# Random trees

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# Outline

Trees are mathematical objects that play an important role in several areas of mathematics and other sciences:

- Combinatorics, graph theory (trees are simple examples of graphs, and can also be used to encode more complicated graphs)
- Probability theory (trees as tools to study Galton-Watson branching processes and other random processes describing the evolution of populations)
- Mathematical biology (population genetics, connections with coalescent processes)
- Theoretical computer science (trees are important cases of data structures, giving ways of storing and organizing data in a computer so that they can be used efficiently)

# Our goal

To understand the properties of “typical” large trees.
A typical tree will be generated randomly :

- Combinatorial tree: By choosing this tree uniformly at random in a certain class of trees (plane trees, Cayley trees, binary trees, etc.) of a given size.
- Galton-Watson tree: By choosing randomly the number of “children” of the root, then recursively the number of children of each child of the root, and so on.

There are many other ways of generating random trees, for instance,

- Binary search trees (used in computer science)
- Preferential attachment models (Barabási-Alberts) used to model the World Wide Web.

# What results are we aiming at?

Limiting distributions for certain characteristics of the tree, when its size tends to infinity:

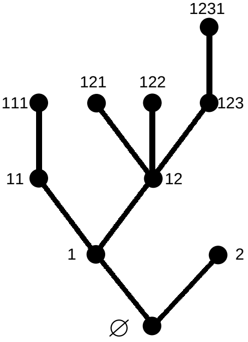
- Height, width of the tree
- Profile of distances in the tree (how many vertices at each generation of the tree)
- More refined genealogical quantities.

Often information about these asymptotic distributions can be derived by studying scaling limits:

$\Rightarrow$ Find a continuous model (continuous random tree) such that the (suitably rescaled) discrete random tree with a large size is close to this continuous model.

Strong analogy with the classical invariance theorems relating random walks to Brownian motion.

# Examples of discrete trees - Plane trees




A plane tree

$\tau = \{\varnothing, 1, 2, 11, 12, \ldots\}$

A plane tree (or rooted ordered tree) is a finite subset $\tau$ of

$$\bigcup_{n=0}^{\infty} \mathbb{N}^n$$

where $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}^0 = \{\varnothing\}$, such that:

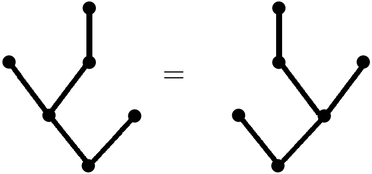
- $\varnothing \in \tau$.
- If $u = u^1 \ldots u^n \in \tau \backslash \{\varnothing\}$ then $u^1 \ldots u^{n-1} \in \tau$.
- For every $u = u^1 \ldots u^n \in \tau$, there exists $k_u(\tau) \geq 0$ such that

  $$u^1 \ldots u^n j \in \tau \text{ iff } 1 \leq j \leq k_u(\tau).$$

($k_u(\tau)$ = number of children of $u$ in $\tau$)

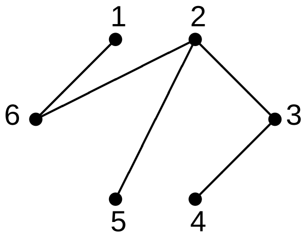
# Other discrete trees

- Unordered rooted trees




Two different plane trees
the same unordered rooted tree

- Cayley trees

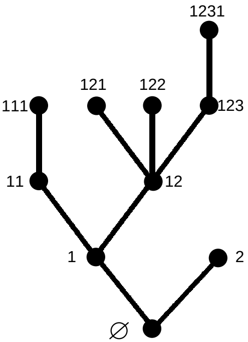



A Cayley tree on 6 vertices
(= connected graph on $\{1, 2, \ldots, 6\}$ with no loop)

- Binary trees : plane trees with 0 or 2 children for each vertex

# 1. Scaling limits of contour functions

$\mu$ = offspring distribution (probability distribution on $\{0, 1, 2, \ldots\}$)
Assume $\mu$ is critical : $\sum_{k=0}^{\infty} k\,\mu(k) = 1$ and $\mu(1) < 1$.




A $\mu$-Galton-Watson tree $\theta$ is a random plane tree such that:

- Each vertex has $k$ children with probability $\mu(k)$.
- The numbers of children of the different vertices are independent.

