SUMMARY OF RESEARCH WORK

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This document emphasizes my research contributions about superprocesses and Brownian snakes, random trees and Lévy processes, and connections with partial differential equations, that were obtained during the period 1990-2005. In addition, Section 6 gives a detailed description of my current research activity, which is concerned with scaling limits of random planar maps and their relations with continuous random trees. The first section below also presents some of my early research work on intersections of Brownian motions and random walks. Numbers in brackets [1], [2], etc. refer to my list of publications at the end of this document.

1 Intersections of Brownian motions and random walks

1.1 Fine properties of Brownian multiple points

Famous papers of Dvoretzky, Erdös and Kakutani [DEK1,DEK2,DEK3] have established the existence of multiple points of planar Brownian motion of any finite order, and even of points of infinite multiplicity. A natural question formulated by Taylor [Ta1] was to study the size of the set of points of a given multiplicity, and in particular to compare the size of the set of n-multiple points with that of the set of $(n+1)$-multiple points. If $h_\alpha(x) = x^2(\log \frac{1}{x})^\alpha$, Taylor conjectured that the h_α-Hausdorff measure of the set of n-multiple points should be zero for $\alpha < n$ but infinite for $\alpha > n$. This conjecture, showing that in a sense there are much more n-multiple points than $(n+1)$-multiple points, was proved in [11]. A key tool there, which turned out to be useful in many other applications, was to approximate the natural measure on the set of n-multiple points (based on the intersection local time, which had been introduced independently by Dynkin [Dy1,Dy2] and Geman-Horowitz-Rosen [GHR]) in terms of the area of the intersection of independent Wiener sausages.

The techniques of [11] were later sharpened in [21] (see also [30]) to yield the exact Hausdorff measure function for the set of n-multiple points of planar Brownian motion (as well as for double points of three-dimensional Brownian motion). The exact Hausdorff measure function turns out to be the function $x^2(\log \frac{1}{x} \log\log\log \frac{1}{x})^n$ in dimension 2 and $x(\log\log \frac{1}{x})^2$ in dimension 3, thus generalizing classical results for the Brownian curve due to Lévy, Ciesielski and Taylor. The same methods can be applied to more general Lévy processes than Brownian motion: The article [22] contains the proof of several conjectures again due to Taylor [Ta2] concerning multiple points of Lévy processes (see also [29] for the proof of a famous Hendricks-Taylor conjecture about the existence of multiple points of Lévy processes).

These Hausdorff measure results for points of finite multiplicity do not say much about points of infinite multiplicity of planar Brownian paths (whose existence was derived in [DEK3]). The paper [17] establishes the somewhat surprising existence of points of infinite multiplicity with any order type. To be specific, if K is any compact subset of the real line $\mathbf{R}$, with an empty interior, there is a point z of the planar Brownian curve such that the set of times at which the Brownian motion is at z is the image of K under an increasing homeomorphism of $\mathbf{R}$. In particular, this implies the existence of points of (infinite) countable multiplicity, a question which was open at that time.

1.2 Asymptotics for the Wiener sausage

If $B = (B_t, t \geq 0)$ is a Brownian motion in $\mathbf{R}^d$, $d \geq 2$, and K is a nonpolar compact subset of $\mathbf{R}^d$, the Wiener sausage S_t^K is the union of the sets $B_s + K$ when s varies over $[0, t]$. To simplify notation, set $S^K := S_1^K$. If m denotes Lebesgue measure on $\mathbf{R}^d$, the asymptotic behavior of $m(S_t^K)$ as $t \to \infty$ is (essentially) equivalent to that of $m(S^{\varepsilon K})$ as $\varepsilon \to 0$, via a scaling argument. A classical result of Kesten, Spitzer and Whitman (see [Sp1], p.40) states that in dimension $d \geq 3$, $t^{-1} m(S_t^K)$ converges a.s. to the Newtonian capacity of K. In dimension 2, $(\log t/t) m(S_t^K)$ converges to π a.s. [11]. The latter convergence essentially follows from an expansion of $E[m(S_t^K)]$ due to Spitzer, in a paper [Sp2] that also connects the expected volume of the Wiener sausage with a heat conduction problem: The quantity $E[m(S_t^K)] - m(K)$ should be interpreted as the amount of heat which has flowed from K into the surrounding medium $\mathbf{R}^d \backslash K$ up to time t, if one assumes that K is held at temperature 1 for all times whereas $\mathbf{R}^d \backslash K$ is at temperature 0 at the initial time.

The articles [6], [25] and [35] provide refinements of the preceding asymptotics for the Wiener sausage. In particular, [25] gives fluctuation results corresponding to the "laws of large numbers" of Kesten-Spitzer-Whitman. In dimension $d \geq 3$, the asymptotic distribution is normal, but, somewhat unexpectedly, it is not in dimension $d = 2$: In that case, the limiting law is that of a renormalized self-intersection local time of planar Brownian motion, which may be defined by the (formal) formula

$$\gamma := \iint_{0 \leq s < t \leq 1} (\delta_0(B_s - B_t) - E[\delta_0(B_s - B_t)])\, ds\, dt.$$

The article [35] goes further in the analysis of the planar Wiener sausage, by providing a full asymptotic expansion for the area $m(S^{\varepsilon K})$ as ε goes to 0. As could be expected, the p-th term in the expansion involves a renormalized self-intersection local time, denoted by γ_p, associated with p-multiple self-intersections of the Brownian path (the existence of these renormalizations had just been established by Dynkin [Dy3]; in particular $\gamma_1 = 1$, $\gamma_2 = \gamma + C$). For every $\varepsilon \in (0, 1)$, set

$$a_\varepsilon = \frac{1}{\pi} \log \frac{1}{\varepsilon} - \frac{1}{\pi} R(K),$$

where $R(K)$ stands for the Robin constant of K (i.e. the logarithm of the logarithmic capacity of K). Then, for every $k \geq 1$,

$$m(S^{\varepsilon K}) = \sum_{p=1}^{k} (-1)^{p+1}\, a_\varepsilon^{-p}\, \gamma_p + R_k(\varepsilon),$$

where the remainder $R_k(\varepsilon)$ is such that $|\log \varepsilon|^k R_k(\varepsilon)$ converges to 0 in L^2, and a.s. when K is star-shaped.

