Bessel processes, the Brownian snake and super-Brownian motion

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Abstract

We prove that, both for the Brownian snake and for super-Brownian motion in dimension one, the historical path corresponding to the minimal spatial position is a Bessel process of dimension -5. We also discuss a spine decomposition for the Brownian snake conditioned on the minimizing path.

1 Introduction

Marc Yor used to say that "Bessel processes are everywhere". Partly in collaboration with Jim Pitman [13, 14], he wrote several important papers, which considerably improved our knowledge of Bessel processes and of their numerous applications. A whole chapter of Marc Yor's celebrated book with Daniel Revuz [15] is devoted to Bessel processes and their applications to Ray-Knight theorems. As a matter of fact, Bessel processes play a major role in the study of properties of Brownian motion, and, in particular, the three-dimensional Bessel process is a key ingredient of the famous Williams decomposition of the Brownian excursion at its maximum. In the present work, we show that Bessel processes also arise in similar properties of super-Brownian motion and the Brownian snake. Informally, we obtain that, both for the Brownian snake and for super-Brownian motion, the (historical) path reaching the minimal spatial position is a Bessel process of negative dimension.

Let us describe our results in a more precise way. We write $(W_s)_{s \geq 0}$ for the Brownian snake whose spatial motion is one-dimensional Brownian motion. Recall that $(W_s)_{s \geq 0}$ is a Markov process taking values in the space of all finite paths in $\mathbb{R}$, and for every $s \geq 0$, write ζ_s for the lifetime of W_s. We let $\mathbb{N}_0$ stand for the σ-finite excursion measure of $(W_s)_{s \geq 0}$ away from the trivial path with initial point 0 and zero lifetime (see Section 2 for the precise normalization of $\mathbb{N}_0$). We let W_* be the minimal spatial position visited by the paths W_s, $s \geq 0$. Then the "law" of W_* under $\mathbb{N}_0$ is given by

$$\mathbb{N}_0(W_* \leq -a) = \frac{3}{2a^2}, \tag{1}$$

for every $a > 0$ (see [9, Section VI.1] or [12, Lemma 2.1]). Furthermore, it is known that, $\mathbb{N}_0$ a.e., there is a unique instant $s_{\mathbf{m}}$ such that $W_* = W_{s_{\mathbf{m}}}(\zeta_{s_{\mathbf{m}}})$. Our first main result (Theorem 5) shows that, conditionally on $W_* = -a$, the random path $a + W_{s_{\mathbf{m}}}$ is a Bessel process of dimension $d = -5$ started from a and stopped upon hitting 0. Because of the relations between the Browian snake and super-Brownian motion, this easily implies a similar result for the unique historical path of one-dimensional super-Brownian motion that attains the minimal spatial value (Corollary 8). Our second result (Theorem 6) provides a "spine decomposition" of the Brownian

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snake under $\mathbb{N}_0$ given the minimizing path $W_{s_{\mathbf{m}}}$. Roughly speaking, this decomposition involves Poisson processes of Brownian snake excursions branching off the minimizing path, which are conditioned not to attain the minimal value W_*. See Theorem 6 for a more precise statement.

Our proofs depend on various properties of the Brownian snake, including its strong Markov property and the "subtree decomposition" of the Brownian snake ([9, Lemma V.5], see Lemma 3 below) starting from an arbitrary finite path w. We also use the explicit distribution of the Brownian snake under $\mathbb{N}_0$ at its first hitting time of a negative level: If $b > 0$ and S_b is the first hitting time of $-b$ by the Brownian snake, the path $b + W_{S_b}$ is distributed under $\mathbb{N}_0(\cdot \mid S_b < \infty)$ as a Bessel process of dimension $d = -3$ started from b and stopped upon hitting 0 (see Lemma 4 below). Another key ingredient (Lemma 1) is a variant of the absolute continuity relations between Bessel processes that were discovered by Yor [17] and studied in a more systematic way in the paper [13] by Pitman and Yor.

Let us briefly discuss connections between our results and earlier work. As a special case of a famous time-reversal theorem due to Williams [16, Theorem 2.5] (see also Pitman and Yor [14, Section 3], and in particular the examples treated in subsection (3.5) of [14]), the time-reversal of a Bessel process of dimension $d = -5$ started from a and stopped upon hitting 0 is a Bessel process of dimension $d = 9$ started from 0 and stopped at its last passage time at a – This property can also be found in [15, Exercise XI.1.23]. Our results are therefore related to the appearance of nine-dimensional Bessel processes in limit theorems derived in [12] and [11]. Note however that in contrast with [12] and [11], Theorem 5 gives an exact identity in distribution and not an asymptotic result. As a general remark, Theorem 6 is related to a number of "spine decompositions" for branching processes that have appeared in the literature in various contexts. We finally note that a strong motivation for the present work came from the forthcoming paper [2], which uses Theorems 5 and 6 to provide a new construction of the random metric space called the Brownian plane [1] and to give a number of explicit calculations of distributions related to this object.

The paper is organized as follows. Section 2 presents a few preliminary results about Bessel processes and the Brownian snake. Section 3 contains the statement and proof of our main results Theorems 5 and 6. Finally Section 4 gives our applications to super-Brownian motion, which are more or less straightforward consequences of the results of Section 3.

2 Preliminaries

2.1 Bessel processes

We will be interested in Bessel processes of negative index. We refer to [13] for the theory of Bessel processes, and we content ourselves with a brief presentation limited to the cases of interest in this work. We let $B = (B_t)_{t \geq 0}$ be a linear Brownian motion and for every $\alpha > 0$, we will consider the nonnegative process $R^{(\alpha)} = (R_t^{(\alpha)})_{t \geq 0}$ that solves the stochastic differential equation

$$\mathrm{d}R_t^{(\alpha)} = \mathrm{d}B_t - \frac{\alpha}{R_t^{(\alpha)}}\,\mathrm{d}t, \tag{2}$$

with a given (nonnegative) initial condition. To be specific, we require that equation (2) holds up to the first hitting time of 0 by $R^{(\alpha)}$,

$$T^{(\alpha)} := \inf\{t \geq 0 : R_t^{(\alpha)} = 0\},$$

and that $R_t^{(\alpha)} = 0$ for $t \geq T^{(\alpha)}$. Note that uniqueness in law and pathwise uniqueness hold for (2).

In the standard terminology (see e.g.[13, Section 2]), the process $R^{(\alpha)}$ is a Bessel process of index $\nu = -\alpha - \frac{1}{2}$, or dimension $d = 1 - 2\alpha$. We will be interested especially in the cases $\alpha = 2$ ($d = -3$) and $\alpha = 3$ ($d = -5$).

For notational convenience, we will assume that, for every $r \geq 0$, there is a probability measure P_r such that both the Brownian motion B and the Bessel processes $R^{(\alpha)}$ start from r under P_r.

