Spatial branching processes: Superprocesses and snakes

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Outline

Spatial branching processes model the evolution of populations where individuals both

- reproduce themselves according to some branching distribution
- move in space according to a certain Markov process (e.g. Brownian motion)

Superprocesses (also called measure-valued branching processes) occur in the limit where:

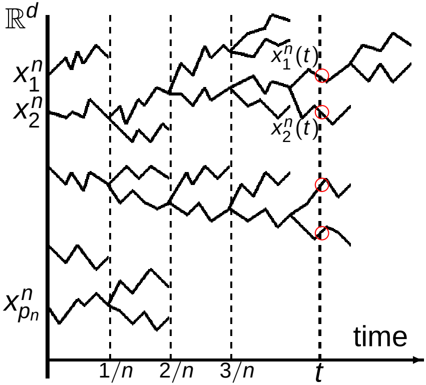
- the population is very large (but each individual has a very small "mass")
- the mean time between two branching events is very small

Related model: Fleming-Viot processes used in population genetics (spatial position = genetic type of the individual)

Why study spatial branching processes, and in particular superprocesses ?

- These objects appear in the asymptotics of many other important probabilistic models:
  - interacting particle systems: voter model, contact process, etc. (Cox, Durrett, Perkins, ...)
  - models from statistical physics: lattice trees, oriented percolation, etc. (Slade, van der Hofstad, Hara, ...)
  - models from mathematical biology, where there is competition between several species (e.g. Lotka-Volterra models)
- Connections with the theory of stochastic partial differential equations.
- Connections with partial differential equations (probabilistic approach to an important class of nonlinear PDEs, cf Dynkin, Kuznetsov, LG, ...)
- Description of asymptotics in models of combinatorics (cf Lecture 3).

1. Branching particle systems and superprocesses



At time $t = 0$, p_n particles located at $x_1^n, x_2^n, \ldots, x_{p_n}^n \in \mathbb{R}^d$.

Particles independently

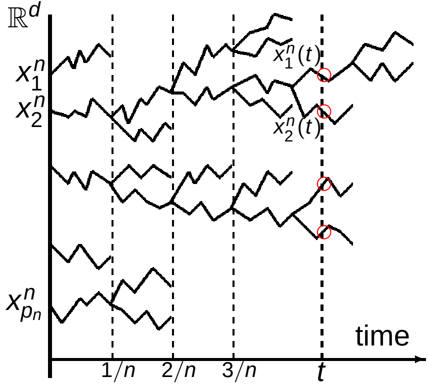
- move in space according to Brownian motion
- die at times $1/n, 2/n, 3/n, \ldots$
- when a particle dies, it gives rise to children according to the offspring distribution γ

For every $t \geq 0$, $x_1^n(t), x_2^n(t), \ldots$ positions of particles alive at time t,

$$Z_t^n = \frac{1}{n} \sum_i \delta_{x_i^n(t)}$$

rescaled sum of Dirac masses at particles alive at time t.

Now let $n \to \infty$ ...



Recall $Z_t^n = \frac{1}{n} \sum_i \delta_{x_i^n(t)}$.

$M_f(\mathbb{R}^d) = \{$finite measures on $\mathbb{R}^d\}$.

Assumptions

- Convergence of initial values:

$$Z_0^n = \frac{1}{n} \sum_{i=1}^{p_n} \delta_{x_i^n} \underset{n\to\infty}{\longrightarrow} \mu \in M_f(\mathbb{R}^d)$$

- The offspring distribution γ has mean 1 and finite variance ρ^2.

Theorem (Watanabe)

Then,

$$(Z_t^n)_{t\geq 0} \xrightarrow[n\to\infty]{\text{(d)}} (Z_t)_{t\geq 0}$$

where $(Z_t)_{t\geq 0}$ is a Markov process with values in $M_f(\mathbb{R}^d)$, called super-Brownian motion.