By taking expectations in the previous expansion, one gets an asymptotic expansion for $E[m(S^{\varepsilon K})]$, or equivalently for $E[m(S_t^K)]$ as $t \to \infty$, which considerably sharpens a classical result of Spitzer [Sp2]. Analogous results in dimension $d \geq 3$ are obtained in [26], thus improving previous expansions due to Spitzer [Sp2] and Kac.

A completely different question, again motivated by the heat conduction problem mentioned above, is to find expansions for $E[m(S_t^K)]$ when $t \to 0$. This problem is treated in [47] in a paper in collaboration with M. van den Berg, under the assumption that K has a smooth boundary: The main theorem of [47] provides the first two terms of the expansion, which involve respectively the Lebesgue measure of the boundary of K and the integral of the mean curvature of the boundary.

1.3 Random walks

Many of the above-mentioned results have analogues for random walks on the lattice $\mathbf{Z}^d$. The articles [12] and [13] treat in an exhaustive manner the problem of finding the aymptotics of the number of intersection points of k independent random walks (with zero mean and finite second moments) in $\mathbf{Z}^d$. A major motivation there was to establish a fluctuation theorem for the range of two-dimensional random walks. The range of random walk had been studied by Dvoretzky and Erdös in a pioneering work [DE], and then in several important papers of Jain and Pruitt (see in particular [JP1,JP2,JP3,JP4]). The existence of a fluctuation theorem for the range of two-dimensional random walks was probably the main open problem left open by the Jain-Pruitt papers. This theorem is proved in [12]: Let R_n be the number of distinct sites visited up to time n by a planar random walk with zero mean and finite second moments. Then,

$$\frac{(\log n)^2}{n}(R_n - E[R_n]) \xrightarrow[n\to\infty]{\text{(d)}} -C\,\gamma,$$

where C is a nonnegative constant and γ is the renormalized self-intersection local time introduced above.

The results of [12] can be applied to invariance principles for weakly self-avoiding random walks: For a suitable choice of the parameters, rescaled weakly self-avoiding random walks in the plane will converge in distribution to the so-called two-dimensional polymer measure (having density $C\exp(-a\,\gamma)$ with respect to Wiener measure). This result was first proved by Stoll [St] using nonstandard analysis, and a simple proof was provided in [65]. In the same spirit, in reply to a question of Slade, the paper [52] studies the existence of exponential moments for the variable γ. The existence of negative exponential moments had been established by Varadhan [Va] and is crucial for the construction of the (two-dimensional) polymer measure. Positive exponential moments allow the construction of models of weakly self-attracting Brownian motion, studied by Brydges and Slade [BS] (see also the second chapter of [Bo]).

The article [38] (in collaboration with Jay Rosen) explores extensions of the results of [12] and [13] to random walks in the domain of attraction of stable laws. Under some mild regularity conditions, this paper gives a fairly complete picture of the limit theorems and fluctuations that can be obtained in that setting.

2 The Brownian snake and superprocesses

In the second half of the eighties, a number of authors, including Ed Perkins and Eugene Dynkin started an extensive study of the measure-valued processes which were named superprocesses by Dynkin. Initially, these processes were introduced as models for the evolution of populations undergoing a spatial motion. Apart from this initial motivation, superprocesses have turned out to be important objects for several other reasons, some of which will described below. In particular, they appear in asymptotics for various models from statistical physics, combinatorics or in the theory of interacting particle systems (see e.g. [CS], [CDP], [DS], [HS], [DuP], [Sl]).

Assuming that the branching phenomenon is independent of the spatial position, a superprocess is typically described by two elements, the spatial motion ξ which is a Markov process and the branching mechanism function ψ. Of particular importance is the quadratic branching case $\psi(u) = c\,u^2$, which appears in the limit of branching particle systems where the offspring distribution has mean one and a finite second moment. In that particular case the total mass process of the superprocess is Feller's branching diffusion.

2.1 The Brownian snake construction

My first contribution to superprocesses [39] was to provide a trajectorial construction in the quadratic branching case, which clearly separates the roles of the branching phenomenon and of the spatial motion. More precisely, the underlying genealogy of the "infinitesimal particles" of the superprocess is first coded by a Brownian excursion (in fact by a Poisson point process of Brownian excursions, which may be generated by a reflected Brownian motion) and then the individual spatial motions can easily be constructed using the prescribed branching structure. The idea that the genealogical structure of Feller's branching diffusion could be coded by a Brownian excursion originated from earlier papers by various authors (in particular Neveu-Pitman [NP1,NP2], see also [14] and [32]) which provided connections between linear Brownian motion and branching processes. Later this idea was made more precise by Aldous in his work [Al1,Al2,Al3] on the continuum random tree (CRT) which is nothing but the continuous tree coded by a normalized Brownian excursion. The article [45], motivated by Aldous' work, gives a simple probabilistic approach of the effective calculation of marginals of the CRT using only properties of Brownian excursions (Aldous' original proof relied on discrete approximations of the CRT). This approach is even simplified in the second chapter of the monograph [71].