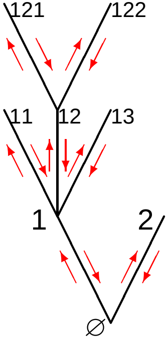
Formally, for each fixed $\tau \in \mathbb{T} :=$ {plane trees},

$$\mathbb{P}(\theta = \tau) = \prod_{u \in \tau} \mu(k_u(\tau)).$$

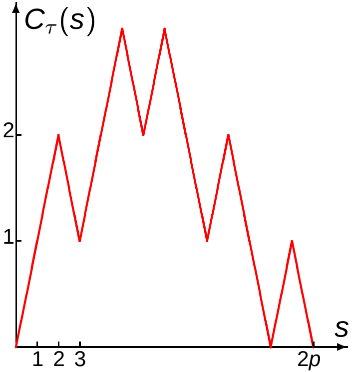
# Important special cases

- Geometric distribution
$$\mu(k) = 2^{-k-1}$$
Then, if $|\tau|$ = number of edges of $\tau$,
$$\mathbb{P}(\theta = \tau) = 2^{-2|\tau|-1}.$$
Consequence: The conditional distribution of $\theta$ given $|\theta| = p$ is uniform over $\{$plane trees with $p$ edges$\}$.
- Poisson distribution
$$\mu(k) = \frac{e^{-1}}{k!}$$
The conditional distribution of $\theta$ given $|\theta| = p$ is uniform over $\{$Cayley trees on $p+1$ vertices$\}$.
(Needs to view a plane tree as a Cayley tree by "forgetting" the order and randomly assigning labels $1, 2, \ldots$ to vertices)

# Coding trees by contour functions




A plane tree $\tau$ with
$p$ edges (or $p+1$ vertices)




and its contour function
$(C_\tau(s), 0 \leq s \leq 2p)$

A plane tree can be coded by its contour function (or Dyck path in combinatorics)

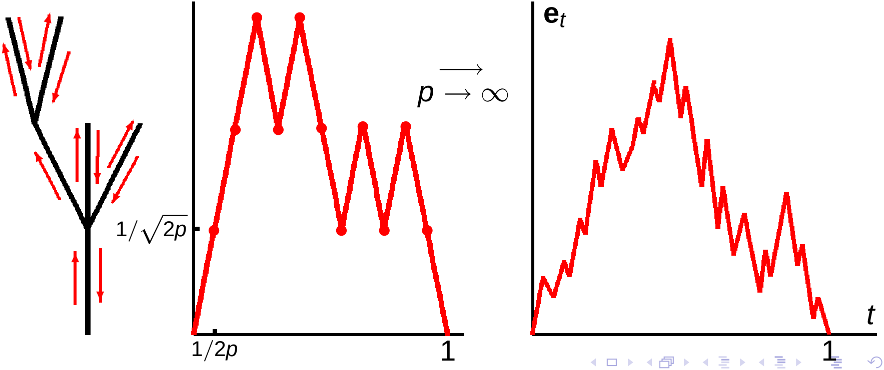
# Aldous' theorem (finite variance case)

## Theorem (Aldous)

*Let $\theta_p$ be a $\mu$-Galton-Watson tree conditioned to have $p$ edges. Then*

$$\left(\frac{1}{\sqrt{2p}}\,C_{\theta_p}(2pt)\right)_{0\le t\le 1} \xrightarrow[p\to\infty]{\text{(d)}} \left(\frac{\sqrt{2}}{\sigma}\,\mathbf{e}_t\right)_{0\le t\le 1}$$

*where $\sigma^2 = \mathrm{var}(\mu)$ and $(\mathbf{e}_t)_{0\le t\le 1}$ is a normalized Brownian excursion.*



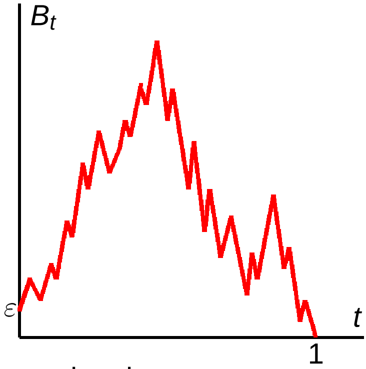
# The normalized Brownian excursion

To construct a normalized Brownian excursion $(\mathbf{e}_t)_{0\leq t\leq 1}$:

- Consider a Brownian motion $(B_t)_{t\geq 0}$ with $B_0 = \varepsilon$.
- Condition on the event

$$\inf\{t \geq 0 : B_t = 0\} = 1$$

- Let $\varepsilon \to 0$.




More intrinsic approaches via Itô's excursion theory.