$Z_t \in M_f(\mathbb{R}^d)$ is supported on "a cloud of particles alive at time t"

Characterizing the law of super-Brownian motion

Notation: $C_b^+(\mathbb{R}^d) = \{$bounded continuous functions $g : \mathbb{R}^d \longrightarrow \mathbb{R}_+\}$
$\langle \mu, g \rangle = \int g \, d\mu$, for $\mu \in M_f(\mathbb{R}^d)$ and $g \in C_b^+(\mathbb{R}^d)$.

Then, for every $g \in C_b^+(\mathbb{R}^d)$,

$$E\Big[\exp(-\langle Z_t, g\rangle \,\Big|\, Z_0 = \mu\Big] = \exp -\langle \mu, u_t \rangle$$

where $(u_t(x), t \geq 0, x \in \mathbb{R}^d)$ is the unique nonnegative solution of

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u - \frac{\rho^2}{2} u^2$$
$$u_0 = g$$

The function $\psi(u) = \frac{\rho^2}{2} u^2$ is called the branching mechanism of Z.

Remark. The law of Z depends on the offspring distribution μ of the approximating system only through the parameter ρ^2.

Other characterizations via martingale problems, more appropriate for models with interactions.

Path properties of super-Brownian motion (Dawson, Perkins, Shiga, ...)

$d = 1$: Then Z_t has a density with respect to Lebesgue measure

$$Z_t(dx) = Y_t(x)\,dx$$

and this density solves the SPDE

$$dY_t = \frac{1}{2}\Delta Y_t\,dt + c\,\sqrt{Y_t}\,dW_t$$

where W is space-time white noise.

$d \geq 2$: Then Z_t is almost surely supported on a set of zero Lebesgue measure, and uniformly spread on its support, in the sense of Hausdorff measure.

2. The Brownian snake approach

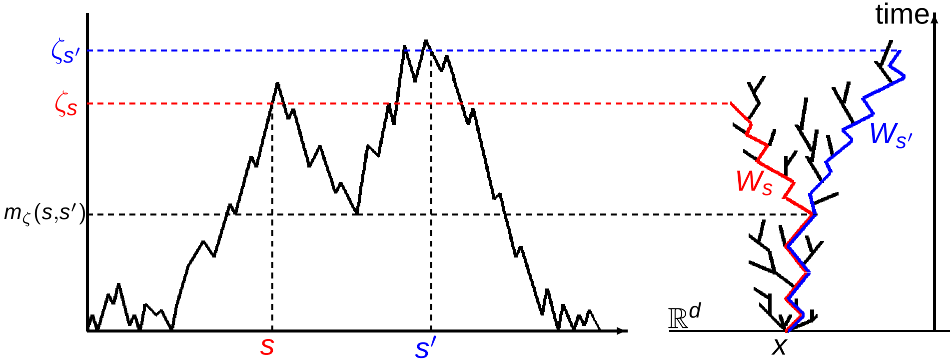
Idea. One can generate the individual particle paths (the "historical paths") of a super-Brownian motion, as the values of a path-valued Markov process called the Brownian snake.

$\longrightarrow$ This construction is closely related to the fact that the underlying genealogical structure of a super-Brownian motion can be coded by Brownian excursions (in the same sense as the CRT is coded by a normalized Brownian excursion, cf Lecture 1).

The construction of the Brownian snake. Fix $x \in \mathbb{R}^d$ and set

$$\begin{aligned}\mathcal{W}_x &= \{\text{finite paths started from } x\} \\ &= \{w : [0, \zeta_w] \longrightarrow \mathbb{R}^d \text{ continuous }, w(0) = x\}.\end{aligned}$$

If $w \in \mathcal{W}_x$, ζ_w is called the lifetime of w.
The terminal point or tip of w is $\widehat{w} = w(\zeta_w)$.