The article [44] proposed a different version of the construction of [39], which turned out to be of crucial importance in the forthcoming applications (particularly to sample path properties of superprocesses and to partial differential equations). If one views the individual spatial motion paths as labelled by the time parameter of the coding Brownian excursion (or reflected Brownian motion), this gives rise to a path-valued **Markov** process, which is the Brownian snake. The behavior of the Brownian snake is quite easy to explain. The value W_s at time s of the Brownian snake is a path of the underlying spatial motion (started at a fixed initial point) with a random lifetime ζ_s. The random process $(\zeta_s)_{s\geq 0}$ evolves like reflected linear Brownian motion. Informally, when ζ_s decreases, the path W_s is shortened from its tip, and when ζ_s increases, the path W_s is extended by adding (independently of the past) small "pieces of paths" following the law of the spatial motion. To summarize, one can generate the full set of historical paths of a superprocess (with quadratic branching) by considering the values taken by the Brownian snake, which is a "nice" path-valued Markov process.

Potential-theoretic applications of the Brownian snake approach are given in [44] and [48]. The article [44] tackles the problem of describing polar sets for the superprocess: This reduces to describing subsets A of the state space of the spatial motion which are such that the set of paths that visit A is polar for the Brownian snake. Since the Brownian snake is a symmetric Markov process, the classical energy criterion leads to sufficient conditions for non-polarity which are stated in [44] for a general spatial motion. In the particular case of super-Brownian motion (i.e. when the spatial motion is Brownian motion in $\mathbf{R}^d$) one recovers the conditions obtained by Perkins [Pe3] and Dynkin [Dy4] (as shown by Dynkin using the connections with partial differential equations, these conditions are both necessary and sufficient in that case). The article [48] pushes further the potential-theoretic study of the Brownian snake by providing a simple formula for the energy of a measure on paths, and then determining the capacitary measure of the set of paths that visit a given subset of the state space (or the set of paths that exit a domain through a given subset of the boundary). As follows from the general theory, these capacitary measures solve simple variational problems on the space of probability measures on paths.

Perhaps one of the main advantages of the Brownian snake approach is that it provides a very clear picture, together with a powerful means of performing effective calculations, for the so-called Integrated Super-Brownian Excursion (or ISE, see [Al4]), which has turned out to be an important object for describing various asymptotics in combinatorics or statistical mechanics models (see in particular [DS], [HS] and [Sl]). As explained in Chapter IV of the monograph [71], ISE is simply the uniform measure on the range of a Brownian snake driven by a normalized Brownian excursion.

2.2 Sample path properties of super-Brownian motion

In the end of the eighties, Perkins and his co-authors [Pe1,Pe2,Pe3,DIP,DP] provided an impressive array of precise sample path properties of super-Brownian motion, concerning in particular the exact Hausdorff measure of the support at a fixed time or of the range of the process. Two remaining open problems were the determination of the exact Haudorff measure function in the critical dimensions ($d = 2$ for the support and $d = 4$ for the range). Both these problems could be solved by using the Brownian snake construction of super-Brownian motion. First the exact Hausdorff measure function for the support in dimension 2 was found to be $\varphi(r) = r^2 \log \frac{1}{r} \log\log\log \frac{1}{r}$ (just as for the planar Brownian path) in the article [57] in collaboration with E. Perkins. The similar problem for the range in dimension 4 is solved in [70], and the corresponding function is $\varphi(r) = r^4 \log \frac{1}{r} \log\log\log \frac{1}{r}$. In both cases, the Brownian snake construction plays a crucial role by providing tools (namely the strong Markov property of the snake) that are not available in other approaches.

The article [58] in collaboration with E. Perkins and J. Taylor gives similar results for the packing measure of the support of super-Brownian motion. In contrast with the case of Hausdorff measures, the exact measure functions turn out to be different from the case of the Brownian path.

Other path properties of super-Brownian motion obtained in particular in [64] and [67] will be described below in connection with applications to partial differential equations. In their PhD work under my supervision, J.F. Delmas and L. Serlet have also given several applications of the Brownian snake approach to various path properties of superprocesses (see [De1,Se1,Se2]).

3 Applications to partial differential equations

It had been known for long that the Laplace functional of superprocesses could be expressed in terms of solutions to a certain semilinear partial differential equation. However it is only with Dynkin's work [Dy4,Dy5,Dy6] at the beginning of the nineties (in particular, with the probabilistic solution of the nonlinear Dirichlet problem, and the connection between polar sets for superprocesses and removable singularities for the associated PDE) that the full strength of these connections became apparent. My own contribution was to observe that in several instances (and particularly for the important problem of the classification of solutions in a domain) the Brownian snake provides a more tractable and "trajectorial" approach for guessing and then proving analytic statements via probabilistic methods. Although this approach seems to be restricted to the quadratic case, and thus to equation $\Delta u = u^2$ or to the associated parabolic PDE (see however the next section), it has turned out that most of the results obtained in that special case could then be extended to more general equations by analytic or probabilistic methods.

In what follows, the word "solution" always means "nonnegative solution".

3.1 Removable singularities

Connections between superprocesses and partial differential equations are reformulated in terms of the Brownian snake in the article [49]. More than a reformulation, the Brownian snake approach significantly simplifies certain constructions. In particular exit measures, which play a crucial role, are obtained from the local times of certain reflected Brownian motions defined from the Brownian snake, whereas Dynkin's (more general) initial construction involved medial limits (later Dynkin [Dy6] circumvented this difficulty by defining a superprocess as the collection of all its exit measures).

