{C}_k^{\lfloor 2kt_1 \rfloor, \lfloor 2kt_2 \rfloor}$ is the generation in θ_k of the last common ancestor to $v_{\lfloor 2kt_1 \rfloor}^k$ and $v_{\lfloor 2kt_2 \rfloor}^k$. From the properties of labels on the tree θ_k, we now see that conditionally on θ_k,

$$(V_k(\lfloor 2kt_1 \rfloor), V_k(\lfloor 2kt_2 \rfloor)) \overset{\text{(d)}}{=}$$

$$(12) \qquad \left(\sum_{i=1}^{\check{C}_k^{\lfloor 2kt_1 \rfloor, \lfloor 2kt_2 \rfloor}} \eta_i + \sum_{i=\check{C}_k^{\lfloor 2kt_1 \rfloor, \lfloor 2kt_2 \rfloor}+1}^{C_k(\lfloor 2kt_1 \rfloor)} \eta_i' \, , \, \sum_{i=1}^{\check{C}_k^{\lfloor 2kt_1 \rfloor, \lfloor 2kt_2 \rfloor}} \eta_i + \sum_{i=\check{C}_k^{\lfloor 2kt_1 \rfloor, \lfloor 2kt_2 \rfloor}+1}^{C_k(\lfloor 2kt_2 \rfloor)} \eta_i'' \right)$$

where the variables $\eta_i, \eta_i', \eta_i''$ are independent and uniformly distributed over $\{-1, 0, 1\}$.

From (9), we have

$$\left((2k)^{-1/2} C_k(\lfloor 2kt_1 \rfloor), (2k)^{-1/2} C_k(\lfloor 2kt_2 \rfloor), (2k)^{-1/2} \check{C}_k^{\lfloor 2kt_1 \rfloor, \lfloor 2kt_2 \rfloor} \right)$$

$$\xrightarrow[k \to \infty]{\text{a.s.}} (\mathbb{e}_{t_1}, \mathbb{e}_{t_2}, m_{\mathbb{e}}(t_1, t_2)).$$

By arguing as in the case $p = 1$, we now deduce from (12) that

$$\left(\frac{C_k(\lfloor 2kt_1 \rfloor)}{\sqrt{2k}}, \frac{C_k(\lfloor 2kt_2 \rfloor)}{\sqrt{2k}}, \left(\frac{9}{8k} \right)^{1/4} V_k(\lfloor 2kt_1 \rfloor), \left(\frac{9}{8k} \right)^{1/4} V_k(\lfloor 2kt_2 \rfloor) \right)$$

$$\xrightarrow[k \to \infty]{\text{(d)}} \left(\mathbb{e}_{t_1}, \mathbb{e}_{t_2}, \sqrt{m_{\mathbb{e}}(t_1, t_2)}\, N + \sqrt{\mathbb{e}_{t_1} - m_{\mathbb{e}}(t_1, t_2)}\, N' \, , \right.$$

$$\left. \sqrt{m_{\mathbb{e}}(t_1, t_2)}\, N + \sqrt{\mathbb{e}_{t_2} - m_{\mathbb{e}}(t_1, t_2)}\, N'' \right)$$

where N, N', N'' are three independent standard normal variables, which are also independent of $\mathbb{e}$. The limiting distribution in the last display is easily identified with that of $(\mathbb{e}_{t_1}, \mathbb{e}_{t_2}, Z_{t_1}, Z_{t_2})$, and this gives the case $p = 2$ in (10). The general case is proved by similar arguments and we leave details to the reader.

To complete the proof of Theorem 4.3, we need a tightness argument. The laws of the processes

$$\left(\frac{1}{\sqrt{2k}} C_k(2kt) \right)_{t \in [0,1]}$$

are tight by Theorem 2.10, and so we need only verify the tightness of the processes

$$\left(\left(\frac{9}{8k} \right)^{1/4} V_k(2kt) \right)_{t \in [0,1]}.$$

This is a consequence of the following lemma, which therefore completes the proof of Theorem 4.3. $\qquad \square$

LEMMA 4.4. *For every integer $p \geq 1$, there exists a constant $K_p < \infty$ such that, for every $k \geq 1$ and every $s, t \in [0, 1]$,*

$$E\left[\left(\frac{V_k(2kt) - V_k(2ks)}{k^{1/4}} \right)^{4p} \right] \leq K_p \, |t - s|^p.$$

PROOF. Simple arguments show that we may restrict our attention to the case when $s = i/(2k)$, $t = j/(2k)$, with $i, j \in \{0, 1, \ldots, 2k\}$. By using the same decomposition as in (12), we have

$$(13) \qquad\qquad V_k(j) - V_k(i) \overset{\text{(d)}}{=} \sum_{n=1}^{d_{gr}(v_i^k, v_j^k)} \eta_n$$

where the random variables η_n are independent and uniform over $\{-1, 0, 1\}$ (and independent of θ_k) and

$$d_{gr}(v_i^k, v_j^k) = C_k(i) + C_k(j) - 2\check{C}_k^{i,j}$$

is the graph distance in the tree θ_k between vertices v_i^k and v_j^k. From (13) and by conditioning with respect to θ_k, we get the existence of a constant K_p' such that

$$E[(V_k(i) - V_k(j))^{4p}] \leq K_p' E[(d_{gr}(v_i^k, v_j^k))^{2p}].$$

So the lemma will be proved if we can verify the bound

(14) $$E[(C_k(i) + C_k(j) - 2\check{C}_k^{i,j})^{2p}] \leq K_p'' |j - i|^p$$

with a constant K_p'' independent of k. By the identity (4), it is enough to prove that this bound holds for $i = 0$. However, the case $i = 0$ is exactly Lemma 2.13. This completes the proof.

5. Planar maps

5.1. Definitions. A map is a combinatorial object, which can be best visualized as a class of graphs embedded in a surface. In these lectures, we will exclusively focus on the case of *plane* (or *planar*) maps, where the surface is the 2-dimensional sphere $\mathbb{S}^2$.

Let us first formalize the notion of map. We will not enter into details, referring the reader to the book by Mohar and Thomassen [**42**] for a very complete exposition. Another useful reference, discussing in depth the different equivalent ways to define maps (in particular through purely algebraic notions) is the book by Lando and Zvonkin [**27**, Chapter 1].

An *oriented edge* in $\mathbb{S}^2$ is a mapping $e : [0, 1] \to \mathbb{S}^2$ that is continuous, and such that either e is injective, or the restriction of e to $[0, 1)$ is injective and $e(0) = e(1)$. In the latter case, e is also called a loop. An oriented edge will always be considered up to reparametrization by a continuous increasing function from $[0, 1]$ to $[0, 1]$, and we will always be interested in properties of edges that do not depend on a particular parameterization. The origin and target of e are the points $e^- = e(0)$ and $e^+ = e(1)$. The reversal of e is the oriented edge $\overline{e} = e(1 - \cdot)$. An *edge* is a pair $\mathbf{e} = \{e, \overline{e}\}$, where e is an oriented edge. The *interior* of $\mathbf{e}$ is defined as $e((0, 1))$.

An *embedded graph* in $\mathbb{S}^2$ is a graph[1] $G = (V, E)$ such that

- V is a (finite) subset of $\mathbb{S}^2$
- E is a (finite) set of edges in $\mathbb{S}^2$
- the vertices incident to $\mathbf{e} = \{e, \overline{e}\} \in E$ are $e^-, e^+ \in V$
- the interior of an edge $\mathbf{e} \in E$ does not intersect V nor the edges of E distinct from $\mathbf{e}$

The support of an embedded graph $G = (V, E)$ is

$$\text{supp}(G) = V \cup \bigcup_{\mathbf{e} = \{e, \overline{e}\} \in E} e([0, 1]).$$

A *face* of the embedding is a connected component of the set $\mathbb{S}^2 \setminus \text{supp}(G)$.

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[1] all the graphs considered here are finite, and are multigraphs in which multiple edges and loops are allowed
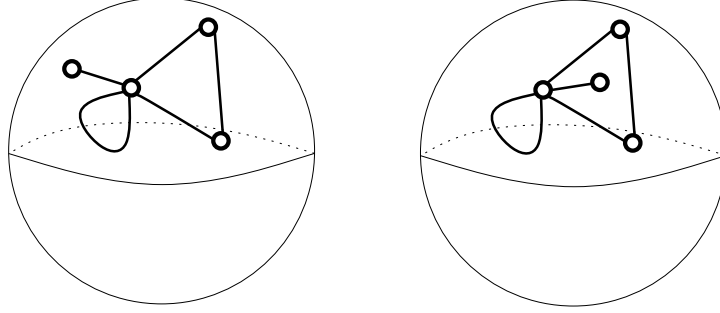
FIGURE 4. Two planar maps, with 4 vertices and 3 faces of degrees 1,3,6 and 1,4,5 respectively

DEFINITION 5.1. *A (planar) map is a connected embedded graph. Equivalently, a map is an embedded graph whose faces are all homeomorphic to the Euclidean unit disk in $\mathbb{R}^2$.*

Topologically, one would say that a map is the 1-skeleton of a CW-complex decomposition of $\mathbb{S}^2$. We will denote maps using bold characters $\mathbf{m}, \mathbf{q}, \ldots$

Let $\mathbf{m} = (V, E)$ be a map, and let $\overrightarrow{E} = \{e \in \mathbf{e} : \mathbf{e} \in E\}$ be the set of all oriented edges of $\mathbf{m}$. Since $\mathbb{S}^2$ is oriented, it is possible to define, for every oriented edge $e \in \overrightarrow{E}$, a unique face f_e of $\mathbf{m}$, located to the left of the edge e. We call f_e the face *incident* to e. Note that the edges incident to a given face form a closed curve in $\mathbb{S}^2$, but not necessarily a Jordan curve (it can happen that $f_e = f_{\overline{e}}$ for some e). The degree of a face f is defined as

$$\deg(f) = \#\{e \in \overrightarrow{E} : f_e = f\}\,.$$

The oriented edges incident to a given face f, are arranged cyclically in counter-clockwise order around the face in what we call the *facial ordering*. With every oriented edge e, we can associate a *corner* incident to e, which is a small simply connected neighborhood of e^- intersected with f_e. Then the corners of two different oriented edges do not intersect.

Of course, the degree of a vertex $u \in V$ is the usual graph-theoretical notion

$$\deg(u) = \#\{e \in \overrightarrow{E} : e^- = u\}\,.$$

Similarly as for faces, the outgoing edges from u are organized cyclically in counterclockwise order around u.

A *rooted* map is a pair $(\mathbf{m}, e)$ where $\mathbf{m} = (V, E)$ is a map and $e \in \overrightarrow{E}$ is a distinguished oriented edge, called the root. We often omit the mention of e in the notation.

5.2. Euler's formula. An important property of maps is the so-called *Euler formula*. If $\mathbf{m}$ is a map, $V(\mathbf{m}), E(\mathbf{m}), F(\mathbf{m})$ denote respectively the sets of all vertices, edges and faces of $\mathbf{m}$. Then,

(15) $$\#V(\mathbf{m}) - \#E(\mathbf{m}) + \#F(\mathbf{m}) = 2\,.$$

This is a relatively easy result in the case of interest (the planar case): One can remove the edges of the graph one by one until a spanning tree $\mathbf{t}$ of the graph is obtained, for which the result is trivial (it has one face, and $\#V(\mathbf{t}) = \#E(\mathbf{t}) + 1$).

5.3. Isomorphism, automorphism and rooting. In the sequel, we will always consider maps "up to deformation" in the following sense.

DEFINITION 5.2. *The maps* $\mathbf{m}, \mathbf{m}'$ *on* $\mathbb{S}^2$ *are isomorphic if there exists an orientation-preserving homeomorphism* h *of* $\mathbb{S}^2$ *onto itself, such that* h *induces a graph isomorphism of* $\mathbf{m}$ *with* $\mathbf{m}'$.
  The rooted maps $(\mathbf{m}, e)$ *and* $(\mathbf{m}', e')$ *are isomorphic if* $\mathbf{m}$ *and* $\mathbf{m}'$ *are isomorphic through a homeomorphism* h *that maps* e *to* e'.

In the sequel, we will almost always identify two isomorphic maps $\mathbf{m}, \mathbf{m}'$. This of course implies that the (non-embedded, combinatorial) graphs associated with $\mathbf{m}, \mathbf{m}'$ are isomorphic, but this is stronger: For instance the two maps of Fig.4 are not isomorphic, since a map isomorphism preserves the degrees of faces.

An *automorphism* of a map $\mathbf{m}$ is an isomorphism of $\mathbf{m}$ with itself. It should be interpreted as a *symmetry* of the map. An important fact is the following.

PROPOSITION 5.1. *An automorphism of* $\mathbf{m}$ *that fixes an oriented edge fixes all the oriented edges.*

Loosely speaking, the only automorphism of a rooted map is the identity. This explains why rooting is an important tool in the combinatorial study of maps, as it "kills the symmetries". The idea of the proof of the previous statement is to see that if e is fixed by the automorphism, then all the edges incident to e^- should also be fixed (since an automorphism preserves the orientation). One can thus progress in the graph (by connectedness) and show that all the edges are fixed.

In a rooted map, the face f_e incident to the root edge e is often called the *external face*, or root face. The other faces are called *internal*. The vertex e^- is called the root vertex.

From now on, unless otherwise specified, all maps will be rooted.

We end this presentation by introducing the notion of *graph distance* in a map $\mathbf{m}$. A *chain* of length $k \geq 1$ is a sequence $e_{(1)}, \ldots, e_{(k)}$ of oriented edges in $\overrightarrow{E}(\mathbf{m})$, such that $e_{(i)}^+ = e_{(i+1)}^-$ for $1 \leq i \leq k-1$, and we say that the chain links the vertices $e_{(1)}^-$ and $e_{(k)}^+$. We also allow, for every vertex $u \in V(\mathbf{m})$, a chain with length 0, starting and ending at u. The *graph distance* $d_{\mathbf{m}}(u, v)$ between two vertices $u, v \in V(\mathbf{m})$ is the minimal k such that there exists a chain with length k linking u and v. A chain with minimal length between two vertices is called a *geodesic chain*.

5.4. The Cori-Vauquelin-Schaeffer bijection. Via the identification of maps up to isomorphisms the set of all maps becomes a countable set. For instance, the set $\mathbf{M}_n$ of all rooted maps with n edges is a finite set: The $2n$ oriented edges should be organized around a finite family of polygons (the faces of the map), and the number of ways to associate the boundary edges of these polygons is finite. A natural question to ask is "what is the cardinality of $\mathbf{M}_n$?".

Tutte answered this question (and many other counting problems for maps), motivated in part by the 4-color problem. He developed a powerful method, the "quadratic method", to solve the apparently ill-defined equations for the generating functions of maps. For recent developments in this direction, see the article by Bousquet-Mélou and Jehanne [6]. The method, however, is a kind of "black box" which solves such counting problems without giving much extra information about

the structure of maps. One obtains

$$\#\mathbf{M}_n = \frac{2}{n+2} 3^n \mathrm{Cat}_n \,,$$

where $\mathrm{Cat}_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number. We also mention the huge literature on the enumeration of maps using matrix integrals, initiating in [**24, 10**], which is particularly popular in the physics literature. See [**27**, Chapter 4] for an introduction to this approach.

Motivated by the very simple form of the formula enumerating $\mathbf{M}_n$, Cori and Vauquelin [**14**] gave in 1981 a bijective approach to this formula. These approaches reached their full power with the work of Schaeffer starting in his 1998 thesis [**47**]. We now describe the bijective approach in the case of quadrangulations.