# An application of Aldous' theorem

Let $h(\theta_p) =$ height of $\theta_p$ ($=$ maximum of contour function). Then

$$\mathbb{P}[h(\theta_p) \geq x\sqrt{p}] \underset{p\to\infty}{\longrightarrow} \mathbb{P}\Big[\max_{0\leq t\leq 1} \mathbf{e}_t \geq \frac{\sigma x}{2}\Big]$$

The RHS is known in the form of a series (Chung 1976)

$$\mathbb{P}\Big[\max_{0\leq t\leq 1} \mathbf{e}_t \geq x\Big] = 2\sum_{k=1}^{\infty}(4k^2x^2 - 1)\exp(-2k^2x^2).$$

Special case $\mu(k) = 2^{-k-1}$ : asymptotic proportion of those trees with $p$ edges whose height is greater than $x\sqrt{p}$.

cf results from theoretical computer science, Flajolet-Odlyzko (1982)

**General idea:**

The limit theorem for the contour gives the "asymptotic shape" of the tree, from which one can derive – or guess – many asymptotics for specific functionals of the tree.

# 2. The CRT and Gromov-Hausdorff convergence

Aldous' theorem suggests that

There exists a continuous random tree which is the universal limit of (rescaled) Galton-Watson trees conditioned to have $n$ edges, and whose "contour function" is the Brownian excursion.

In other words we want to make sense of the convergence

$$\frac{\sigma}{2\sqrt{p}}\,\theta_p \xrightarrow[p\to\infty]{\text{(d)}} \mathcal{T}_{\mathbf{e}}$$

For this, we need:

- to say what kind of an object the limit is (a random real tree)
- to explain how a real tree can be coded by a function (here by **e**)
- to say in which sense the convergence holds (in the Gromov-Hausdorff sense)

# The Gromov-Hausdorff distance

**The Hausdorff distance.** $K_1$, $K_2$ compact subsets of a metric space

$$d_{\mathrm{Haus}}(K_1,K_2) = \inf\{\varepsilon > 0 : K_1 \subset U_\varepsilon(K_2) \text{ and } K_2 \subset U_\varepsilon(K_1)\}$$

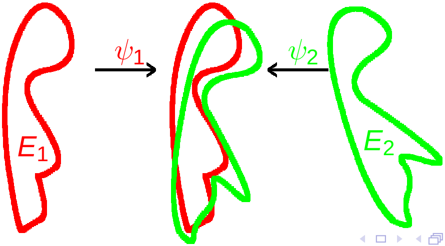
($U_\varepsilon(K_1)$ is the $\varepsilon$-enlargement of $K_1$)

## Definition (Gromov-Hausdorff distance)

If $(E_1,d_1)$ and $(E_2,d_2)$ are two compact metric spaces,

$$d_{\mathrm{GH}}(E_1,E_2) = \inf\{d_{\mathrm{Haus}}(\psi_1(E_1),\psi_2(E_2))\}$$

the infimum is over all isometric embeddings $\psi_1 : E_1 \to E$ and $\psi_2 : E_2 \to E$ of $E_1$ and $E_2$ into the same metric space $E$.



# Gromov-Hausdorff convergence of rescaled trees

**Fact**

*If* $\mathbb{K} = \{$*isometry classes of compact metric spaces*$\}$, *then*

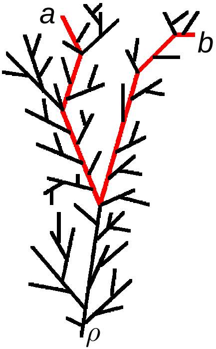
$$(\mathbb{K}, d_{\mathrm{GH}}) \text{ is a separable complete metric space (Polish space)}$$

Equip $\theta_p$ (the Galton-Watson tree conditioned to have $p$ edges) with the graph distance $d_{\mathrm{gr}}$ : $d_{\mathrm{gr}}(v, v')$ is the minimal number of edges on a path from $v$ to $v'$.
$\longrightarrow$ It makes sense to study the convergence of

$$(\theta_p, \frac{1}{\sqrt{p}} d_{\mathrm{gr}})$$

as random variables with values in $\mathbb{K}$.

# The notion of a real tree




**Definition**

A real tree is a (compact) metric space $\mathcal{T}$ such that:

- any two points $a, b \in \mathcal{T}$ are joined by a unique arc
- this arc is isometric to a line segment

It is a rooted real tree if there is a distinguished point $\rho$, called the root.

