

# Spine representations for non-compact models of random geometry\*

Jean-François Le Gall, Armand Riera

Université Paris-Saclay

## Abstract

We provide a unified approach to the three main non-compact models of random geometry, namely the Brownian plane, the infinite-volume Brownian disk, and the Brownian half-plane. This approach allows us to investigate relations between these models, and in particular to prove that complements of hulls in the Brownian plane are infinite-volume Brownian disks.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Preliminaries</b>	<b>5</b>
2.1	Snake trajectories . . . . .	5
2.2	The Brownian snake excursion measure on snake trajectories . . . . .	6
2.3	The positive excursion measure . . . . .	7
2.4	Coding finite or infinite labeled trees . . . . .	8
<b>3</b>	<b>Distributional relations between coding processes</b>	<b>12</b>
3.1	Some explicit distributions . . . . .	12
3.2	A distributional identity for coding triples . . . . .	13
3.3	The coding triple of the infinite-volume Brownian disk . . . . .	16
3.4	The coding triple of the Brownian disk with a given height . . . . .	20
<b>4</b>	<b>From coding triples to random metric spaces</b>	<b>22</b>
4.1	The pseudo-metric functions associated with a coding triple . . . . .	22
4.2	The Brownian plane . . . . .	23
4.3	The pointed Brownian disk with given perimeter and height . . . . .	24
4.4	Infinite-volume Brownian disks . . . . .	27
4.5	The Brownian half-plane . . . . .	31
<b>5</b>	<b>Applications</b>	<b>37</b>
5.1	Infinite Brownian disks in the Brownian plane . . . . .	37
5.2	Horohulls in the Brownian plane . . . . .	42
5.3	Removing a strip from the Brownian half-plane . . . . .	46
<b>6</b>	<b>Consistency with previous definitions</b>	<b>47</b>

---

\*Supported by the ERC Advanced Grant 740943 GEOBROWN

# 1 Introduction

In the recent years, much work has been devoted to the continuous models of random geometry that arise as scaling limits of planar maps, which are discrete graphs embedded in the sphere. The most famous model is the Brownian map or Brownian sphere, which is the limit in the Gromov-Hausdorff sense of large planar maps with  $n$  faces chosen uniformly at random in a suitable class and viewed as metric spaces for the graph distance rescaled by the factor  $n^{-1/4}$ , when  $n \rightarrow \infty$  (see in particular [1, 4, 10, 25, 31, 33]). The rescaling factor  $n^{-1/4}$  is relevant because the typical diameter of a random planar map with  $n$  faces is of order  $n^{1/4}$  when  $n$  is large, and thus the rescaling leads to a compact limit. However, choosing a rescaling factor that tends to 0 at a slower rate than  $n^{-1/4}$  yields a different limiting space, which can be interpreted as an infinite-volume version of the Brownian sphere and is called the Brownian plane [14, 15]. On the other hand, scaling limits of random planar maps with a boundary have also been investigated [5, 8, 9, 11, 20, 32]. In that case, assuming that the planar map has a fixed boundary size equal to  $n$  and a volume (number of faces) of order  $n^2$ , rescaling the graph distance by the factor  $n^{-1/2}$  again leads to a compact limiting space called the Brownian disk. If however the volume grows faster than  $n^2$ , the same rescaling yields a non-compact limit which is the infinite-volume Brownian disk. The so-called Brownian half-plane arises when choosing a rescaling factor that tends to 0 at a slower rate than  $n^{-1/2}$ . A comprehensive discussion of all possible scaling limits of large random quadrangulations with a boundary, including the cases mentioned above, is given in the recent paper of Baur, Miermont and Ray [8]. In this discussion, the Brownian disk and the infinite-volume Brownian disk, the Brownian plane and the Brownian half-plane play a central role. It is worth noting that the Brownian plane is also the scaling limit [14, 12] in the local Gromov-Hausdorff sense of the random lattices called the uniform infinite planar triangulation (UIPT) and the uniform infinite planar quadrangulation (UIPQ), which have been studied extensively since the introduction of the UIPT by Angel and Schramm [7]. Similarly, the Brownian half-plane arises as the scaling limit [8, 19] of the uniform half-plane quadrangulation, which has been introduced and studied in [13, 17]. We finally mention that the preceding models of random geometry are closely related to Liouville quantum gravity surfaces, and the Brownian disk, the Brownian plane and the Brownian half-plane correspond respectively to the quantum disk, the quantum cone and the quantum wedge, see [34, Corollary 1.5], and [19] for the case of the Brownian half-plane.

The main goal of the present article is to provide a unified approach to the three most important non-compact models of random geometry, namely the Brownian plane, the infinite-volume Brownian disk and the Brownian half-plane. As we will discuss below, it is remarkable that these three models can all be constructed in a similar manner from the same infinite Brownian tree equipped with Brownian labels, subject to different conditionings — the precise definition of these conditionings however requires some care especially in the case of the infinite-volume Brownian disk. As an application of these constructions, we are able to get new relations between the different models of random geometry. In particular, we prove that the complement of a hull in the Brownian plane, equipped with its intrinsic metric, is an infinite-volume Brownian disk (this may be viewed as an infinite-volume counterpart of a property derived in [28] for the Brownian sphere). The latter property was in fact a strong motivation for the present study, as it plays a very important role in the forthcoming work [36] concerning isoperimetric bounds in the Brownian plane. We also prove that the “horohull” of radius  $r$  in the Brownian plane, corresponding to the connected component containing the root of the set of points whose “relative distance” to infinity is greater than  $-r$ , is a Brownian disk with height  $r$  (here, a Brownian disk with height  $r$  is obtained by conditioning a free pointed Brownian disk on the event that the distinguished point is at distance exactly  $r$  from the boundary).

Let us now explain our approach in more precise terms. The starting point of our construction is an infinite “Brownian tree”  $\mathfrak{T}_*$  that consists of a spine isometric to  $[0, \infty)$  and two Poisson collections of subtrees grafted respectively to the left side and to the right side of the spine. For our purposes, it is very important to distinguish the left side and the right side because we later need an order structure on the tree. The trees branching off the spine can be obtained as compact  $\mathbb{R}$ -trees coded by Brownian excursions (so they are scaled versions of Aldous’ celebrated CRT). To be specific, in order to define

the subtrees branching off the left side of the tree, one may consider a Poisson point measure

$$\sum_{i \in I} \delta_{(t_i, e_i)},$$

with intensity  $2 \mathbf{1}_{[0, \infty)}(t) dt \mathbf{n}(de)$ , where  $\mathbf{n}(de)$  stands for the Itô measure of positive Brownian excursions, and then declare that, for every  $i \in I$ , the tree  $\mathcal{T}_i$  coded by  $e_i$  is grafted to the left side of the spine at level  $t_i$ . For subtrees branching off the right side, we proceed in the same way, with an independent Poisson point measure. We equip  $\mathfrak{T}_*$  with the obvious choice of a distance (see Section 2.4 below). Then  $\mathfrak{T}_*$  is a non-compact  $\mathbb{R}$ -tree, and, for every  $v \in \mathfrak{T}_*$ , we can define the geodesic line segment  $[[\rho, v]]$  between the root  $\rho$  (bottom of the spine) and  $v$ , and we use the notation  $]]\rho, v[[ = [[\rho, v]] \setminus \{\rho, v\}$ . The tree  $\mathfrak{T}_*$  may be viewed as an “infinite Brownian tree” corresponding to process 2 in Aldous [6].

We then introduce labels on  $\mathfrak{T}_*$ , that is, to each point  $v$  of  $\mathfrak{T}_*$  we assign a real label  $\Lambda_v$ . We let the labels on the spine be given by a three-dimensional Bessel process  $R = (R_t)_{t \geq 0}$  started from 0. Then, conditionally on  $R$ , the labels on the different subtrees are independent, and the labels on a given subtree  $\mathcal{T}_i$  branching off the spine at level  $t_i$  are given by Brownian motion indexed by  $\mathcal{T}_i$  and started from  $R_{t_i}$  at the root of  $\mathcal{T}_i$  (which is the point of the spine at level  $t_i$ ). In other words, labels evolve like linear Brownian motion when moving along a segment of a subtree branching off the spine.

We finally need a last operation, which ensures that we have only nonnegative labels. We let  $\mathfrak{T}$  be the subset of  $\mathfrak{T}_*$  that consists of all  $v \in \mathfrak{T}_*$  such that labels do not vanish along  $]]\rho, v[[$ . So the spine is contained in  $\mathfrak{T}$ , but some of the subtrees branching off the spine in  $\mathfrak{T}_*$  are truncated at points where labels vanish. For each subtree  $\mathcal{T}_i$ , the theory of exit measures gives a way to define a quantity  $\mathcal{Z}_0(\mathcal{T}_i)$  measuring the “number” of branches of  $\mathcal{T}_i$  that are cut in the truncation procedure (or in a more precise manner, the “number” of points  $v$  of  $\mathcal{T}_i$  such that  $\Lambda_v = 0$  but  $\Lambda_w > 0$  for every  $w \in ]]\rho, v[[$ ), and we write  $\mathcal{Z}_0$  for the sum of the quantities  $\mathcal{Z}_0(\mathcal{T}_i)$  for all subtrees  $\mathcal{T}_i$  branching off the spine. We have in fact  $\mathcal{Z}_0 = \infty$  a.s., but a key point of the subsequent discussion is to discuss conditionings of the labeled tree  $\mathfrak{T}$  that ensure that  $\mathcal{Z}_0 < \infty$ .

We are now in a position to define the random metric that will be used in the construction of the non-compact models of random geometry of interest in this work. Set  $\mathfrak{T}^\circ = \{v \in \mathfrak{T} : \Lambda_v > 0\}$ , and for  $v, w \in \mathfrak{T}^\circ$ ,

$$D^\circ(v, w) = \Lambda_v + \Lambda_w - 2 \max \left( \inf_{u \in [v, w]} \Lambda_u, \inf_{u \in [w, v]} \Lambda_u \right),$$

where  $[v, w]$  stands for the set of points visited when going from  $v$  to  $w$  clockwise around the tree (see Section 2.4 for a more precise definition). We slightly modify  $D^\circ(v, w)$  by setting  $\Delta^\circ(v, w) = D^\circ(v, w)$  if the maximum in the last display is positive, and  $\Delta^\circ(v, w) = \infty$  otherwise. Finally, we let  $(\Delta(v, w); v, w \in \mathfrak{T}^\circ)$  be the maximal symmetric function of  $(v, w) \in \mathfrak{T}^\circ \times \mathfrak{T}^\circ$  that is bounded above by  $\Delta^\circ$  and satisfies the triangle inequality. It turns out that the function  $(v, w) \mapsto \Delta(v, w)$  takes finite values and can be extended by continuity to a pseudo-metric on  $\mathfrak{T}$ , and we may thus consider the quotient space of  $\mathfrak{T}$  for the equivalence relation defined by setting  $v \simeq w$  if and only if  $\Delta(v, w) = 0$ . The quotient space  $\mathfrak{T}/\simeq$  equipped with the metric induced by  $\Delta$  is:

1. the Brownian plane under the special conditioning  $\mathcal{Z}_0 = 0$ ;
2. the infinite-volume Brownian disk with perimeter  $z > 0$  under the special conditioning  $\mathcal{Z}_0 = z$ ;
3. the Brownian half-plane under no conditioning (then  $\mathcal{Z}_0 = \infty$  a.s.).

The really new contributions of the present work are cases 2 and 3, because case 1 corresponds to the construction of the Brownian plane in [15] (which is different from the one in [14]): in that case, the conditioning on  $\mathcal{Z}_0 = 0$  turns the process of labels on the spine into a nine-dimensional Bessel process  $X = (X_t)_{t \geq 0}$  started from 0, and the subtrees branching off the spine are conditioned to have positive labels (see Section 4.2 below).

A remarkable feature of the preceding constructions is the fact that labels on  $\mathfrak{T}$  have a nice geometric interpretation in terms of the associated random metric spaces  $\mathfrak{T}/\simeq$ . Precisely, the label  $\Lambda_v$  of a point  $v$  of  $\mathfrak{T}$  is equal to the distance from (the equivalence class of)  $v$  to the set of (equivalence classes of) points of zero label in  $\mathfrak{T}/\simeq$ . The latter set is either a single point in case 1, or a boundary

homeomorphic to the circle in case 2, or a boundary homeomorphic to the line in case 3. Amongst other applications, this interpretation of labels allows us to prove the above-mentioned result about the complement of hulls in the Brownian plane. Write  $\mathbb{BP}_\infty$  for the Brownian plane, and recall that the hull  $B^\bullet(r)$  is defined by saying that its complement  $\check{B}^\bullet(r) := \mathbb{BP}_\infty \setminus B^\bullet(r)$  is the unbounded component of the complement of the closed ball of radius  $r$  centered at the distinguished point (bottom of the spine) of  $\mathbb{BP}_\infty$ . Then Theorem 29 below states that (the closure of)  $\check{B}^\bullet(r)$  equipped with its intrinsic metric is an infinite-volume Brownian disk whose perimeter is the boundary size  $|\partial B^\bullet(r)|$  (see [15] for the definition of this boundary size).

Much of the technical work in the present paper is devoted to making sense of the conditioning  $\mathcal{Z}_0 = z$  in case 2, which is not a trivial matter because  $\mathcal{Z}_0 = \infty$  a.s. Our approach is motivated by the previously mentioned result concerning the distribution of  $\check{B}^\bullet(r)$ . At first, it would seem that our construction of the Brownian plane from an infinite tree  $\mathfrak{T}$  made of a spine equipped with labels  $(X_t)_{t \geq 0}$  (given by a nine-dimensional Bessel process), and labeled subtrees conditioned to have positive labels, would be suited perfectly to analyse the distribution of a hull or of its complement. In fact, it is observed in [15] that the set  $\check{B}^\bullet(r)$  exactly corresponds to a subtree  $\mathfrak{T}_{(r)}$  consisting of the part of the spine of  $\mathfrak{T}$  above level  $L_r := \sup\{t \geq 0 : X_t = r\}$  and of the subtrees branching off the spine of  $\mathfrak{T}$  above level  $L_r$  and truncated at points where labels hit  $r$  — furthermore the boundary size  $|\partial B^\bullet(r)|$  is just the sum of the exit measures at level  $r$  of all these subtrees. However, this representation of  $\check{B}^\bullet(r)$  seems to depend heavily on  $r$ , even if labels are shifted by  $-r$ : in particular, the distribution of the process  $(X_{L_r+t} - r)_{t \geq 0}$  depends on  $r$ . Nevertheless, and perhaps surprisingly, it turns out that, if we condition the boundary size  $|\partial B^\bullet(r)|$  to be equal to a fixed  $z > 0$ , the conditional distribution of the labeled tree  $\mathfrak{T}_{(r)}$  (with labels shifted by  $-r$ ) does not depend on  $r$ , and this leads to the probability measure  $\Theta_z$  which is used in our construction of the infinite-volume Brownian disk. The precise construction of the measures  $\Theta_z$ , which involves an appropriate truncation procedure, is given in Section 3.3, where we also explain in which sense these measures correspond to the conditioning of case 2 above.

As the reader will have guessed from the preceding discussion, some of the technicalities in our proofs are made necessary by the problem of conditioning on events of zero probability. For instance, in order to define the free pointed Brownian disk with perimeter  $z$  and a given height  $r > 0$ , it is relevant to condition the Brownian snake excursion measure  $\mathbb{N}_r$  (see Section 2 for a definition) on the event that the exit measure at 0 is equal to  $z$ . It is not immediately obvious how to make a canonical choice of these conditional distributions, so that they depend continuously on the pair  $(r, z)$ . We deal carefully with these questions in Section 3.

Our proofs also rely on certain explicit distributions, which are of independent interest. In particular, we prove that, in a free pointed Brownian disk of perimeter 1, the density of the distribution of the distance from the distinguished point to the boundary is given by the function

$$p_1(r) := 9r^{-6} \left( r + \frac{2}{3}r^3 - \left(\frac{3}{2}\right)^{1/2} \sqrt{\pi} (1+r^2) \exp\left(\frac{3}{2r^2}\right) \operatorname{erfc}\left(\sqrt{\frac{3}{2r^2}}\right) \right),$$

with the usual notation  $\operatorname{erfc}(\cdot)$  for the complementary error function. See Propositions 3 and 14 below for a short proof, which uses the representation of Brownian disks found in [28] (with some more work, the same formula could also be derived from the representation in [9, 11]).

The paper is organized as follows. Section 2 contains a number of preliminaries, and in particular we introduce the formalism of snake trajectories [3], and the associated Brownian snake excursion measures, to code compact continuous random trees equipped with real labels. We also introduce the notion of a “coding triple” for a non-compact continuous random tree. Such a coding triple consists of a random process representing the labels on the spine, and two random point measures on the space of all pairs  $(t, \omega)$ , where  $t \geq 0$  and  $\omega$  is a snake trajectory (the idea is that, for every such pair, the labeled tree coded by  $\omega$  will be grafted to the left or to the right of the spine at level  $t$ ). The main goal of Section 3 is to define the coding triple associated with the infinite-volume Brownian disk or, in other words, to make sense of the conditioning appearing in case 2 above. Section 4 then gives the construction of the random metric spaces of interest from the corresponding coding triples, starting from the construction of the Brownian plane in [15]. As an important ingredient of our discussion, we consider the free pointed Brownian disk  $\mathbb{D}_z^{(a)}$  with perimeter  $z$  and height  $a$  (recall that the height refers to the distance from the distinguished point to the boundary). The infinite-volume Brownian

disk with perimeter  $z$  is then obtained as the local limit of  $\mathbb{D}_z^{(a)}$  in the Gromov-Hausdorff sense when  $a \rightarrow \infty$ . In an analogous manner, we construct the Brownian half-plane and we verify that it is the tangent cone in distribution of the free Brownian disk at a point chosen uniformly on its boundary. Finally, Section 5 is devoted to our applications to the complement of hulls and to horohulls in the Brownian plane, and Section 6 shows that our definitions of the infinite-volume Brownian disk and of the Brownian half-plane are consistent with previous work.