5.4.1. *Quadrangulations.* A map $\mathbf{q}$ is a quadrangulation if all its faces are of degree 4. We let $\mathbf{Q}_n$ be the set of all (rooted) quadrangulations with n faces. Quadrangulations are a very natural family of maps to consider, in virtue of the fact that there exists a "trivial" bijection between $\mathbf{M}_n$ and $\mathbf{Q}_n$, which can be described as follows.

Let $\mathbf{m}$ be a map with n edges, and imagine that the vertices of $\mathbf{m}$ are colored in black. We then create a new map by adding inside each face of $\mathbf{m}$ a white vertex, and joining this white vertex to every corner of the face f it belongs to, by non-intersecting edges inside the face f. In doing so, notice that some black vertices may be joined to the same white vertex with several edges. Lastly, we erase the interiors of the edges of the map $\mathbf{m}$. We end up with a map $\mathbf{q}$, which is a plane quadrangulation with n faces, each face containing exactly one edge of the initial map. We adopt a rooting convention, for instance, we root $\mathbf{q}$ at the first edge coming after e in counterclockwise order around e^-, where e is the root of $\mathbf{m}$.

Notice that $\mathbf{q}$ also comes with a bicoloration of its vertices in black and white, in which two adjacent vertices have different colors. This says that $\mathbf{q}$ is *bipartite*, and as a matter of fact, every (planar!) quadrangulation is bipartite. So this coloring is superfluous: One can recover it by declaring that the black vertices are those at even distance from the root vertex of $\mathbf{q}$, and the white vertices are those at odd distance from the root vertex.

Conversely, starting from a rooted quadrangulation $\mathbf{q}$, we can recover a bipartite coloration as above, by declaring that the vertices at even distance from the root edge are black. Then, we draw the diagonal linking the two black corners incident to every face of $\mathbf{q}$. Finally, we remove the interior of the edges of $\mathbf{q}$ and root the resulting map $\mathbf{m}$ at the first outgoing diagonal from e^- in clockwise order from the root edge e of $\mathbf{q}$. One checks that this is indeed a left- and right-inverse of the previous mapping from $\mathbf{M}_n$ to $\mathbf{Q}_n$. See Fig.5 below for an illustration of these bijections.

For the record, we state the following useful fact.

PROPOSITION 5.2. *A (planar) map is bipartite if and only if its faces all have even degree.*

5.4.2. *The CVS bijection.* Recall that $\mathbf{Q}_n$ is the set of all rooted quadrangulations with n faces. A simple application of Euler's formula shows that any element of $\mathbf{Q}_n$ has $2n$ edges ($4n$ oriented edges, 4 for each face) and $n+2$ vertices.

Let $\mathbf{T}_n$ be the set of all labeled trees with n edges, as defined in Section 3. If $(\tau, (\ell(u))_{u \in \tau}) \in \mathbf{T}_n$, then τ is a plane tree with n edges, and $\ell : \tau \to \mathbb{Z}$ is a label
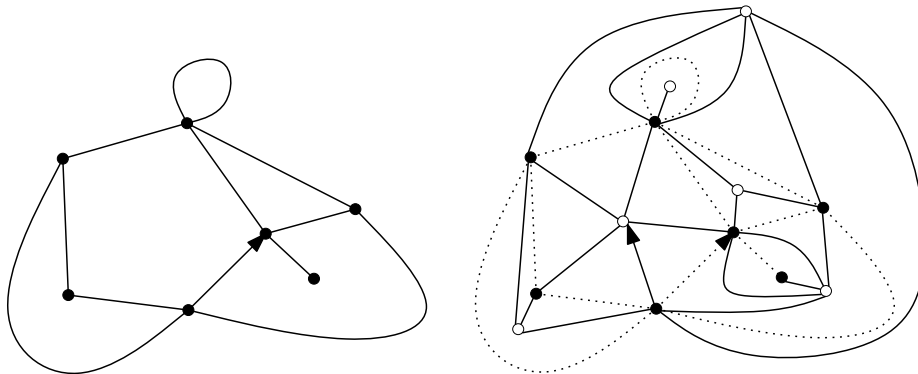
FIGURE 5. The so-called "trivial" bijection

function on τ, such that $\ell(\varnothing) = 0$ and

$$|\ell(u) - \ell(\pi(u))| \leq 1, \qquad \text{for every } u \in \tau \setminus \{\varnothing\}.$$

In order to avoid trivialities, we now assume that $n \geq 1$. It will be convenient here to view a plane tree τ as a planar map, by embedding it in $\mathbb{S}^2$, and rooting it at the edge going from $\varnothing$ to the vertex 1. Let $\varnothing = v_0, v_1, \ldots, v_{2n} = \varnothing$ be the contour exploration of the vertices of the tree τ (see the end of subsection 4.1). For $i \in \{0, 1, \ldots, 2n-1\}$, we let e_i be the oriented edge from v_i to v_{i+1}, and extend the sequences (v_i) and (e_i) to infinite sequences by $2n$-periodicity. With each oriented edge e_i, we can associate a corner around e_i^-, as explained in subsection 4.1. In the remaining part of Section 4, we will often identify the oriented edge e_i with the associated corner, and we adopt the notation $\ell(e_i) = \ell(e_i^-)$. In particular, note that $\ell(e_i) = V_i, 0 \leq i \leq 2n$ is the label contour sequence as defined in Section 3.

For every $i \geq 0$, we define the *successor* of i by

$$s(i) = \inf\{j > i : \ell(e_j) = \ell(e_i) - 1\},$$

with the convention that $\inf \emptyset = \infty$. Note that $s(i) = \infty$ if and only if $\ell(e_i)$ equals $\min\{\ell(v) : v \in \tau\}$. This is a simple consequence of the fact that the integer-valued sequence $(\ell(e_i), i \geq 0)$ can decrease only by taking unit steps.

Consider a point v_* in $\mathbb{S}^2$ that does not belong to the support of τ, and denote by e_∞ a corner around v_*, i.e. a small neighborhood of v_* with v_* excluded, not intersecting the corners $e_i, i \geq 0$. By convention, we set

$$\ell(v_*) = \ell(e_\infty) = \min\{\ell(u) : u \in \tau\} - 1.$$

For every $i \geq 0$, the successor of the corner e_i is then defined by

$$s(e_i) = e_{s(i)}.$$

The CVS construction consists in drawing, for every $i \in \{0, 1, \ldots, 2n-1\}$, an *arc*, which is an edge from the corner e_i to the corner $s(e_i)$ inside $\mathbb{S}^2 \setminus (\{v_*\} \cup \mathrm{supp}\,(\tau))$. See Fig.6 for an illustration of the CVS construction.

LEMMA 5.3. *It is possible to draw the arcs in such a way that the graph with vertex-set $\tau \cup \{v_*\}$ and edge-set consisting of the edges of τ and the arcs is an embedded graph.*
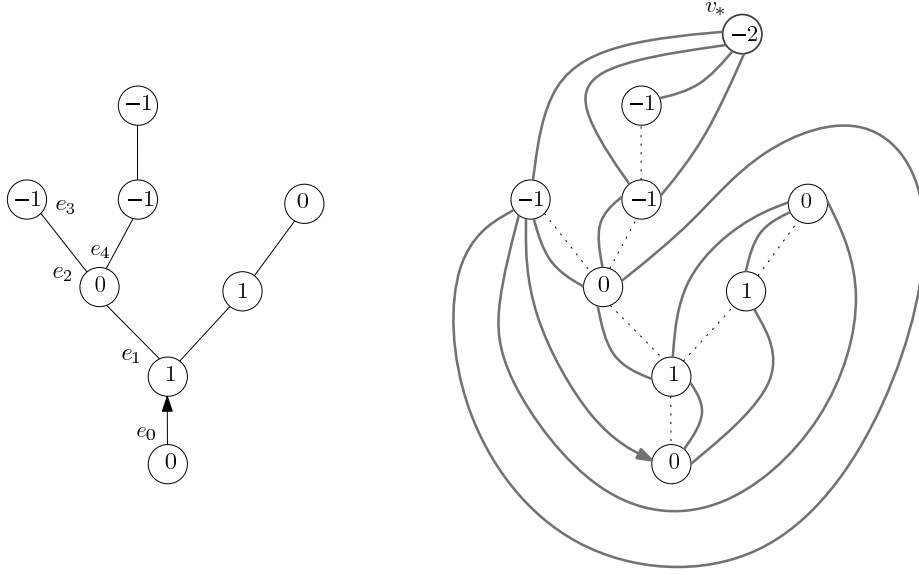
FIGURE 6. Illustration of the Cori-Vauquelin-Schaeffer bijection, in the case $\epsilon = 1$. For instance, e_3 is the successor of e_0, e_2 the successor of e_1, and so on.

PROOF. Since τ is a tree, we can see it as a map with a unique face $\mathbb{S}^2 \backslash \text{supp}(\tau)$. The latter can in turn be seen as an open polygon, bounded by the edges $e_0, e_1, \ldots, e_{2n-1}$ in counterclockwise order. Hence, the result will follow if we can show that the arcs do not cross, i.e. that it is not possible to find pairwise distinct corners $e^{(1)}, e^{(2)}, e^{(3)}, e^{(4)}$ that arise in this order in the cyclic order induced by the contour exploration, and such that $e^{(3)} = s(e^{(1)})$ and $e^{(4)} = s(e^{(2)})$.

If this were the case, then we would have $\ell(e^{(2)}) \geq \ell(e^{(1)})$, as otherwise the successor of $e^{(1)}$ would be between $e^{(1)}$ and $e^{(2)}$. Similarly, $\ell(e^{(3)}) \geq \ell(e^{(2)})$. But by definition, $\ell(e^{(3)}) = \ell(e^{(1)}) - 1$, giving $\ell(e^{(2)}) \geq \ell(e^{(3)}) + 1 \geq \ell(e^{(2)}) + 1$, which is a contradiction. □

We call $\mathbf{q}$ the graph with vertex-set $V(\tau) \cup \{v_*\}$ and edge-set formed by the arcs, now excluding the (interiors of the) edges of τ.

LEMMA 5.4. *The embedded graph $\mathbf{q}$ is a quadrangulation with n faces.*

PROOF. First we check that $\mathbf{q}$ is connected, and hence is a map. But this is obvious since the consecutive successors of any given corner e, given by $e, s(e), s(s(e)), \ldots$, form a finite sequence ending at e_∞. Hence, every vertex in $\mathbf{q}$ can be joined by a chain to v_*, and the graph is connected.

To check that $\mathbf{q}$ is a quadrangulation, let us consider an edge of τ, corresponding to two oriented edges $e, \overline{e}$. Let us first assume that $\ell(e^+) = \ell(e^-) - 1$. Then, the successor of e is incident to e^+ and the preceding construction gives an arc starting from e^- (more precisely from the corner associated with e) and ending at e^+. Next, let e' be the corner following $\overline{e}$ in the contour exploration around τ. Then $\ell(e') = \ell(e^-) = \ell(\overline{e}) + 1$, giving that $s(\overline{e}) = s(s(e'))$. Indeed, $s(e')$ is the first corner coming after e' in contour order and with label $\ell(e') - 1 = \ell(e) - 1$, while $s(s(e'))$

FIGURE 7. Illustration of the proof of Lemma 5.4. In this figure, $l = \ell(e)$

is the first corner coming after e' with label $\ell(e) - 2$. Therefore, it has to be the first corner coming after $\overline{e}$, with label $\ell(e) - 2 = \ell(\overline{e}) - 1$.

We deduce that the arcs joining the corners e to $s(e)$, resp. $\overline{e}$ to $s(\overline{e})$, resp. e' to $s(e')$, resp. $s(e')$ to $s(s(e')) = s(\overline{e})$, form a quadrangle, that contains the edge $\{e, \overline{e}\}$, and no other edge of τ.

If $\ell(e^+) = \ell(e^-) + 1$, the situation is the same by interchanging the roles of e and $\overline{e}$.

The only case that remains is when $\ell(e^+) = \ell(e^-)$. In this case, if e' and e'' are the corners following e and $\overline{e}$ respectively in the contour exploration of τ, then $\ell(e) = \ell(e') = \ell(\overline{e}) = \ell(e'')$, so that $s(e) = s(e')$ on the one hand and $s(\overline{e}) = s(e'')$ on the other hand. We deduce that the edge $\{e, \overline{e}\}$ is the diagonal of a quadrangle formed by the arcs linking e to $s(e)$, e' to $s(e') = s(e)$, $\overline{e}$ to $s(\overline{e})$ and e'' to $s(e'') = s(\overline{e})$. The different cases are summed up in Fig.7.

Now, notice that $\mathbf{q}$ has $2n$ edges (one per corner of τ) and $n + 2$ vertices, so it must have n faces by Euler's formula. So all the faces must be of the form described above. This completes the proof. $\square$

Note that the quadrangulation $\mathbf{q}$ has a distinguished vertex v_*, but for now it is not a rooted quadrangulation. To fix this root, we will need an extra parameter $\epsilon \in \{-1, 1\}$. If $\epsilon = 1$ we let the root edge of $\mathbf{q}$ be the arc linking e_0 with $s(e_0)$, and oriented from $s(e_0)$ from e_0. If $\epsilon = -1$, the root edge is this same arc, but oriented from e_0 to $s(e_0)$.

In this way, we have defined a mapping Φ, from $\mathbf{T}_n \times \{-1, 1\}$ to the set $\mathbf{Q}_n^\bullet$ of pairs $(\mathbf{q}, v_*)$, where $\mathbf{q} \in \mathbf{Q}_n$ and $v_* \in V(\mathbf{q})$. We call such pairs *pointed quadrangulations*.

THEOREM 5.5. *For every $n \geq 1$, the mapping Φ is a bijection from $\mathbf{T}_n \times \{-1, 1\}$ onto $\mathbf{Q}_n^\bullet$.*

We omit the proof of this result. See Chassaing and Schaeffer [**12**, Theorem 4].

COROLLARY 5.6. *We have the following formula for every $n \geq 1$:*

$$\#\mathbf{M}_n = \#\mathbf{Q}_n = \frac{2}{n+2} 3^n \mathrm{Cat}_n$$

PROOF. We first notice that $\#\mathbf{Q}_n^\bullet = (n+2)\#\mathbf{Q}_n$, since every quadrangulation $\mathbf{q} \in \mathbf{Q}_n$ has $n+2$ vertices, each of which induces a distinct element of $\mathbf{Q}_n^\bullet$. On the other hand, it is obvious that

$$\#\mathbf{T}_n \times \{-1,1\} = 2 \cdot 3^n \#\mathbf{A}_n = 2 \cdot 3^n \mathrm{Cat}_n \,.$$

The result follows from Theorem 5.5.   □

The probabilistic counterpart of this can be stated as follows.

COROLLARY 5.7. *Let Q_n be a uniform random element in $\mathbf{Q}_n$, and conditionally given Q_n, let v_* be chosen uniformly at random in $V(Q_n)$. On the other hand, let θ_n be chosen uniformly at random in $\mathbf{T}_n$, and let ϵ be independent of θ_n and uniformly distributed in $\{-1,1\}$. Then $\Phi(\theta_n, \epsilon)$ has the same distribution as (Q_n, v_*).*

The proof is obvious, since the probability that (Q_n, v_*) equals some particular $(\mathbf{q}, v) \in \mathbf{Q}_n^\bullet$ equals $((n+2)\#\mathbf{Q}_n)^{-1} = (\#\mathbf{Q}_n^\bullet)^{-1}$.