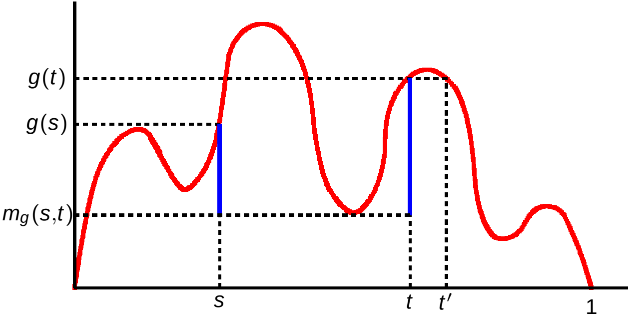
**Remark.** A real tree can have

- infinitely many branching points
- (uncountably) infinitely many leaves

**Fact.** The coding of plane trees by contour functions can be extended to real trees.

# The real tree coded by a function $g$

$g : [0,1] \longrightarrow [0,\infty)$
continuous,
$g(0) = g(1) = 0$




$$m_g(s,t) = m_g(t,s) = \min_{s \le r \le t} g(r)$$
$$d_g(s,t) = g(s) + g(t) - 2m_g(s,t) \qquad t \sim t' \text{ iff } d_g(t,t') = 0$$

**Proposition (Duquesne-LG)**

*$\mathcal{T}_g := [0,1]/\sim$ equipped with $d_g$ is a real tree, called the tree coded by $g$. It is rooted at $\rho = 0$.*

**Remark.** $\mathcal{T}_g$ inherits a “lexicographical order” from the coding.

# Aldous' theorem revisited

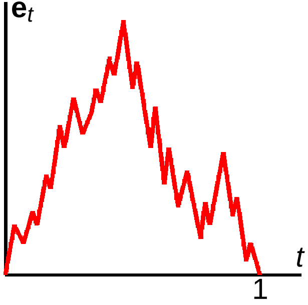
## Theorem

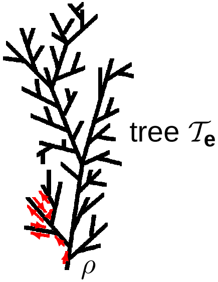
*If $\theta_p$ is a $\mu$-Galton-Watson tree conditioned to have p edges,*

$$(\theta_p, \frac{\sigma}{2\sqrt{p}} d_{\mathrm{gr}}) \xrightarrow[p\to\infty]{\mathrm{(d)}} (\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$$

*in the Gromov-Hausdorff sense.*

The limit $(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$ is the (random) real tree coded by a Brownian excursion $\mathbf{e}$. It is called the CRT (Continuum Random Tree).



# Application to combinatorial trees

By choosing appropriately the offspring distribution, one obtains that the CRT is the scaling limit of

- plane trees
- (ordered) binary trees
- Cayley trees

with size $p$, with the same rescaling $1/\sqrt{p}$ (but different constants).

It is also true that the CRT is the scaling limit of

- unordered binary trees

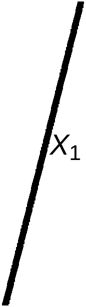
but this is much harder to prove (no connection with Galton-Watson trees), see Marckert-Miermont 2009.

# The stick-breaking construction of the CRT (Aldous)

Consider a sequence $X_1, X_2, \ldots$ of positive random variables such that, for every $n \geq 1$, the vector $(X_1, X_2, \ldots, X_n)$ has density

$$a_n\, x_1(x_1 + x_2) \cdots (x_1 + \cdots + x_n) \exp(-2(x_1 + \cdots + x_n)^2)$$

Then "break" the positive half-line into segments of lengths $X_1, X_2, \ldots$ and paste them together to form a tree :




- The first branch has length $X_1$
- The second branch has length $X_2$ and is attached at a point uniform over the first branch
- The third branch has length $X_3$ and is attached at a point uniform over the union of the first two branches
- And so on

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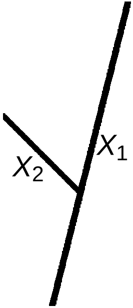


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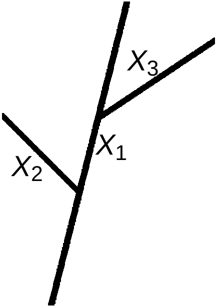


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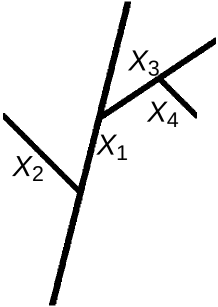