The Brownian snake $(W_s)_{s\geq 0}$ is the Markov process with values in $\mathcal{W}_x = \{\text{finite paths started at } x\}$ such that:

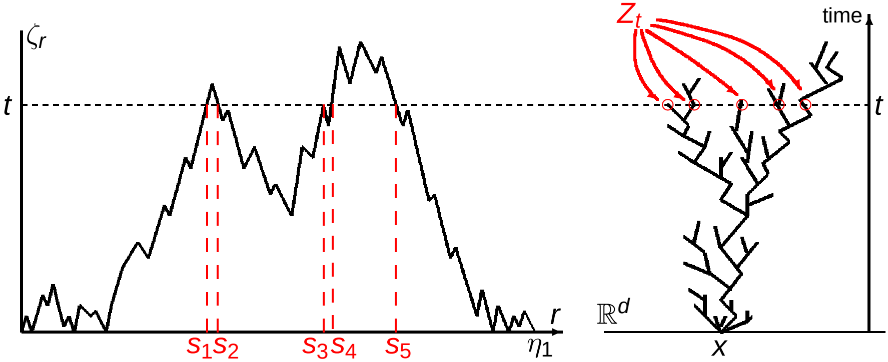
- The lifetime $\zeta_s := \zeta_{W_s}$ is a linear Brownian motion reflected at 0.
- Conditionally on $(\zeta_s)_{s\geq 0}$, $(W_s)_{s\geq 0}$ is time-inhomogeneous Markov, and if $s < s'$,
  - $W_{s'}(t) = W_s(t)$ for every $0 \leq t \leq m_\zeta(s,s') := \min_{[s,s']} \zeta_r$
  - $(W_{s'}(m_\zeta(s,s')+t) - W_{s'}(m_\zeta(s,s')))_{0\leq t\leq \zeta_{s'}-m_\zeta(s,s')}$ is distributed as a Brownian motion in $\mathbb{R}^d$ independent of W_s.

Heuristic description of the Brownian snake $(W_s)_{s\geq 0}$

- For every $s \geq 0$, W_s is a random path in $\mathbb{R}^d$ started at x, with a random lifetime ζ_s.
- The lifetime ζ_s evolves like linear Brownian motion reflected at 0 (a lifetime cannot be negative !)
- When ζ_s decreases, the path W_s is shortened from its tip.
- When ζ_s increases, the path W_s is extended by adding "little pieces" of d-dimensional Brownian motion at its tip.

Why consider such a process ?

In particular, because of its connection with super-Brownian motion.



For every $t \geq 0$, let $L^t = (L^t_s)_{s\geq 0}$ be the local time at level t of $(\zeta_s)_{s\geq 0}$ (the measure $L^t(ds)$ is supported on $\{s \geq 0 : \zeta_s = t\}$).

Theorem

Let $\eta_1 := \inf\{s \geq 0 : L^0_s = 1\}$. *The measure-valued process* $(Z_t)_{t\geq 0}$

$$\langle Z_t, g\rangle = \int_0^{\eta_1} L^t(ds)\, g(W_s(t))$$

is a super-Brownian motion started from δ_x.

Applications

Many results about super-Brownian motion can be stated equivalently and proved more easily in terms of the Brownian snake.
This is true in particular for path properties:

- The values W_s of the Brownian snake are Hölder continuous with exponent $\frac{1}{2} - \varepsilon$. The topological support $supp(Z_t)$ of super-BM cannot move faster: for every $t \geq 0$, $0 < r < r_0(\omega)$,

$$supp(Z_{t+r}) \subset U_{r^{1/2-\varepsilon}}(supp(Z_t))$$

  where $U_\delta(K)$ denotes the δ-enlargement of K.
- If $\widehat{W}_s = W_s(\zeta_s)$ denotes the tip of the path W_s, the map $s \to \widehat{W}_s$ is Hölder continuous with exponent $\frac{1}{4} - \varepsilon$. From the snake approach,

$$\{\widehat{W}_s : 0 \leq s \leq \eta_1\} = \overline{\bigcup_{t \geq 0} supp(Z_t)} =: \mathcal{R}$$

  is the range of Z, that is the set of points touched by the cloud of particles. It follows that: $dim(\mathcal{R}) = 4 \wedge d$

More precise results: Perkins, Dawson, Iscoe, LG, etc.