## 2 Preliminaries

### 2.1 Snake trajectories

Continuous random trees whose vertices are assigned real labels play a fundamental role in this work. The formalism of snake trajectories, which has been introduced in [3], provides a convenient framework to deal with such labeled trees.

A (one-dimensional) finite path  $w$  is a continuous mapping  $w : [0, \zeta] \rightarrow \mathbb{R}$ , where the number  $\zeta = \zeta_{(w)} \geq 0$  is called the lifetime of  $w$ . We let  $\mathcal{W}$  denote the space of all finite paths, which is a Polish space when equipped with the distance

$$d_{\mathcal{W}}(w, w') = |\zeta_{(w)} - \zeta_{(w')}| + \sup_{t \geq 0} |w(t \wedge \zeta_{(w)}) - w'(t \wedge \zeta_{(w')})|.$$

The endpoint or tip of the path  $w$  is denoted by  $\widehat{w} = w(\zeta_{(w)})$ . For  $x \in \mathbb{R}$ , we set  $\mathcal{W}_x = \{w \in \mathcal{W} : w(0) = x\}$ . The trivial element of  $\mathcal{W}_x$  with zero lifetime is identified with the point  $x$  of  $\mathbb{R}$ . We also use the notation  $\mathcal{W}^\infty$ , resp.  $\mathcal{W}_x^\infty$ , for the space of all continuous functions  $w : [0, \infty) \rightarrow \mathbb{R}$ , resp. the set of all such functions with  $w(0) = x$ .

**Definition 1.** *Let  $x \in \mathbb{R}$ . A snake trajectory with initial point  $x$  is a continuous mapping  $s \mapsto \omega_s$  from  $\mathbb{R}_+$  into  $\mathcal{W}_x$  which satisfies the following two properties:*

- (i) *We have  $\omega_0 = x$  and the number  $\sigma(\omega) := \sup\{s \geq 0 : \omega_s \neq x\}$ , called the duration of the snake trajectory  $\omega$ , is finite (by convention  $\sigma(\omega) = 0$  if  $\omega_s = x$  for every  $s \geq 0$ ).*
- (ii) (Snake property) *For every  $0 \leq s \leq s'$ , we have  $\omega_s(t) = \omega_{s'}(t)$  for every  $t \in [0, \min_{s \leq r \leq s'} \zeta_{(\omega_r)}]$ .*

We will write  $\mathcal{S}_x$  for the set of all snake trajectories with initial point  $x$  and  $\mathcal{S} = \bigcup_{x \in \mathbb{R}} \mathcal{S}_x$  for the set of all snake trajectories. If  $\omega \in \mathcal{S}$ , we often write  $W_s(\omega) = \omega_s$  and  $\zeta_s(\omega) = \zeta_{(\omega_s)}$  for every  $s \geq 0$ . The set  $\mathcal{S}$  is a Polish space for the distance

$$d_{\mathcal{S}}(\omega, \omega') = |\sigma(\omega) - \sigma(\omega')| + \sup_{s \geq 0} d_{\mathcal{W}}(W_s(\omega), W_s(\omega')).$$

A snake trajectory  $\omega$  is completely determined by the knowledge of the lifetime function  $s \mapsto \zeta_s(\omega)$  and of the tip function  $s \mapsto \widehat{W}_s(\omega)$ : See [3, Proposition 8]. If  $\omega$  is a snake trajectory, its time reversal  $\check{\omega}$  defined by  $\check{\omega}_s = \omega_{(\sigma(\omega) - s)^+}$  is also a snake trajectory.

Let  $\omega \in \mathcal{S}$  be a snake trajectory and  $\sigma = \sigma(\omega)$ . The lifetime function  $s \mapsto \zeta_s(\omega)$  codes a compact  $\mathbb{R}$ -tree, which will be denoted by  $\mathcal{T}_{(\omega)}$  and called the *genealogical tree* of the snake trajectory. This  $\mathbb{R}$ -tree is the quotient space  $\mathcal{T}_{(\omega)} := [0, \sigma] / \sim$  of the interval  $[0, \sigma]$  for the equivalence relation

$$s \sim s' \text{ if and only if } \zeta_s(\omega) = \zeta_{s'}(\omega) = \min_{s \wedge s' \leq r \leq s \vee s'} \zeta_r(\omega),$$

and  $\mathcal{T}_{(\omega)}$  is equipped with the distance induced by

$$d_{(\omega)}(s, s') = \zeta_s(\omega) + \zeta_{s'}(\omega) - 2 \min_{s \wedge s' \leq r \leq s \vee s'} \zeta_r(\omega).$$

(notice that  $d_{(\omega)}(s, s') = 0$  if and only if  $s \sim s'$ , and see e.g. [30, Section 3] for more information about the coding of  $\mathbb{R}$ -trees by continuous functions). We write  $p_{(\omega)} : [0, \sigma] \rightarrow \mathcal{T}_{(\omega)}$  for the canonical projection. By convention,  $\mathcal{T}_{(\omega)}$  is rooted at the point  $\rho_{(\omega)} := p_{(\omega)}(0)$ , and the volume measure on  $\mathcal{T}_{(\omega)}$

is defined as the pushforward of Lebesgue measure on  $[0, \sigma]$  under  $p_{(\omega)}$ . If  $u, v \in \mathcal{T}_{(\omega)}$ ,  $[[u, v]]$  denotes the geodesic segment between  $u$  and  $v$  in  $\mathcal{T}_{(\omega)}$ , and we also use the notation  $[[u, v[[$  or  $]]u, v]]$  with an obvious meaning.

It will be useful to define also intervals on the tree  $\mathcal{T}_{(\omega)}$ . For  $s, s' \in [0, \sigma]$ , we use the convention  $[s, s'] = [s, \sigma] \cup [0, s']$  if  $s > s'$  (and of course,  $[s, s']$  is the usual interval if  $s \leq s'$ ). If  $u, v \in \mathcal{T}_{(\omega)}$  are distinct, then we can find  $s, s' \in [0, \sigma]$  in a unique way so that  $p_{(\omega)}(s) = u$  and  $p_{(\omega)}(s') = v$  and the interval  $[s, s']$  is as small as possible, and we define  $[u, v] := p_{(\omega)}([s, s'])$ . Informally,  $[u, v]$  is the set of all points that are visited when going from  $u$  to  $v$  in ‘‘clockwise order’’ around the tree. We take  $[u, u] = \{u\}$ .

By property (ii) in the definition of a snake trajectory, the condition  $p_{(\omega)}(s) = p_{(\omega)}(s')$  implies that  $W_s(\omega) = W_{s'}(\omega)$ . So the mapping  $s \mapsto W_s(\omega)$  can be viewed as defined on the quotient space  $\mathcal{T}_{(\omega)}$ . For  $u \in \mathcal{T}_{(\omega)}$ , we set  $\ell_u(\omega) := \widehat{W}_s(\omega)$  whenever  $s \in [0, \sigma]$  is such that  $u = p_{(\omega)}(s)$  (by the previous observation, this does not depend on the choice of  $s$ ). We can interpret  $\ell_u(\omega)$  as a ‘‘label’’ assigned to the ‘‘vertex’’  $u$  of  $\mathcal{T}_{(\omega)}$ . Notice that the mapping  $u \mapsto \ell_u(\omega)$  is continuous on  $\mathcal{T}_{(\omega)}$ , and that, for every  $s \geq 0$ , the path  $W_s(\omega)$  records the labels  $\ell_u(\omega)$  along the ‘‘ancestral line’’  $[[\rho_{(\omega)}, p_{(\omega)}(s)]]$ . We will use the notation  $W_*(\omega) := \min\{\ell_u(\omega) : u \in \mathcal{T}_{(\omega)}\}$ .

We now introduce two important operations on snake trajectories in  $\mathcal{S}$ . The first one is the re-rooting operation (see [3, Section 2.2]). Let  $\omega \in \mathcal{S}$  and  $r \in [0, \sigma(\omega)]$ . Then  $\omega^{[r]}$  is the snake trajectory in  $\mathcal{S}_{\widehat{W}_r(\omega)}$  such that  $\sigma(\omega^{[r]}) = \sigma(\omega)$  and for every  $s \in [0, \sigma(\omega)]$ ,

$$\begin{aligned}\zeta_s(\omega^{[r]}) &= d_{(\omega)}(r, r \oplus s), \\ \widehat{W}_s(\omega^{[r]}) &= \widehat{W}_{r \oplus s}(\omega),\end{aligned}$$

where we use the notation  $r \oplus s = r + s$  if  $r + s \leq \sigma(\omega)$ , and  $r \oplus s = r + s - \sigma(\omega)$  otherwise. By a remark following the definition of snake trajectories, these prescriptions completely determine  $\omega^{[r]}$ . The genealogical tree  $\mathcal{T}_{(\omega^{[r]})}$  may be interpreted as the tree  $\mathcal{T}_{(\omega)}$  re-rooted at the vertex  $p_{(\omega)}(r)$  (see [18, Lemma 2.2] for a precise statement) and vertices of the re-rooted tree receive the same labels as in  $\mathcal{T}_{(\omega)}$ . We sometimes write  $W^{[t]}(\omega)$  instead of  $\omega^{[t]}$ .

The second operation is the truncation of snake trajectories. Let  $x, y \in \mathbb{R}$  with  $y < x$ . For every  $w \in \mathcal{W}_x$ , set

$$\tau_y(w) := \inf\{t \in [0, \zeta_w] : w(t) = y\}$$

with the usual convention  $\inf \emptyset = \infty$  (this convention will be in force throughout this work unless otherwise indicated). Then, if  $\omega \in \mathcal{S}_x$ , we set, for every  $s \geq 0$ ,

$$\eta_s(\omega) = \inf\left\{t \geq 0 : \int_0^t du \mathbf{1}_{\{\zeta_{(\omega_u)} \leq \tau_y(\omega_u)\}} > s\right\}.$$

Note that the condition  $\zeta_{(\omega_u)} \leq \tau_y(\omega_u)$  holds if and only if  $\tau_y(\omega_u) = \infty$  or  $\tau_y(\omega_u) = \zeta_{(\omega_u)}$ . Then, setting  $\omega'_s = \omega_{\eta_s(\omega)}$  for every  $s \geq 0$  defines an element  $\omega'$  of  $\mathcal{S}_x$ , which will be denoted by  $\text{tr}_y(\omega)$  and called the truncation of  $\omega$  at  $y$  (see [3, Proposition 10]). The effect of the time change  $\eta_s(\omega)$  is to ‘‘eliminate’’ those paths  $\omega_s$  that hit  $y$  and then survive for a positive amount of time. We leave it as an exercise for the reader to check that the genealogical tree  $\mathcal{T}_{(\text{tr}_y(\omega))}$  is canonically and isometrically identified to the closed set  $\{v \in \mathcal{T}_{(\omega)} : \ell_u(\omega) > y \text{ for every } u \in [[\rho_{(\omega)}, v]]\}$ , and this identification preserves labels.

## 2.2 The Brownian snake excursion measure on snake trajectories

Let  $x \in \mathbb{R}$ . The Brownian snake excursion measure  $\mathbb{N}_x$  is the  $\sigma$ -finite measure on  $\mathcal{S}_x$  that satisfies the following two properties: Under  $\mathbb{N}_x$ ,

- (i) the distribution of the lifetime function  $(\zeta_s)_{s \geq 0}$  is the Itô measure of positive excursions of linear Brownian motion, normalized so that, for every  $\varepsilon > 0$ ,

$$\mathbb{N}_x\left(\sup_{s \geq 0} \zeta_s > \varepsilon\right) = \frac{1}{2\varepsilon};$$

(ii) conditionally on  $(\zeta_s)_{s \geq 0}$ , the tip function  $(\widehat{W}_s)_{s \geq 0}$  is a Gaussian process with mean  $x$  and covariance function

$$K(s, s') := \min_{s \wedge s' \leq r \leq s \vee s'} \zeta_r.$$

Informally, the lifetime process  $(\zeta_s)_{s \geq 0}$  evolves under  $\mathbb{N}_x$  like a Brownian excursion, and conditionally on  $(\zeta_s)_{s \geq 0}$ , each path  $W_s$  is a linear Brownian path started from  $x$  with lifetime  $\zeta_s$ , which is “erased” from its tip when  $\zeta_s$  decreases and is “extended” when  $\zeta_s$  increases. The measure  $\mathbb{N}_x$  can be interpreted as the excursion measure away from  $x$  for the Markov process in  $\mathcal{W}_x$  called the Brownian snake. We refer to [23] for a detailed study of the Brownian snake. For every  $y < x$ , we have

$$\mathbb{N}_x(W_* \leq y) = \frac{3}{2(x-y)^2}. \quad (1)$$

See e.g. [23, Section VI.1] for a proof.

The measure  $\mathbb{N}_x$  is invariant under the time-reversal operation  $\omega \mapsto \check{\omega}$ . Furthermore, the following scaling property is often useful. For  $\lambda > 0$ , for every  $\omega \in \mathcal{S}_x$ , we define  $\theta_\lambda(\omega) \in \mathcal{S}_{x\sqrt{\lambda}}$  by  $\theta_\lambda(\omega) = \omega'$ , with

$$\omega'_s(t) := \sqrt{\lambda} \omega_{s/\lambda^2}(t/\lambda), \quad \text{for } s \geq 0 \text{ and } 0 \leq t \leq \zeta'_s := \lambda \zeta_{s/\lambda^2}.$$

Then it is a simple exercise to verify that  $\theta_\lambda(\mathbb{N}_x) = \lambda \mathbb{N}_{x\sqrt{\lambda}}$ .

*Exit measures.* Let  $x, y \in \mathbb{R}$ , with  $y < x$ . Under the measure  $\mathbb{N}_x$ , one can make sense of a quantity that “measures the quantity” of paths  $W_s$  that hit level  $y$ . One shows [28, Proposition 34] that the limit

$$L_t^y := \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^2} \int_0^t ds \mathbf{1}_{\{\tau_y(W_s) = \infty, \widehat{W}_s < y + \varepsilon\}} \quad (2)$$

exists uniformly for  $t \geq 0$ ,  $\mathbb{N}_x$  a.e., and defines a continuous nondecreasing function, which is obviously constant on  $[\sigma, \infty)$ . The process  $(L_t^y)_{t \geq 0}$  is called the exit local time at level  $y$ , and the exit measure  $\mathcal{Z}_y$  is defined by  $\mathcal{Z}_y = L_\infty^y = L_\sigma^y$ . Then,  $\mathbb{N}_x$  a.e., the topological support of the measure  $dL_t^y$  is exactly the set  $\{s \in [0, \sigma] : \tau_y(W_s) = \zeta_s\}$ , and, in particular,  $\mathcal{Z}_y > 0$  if and only if one of the paths  $W_s$  hits  $y$ . The definition of  $\mathcal{Z}_y$  is a special case of the theory of exit measures (see [23, Chapter V] for this general theory). We will use the formula for the Laplace transform of  $\mathcal{Z}_y$ : For  $\lambda > 0$ ,

$$\mathbb{N}_x(1 - \exp(-\lambda \mathcal{Z}_y)) = \left( (x-y)\sqrt{2/3} + \lambda^{-1/2} \right)^{-2}. \quad (3)$$

See formula (6) in [15] for a brief justification.

It will be useful to observe that  $\mathcal{Z}_y$  can be defined in terms of the truncated snake  $\text{tr}_y(\omega)$ . To this end, recall the time change  $(\eta_s(\omega))_{s \geq 0}$  used to define  $\text{tr}_y(\omega)$  at the end of Section 2.1, and set  $\tilde{L}_t^y = L_{\eta_t}^y$  for every  $t \geq 0$ . Then  $\tilde{L}_\infty^y = L_\infty^y = \mathcal{Z}_y$ , whereas formula (2) implies that

$$\tilde{L}_t^y = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^2} \int_0^t ds \mathbf{1}_{\{\widehat{W}_s(\text{tr}_y(\omega)) < y + \varepsilon\}} \quad (4)$$

uniformly for  $t \geq 0$ ,  $\mathbb{N}_x$  a.e.