5.4.3. *Interpretation of the labels.* The CVS bijection will be of crucial importance to us when we will deal with metric properties of random elements of $\mathbf{Q}_n$, because the labels on $\mathbf{q}$ that are inherited from a labeled tree through the CVS construction turn out to measure certain distances in $\mathbf{q}$. Recall that the set τ is identified with $V(\mathbf{q}) \setminus \{v_*\}$ if (τ, ℓ) and $\mathbf{q}$ are associated through the CVS bijection (the choice of ϵ is irrelevant here). Hence, the function ℓ is also a function on $V(\mathbf{q}) \setminus \{v_*\}$, and we extend it by letting, as previously, $\ell(v_*) = \min\{\ell(u) : u \in \tau\} - 1$. For simplicity, we write

$$\min \ell = \min\{\ell(u) : u \in \tau\} \,.$$

PROPOSITION 5.8. *For every $v \in V(\mathbf{q})$, we have*

$$(16) \qquad\qquad d_\mathbf{q}(v, v_*) = \ell(v) - \min \ell + 1 \,,$$

where $d_\mathbf{q}$ is the graph distance on $\mathbf{q}$.

PROOF. Let $v \in V(\mathbf{q}) \setminus \{v_*\} = \tau$, and let e be a corner (in τ) incident to v. Then the chain of arcs

$$e \to s(e) \to s^2(e) \to \ldots \to e_\infty$$

is a chain of length $\ell(e) - \ell(e_\infty) = \ell(v) - \ell(v_*)$ between v and v_*. Therefore, $d_\mathbf{q}(v, v_*) \leq \ell(v) - \ell(v_*)$. On the other hand, if $v = v_0, v_1, \ldots, v_d = v_*$ are the consecutive vertices of any chain linking v to v_*, then since $|\ell(e) - \ell(s(e))| = 1$ by definition for any corner e and since the edges of $\mathbf{q}$ all connect a corner to its successor, we get

$$d = \sum_{i=1}^d |\ell(v_i) - \ell(v_{i-1})| \geq |\ell(v_0) - \ell(v_d)| = \ell(v) - \ell(v_*) \,,$$

as desired.   □

Remark. The preceding proof also shows that the chain of arcs $e \to s(e) \to s^2(e) \to \ldots \to e_\infty$ is a geodesic chain linking e^- to v_*. Such a geodesic chain, or more generally a chain of the form $e \to s(e) \to s^2(e) \to \ldots \to s^k(e)$, will be called a successor geodesic chain.

The triangle inequality for $d_{\mathbf{q}}$ (or the second part of the proof) gives the useful bound

(17) $$d_{\mathbf{q}}(u,v) \geq |\ell(u) - \ell(v)|,$$

This bound will be improved in the next subsection.

As a consequence of the proposition, we obtain for instance that the "volume of spheres" around v_* can be interpreted in terms of ℓ: for every $k \geq 0$,

$$|\{v \in V(\mathbf{q}) : d_{\mathbf{q}}(v, v_*) = k\}| = |\{u \in \tau : \ell(u) - \min \ell + 1 = k\}|.$$

5.4.4. *Two useful bounds.* The general philosophy in the forthcoming study of random planar maps is then the following: Information about labels in a random labeled tree, which follows from the results of subsection 3.3 if this tree is uniformly distributed over $\mathbf{T}_n$, allows one to obtain information about distances in the associated quadrangulation. One major problem with this approach is that exact information will only be available for distances to a distinguished vertex v_*. There is no simple expression for the distances between two vertices distinct from v_* in terms of the labels in the tree. However, more advanced properties of the CVS bijection allow to get useful bounds on these distances. Recall that $e_0, e_1, e_2, \ldots$ is the contour sequence of corners (or oriented edges) around a tree $\tau \in \mathbf{A}_n$, starting from the root (see the beginning of subsection 5.4.2). We view $(e_i, i \geq 0)$ as cyclically ordered, and for any two corners e, e' of τ, we let $[e, e']$ be the set of all corners encountered when starting from e, following the cyclic contour order, and stopping when visiting e'.

PROPOSITION 5.9. *Let $((\tau, \ell), \epsilon)$ be an element in $\mathbf{T}_n \times \{-1, 1\}$, and $(\mathbf{q}, v_*) = \Phi(((\tau, \ell), \epsilon))$. Let u, v be two vertices in $V(\mathbf{q}) \setminus \{v_*\}$, and let e, e' be two corners of τ such that $e^- = u, (e')^- = v$.*
(i) *There holds that*

$$d_{\mathbf{q}}(u,v) \leq \ell(u) + \ell(v) - 2 \min_{e'' \in [e, e']} \ell(e'') + 2,$$

(ii) *There holds that*

$$d_{\mathbf{q}}(u,v) \geq \ell(u) + \ell(v) - 2 \min_{w \in [[u,v]]} \ell(w),$$

where $[[u, v]]$ is the set of all vertices lying on the geodesic path from u to v in the tree τ.

PROOF. For simplicity, let $m = \min_{e'' \in [e, e']} \ell(e'')$. Let e'' be the first corner in $[e, e']$ such that $\ell(e'') = m$. The corner $s^k(e)$, whenever it is well defined (i.e. whenever $d_{\mathbf{q}}(e^-, v_*) \geq k$), is called the k-th successor of e. Then e'' is the $(\ell(e) - m)$-th successor of e. Moreover, by definition, $s(e'')$ does not belong to $[e, e']$ since it has lesser label than e'', and necessarily, $s(e'')$ is also the $(\ell(e') - m + 1)$-st successor of e'. Hence, the successor geodesic chain $e \to s(e) \to s^2(e) \to \cdots \to s(e'')$ from $u = e^-$ to $s(e'')^+$, concatenated with the similar geodesic chain from v to $s(e'')^+$ is a path of length

$$\ell(u) + \ell(v) - 2m + 2,$$

and the distance $d_{\mathbf{q}}(u, v)$ is less than or equal to this quantity. This proves (i).

Let us prove (ii). Let $w \in [[u, v]]$ be such that $\ell(w) = \min\{\ell(w') : w' \in [[u, v]]\}$. If $w = u$ or $w = v$ then the statement follows trivially from (17). So we exclude this case. We can then write τ as the union $\tau = \tau_1 \cup \tau_2$ of two connected subgraphs

of τ such that $\tau_1 \cap \tau_2 = \{w\}$, τ_1 contains u but not v and τ_2 contains v but not u. There may be several such decompositions, so we just choose one. We consider a geodesic path γ from u to v in $\mathbf{q}$. If v_* belongs to this path, then this means that $d_{\mathbf{q}}(u,v) = d_{\mathbf{q}}(v_*, u) + d_{\mathbf{q}}(v_*, v)$ and the desired lower bound immediately follows from (16). So we may assume that v_* does not belong to γ. From our choice of τ_1 and τ_2, we can then find two corners $e_{(1)}$ and $e_{(2)}$ of τ such that $e_{(1)}^-$ belongs to τ_1 and $e_{(2)}^-$ belongs to τ_2, $e_{(1)}^-$ and $e_{(2)}^-$ are consecutive points on γ, and the corners $e_{(1)}$ and $e_{(2)}$ are connected by an edge of $\mathbf{q}$. From the latter property, we must have $e_{(2)} = s(e_{(1)})$ or $e_{(1)} = s(e_{(2)})$. Consider only the first case for definiteness (the other one is treated in a similar fashion). Since the contour exploration of vertices of τ must visit w between any visit of $u = e_{(1)}^-$ and any visit of $v = e_{(2)}^-$, the definition of the successor ensures that $\ell(w) \geq \ell(e_{(2)})$ (with equality only possible if $w = e_{(2)}^-$). Then, using (17) once again, we have

$$
\begin{array}{rcl}
d_{\mathbf{q}}(u,v) & = & d_{\mathbf{q}}(u, e_{(2)}^-) + d_{\mathbf{q}}(e_{(2)}^-, v) \\
& \geq & \ell(u) - \ell(e_{(2)}^-) + \ell(v) - \ell(e_{(2)}^-) \\
& \geq & \ell(u) + \ell(v) - 2\ell(w),
\end{array}
$$

giving the desired result. $\qquad\square$

6. Basic convergence results for uniform quadrangulations

For the remaining part of this course, our main goal will be to study the scaling limits of random planar quadrangulations chosen according to the uniform probability measure on $\mathbf{Q}_n$. Thanks to Corollary 5.7, the CVS bijection and the study of scaling limits of random labeled trees will turn out to be useful tools to study this problem. Ultimately, the question we would like to address is to study the convergence in distribution of an appropriately rescaled version of the random metric space $(V(Q_n), d_{Q_n})$, in the sense of the Gromov-Hausdorff topology.

One of the motivations for this problem comes from physics, and we refer the interested reader to [3] for an extensive discussion. In the past 15 years or so, physicists have been starting to view random maps as possible discrete models for a continuum model of random surfaces (called the Euclidean 2-dimensional quantum gravity model), which is still ill-defined from a mathematical point of view. We thus want to investigate whether the scaling limit of Q_n exists in the above sense, and does define a certain random surface. One can also ask the natural question of whether this limiting random surface is *universal*, in the sense that it also arises as the scaling limit of many different models of random maps, for instance, maps chosen uniformly at random in the set of all p-angulations with n faces:

$$
\mathbf{M}_n^p = \{\mathbf{m} : \deg(f) = p \text{ for every } f \in F(\mathbf{m}), \#F(\mathbf{m}) = n\}, \qquad p \geq 3.
$$

Indeed, most of the results that we will describe in the sequel do have analogs in this more general setting [36, 38, 41, 31], thanks to nice generalizations of the CVS bijection that are due to Bouttier, Di Francesco and Guitter [9].

This is of course analogous to the celebrated Donsker Theorem, according to which Brownian motion is the universal scaling limit of discrete random walks, as well as to the fact that the Brownian CRT is the scaling limit of many different models of random trees (see the remarks at the end of subsection 3.4).

6.1. Radius and profile. We will first address a simpler question than the one raised above, which is to determine by what factor we should rescale the distance d_{Q_n} in order to get an interesting scaling limit as $n \to \infty$.

Let $\mathbf{q} \in \mathbf{Q}_n$ be a rooted planar quadrangulation, and v be a vertex of $\mathbf{q}$. As before, let $d_{\mathbf{q}}$ denote the graph distance on the vertex set of $\mathbf{q}$. We define the *radius* of $\mathbf{q}$ seen from v as

$$\mathcal{R}(\mathbf{q}, v) = \max_{u \in V(\mathbf{q})} d_{\mathbf{q}}(u, v),$$

and the *profile* of $\mathbf{q}$ seen from v as the sequence

$$I_{\mathbf{q}, v}(k) = \mathrm{Card}\left\{u \in V(\mathbf{q}) : d_{\mathbf{q}}(u, v) = k\right\}, \qquad k \geq 0$$

which measures the 'volumes' of the spheres centered at v in the graph metric. The profile can be seen as a measure on $\mathbb{Z}_+$ with total volume $n + 2$. Our first limit theorem is the following.

THEOREM 6.1. *Let Q_n be uniformly distributed over $\mathbf{Q}_n$, and conditionally on Q_n, let v_* be chosen uniformly among the $n + 2$ vertices of Q_n. Let also $(\mathbb{e}, Z)$ be as in subsection 4.3.*

(i) We have

$$\left(\frac{9}{8n}\right)^{1/4} \mathcal{R}(Q_n, v_*) \xrightarrow[n \to \infty]{(d)} \sup Z - \inf Z.$$

(ii) If v_{**} is another vertex chosen uniformly in $V(Q_n)$ and independent of v_*,

$$\left(\frac{9}{8n}\right)^{1/4} d_{Q_n}(v_*, v_{**}) \xrightarrow[n \to \infty]{(d)} \sup Z.$$

(iii) Finally, the following convergence in distribution holds for the weak topology on probability measures on $\mathbb{R}_+$:

$$\frac{I_{Q_n, v_*}((8n/9)^{1/4}\cdot)}{n + 2} \xrightarrow[n \to \infty]{(d)} \mathcal{I},$$

where $\mathcal{I}$ is the occupation measure of Z above its infimum, defined as follows: For every non-negative, measurable $g : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\langle \mathcal{I}, g \rangle = \int_0^1 \mathrm{d}t \, g(Z_t - \inf Z).$$

The points (i) and (iii) are due to Chassaing and Schaeffer [**12**], and (ii) is due to Le Gall [**30**], although these references state these properties in a slightly different context, namely, in the case where v_* is the root vertex rather than a uniformly chosen vertex. This indicates that as $n \to \infty$, the root vertex plays no particular role. Some information about the limiting distributions in (i) and (ii) can be found in Delmas [**15**].

Property (ii) identifies the so-called 2-point function of the Brownian map. An important generalization of this result has been obtained by Bouttier and Guitter [**8**], who were able to compute the 3-point function, namely the joint asymptotic distribution of the mutual distances between three vertices chosen uniformly at random in $V(Q_n)$.

PROOF. Let $((T_n, L_n), \epsilon)$ be a uniform random element in $\mathbf{T}_n \times \{-1, 1\}$. Then by Corollary 5.7 we may assume that (Q_n, v_*) equals $\Phi(((T_n, L_n), \epsilon))$, where Φ is the CVS bijection.

Let C_n and V_n be respectively the contour function and the label contour function of (T_n, L_n) (cf. subsections 2.1 and 4.1), and let $u_i^n, 0 \leq i \leq 2n$ be the contour exploration of vertices of T_n as defined in subsection 4.1 (so that $C_n(i) = |u_i^n|$ and $V_n(i) = L_n(u_i^n)$).

By Proposition 5.8, the radius of Q_n viewed from v_* then equals

$$\mathcal{R}(Q_n, v_*) = \max L_n - \min L_n + 1 = \max V_n - \min V_n + 1\,.$$

Property (i) immediately follows from this equality and Theorem 4.3.

As for (ii), we first observe that we may slightly change the hypothesis on the distribution of v_{**}. It clearly suffices to prove the desired convergence when v_{**} is replaced by a vertex that is uniformly chosen among the n vertices of Q_n that are distinct from both v_* and the vertex $\varnothing$ of T_n (recall that $V(Q_n) \setminus \{v_*\} = V(T_n)$).

Now, for $s \in [0, 2n)$, we let $\langle s \rangle = \lceil s \rceil$ if C_n has slope $+1$ immediately after s, and $\langle s \rangle = \lfloor s \rfloor$ otherwise. Then, if $u \in T_n$, we have $u_{\langle s \rangle}^n = u$ if and only if $u \neq \varnothing$ and s is a time when the contour exploration around T_n explores either of the two oriented edges between u and its parent $\pi(u)$. Therefore, for every $u \in T_n \setminus \{\varnothing\}$, the Lebesgue measure of $\{s \in [0, 2n) : u_{\langle s \rangle}^n = u\}$ equals 2. Consequently, if U is a uniform random variable in $[0, 1)$, independent of (T_n, L_n), then $u_{\langle 2nU \rangle}^n$ is uniform in $T_n \setminus \{\varnothing\}$. Hence, it suffices to prove the desired result with $u_{\langle 2nU \rangle}^n$ instead of v_{**}.