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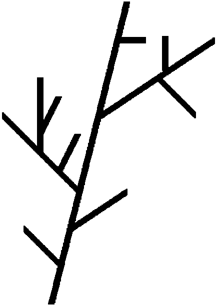

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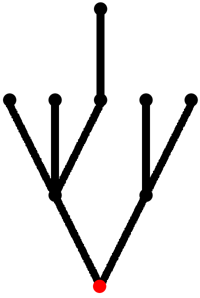
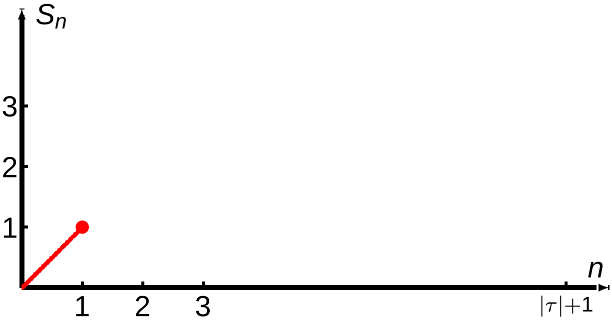
Finally take the completion to get the CRT

# 3. The connection with random walk




With a plane tree $\tau$ associate a discrete path $(S_n)_{0\leq n\leq|\tau|+1}$ :

- Enumerate vertices of $\tau$ in lexicographical order:

$$v_0 = \varnothing, v_1 = 1, v_2, \ldots, v_{|\tau|}.$$

- Define $S_0 = 0$ and, for $0 \leq n \leq |\tau|$,

$$S_{n+1} = S_n + k_{v_n}(\tau) - 1 \quad (k_{v_n}(\tau) \text{ number of children of } v_n \text{ in } \tau)$$

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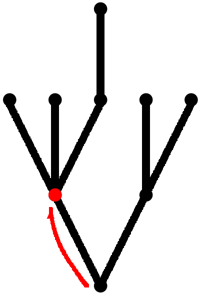


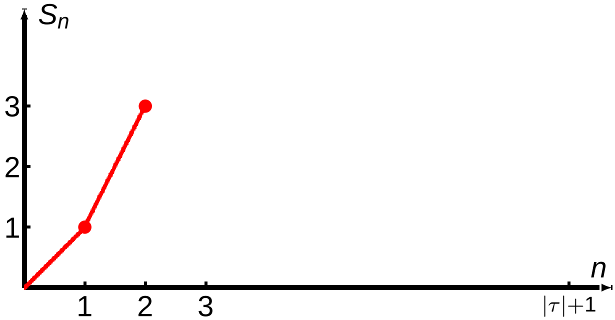


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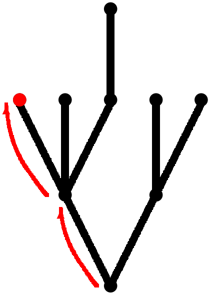


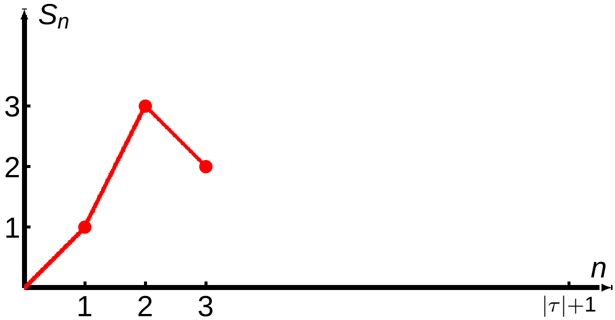


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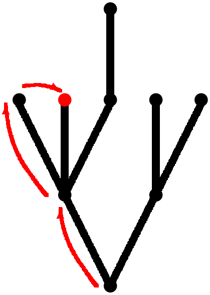


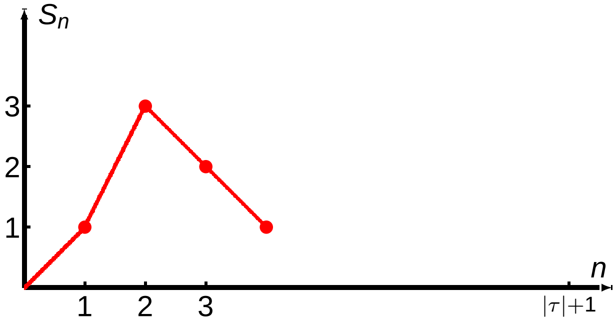


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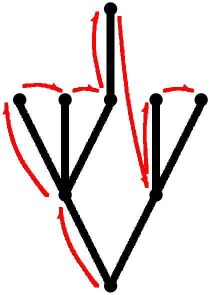