### 2.3 The positive excursion measure

We now introduce another  $\sigma$ -finite measure on  $\mathcal{S}_0$ , which is supported on snake trajectories taking only nonnegative values. For  $\delta \geq 0$ , let  $\mathcal{S}^{(\delta)}$  be the set of all  $\omega \in \mathcal{S}$  such that  $\sup_{s \geq 0} (\sup_{t \in [0, \zeta_s(\omega)]} |\omega_s(t)|) > \delta$ . Also set

$$\mathcal{S}_0^+ = \{\omega \in \mathcal{S}_0 : \omega_s(t) \geq 0 \text{ for every } s \geq 0, t \in [0, \zeta_s(\omega)]\} \cap \mathcal{S}^{(0)}.$$

By [3, Theorem 23], there exists a  $\sigma$ -finite measure  $\mathbb{N}^*$  on  $\mathcal{S}$ , which is supported on  $\mathcal{S}_0^+$  and gives finite mass to the sets  $\mathcal{S}^{(\delta)}$  for every  $\delta > 0$ , such that

$$\mathbb{N}^*(G) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathbb{N}_\varepsilon(G(\text{tr}_0(\omega))),$$

for every bounded continuous function  $G$  on  $\mathcal{S}$  that vanishes on  $\mathcal{S} \setminus \mathcal{S}^{(\delta)}$  for some  $\delta > 0$ . Under  $\mathbb{N}^*$ , each of the paths  $W_s$ ,  $0 < s < \sigma$ , starts from 0, then stays positive during some time interval  $(0, \alpha)$ , and is stopped immediately when it returns to 0, if it does return to 0.

*The re-rooting formula.* We can relate the measure  $\mathbb{N}^*$  to the excursion measures  $\mathbb{N}_x$  of the preceding section via a re-rooting formula which we now state [3, Theorem 28]. Recall the notation  $\omega^{[t]}$  for the snake trajectory  $\omega$  re-rooted at  $t$ . For any nonnegative measurable function  $G$  on  $\mathcal{S}$ , we have

$$\mathbb{N}^* \left( \int_0^\sigma dt G(\omega^{[t]}) \right) = 2 \int_0^\infty dx \mathbb{N}_x \left( \mathcal{Z}_0 G(\text{tr}_0(\omega)) \right). \quad (5)$$

*Conditioning on the exit measure at 0.* In a way analogous to the definition of exit measures, one can make sense of the “quantity” of paths  $W_s$  that return to 0 under  $\mathbb{N}^*$ . To this end, one observes that the limit

$$\mathcal{Z}_0^* := \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^2} \int_0^\sigma ds \mathbf{1}_{\{\widehat{W}_s < \varepsilon\}} \quad (6)$$

exists  $\mathbb{N}^*$  a.e. (this indeed follows from (4), using (5) to relate  $\mathbb{N}^*$  to the distribution of  $\text{tr}_0(\omega)$  under  $\mathbb{N}_x$ ,  $x > 0$ ).

According to [3, Proposition 33], there exists a unique collection  $(\mathbb{N}^{*,z})_{z>0}$  of probability measures on  $\mathcal{S}_0^+$  such that:

(i) We have

$$\mathbb{N}^* = \sqrt{\frac{3}{2\pi}} \int_0^\infty dz z^{-5/2} \mathbb{N}^{*,z}.$$

(ii) For every  $z > 0$ ,  $\mathbb{N}^{*,z}$  is supported on  $\{\mathcal{Z}_0^* = z\}$ .

(iii) For every  $z, z' > 0$ ,  $\mathbb{N}^{*,z'} = \theta_{z'/z}(\mathbb{N}^{*,z})$ .

Informally,  $\mathbb{N}^{*,z} = \mathbb{N}^*(\cdot \mid \mathcal{Z}_0^* = z)$ .

It will be convenient to have a “pointed version” of the measures  $\mathbb{N}^{*,z}$ . We note that  $\mathbb{N}^{*,z}(\sigma) = z^2$  (see the remark after [28, Proposition 15]) and define a probability measure on  $\mathcal{S}_0 \times \mathbb{R}_+$  by setting

$$\overline{\mathbb{N}}^{*,z}(d\omega dt) = z^{-2} \mathbb{N}^{*,z}(d\omega) \mathbf{1}_{[0, \sigma(\omega)]}(t) dt.$$

## 2.4 Coding finite or infinite labeled trees

Many of the random compact (resp. non-compact) metric spaces that we discuss in the present work are coded by triples  $(Z, \mathcal{M}, \mathcal{M}')$  where  $Z = (Z_t)_{t \in [0, h]}$  (resp.  $Z = (Z_t)_{t \in [0, \infty)}$ ) is a finite (resp. infinite) random path, and  $\mathcal{M}$  and  $\mathcal{M}'$  are random point measures on  $[0, h] \times \mathcal{S}$  (resp. on  $[0, \infty) \times \mathcal{S}$ ). Such a triple is called a coding triple, and we interpret it as coding a labeled tree, having a spine isometric to  $[0, h]$  or to  $[0, \infty)$ , in such a way that the path  $Z$  corresponds to labels along the spine, and, for each atom  $(t_i, \omega_i)$  of  $\mathcal{M}$  (resp. of  $\mathcal{M}'$ ), the genealogical tree of  $\omega_i$  corresponds a subtree branching off the left side (resp. off the right side) of the spine at level  $t_i$ . The random metric spaces of interest are then obtained via some identification of vertices in the labeled trees, and equipped with a metric which is determined from the labels.

In this section we explain how coding triples are used to construct labeled trees. We follow closely the presentation given in [15] for a special case.

*The infinite spine case.* We consider a (deterministic) triple  $(w, \mathcal{P}, \mathcal{P}')$  such that:

(i)  $w \in \mathcal{W}^\infty$ ;

(ii)  $\mathcal{P} = \sum_{i \in I} \delta_{(t_i, \omega_i)}$  and  $\mathcal{P}' = \sum_{i \in J} \delta_{(t_i, \omega_i)}$  are point measures on  $(0, \infty) \times \mathcal{S}$  (the indexing sets  $I$  and  $J$  are disjoint), and, for every  $i \in I \cup J$ ,  $\omega_i \in \mathcal{S}_{w(t_i)}$  and  $\sigma(\omega_i) > 0$ ;

(iii) all numbers  $t_i$ ,  $i \in I \cup J$ , are distinct;



(iv) the functions

$$u \mapsto \beta_u := \sum_{i \in I} \mathbf{1}_{\{t_i \leq u\}} \sigma(\omega_i), \quad u \mapsto \beta'_u := \sum_{i \in J} \mathbf{1}_{\{t_i \leq u\}} \sigma(\omega_i).$$

take finite values and are monotone increasing on  $\mathbb{R}_+$ , and tend to  $\infty$  at  $\infty$  (in particular, the sets  $\{t_i : i \in I\}$  and  $\{t_i : i \in J\}$  are dense in  $(0, \infty)$ );

(v) for every  $t > 0$  and  $\varepsilon > 0$ ,

$$\#\left\{i \in I \cup J : t_i \leq t \text{ and } \sup_{0 \leq s \leq \sigma(\omega_i)} |\widehat{W}_s(\omega_i) - w(t_i)| > \varepsilon\right\} < \infty. \quad (7)$$

Such a triple  $(w, \mathcal{P}, \mathcal{P}')$  will be called an infinite spine coding triple. Recall the notation  $\mathcal{T}_{(\omega)}$  for the genealogical tree of the snake trajectory  $\omega$  and  $\rho_{(\omega)}$  for the root of  $\mathcal{T}_{(\omega)}$ . The tree  $\mathcal{T}_\infty$  associated with the coding triple  $(w, \mathcal{P}, \mathcal{P}')$  is obtained from the disjoint union

$$[0, \infty) \cup \left( \bigcup_{i \in I \cup J} \mathcal{T}_{(\omega_i)} \right)$$

by identifying the point  $t_i$  of  $[0, \infty)$  with the root  $\rho_{(\omega_i)}$  of  $\mathcal{T}_{(\omega_i)}$ , for every  $i \in I \cup J$ . The metric  $d_{\mathcal{T}_\infty}$  on  $\mathcal{T}_\infty$  is determined as follows. The restriction of  $d_{\mathcal{T}_\infty}$  to each tree  $\mathcal{T}_{(\omega_i)}$  is the metric  $d_{(\omega_i)}$  on  $\mathcal{T}_{(\omega_i)}$ , and the restriction of  $d_{\mathcal{T}_\infty}$  to the spine  $[0, \infty)$  is the usual Euclidean distance. If  $x \in \mathcal{T}_{(\omega_i)}$  and  $t \in [0, \infty)$ , we take  $d_{\mathcal{T}_\infty}(x, t) = d_{(\omega_i)}(x, \rho_{(\omega_i)}) + |t_i - t|$ . If  $x \in \mathcal{T}_{(\omega_i)}$  and  $y \in \mathcal{T}_{(\omega_j)}$ , with  $i \neq j$ , we take  $d_{\mathcal{T}_\infty}(x, y) = d_{(\omega_i)}(x, \rho_{(\omega_i)}) + |t_i - t_j| + d_{(\omega_j)}(\rho_{(\omega_j)}, y)$ . We note that  $\mathcal{T}_\infty$  is a non-compact  $\mathbb{R}$ -tree. By convention,  $\mathcal{T}_\infty$  is rooted at 0. The tree  $\mathcal{T}_\infty$  is equipped with a volume measure, which is defined as the sum of the volume measures on the trees  $\mathcal{T}_{(\omega_i)}$ ,  $i \in I \cup J$ .

We can also define labels on  $\mathcal{T}_\infty$ . The label  $\Lambda_x$  of  $x \in \mathcal{T}_\infty$  is defined by  $\Lambda_x = w(t)$  if  $x = t$  belongs to the spine  $[0, \infty)$ , and  $\Lambda_x = \ell_x(\omega_i)$  if  $x$  belongs to  $\mathcal{T}_{(\omega_i)}$ , for some  $i \in I \cup J$ . Note that the mapping  $x \mapsto \Lambda_x$  is continuous (use property (7) to check continuity at points of the spine).

For our purposes, it is important to define an order structure on  $\mathcal{T}_\infty$ . To this end, we introduce a ‘‘clockwise exploration’’ of  $\mathcal{T}_\infty$ , which is defined as follows. Write  $\beta_{u-}$  and  $\beta'_{u-}$  for the respective left limits at  $u$  of the functions  $u \mapsto \beta_u$  and  $u \mapsto \beta'_u$  introduced in (iv) above, with the convention  $\beta_{0-} = \beta'_{0-} = 0$ . Then, for every  $s \geq 0$ , there is a unique  $u \geq 0$  such that  $\beta_{u-} \leq s \leq \beta_u$ , and:

- Either we have  $u = t_i$  for some  $i \in I$  (then  $\sigma(\omega_i) = \beta_{t_i} - \beta_{t_i-}$ ), and we set  $\mathcal{E}_s^+ := p_{(\omega_i)}(s - \beta_{t_i-})$ .
- Or there is no such  $i$  and we set  $\mathcal{E}_s^+ = u$ .

We define similarly  $(\mathcal{E}_s^-)_{s \geq 0}$ . For every  $s \geq 0$ , there is a unique  $u \geq 0$  such that  $\beta'_{u-} \leq s \leq \beta'_u$ , and:

- Either we have  $u = t_i$  for some  $i \in J$ , and we set  $\mathcal{E}_s^- := p_{(\omega_i)}(\beta'_{t_i} - s) (= p_{(\omega_i)}(s - \beta'_{t_i-}))$ .
- Or there is no such  $i$  and we set  $\mathcal{E}_s^- = u$ .

Informally,  $(\mathcal{E}_s^+)_{s \geq 0}$  and  $(\mathcal{E}_s^-)_{s \geq 0}$  correspond to the exploration of the left and right side of the tree  $\mathcal{T}_\infty$  respectively. Noting that  $\mathcal{E}_0^+ = \mathcal{E}_0^- = 0$ , we define  $(\mathcal{E}_s)_{s \in \mathbb{R}}$  by setting

$$\mathcal{E}_s := \begin{cases} \mathcal{E}_s^+ & \text{if } s \geq 0, \\ \mathcal{E}_{-s}^- & \text{if } s \leq 0. \end{cases}$$

It is straightforward to verify that the mapping  $s \mapsto \mathcal{E}_s$  from  $\mathbb{R}$  onto  $\mathcal{T}_\infty$  is continuous. We also note that the volume measure on  $\mathcal{T}_\infty$  is the pushforward of Lebesgue measure on  $\mathbb{R}$  under the mapping  $s \mapsto \mathcal{E}_s$ .

This exploration process allows us to define intervals on  $\mathcal{T}_\infty$ , in a way similar to what we did in Section 2.1. Let us make the convention that, if  $s > t$ , the ‘‘interval’’  $[s, t]$  is defined by  $[s, t] = [s, \infty) \cup (-\infty, t]$ . Then, for every  $x, y \in \mathcal{T}_\infty$ , such that  $x \neq y$ , there is a smallest interval  $[s, t]$ , with  $s, t \in \mathbb{R}$ , such that  $\mathcal{E}_s = x$  and  $\mathcal{E}_t = y$ , and we define

$$[x, y] := \{\mathcal{E}_r : r \in [s, t]\}.$$

Note that we have typically  $[x, y] \neq [y, x]$ . Of course, we take  $[x, x] = \{x\}$ . We sometimes also use the self-evident notation  $]x, y[$ . For  $x \in \mathcal{T}_\infty$ , we finally define  $[x, \infty) = \{\mathcal{E}_r : r \in [s, \infty)\}$ , where  $s$  is the largest real such that  $\mathcal{E}_s = x$ , and we define  $(-\infty, x]$  in a similar manner. Note that  $[x, \infty) \cap (-\infty, x] = \llbracket x, \infty \llbracket$  is the range of the geodesic ray starting from  $x$  in  $\mathcal{T}_\infty$ .

*The finite spine case.* It will also be useful to consider the case where  $w = (w(t))_{0 \leq t \leq \zeta}$  is a finite path in  $\mathcal{W}$  with positive lifetime  $\zeta$ , and  $\mathcal{P}$  and  $\mathcal{P}'$  are now point measures supported on  $(0, \zeta] \times \mathcal{S}$ . We then assume that the obvious adaptations of properties (i)—(v) hold, and in particular (iv) is replaced by

(iv)' the functions  $u \mapsto \beta_u := \sum_{i \in I} \mathbf{1}_{\{t_i \leq u\}} \sigma(\omega_i)$ ,  $u \mapsto \beta'_u := \sum_{i \in J} \mathbf{1}_{\{t'_i \leq u\}} \sigma(\omega'_i)$  take finite values and are monotone increasing on  $[0, \zeta]$ .

The same construction yields a (labeled) compact  $\mathbb{R}$ -tree  $(\mathcal{T}, (\Lambda_v)_{v \in \mathcal{T}})$ , with a spine represented by the interval  $[0, \zeta]$ . The distance on  $\mathcal{T}$  is denoted by  $d_{\mathcal{T}}$  and the labels are defined in exactly the same way as in the infinite spine case. The tree  $\mathcal{T}$  has a cyclic order structure induced by a clockwise exploration function  $\mathcal{E}_s$ , which is conveniently defined on the interval  $[0, \beta_\zeta + \beta'_\zeta]$ : Informally  $(\mathcal{E}_s, s \in [0, \beta_\zeta])$  is obtained by concatenating the mappings  $p_{(\omega_i)}$  for all atoms  $(t_i, \omega_i)$  of  $\mathcal{P}$ , in the increasing order of the  $t_i$ 's, and  $(\mathcal{E}_s, s \in [\beta_\zeta, \beta_\zeta + \beta'_\zeta])$  is obtained by concatenating the mappings  $p_{(\omega'_j)}$  for all atoms  $(t'_j, \omega'_j)$  of  $\mathcal{P}'$ , in the decreasing order of the  $t'_j$ 's (in particular,  $\mathcal{E}_0 = \mathcal{E}_{\beta_\zeta + \beta'_\zeta}$  is the root or bottom of the spine and  $\mathcal{E}_{\beta_\zeta}$  is the top of the spine). In order to define ‘‘intervals’’ on the tree  $\mathcal{T}$ , we now make the convention that, if  $s, t \in [0, \beta_\zeta + \beta'_\zeta]$  and  $t < s$ ,  $[s, t] = [s, \beta_\zeta + \beta'_\zeta] \cup [0, t]$ . In that setting, we again refer to  $(w, \mathcal{P}, \mathcal{P}')$  as a (finite spine) coding triple.

Here, in contrast with the infinite spine case, we can also represent the labeled tree  $(\mathcal{T}, (\Lambda_v)_{v \in \mathcal{T}})$  by a snake trajectory  $\omega \in \mathcal{S}_{w(0)}$  such that  $\mathcal{T}_{(\omega)} = \mathcal{T}$ , which is defined as follows. The duration  $\sigma(\omega)$  is equal to  $\beta_\zeta + \beta'_\zeta$ , and, for every  $s \in [0, \beta_\zeta + \beta'_\zeta]$ , the finite path  $\omega_s$  is such that  $\zeta_{(\omega_s)} = d_{\mathcal{T}}(\mathcal{E}_0, \mathcal{E}_s)$  and  $\widehat{\omega}_s = \Lambda_{\mathcal{E}_s}$  (by a remark in Section 2.1, this completely determines  $\omega$ ). The snake trajectory  $\omega$  obtained in this way will be denoted by  $\omega = \Omega(w, \mathcal{P}, \mathcal{P}')$ . We note that the triple  $(w, \mathcal{P}, \mathcal{P}')$  contains more information than  $\omega$ : Roughly speaking, in order to recover this triple from  $\omega$ , we need to know  $s_0 \in (0, \sigma(\omega))$ , such that the ancestral line of  $p_{(\omega)}(s_0)$  in the genealogical tree of  $\omega$  corresponds to the spine.