Since $|s - \langle s \rangle| \leq 1$, Theorem 4.3 entails that

$$
\begin{aligned}
\left(\frac{8n}{9}\right)^{-1/4} d_{Q_n}(v_*, u_{\langle 2nU \rangle}^n) &= \left(\frac{8n}{9}\right)^{-1/4} (L_n(u_{\langle 2nU \rangle}^n) - \min L_n + 1) \\
&= \left(\frac{8n}{9}\right)^{-1/4} (V_n(\langle 2nU \rangle) - \min V_n + 1)\,,
\end{aligned}
$$

converges in distribution to $Z_U - \inf Z$ (here U is also assumed to be independent of $(\mathbbe, Z)$). The fact that $Z_U - \inf Z$ has the same distribution as $\sup Z$, or equivalently as $-\inf Z$, can be derived from the invariance of the CRT under uniform re-rooting, see e.g. [35]. This completes the proof of (ii).

Finally, for (iii) we just note that, for every bounded continuous $g : \mathbb{R}_+ \to \mathbb{R}$,

$$
\begin{aligned}
\frac{1}{n+2} &\sum_{k \in \mathbb{Z}_+} I_{Q_n, v_*}(k)\, g(((8n/9)^{-1/4} k) \\
&= \frac{1}{n+2} \sum_{v \in Q_n} g((8n/9)^{-1/4} d_{Q_n}(v_*, v)) \\
&= E_{**}[g((8n/9)^{-1/4} d_{Q_n}(v_*, v_{**}))] \\
&\underset{n \to \infty}{\longrightarrow} E_U[g(Z_U - \inf Z)] \\
&= \int_0^1 \mathrm{d}t\, g(Z_t - \inf Z)\,,
\end{aligned}
$$

where E_{**} and E_U means that we take the expectation only with respect to v_{**} and U in the corresponding expressions (these are conditional expectations given (Q_n, v_*) and $(\mathbbe, Z)$ respectively). In the penultimate step, we used the convergence established in the course of the proof of (ii). $\square$

6.2. Convergence as metric spaces. We would like to be able to understand the full scaling limit picture for random maps, in a similar way as it was done for trees, where we showed, using Theorem 2.10, that the distances in discrete trees,

once rescaled by $\sqrt{2n}$, converge to the distances in the CRT $(\mathcal{T}_\mathbb{e}, d_\mathbb{e})$. We thus ask if there is an analog of the CRT that arises as the limit of the properly rescaled metric spaces (Q_n, d_{Q_n}). In view of Theorem 6.1, the correct normalization for the distance should be $n^{1/4}$.

Assume that (T_n, L_n) is uniformly distributed over $\mathbb{T}_n$, let ϵ be uniform in $\{-1, 1\}$ and independent of (T_n, L_n), and let Q_n be the random uniform quadrangulation with n faces and with a uniformly chosen vertex v_*, which is obtained from $((T_n, L_n), \epsilon)$ via the CVS bijection. We now follow Le Gall [31][2]. Recall our notation $u_0^n, u_1^n, \ldots, u_{2n}^n$ for the contour exploration of the vertices of T_n, and recall that in the CVS bijection these vertices are also viewed as elements of $V(Q_n) \setminus \{v_*\}$. Define a pseudo-metric on $\{0, \ldots, 2n\}$ by letting $d_n(i, j) = d_{Q_n}(u_i^n, u_j^n)$. A major problem comes from the fact that $d_n(i, j)$ cannot be expressed as a simple functional of (C_n, V_n). The only distances that we are able to handle in an easy way are distances to v_*, through the following rewriting of (16):

$$(18) \qquad d_{Q_n}(v_*, u_i^n) = V_n(i) - \min V_n + 1 \,.$$

We also define, for $i, j \in \{0, 1, \ldots, 2n\}$,

$$d_n^0(i, j) = V_n(i) + V_n(j) - 2 \max \left(\min_{i \leq k \leq j} V_n(k), \min_{j \leq k \leq i} V_n(k) \right) + 2 \,.$$

Here, if $j < i$, the condition $i \leq k \leq j$ means that $k \in \{i, i+1, \ldots, 2n\} \cup \{0, 1, \ldots, j\}$ and similarly for the condition $j \leq k \leq i$ if $i < j$.

As a consequence of Proposition 5.9(i), we have the bound $d_n \leq d_n^0$.

We now extend the function d_n to $[0, 2n]^2$ by letting

$$\begin{aligned} d_n(s, t) &= (\lceil s \rceil - s)(\lceil t \rceil - t) d_n(\lfloor s \rfloor, \lfloor t \rfloor) + (\lceil s \rceil - s)(t - \lfloor t \rfloor) d_n(\lfloor s \rfloor, \lceil t \rceil) \\ (19) &\qquad + (s - \lfloor s \rfloor)(\lceil t \rceil - t) d_n(\lceil s \rceil, \lfloor t \rfloor) + (s - \lfloor s \rfloor)(t - \lfloor t \rfloor) d_n(\lceil s \rceil, \lceil t \rceil) \,, \end{aligned}$$

recalling that $\lfloor s \rfloor = \sup\{k \in \mathbb{Z}_+ : k \leq s\}$ and $\lceil s \rceil = \lfloor s \rfloor + 1$. The function d_n^0 is extended to $[0, 2n]^2$ by the obvious similar formula.

It is easy to check that d_n thus extended is continuous on $[0, 2n]^2$ and satisfies the triangle inequality (although this is not the case for d_n^0), and that the bound $d_n \leq d_n^0$ still holds. We define a rescaled version of these functions by letting

$$D_n(s, t) = \left(\frac{9}{8n} \right)^{1/4} d_n(2ns, 2nt) \,, \qquad 0 \leq s, t \leq 1 \,.$$

We define similarly the functions D_n^0 on $[0, 1]^2$. Then, as a consequence of Theorem 4.3, we have

$$(20) \qquad (D_n^0(s, t), 0 \leq s, t \leq 1) \xrightarrow[n \to \infty]{(d)} (D^0(s, t), 0 \leq s, t \leq 1) \,,$$

for the uniform topology on $C([0, 1]^2, \mathbb{R})$, where by definition

$$(21) \qquad D^0(s, t) = Z_s + Z_t - 2 \max \left(\min_{s \leq r \leq t} Z_r, \min_{t \leq r \leq s} Z_r \right) \,,$$

where if $t < s$ the condition $s \leq r \leq t$ means that $r \in [s, 1] \cup [0, t]$.

We can now state

---

[2]At this point, it should be noted that [31, 34, 32] consider another version of Schaeffer's bijection, where no distinguished vertex v_* has to be considered. This results in considering pairs (T_n, L_n) in which L_n is conditioned to be positive. The scaling limits of such pairs are still tractable, and in fact, are simple functionals of $(\mathbb{e}, Z)$, as shown in [35, 30]. So there will be some differences from our exposition, but these turn out to be unimportant.

PROPOSITION 6.2. *The family of laws of $(D_n(s,t), 0 \leq s, t \leq 1)$, as n varies, is relatively compact for the weak topology on probability measures on $C([0,1]^2, \mathbb{R})$.*

PROOF. Let $s, t, s', t' \in [0,1]$. Then by a simple use of the triangle inequality, and the fact that $D_n \leq D_n^0$,

$$|D_n(s,t) - D_n(s',t')| \leq D_n(s,s') + D_n(t,t') \leq D_n^0(s,s') + D_n^0(t,t'),$$

which allows one to estimate the modulus of continuity at a fixed $\delta > 0$:

$$(22) \qquad \sup_{\substack{|s-s'| \leq \delta \\ |t-t'| \leq \delta}} |D_n(s,t) - D_n(s',t')| \leq 2 \sup_{|s-s'| \leq \delta} D_n^0(s,s').$$

However, the convergence in distribution (20) entails that for every $\varepsilon > 0$,

$$\limsup_{n \to \infty} P\left(\sup_{|s-s'| \leq \delta} D_n^0(s,s') \geq \varepsilon \right) \leq P\left(\sup_{|s-s'| \leq \delta} D^0(s,s') \geq \varepsilon \right),$$

and the latter quantity goes to 0 when $\delta \to 0$ (for any fixed value of $\epsilon > 0$) by the continuity of D^0 and the fact that $D^0(s,s) = 0$. Hence, taking $\eta > 0$ and letting $\varepsilon = \varepsilon_k = 2^{-k}$, we can choose $\delta = \delta_k$ (tacitly depending also on η) such that

$$\sup_{n \geq 1} P\left(\sup_{|s-s'| \leq \delta_k} D_n^0(s,s') \geq 2^{-k} \right) \leq \eta 2^{-k}, \qquad k \geq 1,$$

entailing

$$P\left(\bigcap_{k \geq 1} \left\{ \sup_{|s-s'| \leq \delta_k} D_n^0(s,s') \leq 2^{-k} \right\} \right) \geq 1 - \eta,$$

for all $n \geq 1$. Together with (22), this shows that with probability at least $1 - \eta$, the function D_n belongs to the set of all functions f from $[0,1]^2$ into $\mathbb{R}$ such that $f(0,0) = 0$ and, for every $k \geq 1$,

$$\sup_{\substack{|s-s'| \leq \delta_k \\ |t-t'| \leq \delta_k}} |f(s,t) - f(s',t')| \leq 2^{-k}.$$

The latter set is compact by the Arzelà-Ascoli theorem. The conclusion then follows from Prokhorov's theorem. $\qquad \square$

At this point, we are allowed to say that the random distance functions D_n admit a limit in distribution, up to taking $n \to \infty$ along a subsequence:

$$(23) \qquad (D_n(s,t), 0 \leq s, t \leq 1) \xrightarrow{(d)} (D(s,t), 0 \leq s, t \leq 1)$$

for the uniform topology on $C([0,1]^2, \mathbb{R})$. In fact, we are going to need a little more than the convergence of D_n. From the relative compactness of the components, we see that the closure of the collection of laws of the triplets

$$((2n)^{-1} C_n(2n\cdot), (9/8n)^{1/4} V_n(2n\cdot), D_n), \quad n \geq 1$$

is compact in the space of all probability measures on $C([0,1], \mathbb{R})^2 \times C([0,1]^2, \mathbb{R})$. Therefore, it is possible to choose a subsequence $(n_k, k \geq 1)$ so that this triplet converges in distribution to a limit, which is denoted by $(\mathbbe, Z, D)$ (from Theorem 4.3, this is of course consistent with the preceding notation). The joint convergence to the triplet $(\mathbbe, Z, D)$ gives a coupling of D, D^0 such that $D \leq D^0$, since $D_n \leq D_n^0$ for every n.

Define a random equivalence relation on $[0,1]$ by letting $s \approx t$ if $D(s,t) = 0$. We let $M = [0,1]/\approx$ be the associated quotient space, endowed with the quotient distance, which we still denote by D. The canonical projection $[0,1] \to M$ is denoted by $\mathbf{p}$.

Finally, let $s_* \in [0,1]$ be such that $Z_{s_*} = \inf Z$. It can be proved that s_* is unique a.s., see [**37**] or [**35**], and we will admit this fact (although it is not really needed for the next statement). We set $\rho_* = \mathbf{p}(s_*)$. We can now state the main result of this section.

THEOREM 6.3. *The random pointed metric space (M, D, ρ_*) is the limit in distribution of the spaces $(V(Q_n), (9/8n)^{1/4} d_{Q_n}, v_*)$, for the Gromov-Hausdorff topology, along the subsequence $(n_k, k \geq 1)$. Moreover, we have a.s. for every $x \in M$ and $s \in [0,1]$ such that $\mathbf{p}(s) = x$,*

$$D(\rho_*, x) = D(s_*, s) = Z_s - \inf Z.$$

Note that, in the discrete model, a point at which the minimal label in T_n is attained lies at distance 1 from v_*. Therefore, the point ρ_* should be seen as the continuous analog of the distinguished vertex v_*. The last identity in the statement of the theorem is then of course the continuous analog of (16) and (18).

PROOF. For the purposes of this proof, it is useful to assume, using the Skorokhod representation theorem, that the convergence

$$((2n)^{-1/2} C_n(2n\cdot), (9/8n)^{1/4} V_n(2n\cdot), D_n) \longrightarrow (\mathbb{e}, Z, D)$$

holds a.s. along the subsequence (n_k). In what follows we restrict our attention to values of n in this sequence.

For every n, let $i_*^{(n)}$ be any index in $\{0, 1, \ldots, 2n\}$ such that $V_n(i_*^{(n)}) = \min V_n$. Then for every $v \in V(Q_n)$, it holds that

$$|d_{Q_n}(v_*, v) - d_{Q_n}(u_{i_*^{(n)}}^n, v)| \leq 1$$

because $d_{Q_n}(v_*, u_{i_*^{(n)}}^n) = 1$ (v_* and $u_{i_*^{(n)}}^n$ are linked by an arc in the CVS bijection). Moreover, since $(8n/9)^{-1/4} V_n(2n\cdot)$ converges to Z uniformly on $[0,1]$, and since we know[3] that Z attains its overall infimum at a unique point s_*, it is easy to obtain that $i_*^{(n)}/2n$ converges as $n \to \infty$ towards s_*.

For every integer n, we construct a correspondence $\mathcal{R}_n$ between $V(Q_n)$ and M_n, by putting:

- $(v_*, \rho_*) \in \mathcal{R}_n$;
- $(u_{\lfloor 2ns \rfloor}^n, \mathbf{p}(s)) \in \mathcal{R}_n$, for every $s \in [0,1]$.

We then verify that the distortion of $\mathcal{R}_n$ (with respect to the metrics $(9/8n)^{1/4} d_{Q_n}$ on $V(Q_n)$ and D on M) converges to 0 a.s. as $n \to \infty$. We first observe that

$$\sup_{s \in [0,1]} |(9/8n)^{1/4} d_{Q_n}(v_*, u_{\lfloor 2ns \rfloor}^n) - D(\rho_*, \mathbf{p}(s))|$$

$$\leq (9/8n)^{1/4} + \sup_{s \in [0,1]} |(9/8n)^{1/4} d_{Q_n}(u_{i_*^{(n)}}^n, u_{\lfloor 2ns \rfloor}^n) - D(\rho_*, \mathbf{p}(s))|$$

$$= (9/8n)^{1/4} + \sup_{s \in [0,1]} |D_n(i_*^{(n)}/2n, \lfloor 2ns \rfloor/2n) - D(s_*, s)|,$$

_____

[3]We could also perform the proof without using this fact, but it makes things a little easier

which tends to 0 as $n \to \infty$, by the a.s. uniform convergence of D_n to D, and the fact that $i_*^{(n)}/2n$ converges to s_*. Similarly, we have

$$
\sup_{s,t \in [0,1]} |(9/8n)^{1/4} d_{Q_n}(u_{\lfloor 2ns \rfloor}^n, u_{\lfloor 2nt \rfloor}^n) - D(\mathbf{p}(s), \mathbf{p}(t))|
$$
$$
= \sup_{s,t \in [0,1]} |D_n(\lfloor 2ns \rfloor/2n, \lfloor 2nt \rfloor/2n) - D(s,t)|
$$

which tends to 0 as $n \to \infty$. We conclude that the distortion of $\mathcal{R}_n$ converges to 0 a.s. and that the pointed metric spaces $(V(Q_n), (9/8n)^{-1/4} d_{Q_n}, v_*)$ also converge a.s. to (M, D, ρ_*) in the Gromov-Hausdorff topology.