3. Connections with partial differential equations

Probabilistic potential theory: Classical connections between Brownian motion and the Laplace equation $\Delta u = 0$ or the heat equation $\frac{\partial u}{\partial t} = \Delta u$ (Doob, Kakutani, etc.)

In our setting: Similar remarkable connections between super-Brownian motion or the Brownian snake and semilinear equations of the form $\Delta u = u^\gamma$ or $\frac{\partial u}{\partial t} = \Delta u - u^\gamma$ (Dynkin, Kuznetsov, LG, etc.)

Why study these connections ? Because they

- Allow explicit analytic calculations of probabilistic quantities related to the Brownian snake and super-BM
- Give a probabilistic representation of solutions of PDE that has led to new analytic results

For simple statements of the connections with PDE, needs excursion measures.

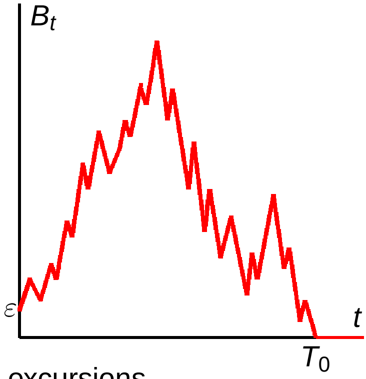
The Itô excursion measure

Consider a Brownian motion $(B_t)_{t\geq 0}$ with $B_0 = \varepsilon$.

Set $T_0 = \inf\{t \geq 0 : B_t = 0\}$.

Let P_ε be the law of $(B_{t\wedge T_0})_{t\geq 0}$
Then,

$$\varepsilon^{-1} P_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \Pi$$

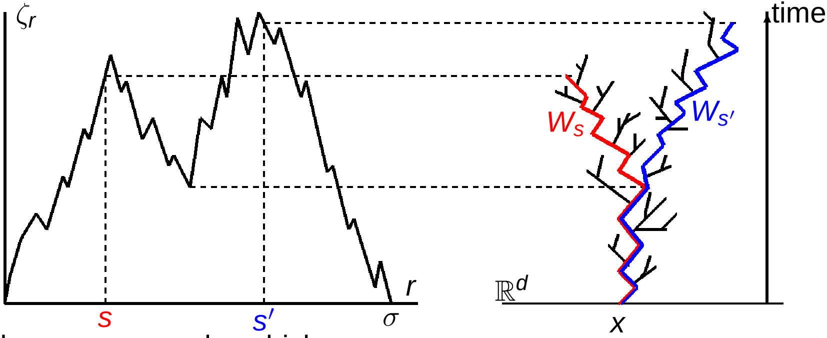


Π is a σ-finite measure on the set of excursions

$$E = \{e : [0,\infty) \longrightarrow [0,\infty) \text{ continuous}, \\ \exists\sigma(e) > 0, e(s) > 0 \text{ iff } 0 < s < \sigma(e)\}$$

Π is called the Itô excursion measure.
(**Note**: $\Pi(\cdot \mid \sigma = 1)$ is the law of the normalized excursion, cf Lect.1)

The excursion measure of the Brownian snake



$\mathbb{N}_x$ is the measure under which:

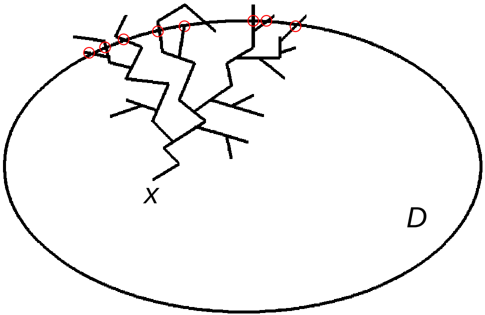
- $(\zeta_s)_{s\geq 0}$ is distributed according to $\Pi(de)$ (the Itô measure)
- Conditionally given $(\zeta_s)_{s\geq 0}$, $(W_s)_{s\geq 0}$ is distributed as the snake driven by $(\zeta_s)_{s\geq 0}$, with initial point x : W_s has lifetime ζ_s, and if $s < s'$, the conditional law of $W_{s'}$ given W_s is as described before.