It will be useful to consider the time-reversal operation on (finite spine) coding triples satisfying our assumptions, which is defined by

$$\mathbf{TR}\left((w(t))_{0 \leq t \leq \zeta}, \sum_{i \in I} \delta_{(t_i, \omega_i)}, \sum_{j \in J} \delta_{(t'_j, \omega'_j)}\right) := \left((w(\zeta - t))_{0 \leq t \leq \zeta}, \sum_{j \in J} \delta_{(\zeta - t'_j, \omega'_j)}, \sum_{i \in I} \delta_{(\zeta - t_i, \omega_i)}\right). \quad (8)$$

We note that the labeled trees associated with the coding triples in the left and right sides of (8) are identified via an isometry that preserves labels and intervals, but the roles of the top and the bottom of the spine are interchanged.

In Section 3 below we investigate relations between different distributions on coding triples, and in Section 4 we explain how to go from (random) coding triples to the random metric spaces of interest in this work.

*Important remark.* Later, when we speak about the tree associated with a coding triple (as we just defined both in the finite and in the infinite spine case), it will always be understood that this includes the labeling on the tree and the clockwise exploration, which is needed to make sense of intervals on the tree.

*Spine decomposition under  $\mathbb{N}_a$ .* Let  $a > 0$ . We conclude this section with a result connecting the measure  $\mathbb{N}_a$  with a (finite spine) coding triple. Arguing under  $\mathbb{N}_a(d\omega)$ , for every  $r \in (0, \sigma)$ , we can define two point measures  $\mathcal{P}_{(r)}$  and  $\mathcal{P}'_{(r)}$  that account for the (labeled) subtrees branching off the ancestral line of  $p_{(\omega)}(r)$  in the genealogical tree  $\mathcal{T}_{(\omega)}$ . Precisely, if  $r$  is fixed, we consider all connected components  $(u_i, v_i)$ ,  $i \in I$ , of the open set  $\{s \in [0, r] : \zeta_s(\omega) > \min_{t \in [s, r]} \zeta_t(\omega)\}$ , and for each  $i \in I$ , we define a snake trajectory  $\omega^i$  by setting  $\sigma(\omega^i) = v_i - u_i$  and, for every  $s \in [0, \sigma(\omega^i)]$ ,

$$\omega_s^i(t) := \omega_{u_i + s}(\zeta_{u_i}(\omega) + t), \quad \text{for } 0 \leq t \leq \zeta_{(\omega_s^i)} := \zeta_{u_i + s}(\omega) - \zeta_{u_i}(\omega).$$

Note that  $\omega^i \in \mathcal{S}_{\widehat{\omega}_{u_i}}$ , and  $\widehat{\omega}_{u_i} = \omega_r(\zeta_{u_i})$  by the snake property. We then set  $\mathcal{P}_{(r)} = \sum_{i \in I} \delta_{(\zeta_{u_i}, \omega^i)}$ . To define  $\mathcal{P}'_{(r)}$ , we proceed in a very similar manner, replacing the interval  $[0, r]$  by  $[r, \sigma]$ .

Recall our notation  $(L_s^0)_{s \in [0, \sigma]}$  for the exit local time at level 0. Let  $M_p(\mathbb{R}_+ \times \mathcal{S})$  stand for the set of all point measures on  $\mathbb{R}_+ \times \mathcal{S}$ .

**Proposition 2.** *Let  $a > 0$  and let  $Y = (Y_t)_{0 \leq t \leq T^Y}$  stand for a linear Brownian motion started from  $a$  and stopped at its first hitting time of 0. Conditionally given  $Y$ , let  $\mathcal{M}$  and  $\mathcal{M}'$  be two independent Poisson point measures on  $\mathbb{R}_+ \times \mathcal{S}$  with intensity*

$$2 \mathbf{1}_{[0, T^Y]}(t) dt \mathbb{N}_{Y_t}(d\omega).$$

Then, for any nonnegative measurable function  $F$  on  $\mathcal{W} \times M_p(\mathbb{R}_+ \times \mathcal{S})^2$ , we have

$$\mathbb{N}_a \left( \int_0^\sigma dL_r^0 F(W_r, \mathcal{P}_{(r)}, \mathcal{P}'_{(r)}) \right) = \mathbb{E} \left[ F(Y, \mathcal{M}, \mathcal{M}') \right].$$

It is straightforward to verify that  $\mathbb{N}_a(d\omega)$  a.e., for every  $r \in (0, \sigma)$ ,  $(W_r, \mathcal{P}_{(r)}, \mathcal{P}'_{(r)})$  is a coding triple in the sense of the previous discussion (finite spine case), and  $\Omega(W_r, \mathcal{P}_{(r)}, \mathcal{P}'_{(r)}) = \omega$ .

*Proof.* We may assume that  $F(W_r, \mathcal{P}_{(r)}, \mathcal{P}'_{(r)}) = F_1(W_r)F_2(\mathcal{P}_{(r)})F_3(\mathcal{P}'_{(r)})$  where  $F_1$  is defined on  $\mathcal{W}$  and  $F_2$  and  $F_3$  are defined on  $M_p(\mathbb{R}_+ \times \mathcal{S})$ . Let  $(\tau_r^0)_{r \geq 0}$  be the inverse local time defined by  $\tau_r^0 = \inf\{s \geq 0 : L_s^0 \geq r\}$ . Then,

$$\mathbb{N}_a \left( \int_0^\sigma dL_r^0 F_1(W_r)F_2(\mathcal{P}_{(r)})F_3(\mathcal{P}'_{(r)}) \right) = \int_0^\infty dr \mathbb{N}_a \left( \mathbf{1}_{\{\tau_r^0 < \infty\}} F_1(W_{\tau_r^0})F_2(\mathcal{P}_{(\tau_r^0)})F_3(\mathcal{P}'_{(\tau_r^0)}) \right).$$

We may now apply the strong Markov property of the Brownian snake [23, Theorem IV.6], noting that  $F_1(W_{\tau_r^0})F_2(\mathcal{P}_{(\tau_r^0)})$  is measurable with respect to the past up to time  $\tau_r^0$ . Using also [23, Lemma V.5], we get, for every  $r > 0$ ,

$$\mathbb{N}_a \left( \mathbf{1}_{\{\tau_r^0 < \infty\}} F_1(W_{\tau_r^0})F_2(\mathcal{P}_{(\tau_r^0)})F_3(\mathcal{P}'_{(\tau_r^0)}) \right) = \mathbb{N}_a \left( \mathbf{1}_{\{\tau_r^0 < \infty\}} F_1(W_{\tau_r^0})F_2(\mathcal{P}_{(\tau_r^0)}) \mathbb{P}^{(W_{\tau_r^0})}(F_3) \right),$$

where, for any finite path  $w$ , we write  $\mathbb{P}^{(w)}$  for the distribution of a Poisson point measure on  $\mathbb{R}_+ \times M_p(\mathcal{S})$  with intensity  $2 \mathbf{1}_{[0, \zeta(w)]}(t) dt \mathbb{N}_{w(t)}(d\omega)$ . From the preceding two displays, we arrive at

$$\mathbb{N}_a \left( \int_0^\sigma dL_r^0 F_1(W_r)F_2(\mathcal{P}_{(r)})F_3(\mathcal{P}'_{(r)}) \right) = \mathbb{N}_a \left( \int_0^\sigma dL_r^0 F_1(W_r)F_2(\mathcal{P}_{(r)}) \mathbb{P}^{(W_r)}(F_3) \right).$$

By the invariance of the excursion measure  $\mathbb{N}_a$  under time-reversal (this is an immediate consequence of the similar property for the Itô measure of Brownian excursions), the right-hand side of the last display is also equal to

$$\mathbb{N}_a \left( \int_0^\sigma dL_r^0 F_1(W_r)F_2(\check{\mathcal{P}}'_{(r)}) \mathbb{P}^{(W_r)}(F_3) \right),$$

where we write  $\check{\mathcal{P}}'_{(r)}(dsd\omega)$  for the image of  $\mathcal{P}'_{(r)}(dsd\omega)$  under the mapping  $(s, \omega) \mapsto (s, \check{\omega})$ . Then the same application of the strong Markov property shows that this equals

$$\mathbb{N}_a \left( \int_0^\sigma dL_r^0 F_1(W_r) \mathbb{P}^{(W_r)}(F_2) \mathbb{P}^{(W_r)}(F_3) \right).$$

Finally, the first-moment formula in [23, Proposition V.3] shows that this quantity is also equal to

$$\mathbb{E} \left[ F_1(Y) \mathbb{P}^{(Y)}(F_2) \mathbb{P}^{(Y)}(F_3) \right]$$

with the notation of the proposition. This completes the proof.  $\square$

Proposition 2 will allow us to relate properties valid under  $\mathbb{N}_a$  to similar properties for the triple  $(Y, \mathcal{M}, \mathcal{M}')$ . Let us illustrate this on an example that will be useful later. Recall from (2) that the exit measure  $\mathcal{Z}_0$  satisfies

$$\mathcal{Z}_0 = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^2} \int_0^\sigma ds \mathbf{1}_{\{\tau_0(W_s)=\infty, \widehat{W}_s < \varepsilon\}}, \quad \mathbb{N}_a \text{ a.e.}$$

Replacing the limit by a liminf, we may assume that  $\mathcal{Z}_0(\omega)$  is defined for every  $\omega \in \mathcal{W}$ . It is a simple matter to verify that,  $\mathbb{N}_a$  a.e. for every  $r$  such that  $\zeta_r = \tau_0(W_r)$ , we have

$$\mathcal{Z}_0 = \int \mathcal{P}_{(r)}(dtd\varpi) \mathcal{Z}_0(\varpi) + \int \mathcal{P}'_{(r)}(dtd\varpi) \mathcal{Z}_0(\varpi). \quad (9)$$

Then, if we define  $\mathcal{Z}^Y := \int \mathcal{M}(dtd\varpi) \mathcal{Z}_0(\varpi) + \int \mathcal{M}'(dtd\varpi) \mathcal{Z}_0(\varpi)$ , we deduce from Proposition 2 that, for any nonnegative measurable function  $\varphi$  on  $\mathbb{R}_+$ , we have  $\mathbb{N}_a(\mathcal{Z}_0 \varphi(\mathcal{Z}_0)) = \mathbb{E}[\varphi(\mathcal{Z}^Y)]$ . More generally, set  $\tau_u^Y = \inf\{t \geq 0 : Y_t = u\}$  and  $\mathcal{Z}_u^Y := \int \mathcal{M}(dtd\varpi) \mathbf{1}_{\{t < \tau_u^Y\}} \mathcal{Z}_u(\varpi) + \int \mathcal{M}'(dtd\varpi) \mathbf{1}_{\{t < \tau_u^Y\}} \mathcal{Z}_0(\varpi)$ , for every  $u \in (0, a)$ . Then we have

$$\mathbb{N}_a(\mathcal{Z}_0 \varphi(\mathcal{Z}_0, \mathcal{Z}_{u_1}^Y, \dots, \mathcal{Z}_{u_p}^Y)) = \mathbb{E}[\varphi(\mathcal{Z}^Y, \mathcal{Z}_{u_1}^Y, \dots, \mathcal{Z}_{u_p}^Y)], \quad (10)$$

for every  $0 < u_1 < \dots < u_p < a$  and any nonnegative measurable function  $\varphi$  on  $\mathbb{R}_+^{p+1}$ .

### 3 Distributional relations between coding processes

#### 3.1 Some explicit distributions

Let us introduce the function

$$\psi(x) = \frac{2}{\sqrt{\pi}}(x^{1/2} + x^{-1/2}) - 2\left(x + \frac{3}{2}\right) e^x \operatorname{erfc}(\sqrt{x}), \quad x > 0.$$

Note that  $\psi(x) = x^{-1} \chi_3(x)$  in the notation of the Appendix below, and thus, by formula (A.3) there,

$$\int_0^\infty e^{-\lambda x} x \psi(x) dx = (1 + \sqrt{\lambda})^{-3}, \quad \lambda \geq 0. \quad (11)$$

Furthermore one checks from the explicit formula for  $\psi$  that  $\psi(x) = \frac{2}{\sqrt{\pi}}x^{-1/2} + O(1)$  as  $x \rightarrow 0$ , and  $\psi(x) = \frac{3}{2\sqrt{\pi}}x^{-5/2} + O(x^{-7/2})$  as  $x \rightarrow \infty$ .

**Proposition 3.** (i) *Let  $a > 0$ . The density of  $\mathcal{Z}_0$  under  $\mathbb{N}_a(\cdot \cap \{\mathcal{Z}_0 \neq 0\})$  is*

$$h_a(z) := \left(\frac{3}{2a^2}\right)^2 \psi\left(\frac{3z}{2a^2}\right), \quad z > 0. \quad (12)$$

(ii) *For every  $z > 0$  and  $a > 0$ , set*

$$p_z(a) := 2 \left(\frac{3}{2}\right)^{3/2} \sqrt{\pi} z^{3/2} a^{-4} \psi\left(\frac{3z}{2a^2}\right).$$

*Then,  $a \mapsto p_z(a)$  defines a probability density on  $(0, \infty)$ , and for every nonnegative measurable function  $g$  on  $[0, \infty)$ ,*

$$z^{-2} \mathbb{N}^{*,z} \left( \int_0^\sigma ds g(\widehat{W}_s) \right) = \int_0^\infty da p_z(a) g(a).$$

**Remark.** The construction of Brownian disks in [28] allows us to interpret (ii) by saying that  $p_z$  is the density of the distribution of the distance from the distinguished point to the boundary in a free pointed Brownian disk with perimeter  $z$ . See Proposition 14 below.

*Proof.* (i) From (3), we have for  $\lambda \geq 0$ ,

$$\mathbb{N}_a\left(1 - \exp(-\lambda \mathcal{Z}_0)\right) = \left(a\sqrt{2/3} + \lambda^{-1/2}\right)^{-2} = \frac{3}{2a^2} \left(1 + \left(\frac{2a^2\lambda}{3}\right)^{-1/2}\right)^{-2},$$

and in particular  $\mathbb{N}_a(\mathcal{Z}_0 \neq 0) = \frac{3}{2a^2}$  in agreement with (1). On the other hand, by formula (A.4) in the Appendix,

$$\int_0^\infty (1 - e^{-\lambda x}) \psi(x) dx = (1 + \lambda^{-1/2})^{-2}.$$

Part (i) follows by comparing the last two displays.

(ii) From (2) and (6), we get the existence of a measurable function  $\Gamma$  on  $\mathcal{S}$  such that  $\mathcal{Z}_0 = \Gamma(\text{tr}_0(\omega))$ ,  $\mathbb{N}_a(d\omega)$  a.e., for any  $a > 0$ , and also  $\mathcal{Z}_0^* = \Gamma(\omega) = \Gamma(\omega^{[t]})$  for every  $t \in [0, \sigma(\omega)]$ ,  $\mathbb{N}^*(d\omega)$  a.e. By applying the re-rooting formula (5) with  $G(\omega) = f(\Gamma(\omega))g(\omega_0)$ , where  $f$  and  $g$  are nonnegative real functions, we get

$$\mathbb{N}^*\left(f(\mathcal{Z}_0^*) \int_0^\sigma ds g(\widehat{W}_s)\right) = 2 \int_0^\infty da g(a) \mathbb{N}_a\left(\mathcal{Z}_0 f(\mathcal{Z}_0)\right).$$

The left-hand side can be written as

$$\sqrt{\frac{3}{2\pi}} \int_0^\infty dz z^{-5/2} f(z) \mathbb{N}^{*,z}\left(\int_0^\sigma ds g(\widehat{W}_s)\right).$$

On the other hand, part (i) allows us to rewrite the right-hand side as

$$2 \int_0^\infty da g(a) \int_0^\infty dz h_a(z) z f(z) = 2 \int_0^\infty dz z f(z) \int_0^\infty da h_a(z) g(a).$$

By comparing the last two displays, we get,  $dz$  a.e.,

$$z^{-2} \mathbb{N}^{*,z}\left(\int_0^\sigma ds g(\widehat{W}_s)\right) = 2\sqrt{\frac{2\pi}{3}} z^{3/2} \int_0^\infty da h_a(z) g(a) = \int_0^\infty da p_z(a) g(a),$$

where  $p_z(a)$  is as in the proposition. A scaling argument shows that this identity indeed holds for every  $z > 0$ . Since  $z^{-2} \mathbb{N}^{*,z}(\sigma) = 1$ ,  $p_z$  is a probability density, which may also be checked directly.  $\square$

### 3.2 A distributional identity for coding triples

As we already explained, coding triples will be used to construct the random metric spaces of interest in this work. The relevant case for the forthcoming construction of the infinite-volume Brownian disk with perimeter  $z > 0$  may be described as follows: we let  $R = (R_t)_{t \in [0, \infty)}$  be a three-dimensional Bessel process started from 0, we assume that, conditionally on  $R$ ,  $\mathcal{P}$  and  $\mathcal{P}'$  are independent Poisson measures with intensity  $2 dt \mathbb{N}_{R_t}(d\omega)$ , and finally we condition on the event  $\mathcal{Z} = z$ , where  $\mathcal{Z}$  denotes the total exit measure at 0 of the atoms of  $\mathcal{P}$  and  $\mathcal{P}'$ . In Section 3.3, we will give a precise meaning to this conditioning and obtain a conditional distribution  $\Theta_z$  on coding triples, which plays an important role in the next sections. Before doing that, we need to develop certain preliminary tools, and we first recall special cases of a well-known time-reversal property for Bessel processes. Let  $R$  be as above and let  $X$  be a Bessel process of dimension 9 started from 0. Then, for every  $a > 0$ ,

- (a) If  $\mathbf{L}_a := \sup\{t \geq 0 : R_t = a\}$ , the process  $(R_{\mathbf{L}_a - t}, 0 \leq t \leq \mathbf{L}_a)$  is distributed as a linear Brownian motion started from  $a$  and stopped upon hitting 0.
- (b) If  $\mathbf{L}_a := \sup\{t \geq 0 : X_t = a\}$ , the process  $(X_{\mathbf{L}_a - t}, 0 \leq t \leq \mathbf{L}_a)$  is distributed as a Bessel process of dimension  $-5$  started from  $a$  and stopped upon hitting 0.