Let us fix $r > 0$ and argue under the probability measure P_r. Fix $\delta \in (0, r)$ and set

$$T_\delta^{(\alpha)} := \inf\{t \geq 0 : R_t^{(\alpha)} = \delta\},$$

and

$$T_\delta := \inf\{t \geq 0 : B_t = \delta\}.$$

The following absolute continuity lemma is very closely related to results of [17] (Lemma 4.5) and [13] (Proposition 2.1), but we provide a short proof for the sake of completeness. If E is a metric space, $C(\mathbb{R}_+, E)$ stands for the space of all continuous functions from $\mathbb{R}_+$ into E, which is equipped with the topology of uniform convergence on every compact interval.

Lemma 1. *For every nonnegative measurable function F on $C(\mathbb{R}_+, \mathbb{R}_+)$,*

$$E_r[F((R_{t \wedge T_\delta^{(\alpha)}}^{(\alpha)})_{t \geq 0})] = \left(\frac{r}{\delta}\right)^\alpha E_r\left[F((B_{t \wedge T_\delta})_{t \geq 0}) \exp\left(-\frac{\alpha(1 + \alpha)}{2} \int_0^{T_\delta} \frac{\mathrm{d}s}{B_s^2}\right)\right].$$

Proof. Write $(\mathcal{F}_t)_{t \geq 0}$ for the (usual augmentation of the) filtration generated by B. For every $t \geq 0$, set

$$M_t := \left(\frac{r}{B_{t \wedge T_\delta}}\right)^\alpha \exp\left(-\frac{\alpha(1 + \alpha)}{2} \int_0^{t \wedge T_\delta} \frac{\mathrm{d}s}{B_s^2}\right).$$

An application of Itô's formula shows that $(M_t)_{t \geq 0}$ is an $(\mathcal{F}_t)$-local martingale. Clearly, $(M_t)_{t \geq 0}$ is bounded by $(r/\delta)^\alpha$ and is thus a uniformly integrable martingale, which converges as $t \to \infty$ to

$$M_\infty = \left(\frac{r}{\delta}\right)^\alpha \exp\left(-\frac{\alpha(1 + \alpha)}{2} \int_0^{T_\delta} \frac{\mathrm{d}s}{B_s^2}\right).$$

We define a probability measure Q absolutely continuous with respect to P_r by setting $Q = M_\infty \cdot P_r$. An application of Girsanov's theorem shows that the process

$$B_t + \alpha \int_0^{t \wedge T_\delta} \frac{\mathrm{d}s}{B_s}$$

is an $(\mathcal{F}_t)$-Brownian motion under Q. It follows that the law of $(B_{t \wedge T_\delta})_{t \geq 0}$ under Q coincides with the law of $(R_{t \wedge T_\delta^{(\alpha)}}^{(\alpha)})_{t \geq 0}$ under P_r. This gives the desired result. $\qquad\square$

The formula of the next lemma is probably known, but we could not find a reference.

Lemma 2. *For every $r > 0$ and $a > 0$,*

$$E_r\left[\exp\left(-3 \int_0^{T^{(2)}} \mathrm{d}t\,(a + R_t^{(2)})^{-2}\right)\right] = 1 - \left(\frac{r}{r + a}\right)^2.$$

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Proof. An application of Itô's formula shows that

$$M_t := \Big(1 - \Big(\frac{R_t^{(2)}}{R_t^{(2)} + a} \Big)^2 \Big) \exp \Big(- 3 \int_0^{t \wedge T^{(2)}} \mathrm{d}s \, (a + R_s^{(2)})^{-2} \Big)$$

is a local martingale. Clearly, M_t is bounded by 1 and is thus a uniformly integrable martingale. Writing $E_r[M_{T^{(2)}}] = E_r[M_0]$ yields the desired result. $\qquad\square$

Remark. An alternative proof of the formula of Lemma 2 will follow from forthcoming calculations: just use formula (4) below with $G = 1$, noting that the left-hand side of this formula is then equal to $\mathbb{N}_0(-b - \varepsilon < W_* \leq -b)$, which is computed using (1). So strictly speaking we do not need the preceding proof. Still it seems a bit odd to use the Brownian snake to prove the identity of Lemma 2, which has to do with Bessel processes only.

2.2 The Brownian snake

We refer to [9] for the general theory of the Brownian snake, and only give a short presentation here. We write $\mathcal{W}$ for the set of all finite paths in $\mathbb{R}$. An element of $\mathcal{W}$ is a continuous mapping $\mathrm{w} : [0, \zeta] \longrightarrow \mathbb{R}$, where $\zeta = \zeta_{(\mathrm{w})} \geq 0$ depends on w and is called the lifetime of w. We write $\widehat{\mathrm{w}} = \mathrm{w}(\zeta_{(\mathrm{w})})$ for the endpoint of w. For $x \in \mathbb{R}$, we set $\mathcal{W}_x := \{\mathrm{w} \in \mathcal{W} : \mathrm{w}(0) = x\}$. The trivial path w such that $\mathrm{w}(0) = x$ and $\zeta_{(\mathrm{w})} = x$ is identified with the point x of $\mathbb{R}$, so that we can view $\mathbb{R}$ as a subset of $\mathcal{W}$. The space $\mathcal{W}$ is equipped with the distance

$$d(\mathrm{w}, \mathrm{w}') = |\zeta_{(\mathrm{w})} - \zeta_{(\mathrm{w}')}| + \sup_{t \geq 0} |\mathrm{w}(t \wedge \zeta_{(\mathrm{w})}) - \mathrm{w}'(t \wedge \zeta_{(\mathrm{w}')})|.$$

The Brownian snake $(W_s)_{s \geq 0}$ is a continuous Markov process with values in $\mathcal{W}$. We will write $\zeta_s = \zeta_{(W_s)}$ for the lifetime process of W_s. The process $(\zeta_s)_{s \geq 0}$ evolves like a reflecting Brownian motion in $\mathbb{R}_+$. Conditionally on $(\zeta_s)_{s \geq 0}$, the evolution of $(W_s)_{s \geq 0}$ can be described informally as follows: When ζ_s decreases, the path W_s is shortened from its tip, and, when ζ_s increases, the path W_s is extended by adding "little pieces of linear Brownian motion" at its tip. See [9, Chapter IV] for a more rigorous presentation.

It is convenient to assume that the Brownian snake is defined on the canonical space $C(\mathbb{R}_+, \mathcal{W})$, in such a way that, for $\omega = (\omega_s)_{s \geq 0} \in C(\mathbb{R}_+, \mathcal{W})$, we have $W_s(\omega) = \omega_s$. The notation $\mathbb{P}_{\mathrm{w}}$ then stands for the law of the Brownian snake started from w.

For every $x \in \mathbb{R}$, the trivial path x is a regular recurrent point for the Brownian snake, and so we can make sense of the excursion measure $\mathbb{N}_x$ away from x, which is a σ-finite measure on $C(\mathbb{R}_+, \mathcal{W})$. Under $\mathbb{N}_x$, the process $(\zeta_s)_{s \geq 0}$ is distributed according to the Itô measure of positive excursions of linear Brownian motion, which is normalized so that, for every $\varepsilon > 0$,

$$\mathbb{N}_x \Big(\sup_{s \geq 0} \zeta_s > \varepsilon \Big) = \frac{1}{2\varepsilon}.$$

We write $\sigma := \sup\{s \geq 0 : \zeta_s > 0\}$ for the duration of the excursion under $\mathbb{N}_x$. In a way analogous to the classical property of the Itô excursion measure [15, Corollary XII.4.3], $\mathbb{N}_x$ is invariant under time-reversal, meaning that $(W_{(\sigma - s) \vee 0})_{s \geq 0}$ has the same distribution as $(W_s)_{s \geq 0}$ under $\mathbb{N}_x$.