In a smooth domain D, the maximal solution of $\Delta u = u^2$ that vanishes everywhere at the boundary except possibly on a given compact set $K \subset D$, can be expressed in terms of the probability that one of the Brownian snake paths exits D at a point of K (cf [48]). In particular, K is a removable boundary singularity if and only if it is a boundary polar set, meaning that the set of paths that exit D at a point of K is polar for the Brownian snake (the study of boundary removable singularities for more general equations of the type $\Delta u = u^p$ was initiated by Gmira and Véron [GV], who proved that singletons are removable if and only if $d \geq \frac{p+1}{p-1}$). Using this observation and potential-theoretic tools, the article [48] proves that K is not boundary polar as soon as K carries a nontrivial measure ν such that

$$\int_D G_D(x_0, y)\Big(\int_{\partial D} P_D(y,z)\,\nu(dz)\Big)^2\,dy < \infty,$$

where G_D is the Green function of D, P_D is its Poisson kernel and x_0 is a fixed point of D whose precise choice is unimportant. This condition had been conjectured by Dynkin [Dy6] to be necessary and sufficient for K not to be boundary polar. The proof of necessity turned out be more difficult and was given in [55] using mainly analytic tools inspired from Baras and Pierre [BP]. It is worth noting that the analogous result for equation $\Delta u = u^p$ was proved later by Dynkin and Kuznetsov [DK1] in the case $1 < p < 2$ and by Marcus and Véron [MV3] in the case $p > 2$.

The paper [55] also contains the proof of another conjecture of Dynkin [Dy6] about solutions of $\Delta u = u^2$ that are bounded above by a harmonic function. Any such solution can be characterized by its minimal harmonic majorant, which is itself associated via the Poisson representation with a finite measure on the boundary. Solutions of $\Delta u = u^2$ that are bounded above by a harmonic function are then in one-to-one correspondence with finite measures on the boundary that do not charge boundary polar sets (see [DK2] for generalizations of this result). The proof again makes heavy use of properties of the Brownian snake and in particular of a version of the so-called special Markov property which is derived in [55] but has other important applications in [57], [70] and [85].

3.2 The classification of solutions

After discussions with Laurent Véron in the beginning of the nineties, I became interested in the problem of the classification of solutions of $\Delta u = u^2$ in a (smooth) domain. Roughly speaking, the problem, which is analogous to the Poisson representation for harmonic functions, is to set a one-to-one correspondence between solutions and their traces on the boundary (defined in a suitable way) and to describe the set of possible traces.

In the case of a planar domain, this problem was completely solved in [62] (for the unit disk, the result was announced in [46]): There is a one-to-one correspondence between solutions of $\Delta u = u^2$ in the planar domain D, and pairs (K, ν) consisting of a compact subset K of ∂D and a Radon measure ν on the complement of K in ∂D. The pair (K, ν) is called the trace of u, and it can easily be characterized from the boundary behavior of u. Informally, K consists of points of the boundary where u blows up strongly, and ν is a generalized boundary value of u on $\partial D \backslash K$. Up to this point the statement is purely analytic, but the proof given in [62] is mainly probabilistic. In addition, it yields a probabilistic representation of the solution with trace (K, ν) which can be stated as follows:

$$u(x) = \mathbf{N}_x\Big(1 - 1_{\{\mathcal{R}^D \cap K = \emptyset\}} \exp(-\langle \nu, z_D \rangle)\Big),$$

where $\mathbf{N}_x$ is the excursion measure of the Brownian snake with initial point $x \in D$, $\mathcal{R}^D$ denotes the range of the snake in D (the union of all Brownian snake paths stopped at their first exit time from D) and z^D is the continuous density of the exit measure from D (the fact that this density exists for a smooth planar domain was established in the paper [51], which also gives a number of other properties of exit measures). Various properties of solutions can be read from this probabilistic representation in a straightforward manner.

Motivated by the results of [62], Marcus and Véron [MV2] extended the one-to-one correspondence between solutions and their traces (K, ν) to the equation $\Delta u = u^p$ in a smooth domain of $\mathbf{R}^d$, in the so-called subcritical case $d < \frac{p+1}{p-1}$ (corresponding to dimensions where singletons are not boundary polar, note that $d = 1$ or 2 are the only possibilities when $p = 2$). In the paper [81] with L. Mytnik, the probabilistic representation has also been extended to this setting for values $p \in (1, 2)$.

The supercritical case $d \geq \frac{p+1}{p-1}$ turns out to be more complicated. In that case, Marcus and Véron [MV3] on one hand, Dynkin and Kuznetsov [DK3] on the other hand proved that it is still possible to define the trace of a solution from its behavior near the boundary, and were able to characterize possible traces. However, a counterexample from [61] (this paper also contains a parabolic version of the results of [62]) shows that a solution is in general not characterized by its trace, and in fact there may exist infinitely many solutions with the same trace. In view of this difficulty, Dynkin and Kuznetsov [DK4] introduced a finer definition of the trace (called the fine trace), and conjectured that solutions should be characterized by their fine trace. This conjecture was first proved by my PhD student B. Mselati [Ms] in the particular case of equation $\Delta u = u^2$, using both analytic tools and probabilistic ingredients involving the Brownian snake. Very recently, Mselati's result was extended to the case of equation $\Delta u = u^p$ for $1 < p \leq 2$, in a series of papers by Dynkin and Kuznetsov (see the book [Dy8] for a presentation of the proof and more references). Recent progress of Marcus and Véron [MV4], [MV5] suggests that the general case should soon be solved.