Let us prove the last statement of the theorem. Using once again the uniform convergence of D_n to D, we obtain that for every $s \in [0,1]$,

$$
\begin{aligned}
D(s_*, s) &= \lim_{n \to \infty} D_n(i_*^{(n)}/2n, \lfloor 2ns \rfloor/2n) \\
&= \lim_{n \to \infty} \left(\frac{8n}{9}\right)^{-1/4} d_{Q_n}(v_*, u_{\lfloor 2ns \rfloor}^n) \\
&= \lim_{n \to \infty} \left(\frac{8n}{9}\right)^{-1/4} (V_n(\lfloor 2ns \rfloor) - \min V_n + 1) \\
&= Z_s - \inf Z,
\end{aligned}
$$

as desired. $\qquad\square$

It is tempting to call (M, D) the "Brownian map", or the "Brownian continuum map", by analogy with the fact that the "Brownian continuum random tree" is the scaling limit of uniformly distributed plane trees with n edges. However, the choice of the subsequence in Theorem 6.3 poses a problem of uniqueness of the limit. As we see in the previous statement, only the distances to ρ_* are *a priori* defined as simple functionals of the process Z. Distances between other points in M seem to be harder to handle. The following conjecture is however very appealing.

CONJECTURE 6.1. *The spaces $(V(Q_n), n^{-1/4} d_{Q_n})$ converge in distribution, for the Gromov-Hausdorff topology.*

Marckert and Mokkadem [37] and Le Gall [31] give a natural candidate for the limit (called the Brownian map in [37]) but until now the convergence result in the above conjecture has not been proved.

7. Identifying the Brownian map

7.1. The Brownian map as a quotient of the CRT. In the previous section, we wrote the scaling limit of rescaled random quadrangulations (along a suitable subsequence) as a quotient space $M = [0,1]/\approx$ where the equivalence relation $\approx$ is defined by $s \approx t$ iff $D(s,t) = 0$. In this section, we provide a more explicit description of this quotient.

Recall the notation of the previous section. In particular, $((T_n, L_n), \epsilon)$ is uniformly distributed over $\mathbf{T}_n \times \{-1, 1\}$, and (Q_n, v_*) is the pointed quadrangulation that is the image of $((T_n, L_n), \epsilon)$ under the CVS bijection. For every $n \geq 1$, $u_0^n, u_1^n, \ldots, u_{2n}^n$ is the contour exploration of the vertices of T_n. Thus, $C_n(k) = |u_k^n|$ and $V_n(k) = L_n(u_k^n)$ for $0 \leq k \leq 2n$.

As in the proof of Theorem 6.3, we may assume that, along the sequence (n_k) we have the almost sure convergence

(24)
$$((2n)^{-1/2}C_n(2ns), (9/8n)^{1/4}V_n(2ns), D_n(s,t))_{s,t\in[0,1]}$$
$$\underset{n\to\infty}{\longrightarrow} (\mathrm{e}_s, Z_s, D(s,t))_{s,t\in[0,1]}$$

uniformly over $[0,1]^2$. Recall from the proof of Theorem 6.3 that this implies the almost sure convergence

$$\left(V(Q_n), \left(\frac{9}{8}\right)^{1/4} d_{Q_n}\right) \underset{n\to\infty}{\longrightarrow} (M, D)$$

in the Gromov-Hausdorff sense, along the sequence (n_k).

As in Section 2 above, introduce the random equivalence relation $\sim_{\mathrm{e}}$ on $[0,1]$ by

$$s \sim_{\mathrm{e}} t \text{ iff } \mathrm{e}_s = \mathrm{e}_t = \min_{s\wedge t\le r\le s\vee t} \mathrm{e}_r$$

and recall that the CRT $\mathcal{T}_{\mathrm{e}}$ is defined as the quotient space $[0,1]/\sim_{\mathrm{e}}$ equipped with the distance d_{e}.

LEMMA 7.1. *We have almost surely for every $s,t \in [0,1]$,*

$$s \sim_{\mathrm{e}} t \Rightarrow D(s,t) = 0 \quad (\Leftrightarrow s \approx t).$$

PROOF. We can use the convergence of the first components in (24) to see that if $s \sim_{\mathrm{e}} t$ and $s < t$ we can find integers $i_n < j_n$ such that $i_n/2n \longrightarrow s$, $j_n/2n \longrightarrow t$, and, for every sufficiently large n (belonging to the sequence (n_k)),

$$C_n(i_n) = C_n(j_n) = \min_{i_n\le k\le j_n} C_n(k).$$

Then, from the definition of the contour function, we must have $u_{i_n}^n = u_{j_n}^n$, and thus $d_n(i_n, j_n) = 0$. Using the convergence (24) again, we conclude that $D(s,t) = 0$.

Consequence. Recall that $p_{\mathrm{e}} : [0,1] \longrightarrow \mathcal{T}_{\mathrm{e}}$ denotes the canonical projection. Then $D(s,t)$ only depends on $p_{\mathrm{e}}(s)$ and $p_{\mathrm{e}}(t)$. We can therefore put for every $a, b \in \mathcal{T}_{\mathrm{e}}$,

$$D(a,b) = D(s,t)$$

where s, resp. t, is an arbitrary representative of a, resp. of b, in $[0,1]$. Then D is (again) a pseudo-distance on $\mathcal{T}_{\mathrm{e}}$. With a slight abuse of notation we keep writing $a \approx b$ iff $D(a,b) = 0$, for $a,b \in \mathcal{T}_{\mathrm{e}}$. Then the Brownian map M can be written as

$$M = [0,1]/\approx = \mathcal{T}_{\mathrm{e}}/\approx$$

where the first equality was the definition of M and the second one corresponds to the fact that there is an obvious canonical isometry between the two quotient spaces.

One may wonder why it is more interesting to write the Brownian map M as a quotient space of the CRT $\mathcal{T}_{\mathrm{e}}$ rather than as a quotient space of $[0,1]$. The point is that it will be possible to give a simple intuitive description of $\approx$ viewed as an equivalence relation on $\mathcal{T}_{\mathrm{e}}$. This is indeed the main goal of the next section.

7.2. Identifying the equivalence relation $\approx$. We noticed in subsection 4.2 that the process Z (the Brownian snake driven by $\mathbbm{e}$) can be viewed as indexed by $\mathcal{T}_{\mathbbm{e}}$. This will be important in what follows: For $a \in \mathcal{T}_{\mathbbm{e}}$, we will write $Z_a = Z_t$ for any choice of t such that $a = p_{\mathbbm{e}}(t)$. We also set $a_* = p_{\mathbbm{e}}(s_*)$: a_* is thus the unique vertex of $\mathcal{T}_{\mathbbm{e}}$ such that

$$Z_{a_*} = \min_{a \in \mathcal{T}_{\mathbbm{e}}} Z_a.$$

We first need to define intervals on the tree $\mathcal{T}_{\mathbbm{e}}$. For simplicity we consider only leaves of $\mathcal{T}_{\mathbbm{e}}$. Recall that a point a of $\mathcal{T}_{\mathbbm{e}}$ is a leaf if $\mathcal{T}_{\mathbbm{e}} \backslash \{a\}$ is connected. Equivalently a vertex a distinct from the root ρ is a leaf if and only $p_{\mathbbm{e}}^{-1}(a)$ is a singleton. Note in particular that a_* is a leaf of $\mathcal{T}_{\mathbbm{e}}$.

Let a and b be two (distinct) leaves of $\mathcal{T}_{\mathbbm{e}}$, and let s and t be the unique elements of $[0,1)$ such that $p_{\mathbbm{e}}(s) = a$ and $p_{\mathbbm{e}}(t) = b$. Assume that $s < t$ for definiteness. We then set

$$[a,b] = p_{\mathbbm{e}}([s,t])$$
$$[b,a] = p_{\mathbbm{e}}([t,1] \cup [0,s]).$$

It is easy to verify that $[a,b] \cap [b,a] = [\![a,b]\!]$ is the line segment between a and b in $\mathcal{T}_{\mathbbm{e}}$.

THEOREM 7.2. *Almost surely, for every distinct $a, b \in \mathcal{T}_{\mathbbm{e}}$,*

$$a \approx b \quad \Leftrightarrow \quad \begin{cases} a, b \text{ are leaves of } \mathcal{T}_{\mathbbm{e}} \text{ and} \\ Z_a = Z_b = \max\left(\min_{c \in [a,b]} Z_c, \min_{c \in [b,a]} Z_c\right) \end{cases}$$

REMARK 7.3. *We know that the minimum of Z over $\mathcal{T}_{\mathbbm{e}}$ is attained at the unique vertex a_*. If a and b are (distinct) leaves of $\mathcal{T}_{\mathbbm{e}} \backslash \{a_*\}$, exactly one of the two intervals $[a,b]$ and $[b,a]$ contains the vertex a_*. Obviously the minimum of Z over this interval is equal to Z_{a_*} and thus cannot be equal to Z_a or Z_b.*

The proof of the implication $\Leftarrow$ in the theorem is easy. Suppose that $a = p_{\mathbbm{e}}(s)$ and $b = p_{\mathbbm{e}}(t)$ with $s < t$ (for definiteness). If

$$Z_a = Z_b = \max\left(\min_{c \in [a,b]} Z_c, \min_{c \in [b,a]} Z_c\right)$$

this means that

$$Z_s = Z_t = \max\left(\min_{r \in [s,t]} Z_r, \min_{r \in [t,1] \cup [0,s]} Z_r\right).$$

The last identity is equivalent to saying that $D^0(s,t) = 0$, and since $D \leq D^0$ we have also $D(s,t) = 0$, or equivalently $a \approx b$.

Unfortunately, the proof of the converse implication is much harder, and we will only give some key ideas of the proof, referring to [**31**] for additional details.

We start with a preliminary lemma. We denote by $\mathrm{vol}(\cdot)$ the mass measure on $\mathcal{T}_{\mathbbm{e}}$, which is simply the image of the Lebesgue measure on $[0,1]$ under the projection $p_{\mathbbm{e}} : [0,1] \longrightarrow \mathcal{T}_{\mathbbm{e}}$.

LEMMA 7.4. *Almost surely, for every $\delta \in (0,1)$, there exists a (random) constant $C_\delta(\omega)$ such that, for every $r > 0$ and every $a \in \mathcal{T}_{\mathbbm{e}}$,*

$$\mathrm{vol}(\{b \in \mathcal{T}_{\mathbbm{e}} : D(a,b) \leq r\}) \leq C_\delta \, r^{4-\delta}.$$

We omit the proof of this lemma. The first ingredient of the proof is a "re-rooting invariance property" of random planar maps, which makes it possible to reduce the proof to the case $a = a_*$. In that case we can use the formula $D(a_*, b) = Z_b - \min Z$ and explicit moment calculations for the Brownian snake (see Corollary 6.2 in [32] for a detailed proof).

Let us come to the proof of the implication $\Rightarrow$ in Theorem 7.2. For simplicity we consider only the case when a and b are leaves of $\mathcal{T}_\mathbb{e}$ (it would be necessary to show also that the equivalence class of any vertex of $\mathcal{T}_\mathbb{e}$ that is not a leaf is a singleton – this essentially follows from Lemma 2.2 in [31]). We let $s, t \in [0, 1]$ be such that $a = p_\mathbb{e}(s)$ and $b = p_\mathbb{e}(t)$, and assume for definiteness that $0 \leq s_* < s < t \leq 1$.

We assume that $a \approx b$, and our goal is to prove that

$$Z_a = Z_b = \min_{c \in [a, b]} Z_c.$$

We already know that $Z_a = Z_b$, because

$$Z_a - \min Z = D(a_*, a) = D(a_*, b) = Z_b - \min Z.$$

First step. We first establish that

$$(25) \qquad Z_a = Z_b = \min_{c \in [\![a, b]\!]} Z_c.$$

To see this, we go back to the discrete picture. We can find $a_n, b_n \in T_n$ such that $a_n \longrightarrow a$ and $b_n \longrightarrow b$ as $n \to \infty$ (strictly speaking these convergences make no sense: What we mean is that $a_n = u_{i_n}^n$, $b_n = u_{j_n}^n$ with $i_n/2n \longrightarrow s$ and $j_n/2n \longrightarrow t$). Then the condition $D(a, b) = 0$ implies that

$$(26) \qquad n^{-1/4} d_{Q_n}(a_n, b_n) \longrightarrow 0.$$

Recall, from Proposition 5.9, the notation $[\![a_n, b_n]\!]$ for the set of vertices lying on the geodesic path from a_n to b_n in the tree T_n. By Proposition 5.9(ii), we have

$$d_{Q_n}(a_n, b_n) \geq L_n(a_n) + L_n(b_n) - 2 \min_{c \in [\![a_n, b_n]\!]} L_n(c).$$

We multiply both sides of this inequality by $n^{-1/4}$ and let n tend to ∞, using (26). Modulo some technical details that we omit (essentially one needs to check that any vertex of $\mathcal{T}_\mathbb{e}$ belonging to $[\![a, b]\!]$ is of the form $p_\mathbb{e}(r)$, where $r = \lim k_n/2n$ and the integers k_n are such that $u_{k_n}^n$ belongs to $[\![a_n, b_n]\!]$), we get that

$$Z_a + Z_b - 2 \min_{c \in [\![a, b]\!]} Z_c \leq 0$$

from which (25) immediately follows.

Second step. We argue by contradiction, assuming that

$$\min_{c \in [a, b]} Z_c < Z_a = Z_b.$$

Let γ_n be a discrete geodesic from a_n to b_n in the quadrangulation Q_n (here we view a_n and b_n as vertices of the quadrangulation Q_n, and this geodesic is of course different from the geodesic from a_n to b_n in the tree T_n). From (26) the maximal distance between a_n (or b_n) and a vertex visited by γ_n is $o(n^{1/4})$ as $n \to \infty$. As a consequence, using the triangle inequality and (16), we have

$$\sup_{u \in \gamma_n} |L_n(u) - L_n(a_n)| = o(n^{1/4})$$

as $n \to \infty$.

To simplify the presentation of the argument, we assume that, for infinitely many values of n, the geodesic path γ_n from a_n to b_n stays in the lexicographical interval $[a_n, b_n]$. This lexicographical interval is defined, analogously to the continuous setting, as the set of all vertices visited by the contour exploration sequence $(u_i^n)_{0 \le i \le 2n}$ between its last visit of a_n and its first visit of b_n. Note that the preceding assumption may not hold, and so the real argument is slightly more complicated than what follows.

We use the previous assumption to prove the following claim. If $x \in [a, b]$, we denote by $\phi_{a,b}(x)$ the last ancestor of x that belongs to $[\![a, b]\!]$ (the condition $x \in [a, b]$ ensures that the ancestral line $[\![\rho, x]\!]$ intersects $[\![a, b]\!]$). Alternatively, $\phi_{a,b}(x)$ is the point of $[\![a, b]\!]$ at minimal $d_{\mathfrak{e}}$-distance of x in the tree $\mathcal{T}_{\mathfrak{e}}$.

Claim. Let $\varepsilon > 0$. For every $c \in [a, b]$ such that

$$\begin{cases} Z_c < Z_a + \varepsilon \\ Z_x > Z_a + \varepsilon/2 \qquad \forall x \in [\![\phi_{a,b}(c), c]\!] \end{cases}$$

we have $D(a, c) \le \varepsilon$.