Under $\mathbb{N}_x$, the paths W_s, $s \in [0, \sigma]$ form a "tree of Brownian paths" with initial point x.

Warning. $\mathbb{N}_x$ is an infinite measure (because so is Π).

Exit points from a domain

Classical theory of relations between Brownian motion and PDEs : A key role is played by the first exit point of Brownian motion from a domain D.
Here one constructs a measure supported on the set of exit points of the paths W_s from D (assuming that the initial point $x \in D$)



For every finite path $w \in \mathcal{W}_x$, set

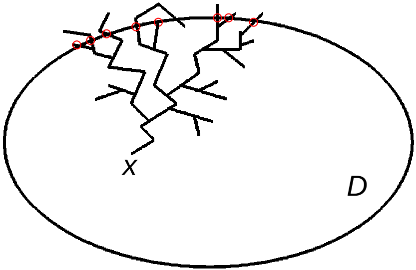
$$\tau(w) = \inf\{t \geq 0 : w(t) \notin D\}$$

and

$$\mathcal{E}^D = \{W_s(\tau(W_s)) : \tau(W_s) < \infty\}$$

(exit points of the paths W_s)

The exit measure of the Brownian snake



$\mathcal{E}^D = \{$exit points of the paths $W_s\}$

Proposition

The formula

$$\langle Z^D, g\rangle = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^\sigma ds\, \mathbf{1}_{\{\tau(W_s)<\zeta_s<\tau(W_s)+\varepsilon\}}\, g(W_s(\tau(W_s)))$$

defines $\mathbb{N}_x$ a.e. a finite measure Z^D supported on $\mathcal{E}^D$.

Z^D is called the exit measure from D (Dynkin)

The key connection with PDE

Theorem (Reformulation of Dynkin 1991)

Let D be a regular domain (in the classical potential-theoretic sense), and $g \in C_b^+(\partial D)$. The formula

$$u(x) = \mathbb{N}_x(1 - \exp -\langle Z^D, g\rangle), \qquad x \in \partial D \tag{1}$$

defines the unique (nonnegative) solution of the Dirichlet problem

$$\Delta u = u^2 \qquad \text{in } D$$
$$u_{|\partial D} = g$$

Remark. Similarity with the probabilistic formula $u(x) = \mathbb{E}_x[g(B_\tau)]$ for the classical Dirichlet problem.

Important point: Formula (1) is very robust with respect to passages to the limit, and yields probabilistic representations for "virtually any" positive solution of $\Delta u = u^2$ in a domain.

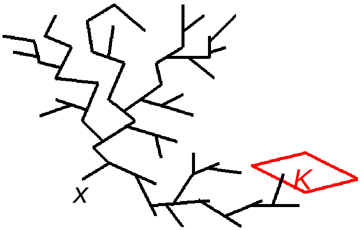
Maximal solutions

Corollary (Dynkin)

Let D be any domain. The formula

$$u(x) = \mathbb{N}_x(\mathcal{E}^D \neq \varnothing), \qquad x \in D$$

gives the maximal nonnegative solution of $\Delta u = u^2$ in D.

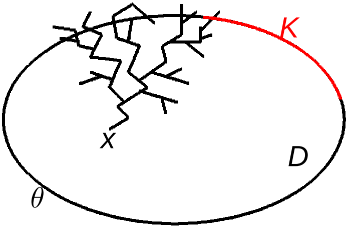


Application. $D = \mathbb{R}^d \backslash K$, K compact

The Brownian snake hits K with positive probability

- $\Leftrightarrow$ There exists a non trivial solution of $\Delta u = u^2$ in $\mathbb{R}^d \backslash K$
- $\Leftrightarrow$ K is **not** a removable singularity for $\Delta u = u^2$
- $\Leftrightarrow$ $cap_{d-4}(K) > 0$ (Baras-Pierre)

The representation of solutions when $d = 2$



D smooth domain in $\mathbb{R}^2$

Fact. If $x \in D$, the exit measure Z^D has $\mathbb{N}_x$ a.e. a continous density with respect to Lebesgue measure on ∂D, denoted by $(z_D(y), y \in \partial D)$.