Both (a) and (b) are special cases of a more general result for Bessel processes, which is itself a consequence of Nagasawa's time-reversal theorem (see [35, Theorem VII.4.5], and [35, Exercise XI.1.23] for the case of interest here, and note that part (a) is due to Williams [37]). As a consequence of (a) and (b), one gets that, for every  $a > 0$ , the process  $(R_{\mathbf{L}_a + t}, t \geq 0)$  is independent of  $(R_t, 0 \leq t \leq \mathbf{L}_a)$ ,

and similarly  $(X_{L_a+t}, t \geq 0)$  is independent of  $(X_t, 0 \leq t \leq L_a)$ . This property is used implicitly in what follows.

Let us introduce some notation needed for the technical results that follow. We fix  $a > 0$  and consider a triple  $(Y, \mathcal{M}, \mathcal{M}')$  distributed as in Proposition 2:  $Y = (Y_t, 0 \leq t \leq T^Y)$  is a linear Brownian motion started from  $a$  and stopped at the first time it hits 0, and, conditionally on  $Y$ ,  $\mathcal{M}$  and  $\mathcal{M}'$  are independent Poisson point measures on  $\mathbb{R}_+ \times \mathcal{S}$  with intensity  $2 \mathbf{1}_{[0, T^Y]}(t) dt \mathbb{N}_{Y_t}(d\omega)$ . We also introduce the point measures  $\widetilde{\mathcal{M}}$  and  $\widetilde{\mathcal{M}}'$  obtained by truncating the atoms of  $\mathcal{M}$  and  $\mathcal{M}'$  at level 0. More precisely, for any nonnegative measurable function  $\Phi$  on  $\mathbb{R}_+ \times \mathcal{S}$ , we set

$$\int \widetilde{\mathcal{M}}(dtd\omega) \Phi(t, \omega) := \int \mathcal{M}(dtd\omega) \Phi(t, \text{tr}_0(\omega))$$

and  $\widetilde{\mathcal{M}}'$  is defined similarly from  $\mathcal{M}'$ . We will be interested in the triple  $(Y, \widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}')$ , which we may view as a coding triple in the sense of Section 2.4.

We define

$$\mathcal{Z}^Y = \int \mathcal{M}(dtd\omega) \mathcal{Z}_0(\omega) + \int \mathcal{M}'(dtd\omega) \mathcal{Z}_0(\omega)$$

in agreement with the end of Section 2.4. We saw that, for any nonnegative measurable function  $\varphi$  on  $\mathbb{R}_+$ , we have  $\mathbb{E}[\varphi(\mathcal{Z}^Y)] = \mathbb{N}_a(\mathcal{Z}_0 \varphi(\mathcal{Z}_0))$ . It then follows from Proposition 3 (i) that the distribution of  $\mathcal{Z}^Y$  has density  $z h_a(z)$ . In particular, we have

$$\mathbb{E}[e^{-\lambda \mathcal{Z}^Y}] = \int_0^\infty z h_a(z) e^{-\lambda z} dz = \left(1 + a\sqrt{2\lambda/3}\right)^{-3}, \quad (13)$$

by (11) and (12).

We write  $(\check{\Theta}_z^{(a)})_{z>0}$  for a regular version of the conditional distributions of the triple  $(Y, \widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}')$  knowing that  $\mathcal{Z}^Y = z$ . The collection  $(\check{\Theta}_z^{(a)})_{z>0}$  is well defined only up to a set of values of  $z$  of zero Lebesgue measure, but we will see later how to make a canonical choice of this collection.

Let us also fix  $r > 0$ . We next consider a triple  $(V, \mathcal{N}, \mathcal{N}')$ , where

- $V = (V_t, 0 \leq t \leq T^V)$  is distributed as a Bessel process of dimension  $-5$  started from  $r + a$  and stopped at the first time it hits  $r$ .
- Conditionally on  $V$ ,  $\mathcal{N}$  and  $\mathcal{N}'$  are independent Poisson point measures on  $\mathbb{R}_+ \times \mathcal{S}$  with intensity

$$2 \mathbf{1}_{[0, T^V]}(t) dt \mathbb{N}_{V_t}(d\omega \cap \{W_* > 0\}),$$

where we recall the notation  $W_*(\omega) = \min\{\widehat{W}_s(\omega) : 0 \leq s \leq \sigma(\omega)\}$ .

We write  $\widetilde{\mathcal{N}}$  and  $\widetilde{\mathcal{N}}'$  for the point measures obtained by truncating the atoms of  $\mathcal{N}$  and  $\mathcal{N}'$  at level  $r$ , in the same way as  $\widetilde{\mathcal{M}}$  was defined above from  $\mathcal{M}$  by truncation at level 0. We also introduce the exit measure

$$\mathcal{Z}^V = \int \mathcal{N}(dtd\omega) \mathcal{Z}_r(\omega) + \int \mathcal{N}'(dtd\omega) \mathcal{Z}_r(\omega).$$

As we will see in the next proof, the distributions of  $\mathcal{Z}^V$  and  $\mathcal{Z}^Y$  are related by the formula

$$\mathbb{E}[h(\mathcal{Z}^V)] = \left(\frac{r+a}{r}\right)^3 \mathbb{E}[h(\mathcal{Z}^Y) e^{-\frac{3}{2r^2} \mathcal{Z}^Y}]. \quad (14)$$

In particular, the distribution of  $\mathcal{Z}^V$  also has a positive density on  $(0, \infty)$ .

We let  $\vartheta_r$  stand for the obvious shift that maps snake trajectories with initial point  $x$  to snake trajectories with initial point  $x - r$ . If  $\mu = \sum_{i \in I} \delta_{(t_i, \omega_i)}$  is a point measure on  $\mathbb{R}_+ \times \mathcal{S}$ , we also write  $\vartheta_r \mu = \sum_{i \in I} \delta_{(t_i, \vartheta_r \omega_i)}$ , by abuse of notation.

**Proposition 4.** *The collection  $(\check{\Theta}_z^{(a)})_{z>0}$  is a regular version of the conditional distributions of  $(V - r, \vartheta_r \widetilde{\mathcal{N}}, \vartheta_r \widetilde{\mathcal{N}}')$  knowing that  $\mathcal{Z}^V = z$ .*

In other words, the conditional distribution of  $(V - r, \vartheta_r \widetilde{\mathcal{N}}, \vartheta_r \widetilde{\mathcal{N}}')$  knowing that  $\mathcal{Z}^V = z$  coincides with the conditional distribution of  $(Y, \widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}')$  knowing that  $\mathcal{Z}^Y = z$ . In particular, the conditional distribution of  $(V - r, \vartheta_r \widetilde{\mathcal{N}}, \vartheta_r \widetilde{\mathcal{N}}')$  knowing that  $\mathcal{Z}^V = z$  does not depend on  $r$ , which is by no means an obvious fact.

*Proof.* Recall our notation  $M_p(\mathbb{R}_+ \times \mathcal{S})$  for the set of all point measures on  $\mathbb{R}_+ \times \mathcal{S}$ . As in the proof of Proposition 2, if  $w$  is a finite path taking nonnegative values, we write  $\mathbb{P}^{(w)}(d\mu)$  for the probability measure on  $M_p(\mathbb{R}_+ \times \mathcal{S})$  which is the distribution of a Poisson point measure on  $\mathbb{R}_+ \times \mathcal{S}$  with intensity  $2 \mathbf{1}_{[0, \zeta(w)]}(t) dt \mathbb{N}_{w(t)}(d\omega)$ . Denoting the generic element of  $M_p(\mathbb{R}_+ \times \mathcal{S}) \times M_p(\mathbb{R}_+ \times \mathcal{S})$  by  $(\mu, \mu')$ , we have the formula

$$\mathbb{P}^{(w)} \otimes \mathbb{P}^{(w)}(\mu(W_* \leq 0) = \mu'(W_* \leq 0) = 0) = \exp\left(-4 \int_0^{\zeta(w)} dt \mathbb{N}_{w(t)}(W_* \leq 0)\right) = \exp\left(-6 \int_0^{\zeta(w)} \frac{dt}{w(t)^2}\right), \quad (15)$$

where in the left-hand side we abuse notation by writing  $\mu(W_* \leq 0)$  instead of  $\mu(\{(t, \omega) \in \mathbb{R}_+ \times \mathcal{S} : W_*(\omega) \leq 0\})$ .

We then introduce a random finite path  $(U_t)_{0 \leq t \leq T^U}$ , which is distributed as a linear Brownian motion started from  $r + a$  and stopped when hitting  $r$  (so  $U - r$  has the same distribution as  $Y$ ). Let  $\mathcal{P}$  and  $\mathcal{P}'$  be random elements of  $M_p(\mathbb{R}_+ \times \mathcal{S})$  such that the conditional distribution of the pair  $(\mathcal{P}, \mathcal{P}')$  given  $U$  is  $\mathbb{P}^{(U)} \otimes \mathbb{P}^{(U)}(d\mu d\mu')$ . Define

$$\mathcal{Z}^U = \int \mathcal{P}(dtd\omega) \mathcal{Z}_r(\omega) + \int \mathcal{P}'(dtd\omega) \mathcal{Z}_r(\omega),$$

and also write  $\widetilde{\mathcal{P}}$ , resp.  $\widetilde{\mathcal{P}}'$ , for the point measure  $\mathcal{P}$ , resp.  $\mathcal{P}'$ , truncated at level  $r$ . Then, the statement of the proposition reduces to showing that the conditional distribution of  $(V, \widetilde{\mathcal{N}}, \widetilde{\mathcal{N}}')$  knowing that  $\mathcal{Z}^V = z$  coincides dz a.e. with the conditional distribution of  $(U, \widetilde{\mathcal{P}}, \widetilde{\mathcal{P}}')$  knowing that  $\mathcal{Z}^U = z$  (an obvious translation argument yields that  $(U - r, \vartheta_r \widetilde{\mathcal{P}}, \vartheta_r \widetilde{\mathcal{P}}', \mathcal{Z}^U)$  has the same law as  $(Y, \widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}', \mathcal{Z}^Y)$  above).

The first step of the proof is to verify that, for any nonnegative measurable functions  $F$  and  $G$  defined on  $\mathcal{W}$  and on  $M_p(\mathbb{R}_+ \times \mathcal{S})^2$  respectively,

$$\begin{aligned} & \mathbb{E}\left[F(U)G(\mathcal{P}, \mathcal{P}') \mid \mathcal{P}(W_* \leq 0) = \mathcal{P}'(W_* \leq 0) = 0\right] \\ &= \mathbb{E}\left[F(U) \mathbb{P}^{(U)} \otimes \mathbb{P}^{(U)}[G(\mu, \mu') \mid \mu(W_* \leq 0) = \mu'(W_* \leq 0) = 0]\right]. \end{aligned} \quad (16)$$

To prove (16), we first apply (15) to get

$$\mathbb{P}(\mathcal{P}(W_* \leq 0) = \mathcal{P}'(W_* \leq 0) = 0) = \mathbb{E}\left[\exp\left(-6 \int_0^{T^U} \frac{dt}{U_t^2}\right)\right] = \left(\frac{r}{r+a}\right)^3$$

where the last equality is easily derived by using Itô's formula to verify that  $U_{t \wedge T^U}^{-3} \exp(-6 \int_0^{t \wedge T^U} U_s^{-2} ds)$  is a martingale. So we have

$$\begin{aligned} & \left(\frac{r}{r+a}\right)^3 \mathbb{E}\left[F(U)G(\mathcal{P}, \mathcal{P}') \mid \mathcal{P}(W_* \leq 0) = \mathcal{P}'(W_* \leq 0) = 0\right] \\ &= \mathbb{E}\left[F(U)G(\mathcal{P}, \mathcal{P}') \mathbf{1}_{\{\mathcal{P}(W_* \leq 0) = \mathcal{P}'(W_* \leq 0) = 0\}}\right] \\ &= \mathbb{E}\left[F(U) \mathbb{P}^{(U)} \otimes \mathbb{P}^{(U)}[G(\mu, \mu') \mathbf{1}_{\{\mu(W_* \leq 0) = \mu'(W_* \leq 0) = 0\}}]\right] \\ &= \mathbb{E}\left[F(U) \exp\left(-6 \int_0^{T^U} \frac{dt}{U_t^2}\right) \mathbb{P}^{(U)} \otimes \mathbb{P}^{(U)}[G(\mu, \mu') \mid \mu(W_* \leq 0) = \mu'(W_* \leq 0) = 0]\right] \end{aligned}$$

using (15) in the last equality. To complete the proof of (16), we just observe that, by classical absolute continuity relations between Brownian motion and Bessel processes, the law of  $U$  under the probability measure

$$\left(\frac{r+a}{r}\right)^3 \exp\left(-6 \int_0^{T^U} \frac{dt}{U_t^2}\right) \cdot \mathbb{P}$$

coincides with the law of  $V$  under  $\mathbb{P}$  (see [26, Lemma 1] for a short proof).

Let us complete the proof of the proposition. By a standard property of Poisson measures and the definition of the pair  $(\mathcal{N}, \mathcal{N}')$ , we have

$$\mathbb{E}[G(\mathcal{N}, \mathcal{N}') \mid V] = \mathbb{P}^{(V)} \otimes \mathbb{P}^{(V)}[G(\mu, \mu') \mid \mu(W_* \leq 0) = \mu'(W_* \leq 0) = 0].$$

It thus follows from (16) that

$$\mathbb{E}\left[F(U)G(\mathcal{P}, \mathcal{P}') \mid \mathcal{P}(W_* \leq 0) = \mathcal{P}'(W_* \leq 0) = 0\right] = \mathbb{E}[F(V) \mathbb{E}[G(\mathcal{N}, \mathcal{N}') \mid V]] = \mathbb{E}[F(V) G(\mathcal{N}, \mathcal{N}')].$$

In particular, for any nonnegative measurable function  $h$  on  $[0, \infty)$ , we have

$$\mathbb{E}\left[F(U)G(\tilde{\mathcal{P}}, \tilde{\mathcal{P}}')h(\mathcal{Z}^U) \mid \mathcal{P}(W_* \leq 0) = \mathcal{P}'(W_* \leq 0) = 0\right] = \mathbb{E}[F(V) G(\tilde{\mathcal{N}}, \tilde{\mathcal{N}}') h(\mathcal{Z}^V)].$$

The left-hand side of the last display is equal to

$$\left(\frac{r+a}{r}\right)^3 \mathbb{E}\left[F(U)G(\tilde{\mathcal{P}}, \tilde{\mathcal{P}}')h(\mathcal{Z}^U) \exp\left(-\frac{3}{2r^2} \mathcal{Z}^U\right)\right]$$

because, on one hand, we saw that  $\mathbb{P}(\mathcal{P}(W_* \leq 0) = \mathcal{P}'(W_* \leq 0) = 0) = \left(\frac{r}{r+a}\right)^3$  and, on the other hand, the special Markov property (see e.g. the appendix of [27]) and (1) show that

$$\mathbb{P}(\mathcal{P}(W_* \leq 0) = \mathcal{P}'(W_* \leq 0) = 0 \mid U, \tilde{\mathcal{P}}, \tilde{\mathcal{P}}') = \exp\left(-\frac{3}{2r^2} \mathcal{Z}^U\right).$$

We can find nonnegative measurable functions  $\varphi_1$  and  $\varphi_2$  on  $[0, \infty)$  such that

$$\mathbb{E}\left[F(U)G(\tilde{\mathcal{P}}, \tilde{\mathcal{P}}') \mid \mathcal{Z}^U\right] = \varphi_1(\mathcal{Z}^U), \quad \mathbb{E}[F(V) G(\tilde{\mathcal{N}}, \tilde{\mathcal{N}}') \mid \mathcal{Z}^V] = \varphi_2(\mathcal{Z}^V),$$

and it follows from the preceding considerations that, for any function  $h$ ,

$$\left(\frac{r+a}{r}\right)^3 \mathbb{E}[\varphi_1(\mathcal{Z}^U) h(\mathcal{Z}^U) \exp\left(-\frac{3}{2r^2} \mathcal{Z}^U\right)] = \mathbb{E}[\varphi_2(\mathcal{Z}^V) h(\mathcal{Z}^V)]. \quad (17)$$

By specializing this identity to the case  $F = 1$ ,  $G = 1$ , we get the relation (14) between the distributions of  $\mathcal{Z}^V$  and  $\mathcal{Z}^U$  (recall that  $\mathcal{Z}^U$  has the same distribution as  $\mathcal{Z}^Y$ ). But then it also follows from (17) that (for arbitrary  $F$  and  $G$ ) we have  $\mathbb{E}[\varphi_1(\mathcal{Z}^V) h(\mathcal{Z}^V)] = \mathbb{E}[\varphi_2(\mathcal{Z}^V) h(\mathcal{Z}^V)]$  for any test function  $h$ , so that  $\varphi_1(\mathcal{Z}^V) = \varphi_2(\mathcal{Z}^V)$  a.s. and  $\varphi_1(z) = \varphi_2(z)$ ,  $dz$  a.e., which completes the proof.  $\square$

For every  $z > 0$ , we let  $\Theta_z^{(a)}$  denote the image of  $\check{\Theta}_z^{(a)}$  under the time-reversal transformation  $\mathbf{TR}$  in (8). Since the process  $(Y_t, 0 \leq t \leq T^Y)$  is mapped by time-reversal to a three-dimensional Bessel process started from 0 and stopped at its last passage at  $a$  (by property (a) stated at the beginning of the section), we could have defined  $\Theta_z^{(a)}$  directly in terms of conditioning a coding triple whose first component is a three-dimensional Bessel process up to a last passage time. The connection with the discussion at the beginning of this section should then be clear:  $\Theta_z^{(a)}$  is the analog of the probability measure  $\Theta_z$  we are aiming at, when the three-dimensional Bessel process is truncated at a last passage time.