3.3 Solutions with boundary blow-up

In the fifties, Keller [Ke] and Osserman [Os] proved that under certain conditions on the function ψ (which hold for $\psi(u) = u^p$, $p > 1$), equation $\Delta u = \psi(u)$ in a smooth domain will have solutions that blow up everywhere at the boundary. This leads to the following two basic questions:

(1) For a general domain D, does there still exist a solution that blows up everywhere at the boundary ?

(2) If the answer to (1) is yes, is the solution with boundary blow-up unique ?

See in particular [BM],[MV1],[Ve1],[Ve3] for an analytic discussion of these questions.

In the special case of equation $\Delta u = u^2$, a complete answer to Problem (1) is given in [64]. More precisely, this paper provides a necessary and sufficient condition (taking the form of a Wiener test) for the maximal solution in D to blow up at a given point z of ∂D. In probabilistic terms, this condition means that the Brownian snake with initial point z will immediately hit D^c (in the sense that there will exist arbitrarily small values of s such that the set $(W_s(t), 0 < t \leq \zeta_s)$ will intersect D^c). It is interesting to note that the analytic form of the main result of [64] has been extended to the equation $\Delta u = u^p$ by Labutin [Lab].

The article [67] provides a parabolic analogue of the results of [64]. From the analytic viewpoint, the problem treated there is to characterize the functions $h(t)$ such that there exists a solution of $\frac{\partial u}{\partial t} + \Delta u = u^2$ in the domain $\{(t, x) \in (0, \infty) \times \mathbf{R}^d : |x| < h(t)\}$ that blows up at the origin. In probabilistic terms this is equivalent to determining the functions $g(t)$ such that for all $t > 0$ small enough the support at time t of super-Brownian motion started at δ_0 will be contained in the ball with radius $g(t)$ centered at the origin. The answer given in [67] takes the form of an integral test analogous to the classical Kolmogorov test. Other parabolic analogues of the Wiener test have been derived by my former students Delmas and Dhersin [DD1,DD2].

As for Problem (2) above, no necessary and sufficient condition has been found until now. Still [49] has provided a sufficient condition (in terms of the Newtonian capacity of the intersection of the boundary with small balls) for the uniqueness of the solution of $\Delta u = u^2$ with boundary blow-up, which is less stringent than the criteria known by analytic methods (see however [MV1]).

4 General branching and Lévy processes

The Brownian snake construction is based on the fact that the genealogy of Feller's branching diffusion can be coded by linear Brownian motion, a fact which has other important consequences, e.g. for the construction and study of Aldous' continuum random tree. It was tempting to look for an analogous description of the genealogy of more general (critical or subcritical) continuous-state branching processes, and to use it to extend the snake construction to superprocesses with a general branching mechanism. A first step in this direction was made in [63], where we used a subordination method to construct superprocesses with a branching mechanism of a particular type (including the stable case $\psi(u) = u^p$, $1 < p < 2$) from the quadratic case. The subordination method of [63] has found applications to path properties of superprocesses with a general branching mechanism in the work of Delmas [De2], and more recently to catalytic superprocesses in the work of Dawson et al [DFM].

4.1 Lévy processes and the height process

A full analogue of the coding of the genealogy of Feller's branching diffusion via linear Brownian motion was provided in the papers [66,68] with Yves Le Jan. These papers consider a branching mechanism function ψ of the general form

$$\psi(u) = \alpha u + \beta u^2 + \int_{(0,\infty)} \pi(dr)\,(e^{-ru} - 1 + ru)$$

where $\alpha \geq 0$, $\beta \geq 0$ and π is a σ-finite measure on $(0,\infty)$ such that $\int \pi(dr)(r \wedge r^2) < \infty$. To avoid considering simpler cases, it is also assumed that at least one of the two conditions $\beta > 0$ and $\int r\,\pi(dr) = \infty$ holds.

Then a coding of the genealogy of the ψ-continuous-state branching process (or equivalently of the superprocess with branching mechanism ψ) is given by the height process H, which is defined as a functional of the Lévy process X with no negative jumps and Laplace functional ψ. More precisely, for every $s \geq 0$, H_s measures the "size" of the set $\{r \in [0,s] : X_r = \inf_{r \leq t \leq s} X_t\}$, and can be obtained through the approximation

$$H_s = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^s 1_{\{X_r \leq \inf_{r \leq t \leq s} X_t + \varepsilon\}}\, dr.$$

Of course when $\psi(u) = u^2$, H is simply reflected Brownian motion and we recover the above-mentioned coding underlying the Brownian snake construction. In general however, H is not a Markov process. Still it enjoys a number of remarkable properties some of which are presented in [66] and in the first chapter of the monograph [79] with Thomas Duquesne. One key result of [66] is the fact that H has a continuous modification if and only $\int^\infty \psi(u)^{-1} du < \infty$, a condition which is equivalent to the a.s. extinction of the underlying branching process.

The height process makes it possible to extend the Brownian snake construction to general branching mechanisms in a straightforward way. It suffices to consider a path-valued Markov process (W_s) (called the Lévy snake) such that the lifetime of W_s is H_s, and the path W_s is shortened when H_s decreases or extended when H_s increases exactly as for the Brownian snake. This construction is developed in [68] and various applications (including some discussion of the connections with partial differential equations) are presented in [79]. In particular, Chapter

IV of this monograph gives several explicit distributions for the Lévy snake, such as the law of the spatial reduced tree in a domain, consisting of the collection of all historical paths that hit the boundary (and are stopped at that moment).