Recall $\mathcal{E}^D = \{$exit points of the paths $W_s\}$

Theorem (LG)

The formula

$$u_{K,\theta}(x) = \mathbb{N}_x\Big(1 - \mathbf{1}_{\{\mathcal{E}^D \cap K = \varnothing\}} \exp -\langle \theta, z_D \rangle\Big)$$

gives a bijection between $\{$positive solutions of $\Delta u = u^2$ in $D\}$ and the set of all pairs (K, θ), where:

- *K is a compact subset of ∂D*
- *θ is a Radon measure on $\partial D \backslash K$*

Extensions of the representation theorem

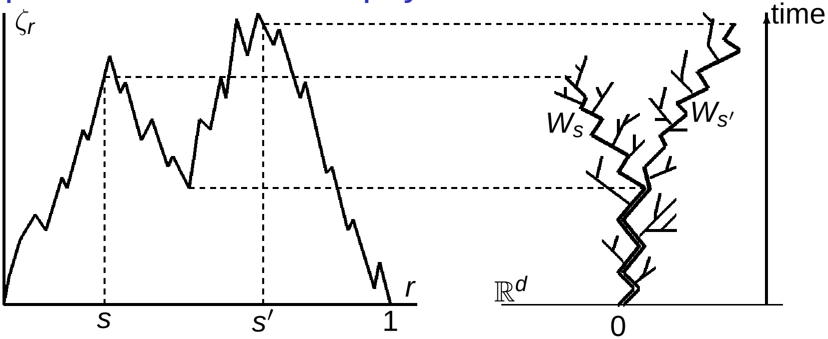
Consider more generally the equation

$$\Delta u = u^p$$

for any $p > 1$, in dimension $d \geq 2$

- Subcritical case $p < \frac{d+1}{d-1}$ (includes $p = 2$, $d = 2$)
  The correspondence betwen solutions and traces (K, θ) remains valid as in the preceding theorem (cf Marcus-Véron (analytic methods), Dynkin-Kuznetsov and LG-Mytnik)
- Supercritical case $p \geq \frac{d+1}{d-1}$
  Needs to introduce a notion of fine trace of a solution (Dynkin)
  Dynkin conjectured a one-to-one correspondence between solutions and admissible fine traces.
  - Proved by Mselati (Memoirs AMS 2003) for $p = 2$ (using the Brownian snake)
  - Proved by Dynkin-Kuznetsov for $1 < p < 2$
  - Still open for $p > 2$ but recent analytic progress by Marcus-Véron

4. Applications: Statistical physics



Consider the Brownian snake (W_s)

- with initial point $x = 0$
- driven by a normalized excursion (condition on $\sigma = 1$)

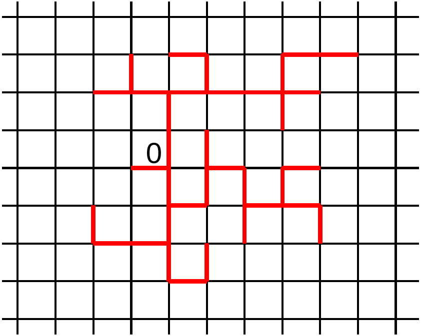
The random probability measure $\mathcal{I}$ on $\mathbb{R}^d$ defined by

$$\langle \mathcal{I}, g \rangle = \int_0^1 ds\, g(\widehat{W}_s) \qquad \text{(recall } \widehat{W}_s = \text{terminal point of } W_s\text{)}$$

is called ISE (for integrated super-Brownian excursion, Aldous).

ISE has appeared in a number of limit theorems for models of statistical physics in high dimensions: Lattice trees, percolation clusters, etc.

A lattice tree is a finite subgraph of $\mathbb{Z}^d$ with no loop.