### 3.3 The coding triple of the infinite-volume Brownian disk

In this section, we define the probability measures  $\Theta_z$ ,  $z > 0$ , which were introduced informally at the beginning of the preceding section. Roughly speaking, the idea is to get  $\Theta_z$  as the limit of  $\Theta_z^{(a)}$  as  $a \rightarrow \infty$ . Proposition 6 below will also show that, for every  $r > 0$ , the collection  $(\Theta_z)_{z>0}$  corresponds to conditional distributions of a coding triple whose first component is a nine-dimensional Bessel process considered after its last passage time at  $r$  (compare with Proposition 4). The latter fact is the key to the identification as infinite-volume Brownian disks of the complement of hulls in the Brownian plane.



We consider a triple  $(X, \mathcal{L}, \mathcal{R})$  such that  $X = (X_t)_{t \geq 0}$  is a nine-dimensional Bessel process started from 0 and, conditionally on  $X$ ,  $\mathcal{L}$  and  $\mathcal{R}$  are two independent Poisson measures on  $\mathbb{R}_+ \times \mathcal{S}$  with intensity

$$2 \, dt \, \mathbb{N}_{X_t}(d\omega \cap \{W_* > 0\}).$$

We also set, for every  $r > 0$ ,

$$L_r := \sup\{t \geq 0 : X_t = r\}. \quad (18)$$

In what follows, we fix  $r > 0$ , and we shall be interested in atoms  $(t, \omega)$  of  $\mathcal{L}$  or  $\mathcal{R}$  such that  $t > L_r$ . More precisely, we introduce a point measure  $\mathcal{L}^{(r)}$  as the image of

$$\mathbf{1}_{(L_r, \infty)}(t) \mathcal{L}(dt \, d\omega)$$

under the mapping  $(t, \omega) \mapsto (t - L_r, \vartheta_r \omega)$  (where  $\vartheta_r$  is the shift operator already used in Proposition 4). In a way similar to the previous section, we define  $\tilde{\mathcal{L}}^{(r)}$  by truncating the atoms of  $\mathcal{L}^{(r)}$  at level 0 (more precisely,  $\tilde{\mathcal{L}}^{(r)}$  is the image of  $\mathcal{L}^{(r)}$  under the mapping  $(t, \omega) \mapsto (t, \text{tr}_0(\omega))$ ). We define similarly  $\mathcal{R}^{(r)}$  and  $\tilde{\mathcal{R}}^{(r)}$  from the point measure  $\mathcal{R}$ . Finally, we set

$$\begin{aligned} \mathcal{Z}^{(r)} &= \int \mathcal{L}^{(r)}(dt \, d\omega) \mathcal{Z}_0(\omega) + \int \mathcal{R}^{(r)}(dt \, d\omega) \mathcal{Z}_0(\omega) \\ &= \int \mathcal{L}(dt \, d\omega) \mathbf{1}_{(L_r, \infty)}(t) \mathcal{Z}_r(\omega) + \int \mathcal{R}(dt \, d\omega) \mathbf{1}_{(L_r, \infty)}(t) \mathcal{Z}_r(\omega) \end{aligned} \quad (19)$$

and we also consider the process  $(X_t^{(r)})_{t \geq 0}$  defined by

$$X_t^{(r)} = X_{L_r+t} - r.$$

By [15, Proposition 1.2], the distribution of  $\mathcal{Z}^{(r)}$  has a density given by

$$k_r(z) := \frac{1}{\sqrt{\pi}} 3^{3/2} 2^{-1/2} r^{-3} z^{1/2} e^{-\frac{3z}{2r^2}}. \quad (20)$$

Our first goal is to verify that the conditional distribution of the triple  $(X^{(r)}, \tilde{\mathcal{L}}^{(r)}, \tilde{\mathcal{R}}^{(r)})$  knowing that  $\mathcal{Z}^{(r)} = z$  does not depend on  $r$ . Note that, for instance, the unconditional distribution of  $X^{(r)}$  depends on  $r$ .

We will deduce the preceding assertion from Proposition 4, but to this end a truncation argument is needed. So we consider  $a > 0$ , and we set

$$L_a^{(r)} := L_{r+a} - L_r = \sup\{t \geq 0 : X_t^{(r)} = a\}.$$

We then set<sup>1</sup>

$$\begin{aligned} \mathcal{L}^{(r, r+a)}(dt \, d\omega) &= \mathbf{1}_{[0, L_a^{(r)}]}(t) \mathcal{L}^{(r)}(dt \, d\omega), \\ \mathcal{L}^{(r+a, \infty)}(dt \, d\omega) &= \mathbf{1}_{(L_a^{(r)}, \infty)}(t) \mathcal{L}^{(r)}(dt \, d\omega), \end{aligned}$$

and we define  $\mathcal{R}^{(r, r+a)}$  and  $\mathcal{R}^{(r+a, \infty)}$  in a similar way from  $\mathcal{R}^{(r)}$ . As previously, we let  $\tilde{\mathcal{L}}^{(r, r+a)}$ ,  $\tilde{\mathcal{L}}^{(r+a, \infty)}$ ,  $\tilde{\mathcal{R}}^{(r, r+a)}$ ,  $\tilde{\mathcal{R}}^{(r+a, \infty)}$  stand for these point measures truncated at level 0. We finally set

$$\begin{aligned} \mathcal{Z}^{(r, r+a)} &= \int \mathcal{L}^{(r, r+a)}(dt \, d\omega) \mathcal{Z}_0(\omega) + \int \mathcal{R}^{(r, r+a)}(dt \, d\omega) \mathcal{Z}_0(\omega) \\ \mathcal{Z}^{(r+a, \infty)} &= \int \mathcal{L}^{(r+a, \infty)}(dt \, d\omega) \mathcal{Z}_0(\omega) + \int \mathcal{R}^{(r+a, \infty)}(dt \, d\omega) \mathcal{Z}_0(\omega). \end{aligned}$$

Obviously  $\mathcal{Z}^{(r)} = \mathcal{Z}^{(r, r+a)} + \mathcal{Z}^{(r+a, \infty)}$ . Also, the random variables  $\mathcal{Z}^{(r, r+a)}$  and  $\mathcal{Z}^{(r+a, \infty)}$  are independent, as a consequence of the independence properties stated at the beginning of Section 3.2 after properties (a) and (b).

<sup>1</sup>Our notation is somewhat misleading since  $\mathcal{L}^{(r+a, \infty)}$  and  $\mathcal{R}^{(r+a, \infty)}$  both depend on  $r$  and not only on  $r+a$ . Since  $r$  is fixed in most of this section, this should not be confusing.

**Lemma 5.** *The collection  $(\Theta_z^{(a)})_{z>0}$  is a regular version of the conditional distributions of the triple*

$$\left( (X_t^{(r)})_{0 \leq t \leq L_a^{(r)}}, \tilde{\mathcal{L}}^{(r,r+a)}, \tilde{\mathcal{R}}^{(r,r+a)} \right)$$

knowing that  $\mathcal{Z}^{(r,r+a)} = z$ .

This lemma is merely a reformulation of Proposition 4. The point is that the time-reversed process  $(X_{L_{r+a}-t})_{0 \leq t \leq L_a^{(r)}}$  is distributed as a Bessel process of dimension  $-5$  started from  $r+a$  and stopped upon hitting  $r$  (by property (b) stated at the beginning of Section 3.2). Recalling our notation  $\mathbf{TR}$  for the time-reversal operation defined in (8), it follows that

$$\left( \mathbf{TR} \left( (X_t^{(r)})_{0 \leq t \leq L_a^{(r)}}, \tilde{\mathcal{L}}^{(r,r+a)}, \tilde{\mathcal{R}}^{(r,r+a)} \right), \mathcal{Z}^{(r,r+a)} \right)$$

has the same distribution as  $((V-r, \vartheta_r \tilde{\mathcal{N}}, \vartheta_r \tilde{\mathcal{N}}'), \mathcal{Z}^V)$ , with the notation introduced before Proposition 4. The result of the lemma now follows from Proposition 4.

Since the distribution of  $\mathcal{Z}^{(r,r+a)}$  is the same as the distribution of  $\mathcal{Z}^V$  in the preceding section, it has a positive density with respect to Lebesgue measure, which we denote by  $g_{r,a}(z)$ . Recalling that  $\mathcal{Z}^Y$  has density  $z h_a(z)$ , (14) gives the explicit expression

$$g_{r,a}(z) = \left( \frac{r+a}{r} \right)^3 e^{-\frac{3z}{2r^2}} z h_a(z) \quad (21)$$

where  $h_a$  is defined in (12).

On the other hand, the distribution of  $\mathcal{Z}^{(r+a,\infty)}$  may be written in the form

$$(1 - \varepsilon_{r,a}) \delta_0(dz) + \Upsilon_{r,a}(dz)$$

where  $\varepsilon_{r,a} \in [0, 1]$  and the measure  $\Upsilon_{r,a}$  is supported on  $(0, \infty)$ . Note that

$$\varepsilon_{r,a} = \Upsilon_{r,a}((0, \infty)) = \mathbb{P}(\mathcal{Z}^{(r+a,\infty)} > 0) = 1 - \left( \frac{a}{r+a} \right)^3$$

where the last equality follows from Lemma 4.2 in [15], using the fact that  $\mathbb{N}_x(0 < W_* \leq r) = \frac{3}{2}((x-r)^{-2} - x^{-2})$  for  $x > r$ . In particular,  $\varepsilon_{r,a} \rightarrow 0$  as  $a \rightarrow \infty$ .

Recall that  $k_r(z)$  denotes the density of  $\mathcal{Z}^{(r)}$  (cf. (20)). Since  $\mathcal{Z}^{(r)} = \mathcal{Z}^{(r,r+a)} + \mathcal{Z}^{(r+a,\infty)}$ , and  $\mathcal{Z}^{(r,r+a)}$  and  $\mathcal{Z}^{(r+a,\infty)}$  are independent, the conditional distributions of  $\mathcal{Z}^{(r+a,\infty)}$  knowing that  $\mathcal{Z}^{(r)} = z$  are defined in a canonical manner by

$$\nu_{r,a}(dz' | z) = \frac{1}{k_r(z)} \left( (1 - \varepsilon_{r,a}) g_{r,a}(z) \delta_0(dz') + g_{r,a}(z - z') \Upsilon_{r,a}(dz') \right).$$

In particular, we have for every  $z > 0$ ,

$$\nu_{r,a}(\{0\} | z) = \frac{(1 - \varepsilon_{r,a}) g_{r,a}(z)}{k_r(z)},$$

and the explicit expression (21) can be used to verify that  $g_{r,a}(z) \rightarrow k_r(z)$  as  $a \rightarrow \infty$ . It follows that

$$\nu_{r,a}(\{0\} | z) \xrightarrow{a \rightarrow \infty} 1. \quad (22)$$

Recall the scaling transformations  $\theta_\lambda$  on snake trajectories defined in Section 2.2. It will also be useful to consider restriction operators which are defined as follows. For every  $a > 0$ ,  $\mathfrak{R}_a$  acts both on  $\mathcal{W}_0^\infty \times M_p(\mathbb{R}_+ \times \mathcal{S})^2$  and on  $\mathcal{W}_0 \times M_p(\mathbb{R}_+ \times \mathcal{S})^2$  by

$$\mathfrak{R}_a : \left( \mathbf{w}, \sum_{i \in I} \delta_{(t_i, \omega_i)}, \sum_{j \in J} \delta_{(t'_j, \omega'_j)} \right) \mapsto \left( (\mathbf{w}(t))_{t \leq \lambda^{(a)}(\mathbf{w})}, \sum_{i \in I, t_i \leq \lambda^{(a)}(\mathbf{w})} \delta_{(t_i, \omega_i)}, \sum_{j \in J, t'_j \leq \lambda^{(a)}(\mathbf{w})} \delta_{(t'_j, \omega'_j)} \right), \quad (23)$$

where  $\lambda^{(a)}(\mathbf{w}) = \sup\{t \geq 0 : \mathbf{w}(t) \leq a\}$  for  $\mathbf{w} \in \mathcal{W}_0^\infty$  or  $\mathbf{w} \in \mathcal{W}_0$ .

**Proposition 6.** We can find a collection  $(\Theta_z)_{z>0}$  of probability measures on  $\mathcal{W}^\infty \times M_p(\mathbb{R}_+ \times \mathcal{S})^2$  that does not depend on  $r$  and is such that, for every  $r > 0$ ,  $(\Theta_z)_{z>0}$  is a regular version of the conditional distributions of the triple

$$(X^{(r)}, \tilde{\mathcal{L}}^{(r)}, \tilde{\mathcal{R}}^{(r)})$$

knowing that  $\mathcal{Z}^{(r)} = z$ . This collection is unique if we impose the additional scaling invariance property: for every  $\lambda > 0$  and  $z > 0$ ,  $\Theta_{\lambda z}$  is the image of  $\Theta_z$  under the scaling transformation

$$\Sigma_\lambda : \left( \mathbb{w}, \sum_{i \in I} \delta_{(t_i, \omega_i)}, \sum_{j \in J} \delta_{(t'_j, \omega'_j)} \right) \mapsto \left( \sqrt{\lambda} \mathbb{w}(\cdot/\lambda), \sum_{i \in I} \delta_{(\lambda t_i, \theta_\lambda(\omega_i))}, \sum_{j \in J} \delta_{(\lambda t'_j, \theta_\lambda(\omega'_j))} \right).$$

*Proof.* Let  $r > 0$ , and let  $(\Theta_{z,r})_{z>0}$  be a regular version of the conditional distributions of  $(X^{(r)}, \tilde{\mathcal{L}}^{(r)}, \tilde{\mathcal{R}}^{(r)})$  knowing  $\mathcal{Z}^{(r)} = z$ . Our first goal is to verify that  $(\Theta_{z,r})_{z>0}$  does not depend on  $r$ , except possibly on a values of  $z$  of zero Lebesgue measure. To this end, let  $c > 0$  and let  $G$  be a measurable function on  $\mathcal{W} \times M_p(\mathbb{R}_+ \times \mathcal{S})^2$  such that  $0 \leq G \leq 1$ . Then, for every  $a \geq c$ , and for every nonnegative measurable function  $f$  on  $(0, \infty)$ ,

$$\begin{aligned} \int dz k_r(z) f(z) \Theta_{z,r}(G \circ \mathfrak{R}_c) &= \mathbb{E}[G((X_t^{(r)})_{0 \leq t \leq L_c^{(r)}}, \tilde{\mathcal{L}}^{(r,r+c)}, \tilde{\mathcal{R}}^{(r,r+c)}) f(\mathcal{Z}^{(r)})] \\ &= \mathbb{E}\left[\mathbb{E}[G((X_t^{(r)})_{0 \leq t \leq L_c^{(r)}}, \tilde{\mathcal{L}}^{(r,r+c)}, \tilde{\mathcal{R}}^{(r,r+c)}) \mid \mathcal{Z}^{(r,r+a)}] f(\mathcal{Z}^{(r,r+a)} + \mathcal{Z}^{(r+a,\infty)})\right] \end{aligned}$$

where we use the fact that  $\mathcal{Z}^{(r+a,\infty)}$  is independent of  $((X_t^{(r)})_{0 \leq t \leq L_c^{(r)}}, \tilde{\mathcal{L}}^{(r,r+c)}, \tilde{\mathcal{R}}^{(r,r+c)})$ ,  $\mathcal{Z}^{(r,r+a)}$  to write the last equality. By Lemma 5, we have

$$\mathbb{E}[G((X_t^{(r)})_{0 \leq t \leq L_c^{(r)}}, \tilde{\mathcal{L}}^{(r,r+c)}, \tilde{\mathcal{R}}^{(r,r+c)}) \mid \mathcal{Z}^{(r,r+a)}] = \Phi(\mathcal{Z}^{(r,r+a)})$$

where  $\Phi(z) = \Theta_z^{(a)}(G \circ \mathfrak{R}_c)$ . Using the explicit distribution of  $\mathcal{Z}^{(r,r+a)}$  and  $\mathcal{Z}^{(r+a,\infty)}$ , we thus get

$$\begin{aligned} \int dz k_r(z) f(z) \Theta_{z,r}(G \circ \mathfrak{R}_c) &= \int dy g_{r,a}(y) \int ((1 - \varepsilon_{r,a})\delta_0 + \Upsilon_{r,a})(dy') f(y + y') \Theta_y^{(a)}(G \circ \mathfrak{R}_c) \\ &= \int dz k_r(z) f(z) \int \nu_{r,a}(dz' \mid z) \Theta_{z-z'}^{(a)}(G \circ \mathfrak{R}_c). \end{aligned}$$

It follows that we have,  $dz$  a.e.,

$$\Theta_{z,r}(G \circ \mathfrak{R}_c) = \int \nu_{r,a}(dz' \mid z) \Theta_{z-z'}^{(a)}(G \circ \mathfrak{R}_c) = \nu_{r,a}(\{0\} \mid z) \Theta_z^{(a)}(F) + \kappa_{r,a}(z)$$

where the ‘‘remainder’’  $\kappa_{r,a}(z)$  is nonnegative and bounded above by  $1 - \nu_{r,a}(\{0\} \mid z)$ . Specializing to integer values of  $a$  and using (22), we get,  $dz$  a.e.,

$$\lim_{\mathbb{N} \ni k \rightarrow \infty} \Theta_z^{(k)}(G \circ \mathfrak{R}_c) = \Theta_{z,r}(G \circ \mathfrak{R}_c).$$

Since the left-hand side does not depend on  $r$ , we conclude that, for every  $r, r' > 0$ , we must have  $\Theta_{z,r}(G \circ \mathfrak{R}_c) = \Theta_{z,r'}(G \circ \mathfrak{R}_c)$ ,  $dz$  a.e., and since this holds for any  $c > 0$  and any function  $G$ , we conclude that  $\Theta_{z,r} = \Theta_{z,r'}$ ,  $dz$  a.e. So, if we take  $\bar{\Theta}_z = \Theta_{z,1}$ , the collection  $(\bar{\Theta}_z)_{z>0}$  satisfies the first part of the statement.