4.2 Discrete and continuous trees

Another major motivation for introducing and studying the height process was to understand the limits of rescaled Galton-Watson trees. It has been known since the work of Lamperti [La] in the sixties that the only possible limits for rescaled Galton-Watson branching processes are the continuous-state branching processes. The genealogy of Galton-Watson branching processes is obviously coded by discrete trees (or forests). One expects that whenever a sequence of rescaled Galton-Watson processes converges in distribution, their genealogies also converge to the continuous branching structure associated with the limiting continuous-state branching process. The height process provides a way to state this convergence in a precise form: Under mild regularity conditions, the *contour processes* coding the genealogical forests of the Galton-Watson processes will converge in distribution in a functional sense to the height process associated with the limit. A weak formulation of this result was already given in [66], but much more precise forms can be found in Chapter II of [79]. As usual, this invariance principle allows one to recover various asymptotics for Galton-Watson trees and processes.

Note that in the case when the limiting process is Feller's branching diffusion, a version of the previous convergence was obtained earlier by Aldous [Al3] who considered the asymptotic behavior of a single Galton-Watson tree conditioned to have n vertices, in the limit $n \to \infty$ (this motivated the definition of the CRT). A generalization of Aldous' result to the case when the offspring distribution is in the domain of attraction of a stable distribution with index α has been recently given by Duquesne [Duq], and the limit of the associated contour processes is then the normalized excursion of the height process with $\psi(u) = u^\alpha$, which should be seen as coding the stable continuous tree (just as the CRT is coded by a normalized Brownian excursion).

More generally, it is not hard to define formally the ψ-continuous tree which is coded by an excursion away from 0 of the height process corresponding to the branching mechanism ψ. Quite remarkably, the properties of the height process allow the explicit calculation of a number of distributions relative to this continuous tree (see Chapter III of [79]). In particular, for a general branching mechanism ψ, one can compute the distribution of the (finite) reduced tree associated with Poissonian marks on the ψ-continuous tree. Specializing to the stable case ($\psi(u) = u^\alpha$, $1 < \alpha < 2$), a scaling argument leads to the explicit form of the finite-dimensional marginals of the stable continuous tree. These explicit calculations have found applications in the recent work of Miermont [Mi] dealing with self-similar fragmentations derived by splitting from the stable tree.

The paper [82], again with Thomas Duquesne, can be viewed as a continuation of the monograph [79]. One major originality of [82] is to use a new formalism for trees, which was inspired by the recent paper [EPW]. Precisely, the random continuous trees describing the genealogy of continuous-state branching processes are viewed in [82] as random variables taking values in the set of compact $\mathbf{R}$-trees, which is equipped with the Gromov-Hausdorff distance. Many properties of the resulting objects (called Lévy trees in [82]), which were somewhat hidden in the height process coding, take a neat form in this new formalism. This is in particular the case for the branching property which says that conditionally given the tree below level a, the subtrees originating from that level are distributed as the atoms of a Poisson measure whose intensity involves a local time measure supported on the vertices at distance a from the root. (Conversely, a simple form of this branching property, called the regenerative property, characterizes Lévy trees, as was shown by student Mathilde Weill [W1]). In addition, the formalism of $\mathbf{R}$-trees makes it possible to state and prove several new properties of Lévy trees which have numerous potential applications. The paper [82] gives explicit calculations of several fractal dimensions for Lévy trees, including their Hausdorff and packing dimension as well as that of the level sets. Further results are obtained in the subsequent paper [90] with Thomas Duquesne. This paper gives in particular the exact Hausdorff measure function for the CRT and for its level sets. Interestingly, the relevant measure function for the CRT, $h(r) = r^2 \log\log(1/r)$, is the same as the one for the range of a d-dimensional Brownian motion when $d \geq 3$. Somewhat less precise results are derived for stable trees.

5 Branching and coalescence

In the last few years I have been interested in certain coalescence models whose asymptotics can be described in terms of branching processes, or conversely which appear in the long time limit of the genealogical structure of branching populations. My interest in the area was prompted by a question of R. Durrett in 1997: Suppose that one starts independent simple random walks at each point of the lattice $\mathbf{Z}^d$ and let any two such random walks coalesce at the first time when they meet. Can one describe the asymptotic distribution as $t \to \infty$ of the set of initial points of those walks which have coalesced with the walk started at the origin before time t ? Moments calculations for the counting measure on this random set involve the consideration of various "coalescence trees" for p random walks with different starting points. Relating these trees to those that appear in the genealogical structure of Brownian excursions (see [45] or Chapter 2 of [71]) leads to the conclusion that the previous random set, suitably rescaled, converges in distribution to the support of super-Brownian motion under an appropriate Palm mesure. This result appears in the article [76] in collaboration with M. Bramson and T. Cox, where we chose to give a different presentation in terms of the voter model (which is dual to systems of coalescing random walks). Rather than using moments calculations, we proposed an approach based on the Cox-Durrett-Perkins invariance principle [CDP] for the voter model. The results of [76] have found an application in the work [Me] of my student Mathieu Merle, who derived the exact asymptotic behaviour of the probability that a distant point x eventually gets opinion 1, for the voter model started initially with a single 1 at the origin.