Recall the notation

$$W_* := \inf_{0 \leq s \leq \sigma} \widehat{W}_s = \inf_{0 \leq s \leq \sigma} \inf_{0 \leq t \leq \zeta_s} W_s(t),$$

and formula (1) determining the law of W_* under $\mathbb{N}_0$. It is known (see e.g. [12, Proposition 2.5]) that $\mathbb{N}_x$ a.e. there is a unique instant $s_{\mathbf{m}} \in [0, \sigma]$ such that $\widehat{W}_{s_{\mathbf{m}}} = W_*$. One of our main objectives is to determine the law of $W_{s_{\mathbf{m}}}$. We start with two important lemmas.

Our first lemma is concerned with the Brownian snake started from $\mathbb{P}_{\mathrm{w}}$, for some fixed $\mathrm{w} \in \mathcal{W}$, and considered up to the first hitting time of 0 by the lifetime process, that is

$$\eta_0 := \inf\{s \geq 0 : \zeta_s = 0\}.$$

Then the values of the Brownian snake between times 0 and η_0 can be classified according to "subtrees" branching off the initial path w. To make this precise, let (α_i, β_i), $i \in I$ be the excursion intervals away from 0 of the process

$$\zeta_s - \min_{0 \leq r \leq s} \zeta_r$$

before time η_0. In other words, the intervals (α_i, β_i) are the connected components of the open set $\{s \in [0, \eta_0] : \zeta_s > \min_{0 \leq r \leq s} \zeta_r\}$. Using the properties of the Brownian snake, it is easy to verify that $\mathbb{P}_{\mathrm{w}}$ a.s. for every $i \in I$, $W_{\alpha_i} = W_{\beta_i}$ is just the restriction of w to $[0, \zeta_{\alpha_i}]$, and the paths W_s, $s \in [\alpha_i, \beta_i]$ all coincide over the time interval $[0, \zeta_{\alpha_i}]$. In order to describe the behavior of these paths beyond time ζ_{α_i} we introduce, for every $i \in I$, the element $W^i = (W^i_s)_{s \geq 0}$ of $C(\mathbb{R}_+, \mathcal{W})$ obtained by setting, for every $s \geq 0$,

$$W^i_s(t) := W_{(\alpha_i + s) \wedge \beta_i}(\zeta_{\alpha_i} + t) , \quad 0 \leq t \leq \zeta^i_s := \zeta_{(\alpha_i + s) \wedge \beta_i} - \zeta_{\alpha_i}.$$

Lemma 3. *Under* $\mathbb{P}_{\mathrm{w}}$*, the point measure*

$$\sum_{i \in I} \delta_{(\zeta_{\alpha_i}, W^i)}(\mathrm{d}t, \mathrm{d}\omega)$$

is a Poisson point measure on $\mathbb{R}_+ \times C(\mathbb{R}_+, \mathcal{W})$ *with intensity*

$$2\, \mathbf{1}_{[0, \zeta_{(\mathrm{w})}]}(t)\, \mathrm{d}t\, \mathbb{N}_{\mathrm{w}(t)}(\mathrm{d}\omega).$$

We refer to [9, Lemma V.5] for a proof of this lemma. Our second lemma deals with the distribution of the Brownian snake under $\mathbb{N}_0$ at the first hitting time of a negative level. For every $b > 0$, we set

$$S_b := \inf\{s \geq 0 : \widehat{W}_s = -b\}$$

with the usual convention $\inf \varnothing = \infty$.

Lemma 4. *The law of the random path* W_{S_b} *under the probability measure* $\mathbb{N}_0(\cdot \mid S_b < \infty)$ *is the law of the process* $(R^{(2)}_t - b)_{0 \leq t \leq T^{(2)}}$ *under* P_b.

This lemma can be obtained as a very special case of Theorem 4.6.2 in [6]. Alternatively, the lemma is also a special case of Proposition 1.4 in [5], which relied on explicit calculations of capacitary distributions for the Brownian snake found in [8]. Let us briefly explain how the result follows from [6]. For every $x > -b$, set

$$u_b(x) := \mathbb{N}_x(S_b < \infty) = \frac{3}{2(x+b)^2}$$

where the second equality is just (1). Following the comments at the end of Section 4.6 in [6], we get that the law of W_{S_b} under the probability measure $\mathbb{N}_0(\cdot \mid S_b < \infty)$ is the distribution of the process X solving the stochastic differential equation

$$\mathrm{d}X_t = \mathrm{d}B_t + \frac{u'_b}{u_b}(X_t)\, \mathrm{d}t , \quad X_0 = 0,$$

and stopped at its first hitting time of $-b$. Since $\frac{u'_b}{u_b}(x) = -\frac{2}{x+b}$ we obtain the desired result.

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3 The main results

Our first theorem identifies the law of the minimizing path $W_{s_{\mathbf{m}}}$.

Theorem 5. *Let $a > 0$. Under $\mathbb{N}_0$, the conditional distribution of $W_{s_{\mathbf{m}}}$ knowing that $W_* = -a$ is the distribution of the process $(R_t^{(3)} - a)_{0 \leq t \leq T^{(3)}}$, where $R^{(3)}$ is a Bessel process of dimension -5 started from a, and $T^{(3)} = \inf\{t \geq 0 : R_t^{(3)} = 0\}$.*

In an integral form, the statement of the theorem means that, for any nonnegative measurable function F on $\mathcal{W}_0$,

$$\mathbb{N}_0(F(W_{s_{\mathbf{m}}})) = 3 \int_0^\infty \frac{\mathrm{d}a}{a^3} \, E_a\Big[F\big((R_t^{(3)} - a)_{0 \leq t \leq T^{(3)}}\big)\Big]$$

where we recall that the process $R^{(3)}$ starts from a under P_a.

Proof. We fix three positive real numbers δ, K, K' such that $\delta < K < K'$, and we let G be a bounded nonnegative continuous function on $\mathcal{W}_0$. For every $\mathrm{w} \in \mathcal{W}_0$, we then set

$$\tau_\delta(\mathrm{w}) := \inf\{t \geq 0 : \mathrm{w}(t) = -\delta\}$$

and $F(\mathrm{w}) := G((\mathrm{w}(t))_{0 \leq t \leq \tau_\delta(\mathrm{w})})$ if $\tau_\delta(\mathrm{w}) < \infty$, $F(\mathrm{w}) := 0$ otherwise.