The claim eventually leads to the desired contradiction: Using the first step of the proof (which ensures that $Z_c \ge Z_a$ for $c \in [\![a, b]\!]$) and the properties of the Brownian snake, one can check that, under the condition

$$\min_{c \in [a,b]} Z_c < Z_a = Z_b,$$

the volume of the set of all vertices c that satisfy the assumptions of the claim is bounded below by a (random) positive constant times ε^2, at least for sufficiently small $\varepsilon > 0$ (see Lemma 2.4 in [**31**] for a closely related statement). The desired contradiction follows since Lemma 7.4 implies that, for every $\delta \in (0, 1)$,

$$\mathrm{vol}(\{c : D(a, c) \le \varepsilon\}) \le C_\delta\, \varepsilon^{4-\delta}.$$

To complete this sketch, we explain why the claim holds. Again, we need to go back to the discrete setting. We consider a vertex $u \in [a_n, b_n]$ such that

  (i) $L_n(u) < L_n(a_n) + \varepsilon n^{1/4}$;
  (ii) $L_n(v) > L_n(a_n) + \frac{\varepsilon}{2}\, n^{1/4}$, $\qquad \forall v \in [\![\phi_{a_n,b_n}^n(u), u]\!]$

where $\phi_{a_n, b_n}^n(u)$ is the last ancestor of u in the tree T_n that belongs to $[\![a_n, b_n]\!]$.

Condition (ii) guarantees that the vertex u lies "between" $[\![a_n, b_n]\!]$ and the geodesic γ_n: If this were not the case, the geodesic γ_n would contain a point in $[\![\phi_{a_n,b_n}^n(u), u]\!]$, which is impossible by (ii) (we already noticed that the label of a vertex of the geodesic γ_n must be $L_n(a_n) + o(n^{1/4})$).

Consider the geodesic path from u to v_* in Q_n that is obtained from the successor geodesic chain $e \to s(e) \to s^2(e) \to \cdots$ starting from any corner e of u in T_n. Since arcs in the CVS bijection do not cross edges of the tree and since we know that the vertex u lies in the area between $[\![a_n, b_n]\!]$ and the geodesic γ_n, the geodesic we have just constructed cannot "cross" $[\![a_n, b_n]\!]$ and so it must intersect γ_n at a vertex w. This vertex w is such that

$$L_n(u) - L_n(w) = d_{Q_n}(u, w).$$

Since w belongs to γ_n, we have $d_{Q_n}(w, a_n) = o(n^{1/4})$, and therefore

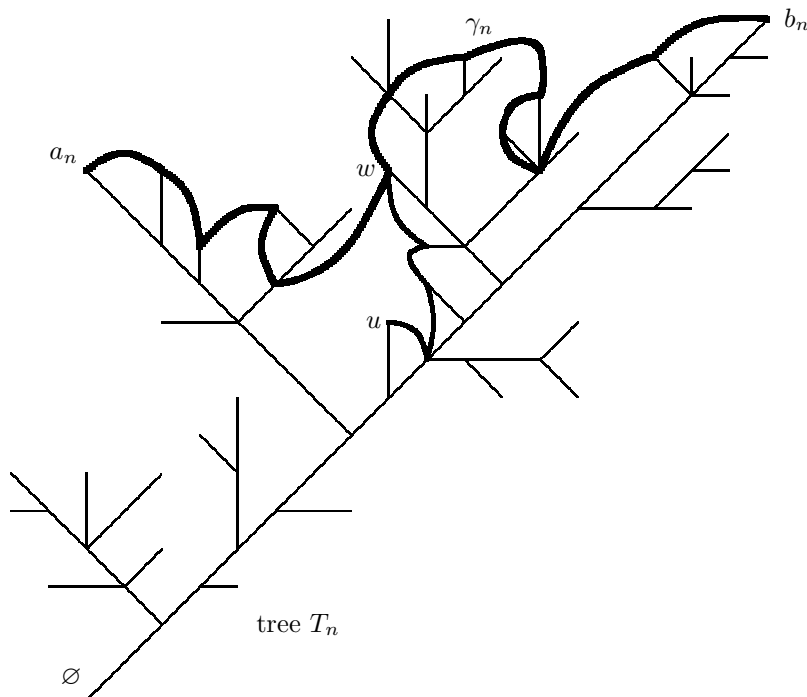$$L_n(u) - L_n(a_n) = d_{Q_n}(u, a_n) + o(n^{1/4}).$$

FIGURE 8. Illustration of the proof: The geodesic path γ_n from a_n to b_n is represented by the thick curves. The thin curves correspond to the beginning of the successor geodesic chain starting from u. This chain does not cross the line segment $[\![a_n, b_n]\!]$ and thus has to meet the path γ_n at some point w.

By (i), we now get
$$d_{Q_n}(u, a_n) \leq \varepsilon n^{1/4} + o(n^{1/4}).$$
We have thus obtained a discrete analog of the claim. To get the continuous version as stated above, we just need to do a careful passage to the limit $n \to \infty$. $\square$

7.3. Hausdorff dimension. The limit in distribution (along a suitable subsequence) in Theorem 6.3 can be written as $(\mathcal{T}_{\mathbb{e}}/\approx, D)$, and the space $\mathcal{T}_{\mathbb{e}}/\approx$ is completely identified: Roughly speaking two vertices a and b of the CRT $\mathcal{T}_{\mathbb{e}}$ are identified if and only if they have the same label $Z_a = Z_b$ and if one can go from a to b following the "contour" of the tree $\mathcal{T}_{\mathbb{e}}$ and visiting only vertices with larger label. In order to prove Conjecture 6.1, it would be necessary to characterize the distance D. Much is known about D (in particular Theorem 6.1 characterizes the distribution of the profile of distances from the distinguished point ρ_*, and one can show that this profile has the same distribution if one replaces ρ_* by a "typical" point of M). Still the characterization of D remains an open problem.

Nevertheless, one can show that the "Brownian map" $(\mathcal{T}_{\mathbb{e}}/\approx, D)$, that is, any of the random metric spaces that can arise as the limit in Theorem 6.3, has Hausdorff dimension 4 and is homeomorphic to the 2-sphere. This was proved in [**31**] and [**34**]. The remainder of these notes will be devoted to the proof of these two results.

THEOREM 7.5. *Almost surely, the space (M, D) has Hausdorff dimension* 4.

The lower bound is an easy consequence of Lemma 7.4. Recall that vol is the image measure of Lebesgue measure on $[0, 1]$ under $p_{\oplus}$. We let Vol be the induced measure on (M, D), that is, the image of Lebesgue measure on $[0, 1]$ under the projection $\mathbf{p} : [0, 1] \to M$. Then Lemma 7.4 implies that a.s., for every $\delta \in (0, 1)$, and every $x \in M$, it holds that

$$\limsup_{r \downarrow 0} \frac{\text{Vol}(B_D(x, r))}{r^{4-\delta}} = 0 \,,$$

where $B_D(x, r) = \{y \in M : D(x, y) < r\}$ is the open ball centered at x with radius r. This last fact, combined with standard density theorems for Hausdorff measures, implies that a.s. the Hausdorff dimension of (M, D) is greater than or equal to $4 - \delta$, for every $\delta \in (0, 1)$.

For the upper bound, we rely on the following easy lemma.

LEMMA 7.6. *Almost surely, for every $\alpha \in (0, 1/4)$, the label process Z is Hölder continuous with exponent α.*

PROOF. This is obtained by the classical Kolmogorov continuity criterion, and moment estimates for Z. Let s, t be such that $0 \le s < t \le 1$, and recall that conditionally given $\oplus$, $Z_s - Z_t$ is a Gaussian random variable with variance $d_{\oplus}(s, t)$. Consequently, for every $p > 0$, there exists $C_p \in (0, \infty)$ such that

$$E[|Z_s - Z_t|^p \,|\, \oplus] = C_p d_{\oplus}(s, t)^{p/2} \,,$$

and since $\oplus$ is a.s. Hölder continuous with exponent 2α, we deduce the existence of a (random) $C'_p \in (0, \infty)$ such that

$$E[|Z_s - Z_t|^p \,|\, \oplus] \le C'_p |s - t|^{p\alpha} \,.$$

The desired Hölder continuity property then follows from an application of the classical Kolmogorov lemma. $\qquad\square$

From this, we deduce that the projection $\mathbf{p} : [0, 1] \to M$ is a.s. Hölder continuous with index $\alpha \in (0, 1/4)$ as well. Indeed, using the fact that $D \le D^0$, where D^0 is defined in (21), we get

$$
\begin{aligned}
D(\mathbf{p}(s), \mathbf{p}(t)) &= D(s, t) \\
&\le Z_s + Z_t - 2 \inf_{s \wedge t \le u \le s \vee t} Z_u \\
&\le 2 \sup_{s \wedge t \le u, v \le s \vee t} |Z_u - Z_v| \\
&\le C''_p |s - t|^{\alpha} \,,
\end{aligned}
$$

for some $C''_p \in (0, \infty)$. The fact that the Hausdorff dimension of (M, D) is bounded above by $1/\alpha$ is then a classical consequence of this last property. This completes the proof of the theorem.

8. The homeomorphism theorem

THEOREM 8.1. *Almost-surely, the Brownian map (M, D) is homeomorphic to the 2-sphere $\mathbb{S}^2$.*

This result was first obtained by Le Gall and Paulin [**34**], by arguing directly on the quotient space $M = \mathcal{T}_{\mathbb{e}}/ \approx$. More precisely, Le Gall and Paulin observe that the equivalence relations $\sim_{\mathbb{e}}$ and $\approx$ may be viewed as equivalence relations on the sphere $\mathbb{S}^2$. Upon showing that the associated classes are closed, arcwise connected, and have connected complements, one can then apply a theorem due to Moore [**43**], showing that under these hypotheses, the quotient $\mathbb{S}^2/ \approx$ is itself homeomorphic to $\mathbb{S}^2$. Here, we will adopt a different approach, introduced in Miermont [**39**], which relies more on the discrete approximations described in these notes. The idea is roughly as follows: Even though the property of being homeomorphic to $\mathbb{S}^2$ is not preserved under Gromov-Hausdorff convergence, this preservation can be deduced under an additional property, called *regular convergence*, introduced by Whyburn. This property says heuristically that the spaces under consideration do not have small bottlenecks, i.e. cycles of vanishing diameters that separate the spaces into two macroscopic components.

In this section, when dealing with elements of the space $\mathbb{K}$ of isometry classes of pointed compact metric spaces, we will often omit to mention the distinguished point, as its role is less crucial than it was in Sections 6 and 7.

8.1. Geodesic spaces and regular convergence. A metric space (X, d) is said to be a *geodesic* metric space if for every $x, y \in X$, there exists an isometry $f : [0, d(x, y)] \to X$ such that $f(0) = x$ and $f(d(x, y)) = y$. Any such f is called a geodesic path between x and y. For instance, real trees are geodesic metric spaces by Definition 3.1. The set $\mathbb{K}_{\text{geo}}$ of isometry classes of (rooted) compact geodesic metric spaces is closed in $(\mathbb{K}, d_{GH})$, as shown in [**11**].

DEFINITION 8.1. *Let $((X_n, d_n), n \geq 1)$ be a sequence of compact geodesic metric spaces, converging to (X, d) in $(\mathbb{K}, d_{GH})$. We say that the convergence is* regular *if for every $\varepsilon > 0$, one can find $\delta > 0$ and $N \in \mathbb{N}$ such that, for every $n > N$, every closed path γ in X_n with diameter at most δ is homotopic to 0 in its ε-neighborhood.*

For instance, let Y_n be the complement in the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ of the open $1/n$-neighborhood of the North pole, and endow Y_n with the intrinsic distance induced from the usual Euclidean metric on $\mathbb{R}^3$ (so that the distance between $x, y \in Y_n$ is the minimal length of a path from x to y in Y_n). Let X_n be obtained by gluing two (disjoint) copies of Y_n along their boundaries, and endow it with the natural intrinsic distance. Then X_n converges in the Gromov-Hausdorff sense to a bouquet of two spheres, i.e. two (disjoint) copies of $\mathbb{S}^2$ whose North poles have been identified. However, the convergence is not regular, because the path γ that consists in the boundary of (either copy of) Y_n viewed as a subset of X_n has vanishing diameter as $n \to \infty$, but is not homotopic to 0 in its ε-neighborhood for any $\varepsilon \in (0, 1)$ and for any n. Indeed, such an ε-neighborhood is a cylinder, around which γ makes one turn.

THEOREM 8.2. *Let $((X_n, d_n), n \geq 1)$ be a sequence of $\mathbb{K}_{\text{geo}}$ that converges regularly to a limit (X, d) that is not reduced to a point. If (X_n, d_n) is homeomorphic to $\mathbb{S}^2$ for every $n \geq 1$, then so is (X, d).*

This theorem is an easy reformulation of a result of Whyburn in the context of Gromov-Hausdorff convergence; see the paper by Begle [**4**]. In the latter, it is assumed that every X_n should be a compact subset of a compact metric space (Z, δ), independent of n, and that X_n converges in the Hausdorff sense to X. This transfers

to our setting, because, if (X_n, d_n) converges to (X, d) in the Gromov-Hausdorff sense, then one can find a compact metric space (Z, δ) containing isometric copies $X'_n, n \geq 1$ and X' of $X_n, n \geq 1$ and X, such that X'_n converges in the Hausdorff sense to X', see for instance [21, Lemma A.1]. In [4], it is also assumed in the definition of regular convergence that for every $\varepsilon > 0$, there exist $\delta > 0$ and $N \in \mathbb{N}$ such that, for every $n \geq N$, any two points of X_n that lie at distance $\leq \delta$ are in a connected subset of X_n of diameter $\leq \varepsilon$. This condition is tautologically satisfied for geodesic metric spaces, which is the reason why we work in this context.

8.2. Quadrangulations seen as geodesic spaces. Theorem 8.2 gives a natural method to prove Theorem 8.1, using the convergence of quadrangulations to the Brownian map, as stated in Theorem 6.3. However, the finite space $(V(Q_n), d_{Q_n})$ is certainly not a geodesic space, nor homeomorphic to the 2-sphere. Hence, we have to modify a little these spaces so that they satisfy the hypotheses of Theorem 8.2. We will achieve this by constructing a particular[4] graphical representation of $\mathbf{q}$.

Let $(X_f, d_f), f \in F(\mathbf{q})$ be disjoint copies of the emptied unit cube "with bottom removed"

$$\mathcal{C} = [0, 1]^3 \setminus \left((0, 1)^2 \times [0, 1) \right),$$

endowed with the intrinsic metric d_f inherited from the Euclidean metric (the distance between two points of X_f is the minimal Euclidean length **of a path in** X_f). Obviously each (X_f, d_f) is a geodesic metric space homeomorphic to a closed disk of $\mathbb{R}^2$. We will write elements of X_f in the form $(s, t, r)_f$, where $(s, t, r) \in \mathcal{C}$ and the subscript f is used to differentiate points of the different spaces X_f. The boundary ∂X_f is then the collection of all points $(s, t, r)_f$ for $(s, t, r) \in ([0, 1]^2 \setminus (0, 1)^2) \times \{0\}$.

Let $f \in F(\mathbf{q})$ and let e_1, e_2, e_3, e_4 be the four oriented edges incident to f enumerated in a way consistent with the counterclockwise order on the boundary (here the labeling of these edges is chosen arbitrarily among the 4 possible labelings preserving the cyclic order). We then define

$$\begin{aligned}
c_{e_1}(t) &= (t, 0, 0)_f & , & \qquad 0 \leq t \leq 1 \\
c_{e_2}(t) &= (1, t, 0)_f & , & \qquad 0 \leq t \leq 1 \\
c_{e_3}(t) &= (1 - t, 1, 0)_f & , & \qquad 0 \leq t \leq 1 \\
c_{e_4}(t) &= (0, 1 - t, 0)_f & , & \qquad 0 \leq t \leq 1.
\end{aligned}$$

In this way, for every oriented edge e of the map $\mathbf{q}$, we have defined a path c_e which goes along one of the four edges of the square ∂X_f, where f is the face located to the left of e.