The article [73] in collaboration with Jean Bertoin is concerned with the Bolthausen-Sznitman coalescent, which has appeared in the probabilistic study of the Sherington-Kirkpatrick model of spin glass theory [BoS]. The Bolthausen-Sznitman coalescent is a special case of the so-called coalescents with multiple collisions introduced by Pitman [Pi], which are Markov processes taking values in the space $\mathcal{P}$ of partitions of $\mathbf{N}$. The main result of [73] states that the Bolthausen-Sznitman coalescent corresponds via time-reversal to the continuous-state branching process Y with branching mechanism $\psi(\lambda) = \lambda \log \lambda$ (Neveu's branching process): Informally, if we pick individuals labelled $1, 2, \ldots$ in the population at time t of Y, the value at time s of the coalescent is obtained by putting in the same class individuals who have the same ancestor at time $t - s$. The proof uses a representation of the genealogy of continuous-state branching processes based on the composition of subordinators (Bochner's subordination).

Up to some point, the paper [80] again with Jean Bertoin, was motivated by the desire to extend the results of [73] to more general coalescents. The class of interest there is the class of exchangeable coalescents: These are Markov processes taking values in $\mathcal{P}$, whose semigroup satisfies a natural exchangeability condition, and which appear as the possible limits for the genealogical structure of large populations with a fixed size (cf the recent work of Möhle and Sagitov [MoS], [Sa]). Pitman's coalescents with multiple collisions are special cases of exchangeable coalescents. Let a *bridge* be defined as a right-continuous process indexed by the time interval $[0, 1]$, starting from 0 and ending at 1, with nondecreasing paths and exchangeable increments. The main result of [80] establishes a one-to-one correspondence between exchangeable coalescents and flows of bridges. This correspondence should be viewed as an infinite-dimensional version of the classical Kingman representation for exchangeable partitions of $\mathbf{N}$. We also give a Poissonian construction of flows of bridges, which is much analogous to the construction of Lévy processes from compound Poisson processes, except that addition is replaced by the composition of mappings.

The analysis of flows of bridges has been pursued in the paper [83], which gives detailed information about the n-point motions of the flow. In the general case, the n-point motion solves a stochastic differential equation with jumps. In the particular case of the Kingman coalescent, one gets a simple diffusion process whose components coalesce when they meet and a similar property holds for the n-point motion of the dual flow: This leads to a very precise description of the flow of bridges associated with the Kingman coalescent. In addition, [83] discusses a remarkable Brownian flow on the circle, which also corrresponds to the Kingman coalescent in the sense that the vector of sizes of the sets of initial values that give rise to the same position at time t coincides with the vector of sizes of blocks of the coalescent at time t, and this simultaneously for all t's.

The recent paper [89], also with Jean Bertoin, discusses various limit theorems that relate coalescent processes with continuous-state branching processes. In particular, we derive a hydrodynamic limit for certain sequences of coalescents with multiple collisions, that shows that the suitably scaled empirical measure associated with sizes of blocks of the coalescent converges to a deterministic limit solving a generalized form of Smoluchovski's coagulation equation.

6 Brownian trees and scaling limits of random planar maps

This section is devoted to my current research work, which is concerned with scaling limits of random planar maps (cf the papers [85],[86] and especially [93],[94]). Recall that a planar map is a proper embedding, without edge crossings, of a connected graph in the two-dimensional sphere. Loops and multiple edges are a priori allowed. The faces of the map are the connected components of the complement of the union of edges. A planar map is rooted if it has a distinguished oriented edge called the root edge, whose origin is called the root vertex. The set of vertices is equipped with the graph distance: If a and a' are two vertices, $d_{gr}(a, a')$ is the minimal number of edges on a path from a to a'. Two rooted planar maps are called equivalent if the second one is the image of the first one under an orientation-preserving homeomorphism of the sphere, which also preserves the root edges. Planar maps are of interest in combinatorics but also in geometry (see e.g. the book [LZ] by Lando and Zvonkin) and in physics. On one hand, enumeration problems for planar maps are related to asymptotics of matrix integrals (see in particular [BIPZ]). On the other hand, planar maps have also been interpreted as models of random geometry, particularly in the setting of the theory of quantum gravity (see the book [ADJ]). The physics literature contains a number of heuristic results about the continuous object that should appear as the scaling limit of planar maps (in a way similar to the convergence of rescaled random walks to Brownian motion, it is believed that this continuous object should not depend on the class of planar maps that is considered). From the mathematical side, interesting properties of infinite random planar maps have been derived (in the case of triangulations) by Angel and Schramm [AS], [An].

My interest for planar maps arose from the pioneering paper [CS] of Chassaing and Schaeffer, which derived certain asymptotic distributions for random rooted quadrangulations (a planar map is a quadrangulation if each face has 4 adjacent edges) in terms of the Brownian snake driven by a normalized Brownian excursion. The reason for the appearance of the Brownian snake is explained by the existence of a nice bijection between rooted quadrangulations and well-labelled trees: A well-labelled tree is a (discrete) rooted ordered tree, whose vertices are assigned integer labels, in such a way that the label of the root is 1, the labels of two neighboring vertices can differ by at most 1, and all labels are positive. If one moves away from the root of the tree along one ray, labels thus evolve like a random walk (with possible jumps 1, 0 or -1) with the constraint that this random walk has to remain positive. Together with Aldous' theorem mentioned above [Al3], this suggests that the continuous analogue of a well-labelled tree should be the tree of paths generated by the one-dimensional Brownian snake (with initial point 0) driven by a normalized Brownian excursion, under the additional constraint that these paths must remain in the positive half-line. Obviously this leads to a very degenerate conditioning: The positivity constraint already gives a degenerate conditioning when one considers a single Brownian path (then the conditioning leads to the Brownian meander) and we are dealing with a continuous tree of Brownian paths started from the origin. The point is that one cannot deal separately with the branching structure and the spatial displacements, since our conditioning will also affect the law of the branching structure.