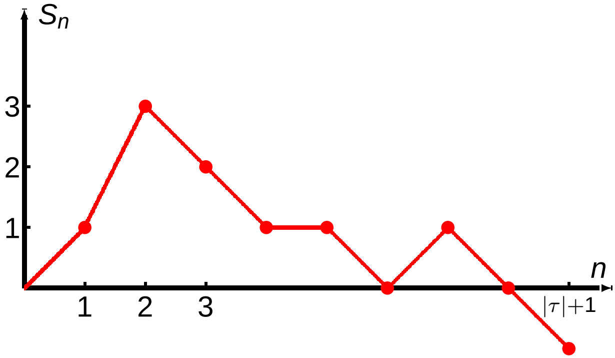


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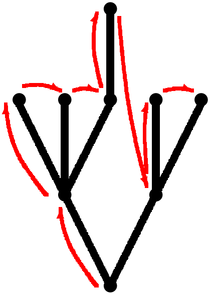
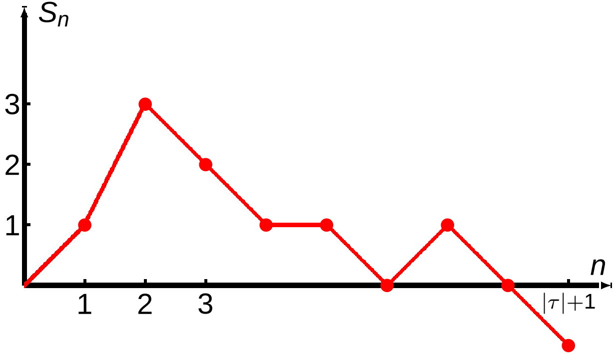
# The random walk associated with a tree




Recall $S_{n+1} = S_n + k_{v_n}(\tau) - 1$ $\quad$ ($k_{v_n}(\tau)$ number of children of $v_n$ in $\tau$)

## Fact

*If $\tau = \theta$ is a Galton-Watson tree with offspring distribution $\mu$, $(S_n)_{0\leq n\leq|\tau|+1}$ is a random walk with jump distribution $\nu(k) = \mu(k+1)$, $k = -1, 0, 1, \ldots$ stopped at its first hitting time of $-1$.*

# Sketch of proof

Recall $S_{n+1} = S_n + k_{v_n}(\tau) - 1 \quad (k_{v_n}(\tau)$ number of children of $v_n$ in $\tau)$

- For a $\mu$-Galton-Watson tree, the r.v. $k_{v_n}(\tau) - 1$ are i.i.d. with distribution $\nu(k) = \mu(k+1)$.
- $$\begin{aligned} S_{|\tau|+1} = \sum_{0\le n\le|\tau|} (k_{v_n}(\tau) - 1) &= \Big(\sum_{0\le n\le|\tau|} k_{v_n}(\tau)\Big) - |\tau| - 1 \\ &= |\tau| - |\tau| - 1 \\ &= -1 \end{aligned}$$
- For $1 \le m \le |\tau|$,

  $$S_m = \sum_{0\le n\le m-1} (k_{v_n}(\tau) - 1) = \sum_{0\le n\le m-1} k_{v_n}(\tau) - m \ge 0$$

  because among all individuals counted in $\sum_{0\le n\le m-1} k_{v_n}(\tau)$, the vertices $v_1, v_2, \ldots, v_m$ all appear.

# An application to the total progeny

**Corollary**

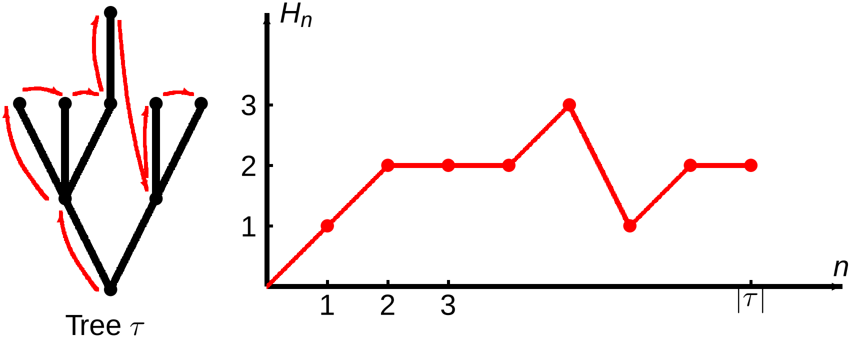
*The total progeny of a $\mu$-Galton-Watson tree has the same distribution as the first hitting time of $-1$ by a random walk with jump distribution $\nu(k) = \mu(k+1)$.*

**Proof.** Just use the identity $|\tau| + 1 = \min\{n \geq 0 : S_n = -1\}$.

Cf Harris (1952), Dwass (1970), etc.