For every real x and every integer $n \geq 1$, write $[x]_n$ for the largest real number of the form $k2^{-n}$, $k \in \mathbb{Z}$, smaller than or equal to x. Using the special form of F and the fact that $S_{[-W_*]_n} \uparrow s_{\mathbf{m}}$ as $n \uparrow \infty$, $\mathbb{N}_0$ a.e., we easily get from the properties of the Brownian snake that $F(W_{S_{[-W_*]_n}}) = F(W_{S_*})$, for all n large enough, $\mathbb{N}_0$ a.e. on the event $\{W_* < -\delta\}$. By dominated convergence, we have then

$$\mathbb{N}_0(F(W_{s_{\mathbf{m}}})\mathbf{1}\{-K' \leq W_* \leq -K\}) = \lim_{n \to \infty} \mathbb{N}_0(F(W_{S_{[-W_*]_n}})\mathbf{1}\{K \leq [-W_*]_n \leq K'\})$$

$$= \lim_{n \to \infty} \sum_{K2^n \leq k \leq K'2^n} \mathbb{N}_0\Big(F(W_{S_{k2^{-n}}})\,\mathbf{1}\{S_{k2^{-n}} < \infty\}\,\mathbf{1}\Big\{\min_{S_{k2^{-n}} \leq s \leq \sigma} \widehat{W}_s > -(k+1)2^{-n}\Big\}\Big). \quad (3)$$

Let $b > \delta$ and $\varepsilon > 0$. We use the strong Markov property of the Brownian snake at time S_b, together with Lemma 3, to get

$$\mathbb{N}_0\Big(F(W_{S_b})\,\mathbf{1}\{S_b < \infty\}\,\mathbf{1}\Big\{\min_{S_b \leq s \leq \sigma} \widehat{W}_s > -b - \varepsilon\Big\}\Big)$$

$$= \mathbb{N}_0\Big(F(W_{S_b})\,\mathbf{1}\{S_b < \infty\} \exp\Big(-2 \int_0^{\zeta_{S_b}} \mathrm{d}t\, \mathbb{N}_{W_{S_b}(t)}(W_* > -b - \varepsilon)\Big)\Big)$$

$$= \mathbb{N}_0\Big(F(W_{S_b})\,\mathbf{1}\{S_b < \infty\} \exp\Big(-3 \int_0^{\zeta_{S_b}} \mathrm{d}t\, (b + \varepsilon + W_{S_b}(t))^{-2}\Big)\Big)$$

$$= \frac{3}{2b^2} E_b\Big[F((R_t^{(2)} - b)_{0 \leq t \leq T^{(2)}}) \exp\Big(-3 \int_0^{T^{(2)}} \mathrm{d}t\, (\varepsilon + R_t^{(2)})^{-2}\Big)\Big] \quad (4)$$

using (1) in the second equality, and Lemma 4 and (1) again in the third one. Recall the definition of the stopping times $T_\delta^{(\alpha)}$ before Lemma 1. From the special form of the function F,

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and then the strong Markov property of the process $R^{(2)}$ at time $T^{(2)}_{b-\delta}$, we obtain that

$$E_b\Big[F((R^{(2)}_t - b)_{0 \le t \le T^{(2)}}) \exp\Big(-3 \int_0^{T^{(2)}} \mathrm{d}t \, (\varepsilon + R^{(2)}_t)^{-2}\Big)\Big]$$

$$= E_b\Big[G((R^{(2)}_t - b)_{0 \le t \le T^{(2)}_{b-\delta}}) \, \exp\Big(-3 \int_0^{T^{(2)}} \mathrm{d}t \, (\varepsilon + R^{(2)}_t)^{-2}\Big)\Big]$$

$$= E_b\Big[G((R^{(2)}_t - b)_{0 \le t \le T^{(2)}_{b-\delta}}) \, \exp\Big(-3 \int_0^{T^{(2)}_{b-\delta}} \mathrm{d}t \, (\varepsilon + R^{(2)}_t)^{-2}\Big)$$

$$\times E_{b-\delta}\Big[\exp\Big(-3 \int_0^{T^{(2)}} \mathrm{d}t \, (\varepsilon + R^{(2)}_t)^{-2}\Big)\Big]\Big]. \tag{5}$$

Using the formula of Lemma 2 and combining (4) and (5), we arrive at

$$\mathbb{N}_0\Big(F(W_{S_b}) \mathbf{1}\{S_b < \infty\} \mathbf{1}\Big\{\min_{S_b \le s \le \sigma} \widehat{W}_s > -b - \varepsilon\Big\}\Big)$$

$$= \frac{3}{2b^2}\Big(1 - \Big(\frac{b-\delta}{b-\delta+\varepsilon}\Big)^2\Big) E_b\Big[G((R^{(2)}_t - b)_{0 \le t \le T^{(2)}_{b-\delta}}) \, \exp\Big(-3 \int_0^{T^{(2)}_{b-\delta}} \mathrm{d}t \, (\varepsilon + R^{(2)}_t)^{-2}\Big)\Big].$$

Hence,

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \mathbb{N}_0\Big(F(W_{S_b}) \mathbf{1}\{S_b < \infty\} \mathbf{1}\Big\{\min_{S_b \le s \le \sigma} \widehat{W}_s > -b - \varepsilon\Big\}\Big)$$

$$= \Big(\frac{3}{b^2(b-\delta)}\Big) E_b\Big[G((R^{(2)}_t - b)_{0 \le t \le T^{(2)}_{b-\delta}}) \, \exp\Big(-3 \int_0^{T^{(2)}_{b-\delta}} \mathrm{d}t \, (R^{(2)}_t)^{-2}\Big)\Big].$$

At this stage we use Lemma 1 twice to see that

$$E_b\Big[G((R^{(2)}_t - b)_{0 \le t \le T^{(2)}_{b-\delta}}) \, \exp\Big(-3 \int_0^{T^{(2)}_{b-\delta}} \mathrm{d}t \, (R^{(2)}_t)^{-2}\Big)\Big]$$

$$= \Big(\frac{b}{b-\delta}\Big)^2 E_b\Big[G((B_t - b)_{0 \le t \le T_{b-\delta}}) \, \exp\Big(-6 \int_0^{T_{b-\delta}} \frac{\mathrm{d}s}{B_s^2}\Big)\Big]$$

$$= \Big(\frac{b}{b-\delta}\Big)^{-1} E_b\Big[G((R^{(3)}_t - b)_{0 \le t \le T^{(3)}_{b-\delta}})\Big]$$

Summarizing, we have

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \mathbb{N}_0\Big(F(W_{S_b}) \mathbf{1}\{S_b < \infty\} \mathbf{1}\Big\{\min_{S_b \le s \le \sigma} \widehat{W}_s > -b - \varepsilon\Big\}\Big) = \frac{3}{b^3} E_b\Big[G((R^{(3)}_t - b)_{0 \le t \le T^{(3)}_{b-\delta}})\Big].$$

Note that the right-hand side of the last display is a continuous function of $b \in (\delta, \infty)$. Furthermore, a close look at the preceding arguments shows that the convergence is uniform when b varies over an interval of the form $[\delta', \infty)$, where $\delta' > \delta$. We can therefore return to (3) and obtain that

$$\mathbb{N}_0(F(W_{s_\mathbf{m}}) \mathbf{1}\{-K' \le W_* \le -K\})$$

$$= \lim_{n \to \infty} \int_K^{K'} \mathrm{d}b \, 2^n \mathbb{N}_0\Big(F(W_{S_{[b]_n}}) \mathbf{1}\{S_{[b]_n} < \infty\} \mathbf{1}\Big\{\min_{S_{[b]_n} \le s \le \sigma} \widehat{W}_s > -[b]_n - 2^{-n}\Big\}\Big)$$

$$= 3 \int_K^{K'} \frac{\mathrm{d}b}{b^3} E_b\Big[G((R^{(3)}_t - b)_{0 \le t \le T^{(3)}_{b-\delta}})\Big].$$

The result of the theorem now follows easily.