We define an equivalence relation $\equiv$ on the disjoint union $\amalg_{f \in F(\mathbf{q})} X_f$, as the coarsest equivalence relation such that, for every oriented edge e of $\mathbf{q}$, and every $t \in [0, 1]$, we have $c_e(t) \equiv c_{\overline{e}}(1 - t)$. By identifying points of the same equivalence class, we glue the oriented sides of the squares ∂X_f pairwise, in a way that is consistent with the map structure. More precisely, the topological quotient $\mathcal{S}_{\mathbf{q}} := \amalg_{f \in F(\mathbf{q})} X_f / \equiv$ is a surface which has a 2-dimensional cell complex structure, whose

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[4]The way we do this is by no means canonical. For instance, the emptied cubes X_f used to fill the faces of $\mathbf{q}$ below could be replaced by unit squares for the l^1 metric. However, our choice avoids the existence of too many geodesic paths between vertices of the map in the surface where it is embedded.

1-skeleton $\mathcal{E}_{\mathbf{q}} := \mathrm{II}_{f \in F(\mathbf{q})} \partial X_f / \equiv$ is a representative of the map $\mathbf{q}$, with faces (2-cells) $X_f \setminus \partial X_f$. In particular, $\mathcal{S}_{\mathbf{q}}$ is homeomorphic to $\mathbb{S}^2$ by [**42**, Lemma 3.1.4]. With an oriented edge e of $\mathbf{q}$ one associates an edge of the graph drawing $\mathcal{E}_{\mathbf{q}}$ in $\mathcal{S}_{\mathbf{q}}$, more simply called an edge of $\mathcal{S}_{\mathbf{q}}$, made of the equivalence classes of points in $c_e([0,1])$ (or $c_{\overline{e}}([0,1])$). We also let $\mathcal{V}_{\mathbf{q}}$ be the 0-skeleton of this complex, i.e. the vertices of the graph — these are the equivalent classes of the corners of the squares ∂X_f. We call them the vertices of $\mathcal{S}_{\mathbf{q}}$ for simplicity.

We next endow the disjoint union $\mathrm{II}_{f \in F(\mathbf{q})} X_f$ with the largest pseudo-metric $D_{\mathbf{q}}$ that is compatible with $d_f, f \in F(\mathbf{q})$ and with $\equiv$, in the sense that $D_{\mathbf{q}}(x,y) \leq d_f(x,y)$ for $x, y \in X_f$, and $D_{\mathbf{q}}(x,y) = 0$ for $x \equiv y$. Therefore, the function $D_{\mathbf{q}} : \mathrm{II}_{f \in F(\mathbf{q})} X_f \times \mathrm{II}_{f \in F(\mathbf{q})} X_f \to \mathbb{R}_+$ is compatible with the equivalence relation $\equiv$, and its quotient mapping defines a pseudo-metric on the quotient space $\mathcal{S}_{\mathbf{q}}$, which is still denoted by $D_{\mathbf{q}}$.

PROPOSITION 8.3. *The space $(\mathcal{S}_{\mathbf{q}}, D_{\mathbf{q}})$ is a geodesic metric space homeomorphic to $\mathbb{S}^2$. Moreover, the space $(\mathcal{V}_{\mathbf{q}}, D_{\mathbf{q}})$ is isometric to $(V(\mathbf{q}), d_{\mathbf{q}})$, and any geodesic path in $\mathcal{S}_{\mathbf{q}}$ between two elements of $\mathcal{V}_{\mathbf{q}}$ is a concatenation of edges of $\mathcal{S}_{\mathbf{q}}$. Last,*

$$d_{GH}((V(\mathbf{q}), d_{\mathbf{q}}), (\mathcal{S}_{\mathbf{q}}, D_{\mathbf{q}})) \leq 3.$$

PROOF. We first check that $D_{\mathbf{q}}$ is a true metric on $\mathcal{S}_{\mathbf{q}}$, i.e. that it separates points. To see this, we use the fact [**11**, Theorem 3.1.27] that $D_{\mathbf{q}}$ admits the constructive expression:

$$D_{\mathbf{q}}(a,b)$$
$$= \inf \left\{ \sum_{i=0}^n d(x_i, y_i) : n \geq 0, x_0 = a, y_n = b, y_i \equiv x_{i+1} \text{ for } 0 \leq i \leq n-1 \right\},$$

where we have set $d(x,y) = d_f(x,y)$ if $x, y \in X_f$ for some f, and $d(x,y) = \infty$ otherwise. It follows that, for $a \in X_f \setminus \partial X_f$ and $b \neq a$, $D_{\mathbf{q}}(a,b) > \min(d(a,b), d_f(a, \partial X_f)) > 0$, so a and b are separated.

To verify that $D_{\mathbf{q}}$ is a a true metric on $\mathcal{S}_{\mathbf{q}}$, it remains to treat the case where $a \in \partial X_f, b \in \partial X_{f'}$ for some $f, f' \in F(\mathbf{q})$. The crucial observation is that a shortest path in X_f between two points of ∂X_f is entirely contained in ∂X_f. It is then a simple exercise to check that if a, b are in distinct equivalence classes, the distance $D_{\mathbf{q}}(a,b)$ will be larger than the length of some fixed non-trivial path with values in $\mathcal{E}_{\mathbf{q}}$. More precisely, if (the equivalence classes of) a, b belong to the same edge of $\mathcal{S}_{\mathbf{q}}$, then we can find representatives a', b' in the same X_f and we will have $D_{\mathbf{q}}(a,b) \geq d_f(a', b')$. If the equivalence class of a is not a vertex of $\mathcal{S}_{\mathbf{q}}$ but that of b is, then $D_{\mathbf{q}}(a,b)$ is at least equal to the distance of $a \in X_f$ to the closest corner of the square ∂X_f. Finally, if the (distinct) equivalence classes of a, b are both vertices, then $D_{\mathbf{q}}(a,b) \geq 1$. One deduces that $D_{\mathbf{q}}$ is a true distance on $\mathcal{S}_{\mathbf{q}}$, which makes it a geodesic metric space by [**11**, Corollary 3.1.24]. Since $\mathcal{S}_{\mathbf{q}}$ is a compact topological space, the metric $D_{\mathbf{q}}$ induces the quotient topology on $\mathcal{S}_{\mathbf{q}}$ by [**11**, Exercise 3.1.14], hence $(\mathcal{S}_{\mathbf{q}}, D_{\mathbf{q}})$ is homeomorphic to $\mathbb{S}^2$.

From the observations in the last paragraph, a shortest path between vertices of $\mathcal{S}_{\mathbf{q}}$ takes values in $\mathcal{E}_{\mathbf{q}}$. Since an edge of $\mathcal{S}_{\mathbf{q}}$ is easily checked to have length 1 for the distance $D_{\mathbf{q}}$, such a shortest path will have the same length as a geodesic path for the (combinatorial) graph distance between the two vertices. Hence $(\mathcal{V}_{\mathbf{q}}, D_{\mathbf{q}})$ is indeed isometric to $(V(\mathbf{q}), d_{\mathbf{q}})$. The last statement follows immediately from this

and the fact that $\operatorname{diam}(X_f, d_f) \leq 3$, entailing that $\mathcal{V}_\mathbf{q}$ is 3-dense in $(\mathcal{S}_\mathbf{q}, D_\mathbf{q})$, i.e. its 3-neighborhood in $(\mathcal{S}_\mathbf{q}, D_\mathbf{q})$ equals $\mathcal{S}_\mathbf{q}$.   □

In view of the proposition, we can view $D_\mathbf{q}$ as an extension to $\mathcal{S}_\mathbf{q}$ of the graph distance $d_\mathbf{q}$ on $V(\mathbf{q})$. For this reason, we will denote $D_\mathbf{q}$ by $d_\mathbf{q}$ from now on, which should not set any ambiguity.

8.3. Proof of the homeomorphism theorem. We now work in the setting of the beginning of subsection 7.1. Recall that the uniform pointed quadrangulation (Q_n, v_*) is encoded by a uniform random element (T_n, L_n) of $\mathbf{T}_n$ via the CVS bijection (the parameter $\epsilon \in \{-1, 1\}$ will play no role here), and that C_n and V_n are the contour and label processes of (T_n, L_n). We assume that the amost sure convergence (24) holds uniformly on $[0, 1]^2$, along the sequence (n_k), which is fixed. In what follows, all convergences as $n \to \infty$ hold along this sequence, or along some further subsequence.

We can also assume that $(V(Q_n), d_{Q_n})$ is actually the (isometric) space $(\mathcal{V}_{Q_n}, d_{Q_n})$, i.e. the subspace of vertices of the space $(\mathcal{S}_{Q_n}, d_{Q_n})$ constructed in the previous subsection. Recalling from subsection 5.4.2 that, in the CVS bijection, each edge of the tree T_n lies in exactly one face of Q_n, we may and will assume that T_n is also embedded in the surface $\mathcal{S}_{Q_n}$, in such a way that the set of its vertices is $\mathcal{V}_{Q_n} \setminus \{v_*\}$, where $v_* \in V(Q_n)$ is identified with its counterpart in $\mathcal{V}_{Q_n}$, and that each edge of T_n lies entirely in the corresponding face of $\mathcal{S}_{Q_n}$ via the CVS bijection.

We will rely on the following lemma. Let $\operatorname{Sk}(\mathcal{T}_\mathbb{e})$ be the complement of the set of leaves in the CRT $\mathcal{T}_\mathbb{e}$. Equivalently, $\operatorname{Sk}(\mathcal{T}_\mathbb{e})$ is the set of all points $a \in \mathcal{T}_\mathbb{e}$ such that $\mathcal{T}_\mathbb{e} \setminus \{a\}$ is disconnected, and it also coincides with the set of all $a \in \mathcal{T}_\mathbb{e}$ that can be written $a = p_\mathbb{e}(s) = p_\mathbb{e}(s')$ for some $0 \leq s < s' < 1$. The set $\operatorname{Sk}(\mathcal{T}_\mathbb{e})$ is called the *skeleton* of $\mathcal{T}_\mathbb{e}$.

LEMMA 8.4. *The following property is true with probability* 1. *Let* $a \in \operatorname{Sk}(\mathcal{T}_\mathbb{e})$, *and let* $s \in (0, 1)$ *be such that* $a = p_\mathbb{e}(s)$. *Then for every* $\varepsilon > 0$, *there exists* $t \in (s, (s + \varepsilon) \wedge 1)$ *such that* $Z_t < Z_s$.

This lemma is a consequence of [**34**, Lemma 3.2] (see also [**31**, Lemma 2.2] for a slightly weaker statement). The proof relies on a precise study of the label function Z, and we refer the interested reader to [**34**]. Note that this result (and the analogous statement derived by time-reversal) implies that a.s., if $a \in \operatorname{Sk}(\mathcal{T}_\mathbb{e})$, then in each component of $\mathcal{T}_\mathbb{e} \setminus \{a\}$, one can find points b that are arbitrarily close to a and such that $Z_b < Z_a$.

LEMMA 8.5. *Almost surely, for every* $\varepsilon > 0$, *there exists* $\delta \in (0, \varepsilon)$ *such that, for* n *large enough, any simple loop* γ_n *made of edges of* $\mathcal{S}_{Q_n}$, *with diameter* $\leq n^{1/4}\delta$, *splits* $\mathcal{S}_{Q_n}$ *in two Jordan domains, one of which has diameter* $\leq n^{1/4}\varepsilon$.

PROOF. We argue by contradiction. Assume that, with positive probability, along some (random) subsequence of (n_k) there exist simple loops γ_n made of edges of $\mathcal{S}_{Q_n}$, with diameters $o(n^{1/4})$ as $n \to \infty$, such that the two Jordan domains bounded by γ_n are of diameters $\geq n^{1/4}\varepsilon$, where $\varepsilon > 0$ is some fixed constant. From now on we argue on this event. By abuse of notation we will sometimes identify the chain γ_n with the set of vertices it visits, or with the union of its edges, in a way that should be clear from the context.

By the Jordan curve theorem, the path γ_n splits $\mathcal{S}_{Q_n}$ into two Jordan domains, which we denote by $\mathcal{D}_n$ and $\mathcal{D}'_n$. Since the diameters of both these domains are at

least $n^{1/4}\varepsilon$, and since every point in $\mathcal{S}_{Q_n}$ is at distance at most 3 from some vertex, we can find vertices y_n and y'_n belonging to $\mathcal{D}_n$ and $\mathcal{D}'_n$ respectively, and which lie at distance at least $n^{1/4}\varepsilon/4$ from γ_n. Since $V(Q_n) = T_n \cup \{v_*\}$, we can always assume that y_n and y'_n are distinct from v_*. Now, consider the geodesic path from y_n to y'_n in T_n, and let x_n be the first vertex of this path that belongs to γ_n.

In the contour exploration around T_n, the vertex x_n is visited at least once in the interval between y_n and y'_n, and another time in the interval between y'_n and y_n. More precisely, let j_n and j'_n be such that $y_n = u^n_{j_n}, y'_n = u^n_{j'_n}$, and assume first that $j_n < j'_n$ for infinitely many n. For such n, we can find integers $i_n \in (j_n, j'_n)$ and $i'_n \in (0, j_n) \cup (j'_n, 2n)$ such that $x_n = u^n_{i_n} = u^n_{i'_n}$. Up to further extraction, we may and will assume that

$$(27) \qquad \frac{i_n}{2n} \to s\,, \qquad \frac{i'_n}{2n} \to s'\,, \qquad \frac{j_n}{2n} \to t\,, \qquad \frac{j'_n}{2n} \to t'\,,$$

for some $s, s', t, t' \in [0,1]$ such that $t \le s \le t'$ and $s' \in [0,t] \cup [t,1]$. Since

$$d_{Q_n}(x_n, y_n) \wedge d_{Q_n}(x_n, y'_n) \ge n^{1/4}\varepsilon/4\,,$$

we deduce from (24) that $D(s,t), D(s',t), D(s,t'), D(s',t') > 0$, and in particular, s, s', t, t' are all distinct. Since $u^n_{i_n} = u^n_{i'_n}$, we conclude that $s \sim_{\oplus} s'$, so that $p_{\oplus}(s) \in \mathrm{Sk}(\mathcal{T}_{\oplus})$. One obtains the same conclusion by a similar argument if $j_n > j'_n$ for every n large. We let $x = p_{\oplus}(s)$ and $y = p_{\oplus}(t)$. Note that $y \ne x$ because $D(s,t) > 0$ (recall Lemma 7.1).

Since $x \in \mathrm{Sk}(\mathcal{T}_{\oplus})$, by Theorem 7.2 we deduce that $D(a_*, x) = D(s_*, s) > 0$, where $a_* = p_{\oplus}(s_*)$ is as before the a.s. unique leaf of $\mathcal{T}_{\oplus}$ where Z attains its minimum. In particular, we obtain by (18), (24) and the fact that $\mathrm{diam}\,(\gamma_n) = o(n^{1/4})$ that

$$\liminf_{n\to\infty} n^{-1/4} d_{Q_n}(v_*, \gamma_n) = \liminf_{n\to\infty} n^{-1/4} d_{Q_n}(v_*, x_n) > 0\,.$$

Therefore, for n large enough, v_* does not belong to γ_n, and for definiteness, we will assume that for such n, $\mathcal{D}_n$ is the component of $\mathcal{S}_{Q_n} \setminus \gamma_n$ that does not contain v_*.