# Proof of Aldous' theorem I

Needs the "height function" associated with a plane tree $\tau$.



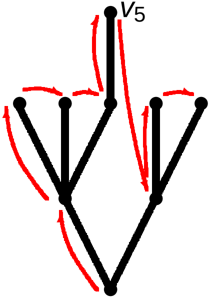

If $\tau = \{v_0, v_1, \ldots, v_{|\tau|}\}$ (in lexicogr. order), $H_n = |v_n|$ (generation of $v_n$).

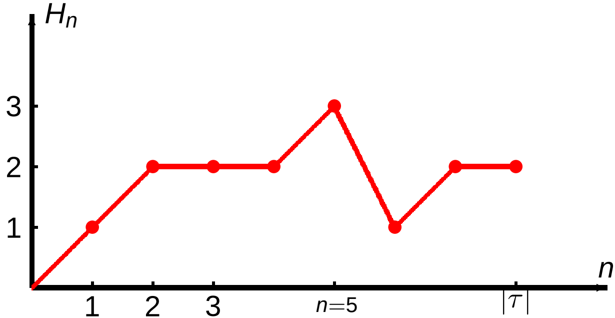
**Lemma (Key formula)**

*If $S$ is the random walk associated with $\tau$, then for $0 \leq n \leq |\tau|$,*

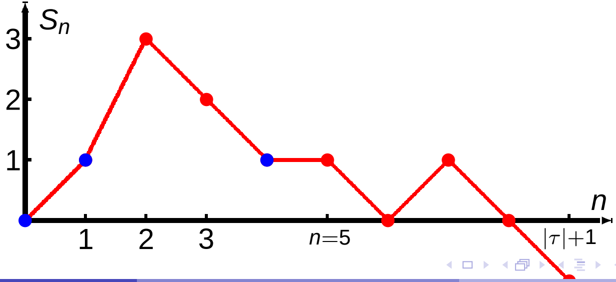
$$H_n = \#\{j \in \{0, 1, \ldots, n-1\} : S_j = \min_{j \leq i \leq n} S_i\}.$$

The key formula: $H_n = \#\{j \in \{0,1,\ldots,n-1\} : S_j = \min_{j \le i \le n} S_i\}$.




For $n = 5$, 3 values of $j \le n-1$ such that $S_j = \min_{j \le i \le n} S_i$.



# Proof of Aldous' theorem II

From the key formula,

$$H_n = \#\{j \in \{0, 1, \ldots, n-1\} : S_j = \min_{j \leq i \leq n} S_i\}$$

one can deduce that, for a Galton-Watson tree $\tau$ conditioned to have $|\tau| = p$,

$$H_n \approx \frac{2}{\sigma^2} S_n^{(p)} \quad , 0 \leq n \leq p$$

where $S^{(p)}$ is distributed as $S$ conditioned to hit $-1$ at time $p+1$.
By a conditional version of Donsker's theorem,

$$\left(\frac{1}{\sigma\sqrt{p}} S_{[pt]}^{(p)}\right)_{0 \leq t \leq 1} \xrightarrow[p \to \infty]{\text{(d)}} (\mathbf{e}_t)_{0 \leq t \leq 1}.$$

Finally, argue that

$$C_{\theta_p}(2pt) \approx H_{[pt]} \approx \frac{2}{\sigma^2} S_{[pt]}^{(p)}.$$

# 4. More general offspring distributions

What happens if we remove the assumption that $\mu$ has finite variance ?

$\longrightarrow$ Get more general continuous random trees

$\longrightarrow$ Coded by "excursions" which are no longer Brownian

## Assumption $(A_\alpha)$

*The offspring distribution* $\mu$ *has mean* 1 *and is such that*

$$\mu(k) \underset{k\to\infty}{\sim} c\, k^{-1-\alpha}$$

*for some* $\alpha \in (1,2)$ *and* $c > 0$

In particular, $\mu$ is in the domain of attraction of a stable distribution with index $\alpha$.