$\square$

We turn to a statement describing the structure of subtrees branching off the minimizing path $W_{s_{\mathbf{m}}}$. In a sense, this is similar to Lemma 3 above (except that we will need to consider separately subtrees branching *before* and *after* time $s_{\mathbf{m}}$, in the time scale of the Brownian snake). Since $s_{\mathbf{m}}$ is not a stopping time of the Brownian snake, it is of course impossible to use the strong Markov property in order to apply Lemma 3. Still this lemma will play an important role.

We argue under the excursion measure $\mathbb{N}_0$ and, for every $s \geq 0$, we set

$$\hat{\zeta}_s := \zeta_{(s_{\mathbf{m}}+s)\wedge\sigma} \, , \quad \check{\zeta}_s := \zeta_{(s_{\mathbf{m}}-s)\vee 0}.$$

We let $(\hat{a}_i, \hat{b}_i)$, $i \in I$ be the excursion intervals of $\hat{\zeta}_s$ above its past minimum. Equivalently, the intervals $(\hat{a}_i, \hat{b}_i)$, $i \in I$ are the connected components of the set

$$\left\{ s \geq 0 : \hat{\zeta}_s > \min_{0 \leq r \leq s} \hat{\zeta}_r \right\}.$$

Similarly, we let $(\check{a}_j, \check{b}_j)$, $j \in J$ be the excursion intervals of $\check{\zeta}_s$ above its past minimum. We may assume that the indexing sets I and J are disjoint. In terms of the tree $\mathcal{T}_\zeta$ coded by the excursion $(\zeta_s)_{0 \leq s \leq \sigma}$ under $\mathbb{N}_0$ (see e.g. [10, Section 2]), each interval $(\hat{a}_i, \hat{b}_i)$ or $(\check{a}_j, \check{b}_j)$ corresponds to a subtree of $\mathcal{T}_\zeta$ branching off the ancestral line of the vertex associated with $s_{\mathbf{m}}$. We next consider the spatial displacements corresponding to these subtrees. For every $i \in I$, we let $W^{(i)} = (W_s^{(i)})_{s \geq 0} \in C(\mathbb{R}_+, \mathcal{W})$ be defined by

$$W_s^{(i)}(t) = W_{s_{\mathbf{m}}+(\hat{a}_i+s)\wedge\hat{b}_i}(\zeta_{s_{\mathbf{m}}+\hat{a}_i} + t) \, , \quad 0 \leq t \leq \zeta_{s_{\mathbf{m}}+(\hat{a}_i+s)\wedge\hat{b}_i} - \zeta_{s_{\mathbf{m}}+\hat{a}_i}.$$

Similarly, for every $j \in J$,

$$W_s^{(j)}(t) = W_{s_{\mathbf{m}}-(\check{a}_j+s)\wedge\check{b}_j}(\zeta_{s_{\mathbf{m}}-\check{a}_j} + t) \, , \quad 0 \leq t \leq \zeta_{s_{\mathbf{m}}-(\check{a}_j+s)\wedge\check{b}_j} - \zeta_{s_{\mathbf{m}}-\check{a}_j}.$$

We finally introduce the point measures on $\mathbb{R}_+ \times C(\mathbb{R}_+, \mathcal{W})$ defined by

$$\hat{\mathcal{N}} = \sum_{i \in I} \delta_{(\zeta_{s_{\mathbf{m}}+\hat{a}_i}, W^{(i)})} \, , \quad \check{\mathcal{N}} = \sum_{j \in J} \delta_{(\zeta_{s_{\mathbf{m}}-\check{a}_j}, W^{(j)})}.$$

If $\omega = (\omega_s)_{s \geq 0}$ belongs to $C(\mathbb{R}_+, \mathcal{W})$, we set $\omega_* := \inf\{\omega_s(t) : s \geq 0, 0 \leq t \leq \zeta_{(\omega_s)}\}$.

Theorem 6. *Under $\mathbb{N}_0$, conditionally on the minimizing path $W_{s_{\mathbf{m}}}$, the point measures $\hat{\mathcal{N}}(\mathrm{d}t, \mathrm{d}\omega)$ and $\check{\mathcal{N}}(\mathrm{d}t, \mathrm{d}\omega)$ are independent and their common conditional distribution is that of a Poisson point measure with intensity*

$$2 \, \mathbf{1}_{[0,\zeta_{s_{\mathbf{m}}}]}(t) \, \mathbf{1}_{\{\omega_* > \widehat{W}_{s_{\mathbf{m}}}\}} \, \mathrm{d}t \, \mathbb{N}_{W_{s_{\mathbf{m}}}(t)}(\mathrm{d}\omega).$$

Clearly, the constraint $\omega_* > \widehat{W}_{s_{\mathbf{m}}}$ corresponds to the fact that none of the spatial positions in the subtrees branching off the ancestral line of $p_\zeta(s_{\mathbf{m}})$ can be smaller than $W_* = \widehat{W}_{s_{\mathbf{m}}}$, by the very definition of W_*.

Proof. We will first argue that the conditional distribution of $\hat{\mathcal{N}}$ given $W_{s_{\mathbf{m}}}$ is as described in the theorem. To this end, we fix again δ, K, K' such that $0 < \delta < K < K'$, and we use the notation $\tau_\delta(\mathrm{w})$ introduced in the proof of Theorem 5. On the event where $W_* < -\delta$, we also set

$$\hat{\mathcal{N}}_\delta = \sum_{\substack{i \in I \\ \zeta_{s_{\mathbf{m}}+\hat{a}_i} \leq \tau_\delta(W_{s_{\mathbf{m}}})}} \delta_{(\zeta_{s_{\mathbf{m}}+\hat{a}_i}, W^{(i)})}.$$

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Informally, considering only the subtrees that occur after $s_{\mathbf{m}}$ in the time scale of the Brownian snake, $\hat{\mathcal{N}}_\delta$ corresponds to those subtrees that branch off the minimizing path $W_{s_{\mathbf{m}}}$ before this path hits the level $-\delta$.