It remains to obtain the scaling invariance property. To this end, we first observe that the process

$$X_t^{\{\lambda\}} := \sqrt{\lambda} X_{t/\lambda}$$

remains a nine-dimensional Bessel process started from 0. Furthermore, with an obvious notation, we have  $L_{r\sqrt{\lambda}}^{\{\lambda\}} = \lambda L_r$  for every  $r > 0$ . Then, it is straightforward to verify that the image of  $\mathcal{L}$  under the transformation

$$\sum_{i \in I} \delta_{(t_i, \omega_i)} \mapsto \sum_{i \in I} \delta_{(\lambda t_i, \theta_\lambda(\omega_i))} \quad (24)$$

is, conditionally on  $X^{\{\lambda\}}$ , a Poisson point measure with intensity

$$2 \, dt \mathbb{N}_{X_t^{\{\lambda\}}} (d\omega \cap \{W_* > 0\}).$$

It follows that the image of  $\mathcal{L}^{(r)}$  under the scaling transformation (24) has the same distribution as  $\mathcal{L}^{(r\sqrt{\lambda})}$ .

We also note that, for every  $x > 0$ , we have  $\mathcal{Z}_0(\theta_\lambda(\omega)) = \lambda \mathcal{Z}_0(\omega)$ ,  $\mathbb{N}_x(d\omega)$  a.e. By combining the preceding observations, we get that, for every  $r > 0$ , the image of the triple  $(X^{(r)}, \tilde{\mathcal{L}}^{(r)}, \tilde{\mathcal{R}}^{(r)})$  under the scaling transformation  $\Sigma_\lambda$  has the same distribution as  $(X^{(r\sqrt{\lambda})}, \tilde{\mathcal{L}}^{(r\sqrt{\lambda})}, \tilde{\mathcal{R}}^{(r\sqrt{\lambda})})$ , and moreover the exit measure at 0 associated with  $\Sigma_\lambda(X^{(r)}, \tilde{\mathcal{L}}^{(r)}, \tilde{\mathcal{R}}^{(r)})$  is  $\lambda \mathcal{Z}^{(r)}$ . By considering conditional distributions with respect to  $\mathcal{Z}^{(r)}$  and using the first part of the proof, we obtain that  $\Sigma_\lambda(\bar{\Theta}_z) = \bar{\Theta}_{\lambda z}$  for a.e.  $z > 0$ . A Fubini type argument allows us to single out  $z_0 > 0$  such that the equality  $\bar{\Theta}_{\lambda z_0} = \Sigma_\lambda(\bar{\Theta}_{z_0})$  holds for a.e.  $\lambda > 0$ . We then define, for every  $z > 0$ ,

$$\Theta_z = \Sigma_{z/z_0}(\bar{\Theta}_{z_0}).$$

Clearly the collection  $(\Theta_z)_{z>0}$  is also a regular version of the conditional distributions of the triple  $(X^{(r)}, \tilde{\mathcal{L}}^{(r)}, \tilde{\mathcal{R}}^{(r)})$  knowing that  $\mathcal{Z}^{(r)} = z$  (for any  $r > 0$ ). Furthermore, by construction, the equality  $\Sigma_\lambda(\Theta_z) = \Theta_{\lambda z}$  holds for every  $z > 0$  and  $\lambda > 0$ . This completes the proof, except for the uniqueness statement, which is easy and left to the reader.  $\square$

From now on,  $(\Theta_z)_{z>0}$  is the unique collection satisfying the properties stated in Proposition 6. Thanks to the scaling invariance property, we can in fact define this collection without appealing to any conditioning. We consider the triple  $(X^{(r)}, \tilde{\mathcal{L}}^{(r)}, \tilde{\mathcal{R}}^{(r)})$  as defined at the beginning of the section, and recall the notation  $\mathcal{Z}^{(r)}$  and the scaling operators  $\Sigma_\lambda$  in Proposition 6.

**Proposition 7.** *Let  $r > 0$  and  $z > 0$ . Then  $\Theta_z$  is the distribution of  $\Sigma_{z/\mathcal{Z}^{(r)}}(X^{(r)}, \tilde{\mathcal{L}}^{(r)}, \tilde{\mathcal{R}}^{(r)})$ .*

*Proof.* Let  $F$  be a nonnegative measurable function on  $\mathcal{W}^\infty \times M_p(\mathbb{R}_+ \times \mathcal{S})^2$ , and recall that the distribution of  $\mathcal{Z}^{(r)}$  has density  $k_r(z)$ . Then, using Proposition 6,

$$\mathbb{E}\left[F\left(\Sigma_{z/\mathcal{Z}^{(r)}}(X^{(r)}, \tilde{\mathcal{L}}^{(r)}, \tilde{\mathcal{R}}^{(r)})\right)\right] = \int_0^\infty dy k_r(y) \Theta_y(F \circ \Sigma_{z/y}) = \Theta_z(F),$$

since the image of  $\Theta_y$  under  $\Sigma_{z/y}$  is  $\Theta_z$ .  $\square$

Proposition 7 is useful to derive almost sure properties of coding triples distributed according to  $\Theta_z$ . We give an important example.

**Corollary 8.** *Let  $z > 0$ , and let  $\mathcal{T}$  be the labeled tree associated with a coding triple distributed according to  $\Theta_z$ . Write  $(\Lambda_v)_{v \in \mathcal{T}}$  for the labels on  $\mathcal{T}$  and  $(\mathcal{E}_s)_{s \in \mathbb{R}}$  for the clockwise exploration of  $\mathcal{T}$ . Then,*

$$\lim_{|s| \rightarrow \infty} \Lambda_{\mathcal{E}_s} = \infty, \quad a.s.$$

*Proof.* By Proposition 7, it suffices to prove the similar statement for the labeled tree associated with  $(X^{(r)}, \tilde{\mathcal{L}}^{(r)}, \tilde{\mathcal{R}}^{(r)})$ , or even for the labeled tree associated with  $(X, \mathcal{L}, \mathcal{R})$ . In the latter case this follows from [15, Lemma 3.3].  $\square$

### 3.4 The coding triple of the Brownian disk with a given height

The fact that the collection  $(\Theta_z)_{z>0}$  has been uniquely defined will now allow us to make a canonical choice for the conditional distributions  $(\Theta_z^{(a)})_{z>0}$  (until now, these conditional distributions were only defined up to a set of values of  $z$  of zero Lebesgue measure). This will be important later as we use  $\Theta_z^{(a)}$  to construct the free pointed Brownian disk with perimeter  $z$  and height  $a$ .

Recall the restriction operator  $\mathfrak{R}_a$  introduced in (23), and the notation  $\lambda^{(a)}(w) = \sup\{t \geq 0 : w(t) \leq a\}$  for  $w \in \mathcal{W}_0^\infty$  or  $w \in \mathcal{W}_0$ .

**Proposition 9.** Let  $a > 0$ , and define a function  $W_{*,a} : \mathcal{W}_0^\infty \times M_p(\mathbb{R}_+ \times \mathcal{S})^2 \longrightarrow \mathbb{R}_+ \cup \{\infty\}$  by

$$W_{*,(a)}\left(\mathbb{w}, \sum_{i \in I} \delta_{(t_i, \omega_i)}, \sum_{j \in J} \delta_{(t'_j, \omega'_j)}\right) = \min\left(\inf_{i \in I, t_i > \lambda^{(a)}(\mathbb{w})} W_*(\omega_i), \inf_{j \in J, t'_j > \lambda^{(a)}(\mathbb{w})} W_*(\omega'_j)\right).$$

Then, we have  $\Theta_z(W_{*,(a)} > 0) = \sqrt{\pi} 2^{1/2} 3^{-3/2} a^3 z^{1/2} h_a(z)$  and  $\Theta_z(W_{*,(a)} > 0) \longrightarrow 1$  as  $a \rightarrow \infty$ . Furthermore, we can choose the collection  $(\Theta_z^{(a)})_{z>0}$  so that, for every  $z > 0$ ,  $\Theta_z^{(a)}$  is the pushforward of  $\Theta_z(\cdot | W_{*,(a)} > 0)$  under  $\mathfrak{R}_a$ .

*Proof.* Let  $r > 0$  and  $a > 0$ . For test functions  $f$  and  $F$  defined on  $\mathbb{R}_+$  and on  $\mathcal{W} \times M_p(\mathbb{R}_+ \times \mathcal{W})^2$  respectively, we have from Proposition 6,

$$\begin{aligned} & \int dz k_r(z) f(z) \Theta_z\left(F \circ \mathfrak{R}_a \mathbf{1}_{\{W_{*,(a)} > 0\}}\right) \\ &= \mathbb{E}\left[f(\mathcal{Z}^{(r)}) F\left((X_t^{(r)})_{0 \leq t \leq L_a^{(r)}}, \tilde{\mathcal{L}}^{(r,r+a)}, \tilde{\mathcal{R}}^{(r,r+a)}\right) \mathbf{1}_{\{\mathcal{Z}^{(r+a,\infty)} = 0\}}\right] \\ &= \mathbb{E}\left[f(\mathcal{Z}^{(r,r+a)}) F\left((X_t^{(r)})_{0 \leq t \leq L_a^{(r)}}, \tilde{\mathcal{L}}^{(r,r+a)}, \tilde{\mathcal{R}}^{(r,r+a)}\right)\right] \times \mathbb{P}(\mathcal{Z}^{(r+a,\infty)} = 0) \\ &= \int dz g_{r,a}(z) f(z) \Theta_z^{(a)}(F) \times \mathbb{P}(\mathcal{Z}^{(r+a,\infty)} = 0), \end{aligned}$$

using Lemma 5 in the last equality. It follows that we have  $dz$  a.e.,

$$k_r(z) \Theta_z\left(F \circ \mathfrak{R}_a \mathbf{1}_{\{W_{*,(a)} > 0\}}\right) = g_{r,a}(z) \mathbb{P}(\mathcal{Z}^{(r+a,\infty)} = 0) \Theta_z^{(a)}(F).$$

For  $F = 1$ , we get that the equality  $k_r(z) \Theta_z(W_{*,(a)} > 0) = g_{r,a}(z) \mathbb{P}(\mathcal{Z}^{(r+a,\infty)} = 0) = g_{r,a}(z) \left(\frac{a}{r+a}\right)^3$  holds  $dz$  a.e., but then, by a scaling argument using also the monotonicity of  $\Theta_z(W_{*,(a)} > 0)$  in the variable  $a$ , it must hold for every  $z > 0$  and  $a > 0$ . It follows that  $\Theta_z(W_{*,(a)} > 0) = (k_r(z))^{-1} g_{r,a}(z) \left(\frac{a}{r+a}\right)^3$ , and the explicit formulas for  $k_r(z)$  and  $g_{r,a}(z)$  give the first assertion of the proposition. Furthermore, the previous display gives

$$\Theta_z^{(a)}(F) = \frac{\Theta_z\left(F \circ \mathfrak{R}_a \mathbf{1}_{\{W_{*,(a)} > 0\}}\right)}{\Theta_z(W_{*,(a)} > 0)},$$

$dz$  a.e. The second assertion follows.  $\square$

In what follows, we assume that, for every  $a > 0$ , the collection  $(\Theta_z^{(a)})_{z>0}$  is chosen as in the preceding proposition, and that the collection  $(\check{\Theta}_z^{(a)})_{z>0}$  is then derived from  $(\Theta_z^{(a)})_{z>0}$  via the time-reversal operation. From the scaling properties of  $(\Theta_z)_{z>0}$ , one checks that, for every  $\lambda > 0$ , the pushforward of  $\Theta_z^{(a)}$  under the scaling operator  $\Sigma_\lambda$  is  $\Theta_{\lambda z}^{(\sqrt{\lambda}a)}$ .

The following corollary, which relates the measures  $\Theta_z^{(a)}$  when  $a$  varies (and  $z$  is fixed) is an immediate consequence of Proposition 9. Before stating this corollary, we note that both  $\mathfrak{R}_a$  and  $W_{*,(a)}$  still make sense as mappings defined on  $\mathcal{W}_0 \times M_p(\mathbb{R}_+ \times \mathcal{S})^2$ .

**Corollary 10.** Let  $0 < a < a'$ . Then we have

$$\Theta_z^{(a')}(W_{*,(a)} > 0) = \frac{a^3 h_a(z)}{a'^3 h_{a'}(z)},$$

and  $\Theta_z^{(a)}$  is the pushforward of  $\Theta_z^{(a')}(\cdot | W_{*,(a)} > 0)$  under  $\mathfrak{R}_a$ .

We now use the collection  $(\check{\Theta}_z^{(a)})_{z>0}$  to construct a regular version of the conditional distributions of  $\text{tr}_0(\omega)$  under  $\mathbb{N}_a$  knowing  $\mathcal{Z}_0 = z$ , for every  $a > 0$  and  $z > 0$ . This regular version is a priori unique up to sets of values of  $z$  of zero Lebesgue measure, but for our purposes it is important that the conditional distribution is defined for every  $z > 0$ .

We fix  $a > 0$  and consider a triple  $(Y^{(z)}, \tilde{\mathcal{M}}^{(z)}, \tilde{\mathcal{M}}'^{(z)})$  distributed according to  $\check{\Theta}_z^{(a)}$ . As explained at the end of Section 2.4 (finite spine case), we can use this triple to construct a snake trajectory, which belongs to  $\mathcal{S}_a$  and is denoted by  $\Omega(Y^{(z)}, \tilde{\mathcal{M}}^{(z)}, \tilde{\mathcal{M}}'^{(z)})$ . We write  $\mathbb{N}_a^{(z)}$  for the distribution of the snake trajectory  $\Omega(Y^{(z)}, \tilde{\mathcal{M}}^{(z)}, \tilde{\mathcal{M}}'^{(z)})$ .