# A "stable" version of Aldous' theorem

## Theorem (Duquesne)

*Under Assumption $(A_\alpha)$, let $(C^p(n))_{0\leq n\leq 2p}$ be the contour function of a $\mu$-Galton-Watson tree conditioned to have p edges. Then,*

$$\left(\frac{1}{p^{1-1/\alpha}}\,C^p(2pt)\right)_{0\leq t\leq 1} \underset{p\to\infty}{\overset{\text{(d)}}{\longrightarrow}} (c\,\mathbf{e}^\alpha_t)_{0\leq t\leq 1},$$

*where $\mathbf{e}^\alpha$ is defined in terms of the normalized excursion $X^\alpha$ of a stable process with index $\alpha$ and nonnegative jumps:*

$$\mathbf{e}^\alpha_t = \text{“}\mathit{measure}\text{”}\{s\in[0,t] : X^\alpha_s = \inf_{s\leq r\leq t} X^\alpha_r\}.$$

The last formula is to be understood in a "local time" sense:

$$\mathbf{e}^\alpha_t = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t ds\,\mathbf{1}_{\{X^\alpha_s < \inf_{s\leq r\leq t} X^\alpha_r + \varepsilon\}}.$$

# Why the process $\mathbf{e}^\alpha$ ?

As in the proof of Aldous' theorem,

$$C^p_{2pt} \approx H^p_{[pt]}$$

where (key formula), for $0 \leq n \leq p$,

$$H^p_n = \#\{j \in \{0,1,\ldots,n-1\} : S^p_j = \min_{j \leq i \leq n} S^p_i\} \tag{1}$$

and $S^p$ is distributed as a random walk with jump distribution $\nu(k) = \mu(k+1)$, conditioned to hit $-1$ at time $p+1$.
By the assumption on $\mu$,

$$\Big(n^{-1/\alpha}\, S^p_{[pt]}\Big)_{0 \leq t \leq 1} \xrightarrow[p\to\infty]{\text{(d)}} \Big(c\, X^\alpha_t\Big)_{0 \leq t \leq 1}.$$

Then pass to the limit in (1): The (suitable rescaled) RHS of (1) converges to

$$\text{“}measure\text{”}\{s \in [0,t] : X^\alpha_s = \inf_{s \leq r \leq t} X^\alpha_r\}.$$

# Convergence of trees

## Corollary

*Under Assumption $(A_\alpha)$, let $\theta_p$ be a $\mu$-Galton-Watson tree conditioned to have p edges. Then, if $d_{gr}$ denotes the graph distance on $\theta_p$,*

$$(\theta_p, \frac{1}{p^{1-1/\alpha}} d_{\rm gr}) \xrightarrow[p\to\infty]{\rm (d)} (\mathcal{T}_{\mathbf{e}^\alpha}, c\, d_{\mathbf{e}^\alpha})$$

*in the Gromov-Hausdorff sense.*
*Here $(\mathcal{T}_{\mathbf{e}^\alpha}, d_{\mathbf{e}^\alpha})$ is the tree coded by the "stable excursion" $\mathbf{e}^\alpha$.*

The random tree $(\mathcal{T}_{\mathbf{e}^\alpha}, d_{\mathbf{e}^\alpha})$ is called the stable tree with index $\alpha$.

Can investigate probabilistic and fractal properties of stable trees in detail (Duquesne, LG). For instance,

$$\dim \mathcal{T}_{\mathbf{e}^\alpha} = \frac{\alpha}{\alpha - 1}$$

and level sets of $\mathcal{T}_{\mathbf{e}^\alpha}$ have dimension $\frac{1}{\alpha-1}$.

# Extensions

For any " branching mechanism function" $\psi$ of the form

$$\psi(u) = a\,u + b\,u^2 + \int_{(0,\infty)} \pi(dr)\,(e^{-ru} - 1 + ru)$$

where $a, b \geq 0$ and $\int_{(0,\infty)} \pi(dr)\,(r \wedge r^2) < \infty$,

one can define a $\psi$-Lévy tree, which is a continuous random tree:

- if $\psi(u) = u^2$, this is Aldous' CRT
- if $\psi(u) = u^\alpha$, this is the stable tree with index $\alpha$.

Lévy trees are

- closely related to the Lévy process with Laplace exponent $\psi$
- the possible scaling limits of (sub)critical Galton-Watson trees
- characterized by a branching property analogous to the discrete case (Weill).