Next, let Φ be a bounded nonnegative measurable function on the space of all point measures on $\mathbb{R}_+ \times C(\mathbb{R}_+, \mathcal{W})$ – we should restrict this space to point measures satisfying appropriate σ-finiteness conditions, but we omit the details – and let Ψ be a bounded continuous function on $C(\mathbb{R}_+, \mathcal{W})$. To simplify notation, we write $W_{\leq s_{\mathbf{m}}}$ for the process $(W_{s \wedge s_{\mathbf{m}}})_{s \geq 0}$ viewed as a random element of $C(\mathbb{R}_+, \mathcal{W})$, and we use the similar notation $W_{\leq S_b}$. For every $b > 0$, let the point measure $\hat{\mathcal{N}}_\delta^{(b)}$ be defined (only on the event where $S_b < \infty$) in a way analogous to $\hat{\mathcal{N}}_\delta$ but replacing the path $W_{s_{\mathbf{m}}}$ with the path W_{S_b}: To be specific, $\hat{\mathcal{N}}_\delta^{(b)}$ accounts for those subtrees (occurring after S_b in the time scale of the Brownian snake) that branch off W_{S_b} before this path hits $-\delta$.

As in (3), we have then

$$\mathbb{N}_0\Big(\Psi(W_{\leq s_{\mathbf{m}}})\mathbf{1}\{-K' \leq W_* \leq -K\}\,\Phi(\hat{\mathcal{N}}_\delta)\Big) = \lim_{n \to \infty} \sum_{K2^n \leq k \leq K'2^n} \mathbb{N}_0\Big(\Psi(W_{\leq S_{k2^{-n}}})\,\mathbf{1}\{S_{k2^{-n}} < \infty\}$$
$$\mathbf{1}\Big\{\min_{S_{k2^{-n}} \leq s \leq \sigma} \widehat{W}_s > -(k+1)2^{-n}\Big\}\,\Phi(\hat{\mathcal{N}}_\delta^{(k2^{-n})})\Big). \tag{6}$$

The point in (6) is the fact that, $\mathbb{N}_0$ a.e., if n is sufficiently large, and if $k \geq K2^{-n}$ is the largest integer such that $S_{k2^{-n}} < \infty$, the paths $W_{s_{\mathbf{m}}}$ and $W_{S_{k2^{-n}}}$ are the same up to a time which is greater than $\tau_\delta(W_{s_{\mathbf{m}}})$, and the point measures $\hat{\mathcal{N}}_\delta$ and $\hat{\mathcal{N}}_\delta^{(k2^{-n})}$ coincide.

Next fix $b > \delta$ and, for $\varepsilon > 0$, consider the quantity

$$\mathbb{N}_0\Big(\Psi(W_{\leq S_b})\,\mathbf{1}\{S_b < \infty\}\mathbf{1}\Big\{\min_{S_b \leq s \leq \sigma} \widehat{W}_s > -b - \varepsilon\Big\}\,\Phi(\hat{\mathcal{N}}_\delta^{(b)})\Big). \tag{7}$$

To evaluate this quantity, we again apply the strong Markov property of the Brownian snake at time S_b. For notational convenience, we suppose that, on a certain probability space, we have a random point measure $\mathcal{M}$ on $\mathbb{R}_+ \times C(\mathbb{R}_+, \mathcal{W})$ and, for every $\mathrm{w} \in \mathcal{W}_0$, a probability measure Π_{w} under which $\mathcal{M}(\mathrm{d}t, \mathrm{d}\omega)$ is Poisson with intensity

$$2\,\mathbf{1}_{[0,\zeta_{\mathrm{w}}]}(t)\,\mathrm{d}t\,\mathbb{N}_{\mathrm{w}(t)}(\mathrm{d}\omega).$$

By the strong Markov property at S_b and Lemma 3, the quantity (7) is equal to

$$\mathbb{N}_0\Big(\Psi(W_{\leq S_b})\,\mathbf{1}\{S_b < \infty\}\,\Pi_{W_{S_b}}\Big(\mathbf{1}\{\mathcal{M}(\{(t,\omega) : \omega_* \leq -b - \varepsilon\}) = 0\}\,\Phi(\mathcal{M}_{\leq \tau_\delta(W_{S_b})})\Big)\Big),$$

where $\mathcal{M}_{\leq \tau_\delta(W_{S_b})}$ denotes the restriction of the point measure $\mathcal{M}$ to $[0, \tau_\delta(W_{S_b})] \times C(\mathbb{R}_+, \mathcal{W})$. Write $W_{S_b}^{(\delta)}$ for the restriction of the path W_{S_b} to $[0, \tau_\delta(W_{S_b})]$. We have then

$$\Pi_{W_{S_b}}\Big(\mathbf{1}\{\mathcal{M}(\{(t,\omega) : \omega_* \leq -b - \varepsilon\}) = 0\}\,\Phi(\mathcal{M}_{\leq \tau_\delta(W_{S_b})})\Big)$$
$$= \Pi_{W_{S_b}}(\mathcal{M}(\{(t,\omega) : \omega_* \leq -b - \varepsilon\}) = 0)\,\Pi_{W_{S_b}}\Big(\Phi(\mathcal{M}_{\leq \tau_\delta(W_{S_b})})\,\Big|\,\mathcal{M}(\{(t,\omega) : \omega_* \leq -b - \varepsilon\}) = 0\Big)$$
$$= \Pi_{W_{S_b}}(\mathcal{M}(\{(t,\omega) : \omega_* \leq -b - \varepsilon\}) = 0)\,\Pi_{W_{S_b}^{(\delta)}}\Big(\Phi(\mathcal{M})\,\Big|\,\mathcal{M}(\{(t,\omega) : \omega_* \leq -b - \varepsilon\}) = 0\Big),$$

using standard properties of Poisson measures in the last equality. Summarizing, we see that the quantity (7) coincides with

$$\mathbb{N}_0\Big(\Psi(W_{\leq S_b})\,H(W_{S_b}, b+\varepsilon)\,\mathbf{1}\{S_b < \infty\}\,\Pi_{W_{S_b}}(\mathcal{M}(\{(t,\omega) : \omega_* \leq -b-\varepsilon\}) = 0)\Big), \qquad (8)$$

where, for every $\mathrm{w} \in \mathcal{W}_0$ such that $\tau_\delta(\mathrm{w}) < \infty$, for every $a > \delta$, $H(\mathrm{w}, a) := \widetilde{H}((\mathrm{w}(t))_{0 \leq t \leq \tau_\delta(\mathrm{w})}, a)$, and the function $\widetilde{H}$ is given by

$$\widetilde{H}(\mathrm{w}, a) := \Pi_{\mathrm{w}}\Big(\Phi(\mathcal{M}) \,\Big|\, \mathcal{M}(\{(t,\omega) : \omega_* \leq -a\}) = 0\Big),$$

this definition making sense if $\mathrm{w} \in \mathcal{W}_0$ does not hit $-a$. By the strong Markov property at S_b and again Lemma 3, the quantity (8) is also equal to

$$\mathbb{N}_0\Big(\Psi(W_{\leq S_b})\,H(W_{S_b}, b+\varepsilon)\,\mathbf{1}\{S_b < \infty\}\,\mathbf{1}\Big\{\min_{S_b \leq s \leq \sigma} \widehat{W}_s > -b-\varepsilon\Big\}\Big).$$