A lattice tree in $\mathbb{Z}^2$ with 36 vertices

Question. What can we say about the shape (for instance the diameter) of a typical large lattice tree in $\mathbb{Z}^d$?
$\longrightarrow$ Very hard question if d is small (self-avoiding constraint)

Let

$$\mathcal{T}_n = \{\text{lattice trees with } n \text{ vertices in } \mathbb{Z}^d \text{ containing } 0\}.$$

Let T_n be chosen uniformly over $\mathcal{T}_n$ and let X_n be the random measure that assigns mass $\frac{1}{n}$ to each point of the form $c\,n^{-1/4}\,x$, x vertex of T_n.
(X_n is uniformly spread over the rescaled tree $c\,n^{-1/4}T_n$)

Theorem (Derbez-Slade)

If d is large enough, we can choose $c = c_d > 0$ so that

$$X_n \xrightarrow[n\to\infty]{(d)} \mathcal{I}$$

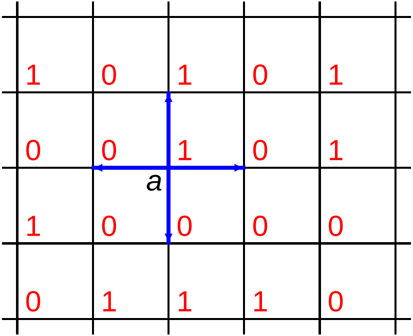
where $\mathcal{I}$ is ISE.

Informally, a typical large lattice tree (suitably rescaled) looks like the support of ISE, or equivalently the range of a Brownian snake driven by a normalized Brownian excursion.

Conjecture. The preceding theorem holds for $d > 8$ (but not for $d \leq 8$).

5. Applications: Interacting particle systems

The voter model.



At each point of $\mathbb{Z}^d$ sits an individual who can have opinion 0 or 1.

For each $a \in \mathbb{Z}^d$, after an exponential time with parameter 1, the individual sitting at a

- chooses one of his neighbors at random
- adopts his opinion

And so on.

Question. How do opinions propagates in space ?

5. Applications: Interacting particle systems

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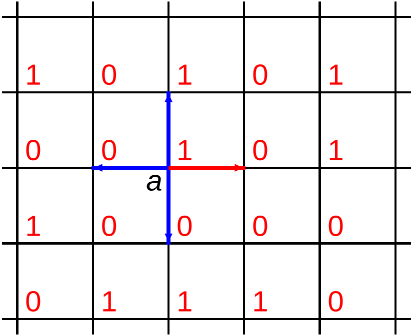

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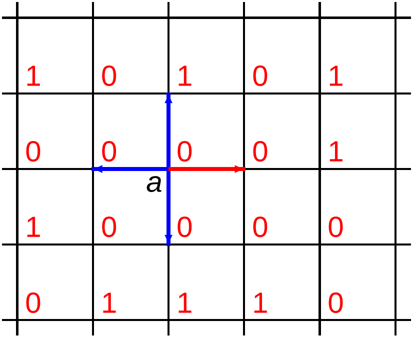

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Question. How do opinions propagates in space ?

Suppose $d \geq 2$. Write $\xi_t(a)$ for the opinion of a at time t.
Suppose that

$$\xi_0(a) = \begin{cases} 0 & \text{if } a \neq 0 \\ 1 & \text{if } a = 0 \end{cases}$$

(At time $t = 0$ only the origin has opinion 1)

Set $\mathcal{V}_t = \{a \in \mathbb{Z}^d : \xi_t(a) = 1\}$.
Bramson-Griffeath: estimates for $P(\mathcal{V}_t \neq \varnothing)$ (opinion 1 survives).

Theorem (Bramson-Cox-LG)

The law of $\frac{1}{\sqrt{t}}\mathcal{V}_t$ *conditional on* $\{\mathcal{V}_t \neq \varnothing\}$ *converges as* $t \to \infty$ *to the law of the random set*

$$\{W_s(1) : s \geq 0, \zeta_s \geq 1\}$$

under the conditional measure $\mathbb{N}_0(\cdot \mid \sup_{s \geq 0} \zeta_s > 1)$.

Asymptotically interactions disappear and opinions propagate like a spatial branching process: see also Cox-Durrett-Perkins, ...