The definition of the conditioned Brownian tree is made rigorous in [85] via several different approaches that all lead to the same object. These approaches also provide a description of the conditioned snake, which is much analogous to a famous result of Verwaat linking the Brownian bridge and the Brownian excursion. Roughly speaking, the conditioned Brownian tree is obtained by re-rooting the underlying continuous genealogical tree at the vertex corresponding to the minimum of the spatial positions, and then shifting the spatial paths so that this vertex is at the origin of the line. In terms of ISE (see section 2.1 above), if we want to define one-dimensional ISE conditioned to put no mass on the negative half-line, the most natural way is to condition it to put no mass on $]-\infty,-\varepsilon[$ and then to let ε go to 0. The results of [85] show that this is equivalent to shifting the unconditioned ISE to the right, so that the left-most point of its support becomes the origin. The paper [85] also contains a number of explicit estimates and calculations. For instance the probability that ISE put no mass on $]-\infty,-\varepsilon[$ is shown to behave like $2\varepsilon^4/21$ when $\varepsilon \to 0$ (similar estimates had been obtained earlier in [AW]).

The paper [86] establishes an invariance principle for conditioned Brownian trees, which confirms, in a greater generality, a conjecture of [MM]. Precisely one considers a Galton-Watson tree whose offspring distribution is critical and has exponential moments, which is conditioned to have exactly n vertices (in the special case of the geometric distribution, this gives rise to a tree that is uniformly distributed over the set of rooted ordered trees with n vertices). One combines this branching structure with a spatial displacement which is a symmetric random walk with bounded jump size on $\mathbf{Z}$. Assuming that the root is at the origin of $\mathbf{Z}$, the spatial tree is then conditioned to remain on the positive side. The main theorem of [86] shows that the scaling limit of this conditioned discrete tree when $n \to \infty$ is given by the conditioned Brownian tree discussed in [85]. Even though this result is not a surprise, the proof involves significant technical difficulties. The asymptotics obtained in [CS] for planar quadrangulations readily follow from this invariance principle.

The approach of [86] can be extended to yield asymptotics for more general (bipartite) planar maps. This was done my student Mathilde Weill [W2] using a bijection between bipartite planar maps and certain two-type trees that had been obtained by the physicists Bouttier, Di Francesco and Guitter [BDG] (results similar to those of [W2], but in a slightly different context, had appeared in the earlier paper of Marckert and Miermont [MMi]). The results of [W2] apply in particular to $2p$-angulations, that is to planar maps where each face has $2p$ adjacent edges.

The paper [93] addresses the key problem of the convergence of rescaled random planar maps towards a limiting continuous object. For every given planar map M with n faces, one equips the set of its vertices with the graph distance scaled by the factor $n^{-1/4}$ (the fact that this is the correct scaling follows from [CS]), and the aim is to study the resulting compact metric space when the number of faces of the map tends to infinity. Assuming that the map M is chosen uniformly over the set of all rooted $2p$-angulations with n faces, [93] investigates the convergence in distribution when n tends to infinity of the associated random metric spaces, in the sense of the Gromov-Hausdorff distance between compact metric spaces. A similar question was already discussed in the paper [MM] by Marckert and Mokkadem, but there the existence of the limit was in the sense of the convergence of coding functions, and in particular the limiting behavior of distances between two points other than the root vertex was not considered. Although [93] fails to prove the desired convergence in distribution, it gives a compactness statement (so there exist sequential limits) and more importantly, it identifies the possible limiting metric spaces up to homeomorphism. Any such limiting metric space can be written as a quotient of the CRT equipped with Brownian labels, for the following equivalence relation: Two vertices of the CRT are equivalent iff they have the same label equal say to ℓ, and if when going from the first vertex to the second one around the tree one only meets vertices with label greater than or equal to ℓ. From the bijections between trees and planar maps, it is not hard to see that vertices that are equivalent for the preceding equivalence relation must be identified in the continuous limit. The difficult part of the proof in [93] is to verify that these are the only pairs of vertices that need to be identified.

Although [93] fails to identify the distance on the limiting space, it shows that this distance must be bounded above by another distance for which there is an explicit formula, and one conjectures that this upper bound is indeed an equality. Moreover, there is enough information to check that the Hausdorff dimension of the limiting space (for any sequential limit) is equal to 4, a fact that had appeared in the physics literature.

The paper [94] with Frédéric Paulin applies the results of [93] to show that any scaling limit of random planar maps (in the setting described above) is a.s. homeomorphic to the two-dimensional sphere. This result, which has been conjectured earlier (O. Schramm, personal communication), has interesting combinatorial consequences: Let M_n be uniformly distributed over the set of all $2p$-angulations with n faces, and recall that the diameter of M_n is of order $n^{1/4}$. Then with a probability close to one when $n \to \infty$, one cannot find small "bottlenecks" of size $o(n^{1/4})$ in the map M_n such that both sides of the bottleneck have a diameter which is also of order $n^{1/4}$. The proof of the main result of [94] is based on the identification of the limiting space made in [93], and on an analysis of properties of certain random geodesic laminations of the unit disk, which is of independent interest.

My current research work is oriented towards proving the uniqueness of the limit appearing in the main result of [93], and studying various geometric properties of the limiting space (both questions are indeed related). This limiting space, called the Brownian map in [MM], seems to be a fascinating object both from the probabilistic and the geometric viewpoint. The recent work of Miermont suggests that the same object also appears as the scaling limit of very general planar maps including triangulations, thus confirming the universality property found in the physics literature.

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