We may now come back to (6), and get from the previous observations that

$$\mathbb{N}_0\Big(\Psi(W_{\leq s_{\mathbf{m}}})\mathbf{1}\{-K' \leq W_* \leq -K\}\,\Phi(\hat{\mathcal{N}}_\delta)\Big)$$

$$= \lim_{n \to \infty} \sum_{K2^n \leq k \leq K'2^n} \mathbb{N}_0\Big(\Psi(W_{\leq S_{k2^{-n}}})\,H(W_{S_{k2^{-n}}}, (k+1)2^{-n})$$

$$\mathbf{1}\{S_{k2^{-n}} < \infty\}\,\mathbf{1}\Big\{\min_{S_{k2^{-n}} \leq s \leq \sigma} \widehat{W}_s > -(k+1)2^{-n}\Big\}\Big)$$

$$= \lim_{n \to \infty} \mathbb{N}_0\Big(\Psi(W_{\leq S_{[-W_*]_n}})\,H(W_{S_{[-W_*]_n}}, [-W_*]_n - 2^{-n})\,\mathbf{1}\{K \leq [-W_*]_n \leq K'\}\Big)$$

$$= \mathbb{N}_0\Big(\Psi(W_{\leq s_{\mathbf{m}}})\,H(W_{s_{\mathbf{m}}}, -W_*)\,\mathbf{1}\{-K' \leq W_* \leq -K\}\Big).$$

To verify the last equality, recall that the paths $W_{S_{[-W_*]_n}}$ and $W_{s_{\mathbf{m}}}$ coincide up to their first hitting time of $-\delta$, for all n large enough, $\mathbb{N}_0$ a.e., and use also the fact that the function $H(\mathrm{w}, a)$ is Lipschitz in the variable a on every compact subset of (δ, ∞), uniformly in the variable w.

From the definition of H, we have then

$$\mathbb{N}_0\Big(\Psi(W_{\leq s_{\mathbf{m}}})\mathbf{1}\{-K' \leq W_* \leq -K\}\,\Phi(\hat{\mathcal{N}}_\delta)\Big)$$

$$= \mathbb{N}_0\Big(\Psi(W_{\leq s_{\mathbf{m}}})\mathbf{1}\{-K' \leq W_* \leq -K\}\,\Pi_{W_{s_{\mathbf{m}}}^{(\delta)}}\Big(\Phi(\mathcal{M}) \,\Big|\, \mathcal{M}(\{(t,\omega) : \omega_* \leq W_*\}) = 0\Big)\Big),$$

where $W_{s_{\mathbf{m}}}^{(\delta)}$ denotes the restriction of $W_{s_{\mathbf{m}}}$ to $[0, \tau_\delta(W_{s_{\mathbf{m}}})]$. From this, and since $W_* = \widehat{W}_{s_{\mathbf{m}}}$, we obtain that the conditional distribution of $\hat{\mathcal{N}}_\delta$ given $W_{\leq s_{\mathbf{m}}}$ is (on the event where $W_* < -\delta$) the law of a Poisson point measure with intensity

$$2\,\mathbf{1}_{[0, \tau_\delta(W_{s_{\mathbf{m}}})]}(t)\,\mathbf{1}_{\{\omega_* > \widehat{W}_{s_{\mathbf{m}}}\}}\,\mathrm{d}t\,\mathbb{N}_{W_{s_{\mathbf{m}}}(t)}(\mathrm{d}\omega).$$

Since δ is arbitrary, it easily follows that the conditional distribution of $\hat{\mathcal{N}}$ given $W_{\leq s_{\mathbf{m}}}$ is that of a Poisson measure with intensity

$$2\,\mathbf{1}_{[0, \zeta_{W_{s_{\mathbf{m}}}}]}(t)\,\mathbf{1}_{\{\omega_* > \widehat{W}_{s_{\mathbf{m}}}\}}\,\mathrm{d}t\,\mathbb{N}_{W_{s_{\mathbf{m}}}(t)}(\mathrm{d}\omega).$$

Note that this conditional distribution only depends on $W_{s_{\mathbf{m}}}$, meaning that $\hat{\mathcal{N}}$ is conditionally independent of $W_{\leq s_{\mathbf{m}}}$ given $W_{s_{\mathbf{m}}}$.

Since the measure $\mathbb{N}_0$ is invariant under time-reversal, we also get that the conditional distribution of $\check{\mathcal{N}}$ given $W_{s_{\mathbf{m}}}$ is the same as the conditional distribution of $\hat{\mathcal{N}}$ given $W_{s_{\mathbf{m}}}$. Finally, $\check{\mathcal{N}}$ is a measurable function of $W_{\leq s_{\mathbf{m}}}$ and since $\hat{\mathcal{N}}$ is conditionally independent of $W_{\leq s_{\mathbf{m}}}$ given $W_{s_{\mathbf{m}}}$, we get that $\check{\mathcal{N}}$ and $\hat{\mathcal{N}}$ are conditionally independent given $W_{s_{\mathbf{m}}}$. $\qquad\square$

4 Applications to super-Brownian motion

We will now discuss applications of the preceding results to super-Brownian motion. Let μ be a (nonzero) finite measure on $\mathbb{R}$. We denote the topological support of μ by $\mathrm{supp}(\mu)$ and always assume that
$$m := \inf \mathrm{supp}(\mu) > -\infty.$$

We then consider a super-Brownian motion $X = (X_t)_{t \geq 0}$ with quadratic branching mechanism $\psi(u) = 2u^2$ started from μ. The particular choice of the normalization of ψ is motivated by the connection with the Brownian snake. Let us recall this connection following Section IV.4 of [9]. We consider a Poisson point measure $\mathcal{P}(\mathrm{d}x, \mathrm{d}\omega)$ on $\mathbb{R} \times C(\mathbb{R}_+, \mathcal{W})$ with intensity
$$\mu(\mathrm{d}x)\,\mathbb{N}_x(\mathrm{d}\omega).$$

Write
$$\mathcal{P}(\mathrm{d}x, \mathrm{d}\omega) = \sum_{i \in I} \delta_{(x^i, \omega^i)}(\mathrm{d}x, \mathrm{d}\omega)$$

and for every $i \in I$, let $\zeta_s^i = \zeta_{(\omega_s^i)}$, $s \geq 0$, stand for the lifetime process associated with ω^i. Also, for every $r \geq 0$ and $s \geq 0$, let $\ell_s^r(\zeta^i)$ be the local time at level r and at time s of the process ζ^i. We may and will construct the super-Brownian motion X by setting $X_0 = \mu$ and for every $r > 0$, for every nonnegative measurable function φ on $\mathbb{R}$,

$$\langle X_r, \varphi \rangle = \sum_{i \in I} \int_0^\infty \mathrm{d}_s \ell_s^r(\zeta^i)\, \varphi(\widehat{\omega}_s^i), \tag{9}$$

where the notation $\mathrm{d}_s \ell_s^r(\zeta^i)$ refers to integration with respect to the increasing function $s \to \ell_s^r(\zeta^i)$.