Now, we let $L_n^+ = L_n - \min L_n + 1$, and in the rest of this proof, we call $L_n^+(v) = d_{Q_n}(v_*, v)$ the *label of the vertex v in Q_n*. Let $l_n = d_{Q_n}(v_*, \gamma_n) = \min_{v \in \gamma_n} L_n^+(v)$ be the minimal distance from v_* to a point visited by γ_n. Note that, for every vertex $v \in \mathcal{D}_n$, the property $L_n^+(v) \ge l_n$ holds, since any geodesic chain from v_* to v in Q_n has to cross γ_n.

Recalling that the vertex x_n was chosen so that the simple path in T_n from x_n to y_n lies entirely in $\mathcal{D}_n$, we conclude that the labels of vertices on this path are all greater than or equal to l_n. By passing to the limit, one concludes that for every c in the path $[\![x,y]\!]$ in $\mathcal{T}_{\oplus}$, there holds that $Z_c \ge Z_x$. Since the process Z evolves like Brownian motion along line segments of the tree $\mathcal{T}_{\oplus}$, we deduce that for every $c \in [\![x,y]\!]$ close enough to x, we have in fact $Z_c > Z_x$. From the interpretation of line segments in $\mathcal{T}_{\oplus}$ in terms of the coding function $\oplus$ (see the end of subsection 3.2), we can find $\overline{s} \in (0,1)$ such that $p_{\oplus}(\overline{s}) = x$, and such that, for every $u > \overline{s}$ sufficiently close to $\overline{s}$, the intersection of $[\![x, p_{\oplus}(u)]\!]$ with $[\![x,y]\!]$ will be of the form $[\![x, p_{\oplus}(r)]\!]$ for some $r \in (\overline{s}, u]$. By Lemma 8.4, and the fact that $Z_c \ge Z_x$ for every $c \in [\![x,y]\!]$ close enough to x, we can find $u > \overline{s}$ encoding a point $a = p_{\oplus}(u)$ and some $\eta > 0$ such that $Z_a \le Z_x - (9/8)^{1/4}\eta$, and such that $[\![x,a]\!] \cap [\![x,y]\!] = [\![x,b]\!]$ for some $b \ne x$ such that $Z_b \ge Z_x + (9/8)^{1/4}\eta$.
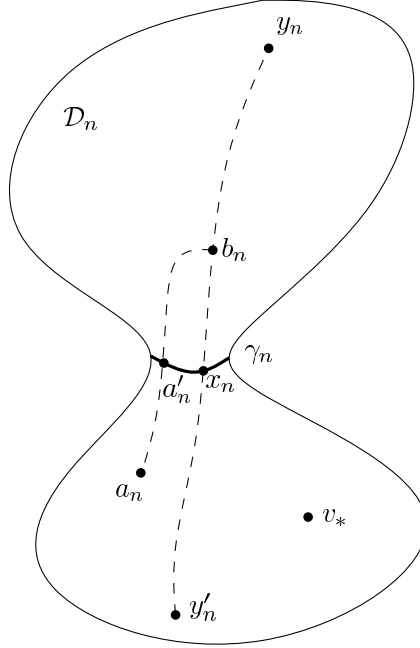
FIGURE 9. Illustration of the proof. The surface $\mathcal{S}_{Q_n}$ is depicted as a sphere with a bottleneck circled by γ_n (thick line). The dashed lines represent paths of T_n that are useful in the proof: One enters the component $\mathcal{D}_n$, and the other goes out after entering, identifying in the limit a point of the skeleton with another.

We then go back once again to the discrete approximations of the Brownian map, by considering k_n such that $k_n/2n$ converges to u. From the fact that $Z_a < Z_x$, we deduce that the vertex $a_n = u_{k_n}^n$ has label $L_n^+(a_n) < l_n$ for every n large enough. Indeed, the convergence (24) and the fact that $\operatorname{diam}(\gamma_n) = o(n^{1/4})$ imply that $(9/8n)^{1/4} l_n \to Z_x - \inf Z$. Consequently, the point a_n does not belong to $\mathcal{D}_n$. Moreover, the path in T_n from a_n to x_n meets the path from x_n to y_n at a point b_n such that $L_n^+(b_n) \geq l_n + \eta n^{1/4}$. The path from a_n to b_n has to cross the loop γ_n at some vertex, and we let a_n' be the first such vertex. By letting $n \to \infty$ one last time, we find a vertex $a' \in \mathcal{T}_{\mathbb{e}}$, which in the appropriate sense is the limit of a_n' as $n \to \infty$, such that $[\![a', x]\!]$ meets $[\![x, y]\!]$ at b. In particular, $a' \neq x$. But since a_n' and x_n are both on γ_n, we deduce that $D(a', x) = 0$. This contradicts Theorem 7.2 because x is not a leaf of $\mathcal{T}_{\mathbb{e}}$. This contradiction completes the proof of the lemma. $\square$

We claim that Lemma 8.5 suffices to verify that the convergence of $(V(Q_n), (9/8n)^{1/4} d_{Q_n})$ to (M, D) is regular, and hence to conclude by Theorem 8.2 that the limit (M, D) is a topological sphere. To see this, we first choose $\varepsilon < \operatorname{diam}(M)/3$ to avoid trivialities. Let γ_n be a loop in $\mathcal{S}_{Q_n}$ with diameter $\leq n^{1/4}\delta$. Consider the union of the closures of faces of $\mathcal{S}_{Q_n}$ that are visited by γ_n. The boundary of this union is a collection $\mathcal{L}$ of pairwise disjoint simple loops made of edges of $\mathcal{S}_{Q_n}$. If x, y belong to the preceding union of faces, the fact that a face of $\mathcal{S}_{Q_n}$ has diameter

less than 3 implies that there exist points x' and y' of γ_n at distance at most 3 from x and y respectively. Therefore, the diameters of the loops in $\mathcal{L}$ all are $\leq n^{1/4}\delta + 6$.

By the Jordan Curve Theorem, each of these loops splits $\mathcal{S}_{Q_n}$ into two simply connected components. By definition, one of these two components contains γ_n entirely. By Lemma 8.5, one of the two components has diameter $\leq n^{1/4}\varepsilon$. If we show that the last two properties hold simultaneously for one of the two components associated with (at least) one of the loops in $\mathcal{L}$, then obviously γ_n will be homotopic to 0 in its ε-neighborhood in $(\mathcal{S}_{Q_n}, n^{-1/4}d_{Q_n})$. So assume the contrary: The component not containing γ_n associated with every loop of $\mathcal{L}$ is of diameter $\leq n^{1/4}\varepsilon$. If this holds, then any point in $\mathcal{S}_{Q_n}$ must be at distance at most $n^{1/4}\varepsilon + 3$ from some point in γ_n. Take x, y such that $d_{Q_n}(x, y) = \operatorname{diam}(\mathcal{S}_{Q_n})$. Then there exist points x' and y' in γ_n at distance at most $n^{1/4}\varepsilon + 3$ respectively from x and y, and we conclude that $d_{Q_n}(x', y') \geq \operatorname{diam}(\mathcal{S}_{Q_n}) - 6 - 2n^{1/4}\varepsilon > n^{1/4}\delta \geq \operatorname{diam}(\gamma_n)$ for n large enough by our choice of ε. This contradiction completes the proof.

Note added in proof. The uniqueness problem for the Brownian map has been solved in two very recent papers of the authors: See the preprints arxiv:1104.1606 and arxiv:1105.4842. Consequently, Conjecture 6.1 is now a theorem, and analogs of this result hold for more general random planar maps such as triangulations.

References

[1] D. Aldous. The continuum random tree I. *Ann. Probab.* 19:1–28, 1991.

[2] D. Aldous. The continuum random tree III. *Ann. Probab.* 21:248–289, 1993.

[3] J. Ambjørn, B. Durhuus, and T. Jonsson. *Quantum geometry. A statistical field theory approach.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1997.

[4] E.G. Begle. Regular convergence. *Duke Math. J.*, 11: 441–450, 1944.

[5] J. Bettinelli. The topology of scaling limits of positive genus random quadrangulations. Preprint, 2010, arXiv:1012.3726

[6] M. Bousquet-Mélou and A. Jehanne. Polynomial equations with one catalytic variable, algebraic series and map enumeration. *J. Combin. Theory Ser. B*, 96(5):623–672, 2006.

[7] J. Bouttier. *Physique Statistique des Surfaces Aléatoires et Combinatoire Bijective des Cartes Planaires.* PhD thesis, Université Paris 6, 2005.

[8] J. Bouttier and E. Guitter. The three-point function of planar quadrangulations. *J. Stat. Mech. Theory Exp.* 2008, no. 7, P07020, 39 pp.

[9] J. Bouttier, P. Di Francesco, and E. Guitter. Planar maps as labeled mobiles. *Electron. J. Combin.*, 11:no. 69, 27 pp. (electronic), 2004.

[10] E. Brézin, C. Itzykson, G. Parisi, and J. B. Zuber. Planar diagrams. *Comm. Math. Phys.*, 59(1):35–51, 1978.

[11] D. Burago, Y. Burago, and S. Ivanov. *A Course in Metric Geometry.* Graduate Studies in Mathematics, vol. 33. AMS, Boston, 2001.

[12] P. Chassaing and G. Schaeffer. Random planar lattices and integrated superBrownian excursion. *Probab. Theory Related Fields*, 128(2):161–212, 2004.

[13] K.L. Chung. Excursions in Brownian motion. *Ark. Mat.* 14:155–177, 1976.

[14] R. Cori and B. Vauquelin. Planar maps are well labeled trees. *Canad. J. Math.*, 33(5):1023–1042, 1981.

[15] J.-F. Delmas. Computation of moments for the length of the one dimensional ISE support. *Electron. J. Probab.*, 8:no. 17, 15 pp. (electronic), 2003.

[16] B. Duplantier and S. Sheffield. Liouville quantum gravity and KPZ. *Invent. Math.*, 185:333–393, 2011.

[17] T. Duquesne. A limit theorem for the contour process of conditioned Galton-Watson trees. *Ann. Probab.*, 31:996–1027, 2003.

[18] T. Duquesne and J.-F. Le Gall. Random Trees, Lévy Processes and Spatial Branching Processes. *Astérisque*, 281, 2002.

[19] T. Duquesne and J.-F. Le Gall. Probabilistic and fractal aspects of Lévy trees. *Probab. Th. Rel. Fields*, 131:553–603, 2005.

[20] S.N. Evans and A. Winter. Subtree prune and re-graft: A reversible tree-valued Markov process. *Ann. Probab.*, 34:918–961, 2006.

[21] A. Greven, P. Pfaffelhuber, and A. Winter. Convergence in distribution of random metric measure spaces (Λ-coalescent measure trees). *Probab. Theory Related Fields*, 145: 285–322, 2009.

[22] M. Gromov. *Metric Structures for Riemannian and Non-Riemannian Spaces*. Birkhäuser 1999.

[23] B. Haas and G. Miermont. Scaling limits of Markov branching trees, with applications to Galton-Watson and random unordered trees. Preprint, arXiv:1003.3632

[24] G. 't Hooft. A planar diagram theory for strong interactions. *Nucl. Phys. B*, 72:461–473, 1974.

[25] S. Janson and J.-F. Marckert. Convergence of discrete snakes. *J. Theoret. Probab.*, 18:615–647, 2005.

[26] W.D. Kaigh. An invariance principle for random walk conditioned by a late return to zero. *Ann. Probab.*, 4:115–121, 1976.

[27] S. K. Lando and A. K. Zvonkin. *Graphs on surfaces and their applications*, volume 141 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2004.

[28] J.-F. Le Gall. *Spatial Branching Processes, Random Snakes and Partial Differential Equations*. Birkhäuser 1999.

[29] J.-F. Le Gall. Random trees and applications. *Probab. Surveys*, 2:245-311, 2005.

[30] J.-F. Le Gall. A conditional limit theorem for tree-indexed random walk. *Stochastic Process. Appl.*, 116(4):539–567, 2006.

[31] J.-F. Le Gall. The topological structure of scaling limits of large planar maps. *Invent. Math.*, 169:621–670, 2007.

[32] J.-F. Le Gall. Geodesics in large planar maps and in the Brownian map. *Acta Mathematica*, 205:287–360, 2010.

[33] J.-F. Le Gall and G. Miermont. Scaling limits of random planar maps with large faces. *Ann. Probab.*, 39(1):1–69, 2011.

[34] J.-F. Le Gall and F. Paulin. Scaling limits of bipartite planar maps are homeomorphic to the 2-sphere. *Geom. Funct. Anal.*, 18(3):893–918, 2008.

[35] J.-F. Le Gall and M. Weill. Conditioned Brownian trees. *Ann. Inst. H. Poincaré Probab. Statist.*, 42(4):455–489, 2006.

[36] J.-F. Marckert and G. Miermont. Invariance principles for random bipartite planar maps. *Ann. Probab.*, 35(5):1642–1705, 2007.

[37] J.-F. Marckert and A. Mokkadem. Limit of normalized random quadrangulations: the Brownian map. *Ann. Probab.*, 34(6):2144–2202, 2006.

[38] G. Miermont. An invariance principle for random planar maps. In *Fourth Colloquium on Mathematics and Computer Sciences CMCS'06*, Discrete Math. Theor. Comput. Sci. Proc., AG, p. 39–58 (electronic). Nancy, 2006.

[39] G. Miermont. On the sphericity of scaling limits of random planar quadrangulations. *Electron. Commun. Probab.*, 13:248–257, 2008.

[40] G. Miermont. Tessellations of random maps of arbitrary genus. *Ann. Sci. Éc. Norm. Supér.*, 42(5):725–781, 2009.

[41] G. Miermont and M. Weill. Radius and profile of random planar maps with faces of arbitrary degrees. *Electron. J. Probab.*, 13: 79–106, 2008.

[42] B. Mohar and C. Thomassen. *Graphs on surfaces*. Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, 2001.

[43] R.L. Moore. Concerning upper-semicontinuous collections of continua. *Trans. Amer. Math. Soc.* 25:416-428, 1925.

[44] J. Neveu. Arbres et processus de Galton-Watson. *Ann. Inst. Henri Poincaré Probab. Stat.* 22:199–207, 1986.

[45] J.W. Pitman. Combinatorial stochastic processes. Ecole d'été de probabilités de Saint-Flour 2002. *Lecture Notes Math.* 1875. Springer, Berlin, 2006.

[46] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion*. Springer 1991.

[47] G. Schaeffer. *Conjugaison d'arbres et cartes combinatoires aléatoires*. PhD thesis, Université Bordeaux I, 1998.

[48] O. Schramm. Conformally invariant scaling limits: an overview and a collection of problems. In Proceedings of the International Congress of Mathematicians (Madrid 2006), Vol.I, pp. 513–543. European Math. Soc., Zürich, 2007.

[49] F. Spitzer. *Principles of Random Walk*. Van Nostrand 1963.

[50] R.P. Stanley. *Enumerative Combinatorics, Vol.2*. Cambridge University Press, Cambridge, 1999.

[51] W.T. Tutte. A census of planar maps. *Canad. J. Math.*, 15:249–271, 1963.

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