A major advantage of the Brownian snake construction is the fact that it also yields an immediate definition of the historical super-Brownian motion $Y = (Y_r)_{r \geq 0}$ associated with X (we refer to [4] or [7] for the general theory of historical superprocesses). For every $r \geq 0$, Y_r is a finite measure on the subset of $\mathcal{W}$ consisting of all stopped paths with lifetime r. We have $Y_0 = \mu$ and for every $r > 0$,

$$\langle Y_r, \Phi \rangle = \sum_{i \in I} \int_0^\infty \mathrm{d}_s \ell_s^r(\zeta^i)\, \Phi(\omega_s^i), \tag{10}$$

for every nonnegative measurable function Φ on $\mathcal{W}$. Note the relation $\langle X_r, \varphi \rangle = \int Y_r(\mathrm{dw})\,\varphi(\widehat{\mathrm{w}})$.

The range $\mathcal{R}^X$ is the closure in $\mathbb{R}$ of the set
$$\bigcup_{r \geq 0} \mathrm{supp}(X_r),$$

and, similarly, we define $\mathcal{R}^Y$ as the closure in $\mathcal{W}$ of
$$\bigcup_{r \geq 0} \mathrm{supp}(Y_r).$$

We note that
$$\mathcal{R}^X = \mathrm{supp}(\mu) \cup \left(\bigcup_{i \in I} \{\widehat{\omega}_s^i : s \geq 0\} \right)$$

and

$$\mathcal{R}^Y = \mathrm{supp}(\mu) \cup \Big(\bigcup_{i \in I} \{\omega_s^i : s \geq 0\} \Big).$$

We set

$$m_X := \inf \mathcal{R}^X.$$

From the preceding formulas and the uniqueness of the minimizing path in the case of the Brownian snake, it immediately follows that there is a unique stopped path $\mathrm{w}_{\min} \in \mathcal{R}^Y$ such that $\widehat{\mathrm{w}}_{\min} = m_X$. Our goal is to describe the distribution of $\mathrm{w}_{\min}$. We first observe that the distribution of m_X is easy to obtain from (1) and the Brownian snake representation: We have obviously $m_X \leq m$ and, for every $x < m$,

$$P(m_X \geq x) = \exp\Big(-\frac{3}{2} \int \frac{\mu(\mathrm{d}u)}{(u - x)^2} \Big). \tag{11}$$

Note that this formula is originally due to [3, Theorem 1.3]. It follows that

$$P(m_X = m) = \exp\Big(-\frac{3}{2} \int \frac{\mu(\mathrm{d}u)}{(u - m)^2} \Big).$$

Therefore, if $\int (u - m)^{-2} \mu(\mathrm{d}u) < \infty$, the event $\{m_X = m\}$ occurs with positive probability. If this event occurs, $\mathrm{w}_{\min}$ is just the trivial path m with zero lifetime.

Proposition 7. *The joint distribution of the pair* $(\mathrm{w}_{\min}(0), m_X)$ *is given by the formulas*

$$P(\mathrm{w}_{\min}(0) \leq a, \, m_X \leq x) = 3 \int_{-\infty}^{x} \mathrm{d}y \, \Big(\int_{[m,a]} \frac{\mu(\mathrm{d}u)}{(u - y)^3} \Big) \exp\Big(-\frac{3}{2} \int \frac{\mu(\mathrm{d}u)}{(u - y)^2} \Big),$$

for every $a \in [m, \infty)$ *and* $x \in (-\infty, m)$, *and*

$$P(m_X = m) = P(m_X = m, \mathrm{w}_{\min}(0) = m) = \exp\Big(-\frac{3}{2} \int \frac{\mu(\mathrm{d}u)}{(u - m)^2} \Big).$$

Proof. Fix $a \in [m, \infty)$, and let μ', respectively μ'' denote the restriction of μ to $[m, a]$, resp. to (a, ∞). Define X', respectively X'', by setting $X_0' = \mu'$, resp. $X_0'' = \mu''$, and restricting the sum in the right-hand side of (9) to indices $i \in I$ such that $x^i \in [m, a]$, resp. $x^i \in (a, \infty)$. Define Y' and Y'' similarly using (10) instead of (9). Then X', respectively X'' is a super-Brownian motion started from μ', resp. from μ'', and Y', resp. Y'' is the associated historical super-Brownian motion. Furthermore, (X', Y') and (X'', Y'') are independent.

By (11), the law of $m_{X'}$ has a density on $(-\infty, m)$ given by

$$f_{m_{X'}}(y) = 3\Big(\int_{[m,a]} \frac{\mu(\mathrm{d}u)}{(u - y)^3} \Big) \exp\Big(-\frac{3}{2} \int_{[m,a]} \frac{\mu(\mathrm{d}u)}{(u - y)^2} \Big), \quad y \in (-\infty, m).$$

On the other hand, if $x \in (-\infty, m)$,

$$P(\mathrm{w}_{\min}(0) \leq a, \, m_X \leq x) = P(m_{X'} \leq x, \, m_{X''} > m_{X'}) = \int_{-\infty}^{x} \mathrm{d}y \, f_{m_{X'}}(y) \, P(m_{X''} > y),$$

and we get the first formula of the proposition using (11) again. The second formula is obvious from the remarks preceding the proposition. □

Together with Proposition 7, the next corollary completely characterizes the law of $w_{\min}$. Recall that the case where $m_X = m$ is trivial, so that we do not consider this case in the following statement.

Corollary 8. *Let $x \in (-\infty, m)$ and $a \in [m, \infty)$. Then conditionally on $m_X = x$ and $w_{\min}(0) = a$, the path $w_{\min}$ is distributed as the process $(x + R_t^{(3)})_{0 \leq t \leq T^{(3)}}$ under P_{a-x}.*

Proof. On the event $\{m_X < m\}$, there is a unique index $i_{\min} \in I$ such that

$$m_X = \min\{\widehat{\omega}_s^{i_{\min}} : s \geq 0\}.$$

Furthermore, if $s_{\min}$ is the unique instant such that $m_X = \widehat{\omega}_{s_{\min}}^{i_{\min}}$, we have $w_{\min} = \omega_{s_{\min}}^{i_{\min}}$, and in particular $x_{i_{\min}} = w_{\min}(0)$.

Standard properties of Poisson measures now imply that, conditionally on $m_X = x$ and $w_{\min}(0) = a$, $\omega^{i_{\min}}$ is distributed according to $\mathbb{N}_a(\cdot \mid W_* = x)$. The assertions of the corollary then follow from Theorem 5. $\square$

We could also have obtained an analog of Theorem 6 in the superprocess setting. The conditional distribution of the process X (or of Y) given the minimizing path $w_{\min}$ is obtained by the sum of two contributions. The first one (present only if $\hat{w}_{\min} < m$) corresponds to the minimizing "excursion" $\omega^{i_{\min}}$ introduced in the previous proof, whose conditional distribution given $w_{\min}$ is described by Theorem 6. The second one is just an independent super-Brownian motion $\widetilde{X}$ started from μ and conditioned on the event $m_{\widetilde{X}} \geq \hat{w}_{\min}$. We leave the details of the statement to the reader.

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