**Proposition 11.** *The collection  $(\mathbb{N}_a^{(z)})_{z>0}$  forms a regular version of the conditional distributions of  $\text{tr}_0(\omega)$  under  $\mathbb{N}_a$  knowing that  $\mathcal{Z}_0 = z$ .*

*Proof.* Recall the notation introduced before Proposition 2: Under the measure  $\mathbb{N}_a(d\omega)$ , we can consider, for every  $s \in (0, \sigma(\omega))$ , the point measure  $\mathcal{P}_{(s)}$  (resp.  $\mathcal{P}'_{(s)}$ ) that gives the snake trajectories associated with the subtrees branching off the left side (resp. off the right side) of the ancestral line of the vertex  $p_{(\omega)}(s)$  in the genealogical tree of  $\omega$ . Also use the notation  $\tilde{\mathcal{P}}_{(s)}$  (resp.  $\tilde{\mathcal{P}}'_{(s)}$ ) for the point measure  $\mathcal{P}_{(s)}$  (resp.  $\mathcal{P}'_{(s)}$ ) “truncated at level 0”. This makes sense if  $s$  is such that  $W_s(t) > 0$  for  $0 \leq t < \zeta_s$ , which is the case we will consider. From Proposition 2, and using also (9), we have, for every nonnegative measurable functions  $f$  and  $F$  defined on  $\mathbb{R}_+$  and on  $\mathcal{W} \times M_p(\mathbb{R}_+ \times \mathcal{S})^2$  respectively,

$$\mathbb{N}_a \left( \int_0^\sigma dL_s^0 f(\mathcal{Z}_0) F(W_s, \tilde{\mathcal{P}}_{(s)}, \tilde{\mathcal{P}}'_{(s)}) \right) = \mathbb{E} \left[ f(\mathcal{Z}^Y) F(Y, \tilde{\mathcal{M}}, \tilde{\mathcal{M}}') \right], \quad (25)$$

where  $(Y, \tilde{\mathcal{M}}, \tilde{\mathcal{M}}')$  and  $\mathcal{Z}^Y$  are as in Section 3.2. Notice that,  $dL_s^0$  a.e., we have  $\text{tr}_0(\omega) = \Omega(W_s, \tilde{\mathcal{P}}_{(s)}, \tilde{\mathcal{P}}'_{(s)})$ . Hence the previous identity also gives, for every nonnegative measurable function  $H$  on  $\mathcal{S}$ ,

$$\mathbb{N}_a(\mathcal{Z}_0 f(\mathcal{Z}_0) H(\text{tr}_0(\omega))) = \mathbb{E}[f(\mathcal{Z}^Y) H(\Omega(Y, \tilde{\mathcal{M}}, \tilde{\mathcal{M}}'))]. \quad (26)$$

Since the density of  $\mathcal{Z}^Y$  is  $zh_a(z)$  and  $\check{\Theta}_z^{(a)}$  is the conditional distribution of  $(Y, \mathcal{M}, \mathcal{M}')$  given  $\mathcal{Z}^Y = z$ , the right-hand side can be written as

$$\int dz zh_a(z) f(z) \check{\Theta}_z^{(a)}(H \circ \Omega) = \int dz zh_a(z) f(z) \mathbb{N}_a^{(z)}(H),$$

by the very definition of  $\mathbb{N}_a^{(z)}$ . The statement of the proposition follows.  $\square$

From the scaling properties of the measures  $\Theta_z^{(a)}$ , we immediately get that, for every  $\lambda > 0$ , the pushforward of  $\mathbb{N}_a^{(z)}$  under the scaling transformation  $\theta_\lambda$  is  $\mathbb{N}_{a\sqrt{\lambda}}^{(\lambda z)}$ .

In view of further applications, we also note that the definition of the exit local time at 0 makes sense under  $\mathbb{N}_a^{(z)}$ . Precisely, one gets that,  $\mathbb{N}_a^{(z)}(d\omega)$  a.e., the limit

$$\tilde{L}_t^0 := \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^2} \int_0^t ds \mathbf{1}_{\{\widehat{W}_s(\omega) < \varepsilon\}} \quad (27)$$

exists uniformly for  $t \geq 0$ , and  $\tilde{L}_\infty^0 = \tilde{L}_\sigma^0 = z$ . If  $\mathbb{N}_a^{(z)}$  is replaced by  $\mathbb{N}_a$  (and  $\widehat{W}_s(\omega)$  by  $\widehat{W}_s(\text{tr}_0(\omega))$ ) this is just formula (4) in Section 2.2. So (27) is a conditional version of (4), which must therefore hold  $\mathbb{N}_a^{(z)}$  a.e., at least for a.e. value of  $z$ . But then a scaling argument, using also the way we have defined the conditional distributions  $\mathbb{N}_a^{(z)}$  and Corollary 10, shows that (27) indeed holds for every  $z > 0$ . We omit the details.

## 4 From coding triples to random metric spaces

### 4.1 The pseudo-metric functions associated with a coding triple

Let  $(w, \mathcal{P}, \mathcal{P}')$  be a coding triple satisfying the assumptions of Section 2.4 in the infinite spine case, and let  $(\mathcal{T}, (\Lambda_v)_{v \in \mathcal{T}})$  be the associated labeled tree. We suppose here that labels take nonnegative values,  $\Lambda_v \geq 0$  for every  $v \in \mathcal{T}$ , and we set  $\mathcal{T}^\circ := \{v \in \mathcal{T} : \Lambda_v > 0\}$  and  $\partial\mathcal{T} = \mathcal{T} \setminus \mathcal{T}^\circ$ . We assume that  $\partial\mathcal{T}$  is not empty and that all points of  $\partial\mathcal{T}$  are leaves (points whose removal does not disconnect  $\mathcal{T}$ ). In particular,  $\mathcal{T}^\circ$  is dense in  $\mathcal{T}$ . We denote the clockwise exploration of  $\mathcal{T}$  by  $(\mathcal{E}_t)_{t \in \mathbb{R}}$ , and we assume that either  $\Lambda_{\mathcal{E}_t} \rightarrow \infty$  as  $|t| \rightarrow \infty$ , or the set  $\{t \in \mathbb{R} : \Lambda_{\mathcal{E}_t} = 0\}$  intersects both intervals  $[K, \infty)$  and  $(-\infty, -K]$ , for every  $K > 0$ . This ensures that  $\inf_{w \in [u, v]} \Lambda_w$  is attained for every “interval”  $[u, v]$  of  $\mathcal{T}$ .

We define, for every  $u, v \in \mathcal{T}^\circ$ ,

$$\Delta^\circ(u, v) := \begin{cases} \Lambda_u + \Lambda_v - 2 \max \left( \inf_{w \in [u, v]} \Lambda_w, \inf_{w \in [v, u]} \Lambda_w \right) & \text{if } \max \left( \inf_{w \in [u, v]} \Lambda_w, \inf_{w \in [v, u]} \Lambda_w \right) > 0, \\ +\infty & \text{otherwise.} \end{cases} \quad (28)$$

We then let  $\Delta(u, v)$ ,  $u, v \in \mathcal{T}^\circ$  be the maximal symmetric function on  $\mathcal{T}^\circ \times \mathcal{T}^\circ$  that is bounded above by  $\Delta^\circ$  and satisfies the triangle inequality:

$$\Delta(u, v) = \inf_{u_0=u, u_1, \dots, u_p=v} \sum_{i=1}^p \Delta^\circ(u_{i-1}, u_i) \quad (29)$$

where the infimum is over all choices of the integer  $p \geq 1$  and of the finite sequence  $u_0, u_1, \dots, u_p$  in  $\mathcal{T}$  such that  $u_0 = u$  and  $u_p = v$ . Then  $\Delta(u, v) < \infty$  for every  $u, v \in \mathcal{T}^\circ$ . Indeed, a compactness argument shows that we can find finitely many points  $u_0 = u, u_1, \dots, u_{p-1}, u_p = v$  belonging to the geodesic segment  $[[u, v]]$  of  $\mathcal{T}$  and such that  $\Delta^\circ(u_{i-1}, u_i) < \infty$  for every  $1 \leq i \leq p$ .

Furthermore, the mapping  $(u, v) \mapsto \Delta(u, v)$  is continuous on  $\mathcal{T}^\circ \times \mathcal{T}^\circ$  (observe that  $\Delta^\circ(u_n, u) \rightarrow 0$  if  $u_n \rightarrow u$  in  $\mathcal{T}^\circ$ , and use the triangle inequality). We note the trivial bound  $\Delta^\circ(u, v) \geq |\Lambda_u - \Lambda_v|$ , which also implies

$$\Delta(u, v) \geq |\Lambda_u - \Lambda_v|. \quad (30)$$

We will call  $\Delta^\circ(u, v)$  and  $\Delta(u, v)$  the *pseudo-metric functions associated with the triple*  $(w, \mathcal{P}, \mathcal{P}')$ . From now on, let us assume that the function  $(u, v) \mapsto \Delta(u, v)$  has a continuous extension to  $\mathcal{T} \times \mathcal{T}$ , which is therefore a pseudo-metric on  $\mathcal{T}$ . We will be interested in the resulting quotient metric space  $\mathcal{T}/\approx$  where the equivalence relation  $\approx$  is defined by saying that  $u \approx v$  if and only if  $\Delta(u, v) = 0$ . By abuse of notation, we will write  $\mathcal{T}/\{\Delta = 0\}$  instead of  $\mathcal{T}/\approx$ . We write  $\Pi$  for the canonical projection from  $\mathcal{T}$  onto  $\mathcal{T}/\{\Delta = 0\}$ . We also write  $\Lambda_x = \Lambda_u$  when  $x \in \mathcal{T}/\{\Delta = 0\}$  and  $u \in \mathcal{T}$  are such that  $x = \Pi(u)$  (this is unambiguous by (30)).

If  $x \in \mathcal{T}/\{\Delta = 0\}$  is such that  $\Lambda_x > 0$ , we can define a geodesic path starting from  $x$  in the following way. We pick  $u \in \mathcal{T}$  such that  $\Pi(u) = x$  and then  $s \in \mathbb{R}$  such that  $\mathcal{E}_s = u$ . We then define  $\gamma^{(s)} = (\gamma_r^{(s)})_{0 \leq r \leq \Lambda_x}$  by setting  $\gamma_r^{(s)} = \Pi(\mathcal{E}_{\eta_r^{(s)}})$ , with

$$\eta_r^{(s)} := \begin{cases} \inf\{t \geq s : \Lambda_{\mathcal{E}_t} = \Lambda_x - r\} & \text{if } \inf\{\Lambda_{\mathcal{E}_t} : t \geq s\} \leq \Lambda_x - r, \\ \inf\{t \leq s : \Lambda_{\mathcal{E}_t} = \Lambda_x - r\} & \text{if } \inf\{\Lambda_{\mathcal{E}_t} : t \geq s\} > \Lambda_x - r. \end{cases}$$

It is then a simple matter to verify that  $\gamma^{(s)}$  is a geodesic path in  $(\mathcal{T}/\{\Delta = 0\}, \Delta)$ , which starts from  $x$  and ends at a point belonging to  $\Pi(\partial\mathcal{T})$ . On the other hand, the bound (30) shows that  $\Delta(x, y) \geq \Lambda_x$  if  $y \in \Pi(\partial\mathcal{T})$ . It follows that  $\Delta(x, \Pi(\partial\mathcal{T})) = \Lambda_x$  for every  $x \in \mathcal{T}/\{\Delta = 0\}$ . The path  $\gamma^{(s)}$  is called a simple geodesic (see e.g. [25, Section 2.6] for the analogous definition in the Brownian map).

We finally note that  $\mathcal{T}/\{\Delta = 0\}$  is a length space, meaning that the distance between two points is equal to the infimum of the lengths of paths connecting these two points. To get this property, just notice that, if  $u, v \in \mathcal{T}^\circ$  and  $\Delta^\circ(u, v) < \infty$ , then  $\Delta^\circ(u, v)$  coincides with the length of a path from  $\Pi(u)$  to  $\Pi(v)$ , that is obtained by concatenating two simple geodesics starting from  $\Pi(u)$  and  $\Pi(v)$  respectively, up to the time when they merge. More explicitly, if  $\Delta^\circ(u, v) = \Lambda_u + \Lambda_v - \inf_{w \in [u, v]} \Lambda_w$ , and if the reals  $s'$  and  $s''$  are such that  $\mathcal{E}_{s'} = u$ ,  $\mathcal{E}_{s''} = v$  and  $[u, v] = \{\mathcal{E}_r : r \in [s', s'']\}$ , then the concatenation of  $(\Pi(\gamma_r^{(s')}), 0 \leq r \leq \Lambda_u - \inf_{w \in [u, v]} \Lambda_w)$  and  $(\Pi(\gamma_r^{(s'')}), 0 \leq r \leq \Lambda_v - \inf_{w \in [u, v]} \Lambda_w)$  gives a continuous path from  $\Pi(u)$  to  $\Pi(v)$  with length  $\Delta^\circ(u, v)$ , which furthermore is contained in  $\Pi([u, v])$ .

## 4.2 The Brownian plane

As an illustration of the procedure described in the previous section, and in view of further developments, we briefly recall the construction of the Brownian plane given in [15]. We consider a (random) coding triple  $(X, \mathcal{L}, \mathcal{R})$  distributed as in Section 3.3:

- $X = (X_t)_{t \geq 0}$  is a nine-dimensional Bessel process started from 0.
- Conditionally on  $X$ ,  $\mathcal{L}$  and  $\mathcal{R}$  are independent Poisson point measures on  $\mathbb{R}_+ \times \mathcal{S}$  with intensity

$$2 dt \mathbb{N}_{X_t}(d\omega \cap \{W_* > 0\}).$$

It is easy to verify that the assumptions of Section 2.4 hold a.s. for  $(X, \mathcal{L}, \mathcal{R})$ , and thus we can associate an infinite labeled tree  $(\mathcal{T}_\infty^p, (\Lambda_v)_{v \in \mathcal{T}_\infty^p})$  with this coding triple. The assumptions of the beginning of Section 4.1 also hold (notice that the condition  $\lim_{|s| \rightarrow \infty} \Lambda_{\varepsilon_s} = \infty$  holds by [15, Lemma 3.3]), and we introduce the two pseudo-metric functions  $\Delta^{p,\circ}(u, v)$  and  $\Delta^p(u, v)$  defined for  $u, v \in \mathcal{T}_\infty^{p,\circ} := \{v \in \mathcal{T}_\infty^p : \Lambda_v > 0\}$  via formulas (28) and (29). In that case, since the root of  $\mathcal{T}_\infty^p$  is the only point with zero label, it is easy to see that at least one of the two infima  $\inf_{w \in [u, v]} \Lambda_w$  and  $\inf_{w \in [v, u]} \Lambda_w$  is positive, for any  $u, v \in \mathcal{T}_\infty^{p,\circ}$ . Furthermore, it is immediate to obtain that  $\Delta^{p,\circ}(u, v)$  and  $\Delta^p(u, v)$  can be extended continuously to  $\mathcal{T}_\infty^p$  — in fact in that case we can define  $\Delta^{p,\circ}(u, v)$  for every  $u, v \in \mathcal{T}_\infty^p$  by the quantity in the first line of (28), and use formula (29) to define  $\Delta^p(u, v)$  for every  $u, v \in \mathcal{T}_\infty^p$ . One can prove [15, Section 3.2] that, for any  $u, v \in \mathcal{T}_\infty^p$ ,  $\Delta^p(u, v) = 0$  if and only if  $\Delta^{p,\circ}(u, v) = 0$ .

The Brownian plane  $\mathbb{B}\mathbb{P}_\infty$  is defined as the quotient space  $\mathcal{T}_\infty^p / \{\Delta^p = 0\}$  equipped with the distance induced by  $\Delta^p$  (for which we keep the same notation  $\Delta^p$ ) and with the volume measure which is the pushforward of the volume measure on  $\mathcal{T}_\infty^p$  under the canonical projection. We note that  $\mathbb{B}\mathbb{P}_\infty$  comes with a distinguished point  $\rho$ , which is the image of the root of  $\mathcal{T}_\infty^p$  under the canonical projection. Furthermore, we have  $\Delta^p(\rho, x) = \Lambda_x$  for every  $x \in \mathbb{B}\mathbb{P}_\infty$ .

The Brownian plane is scale invariant in the following sense. If  $E$  is a pointed measure metric space and  $\lambda > 0$ , we write  $\lambda \cdot E$  for the same space  $E$  with the metric multiplied by the factor  $\lambda$  and the volume measure multiplied by the factor  $\lambda^4$  (and the same distinguished point). Then, for every  $\lambda > 0$ ,  $\lambda \cdot \mathbb{B}\mathbb{P}_\infty$  has the same distribution as  $\mathbb{B}\mathbb{P}_\infty$ .

### 4.3 The pointed Brownian disk with given perimeter and height

In this section, we explain how a free pointed Brownian disk with perimeter  $z$  and height  $a$  is constructed from the measure  $\mathbb{N}_a^{(z)}$  defined in Section 3.4. This is basically an adaptation of [28], but we provide some details in view of further developments.

We start with a preliminary result. Recall the notation  $h_a(z)$  and  $p_z(a)$  in Proposition 3.

**Proposition 12.** *For any nonnegative measurable functions  $G$  and  $f$  defined respectively on  $\mathcal{S}$  and on  $\mathbb{R}_+$ , for every  $z > 0$ , we have*

$$z^{-2} \mathbb{N}^{*,z} \left( \int_0^\sigma dt G(W^{[t]}) f(\widehat{W}_t) \right) = \int_0^\infty da p_z(a) f(a) \mathbb{N}_a^{(z)}(G).$$

*Proof.* We may assume that both  $G$  and  $f$  are bounded and continuous. Then the argument is very similar to the proof of Proposition 3 (ii) (which we recover when  $G = 1$ ). Let  $g$  be a nonnegative measurable function on  $\mathbb{R}_+$ . We use the re-rooting formula (5), and then Proposition 3 (i), to get

$$\begin{aligned} \mathbb{N}^* \left( \int_0^\sigma dt G(W^{[t]}) f(\widehat{W}_t) g(\mathcal{Z}_0^*) \right) &= 2 \int_0^\infty da \mathbb{N}_a \left( \mathcal{Z}_0 G(\text{tr}_0(\omega)) f(a) g(\mathcal{Z}_0) \right) \\ &= 2 \int_0^\infty da f(a) \int_0^\infty dz h_a(z) z g(z) \mathbb{N}_a(G(\text{tr}_0(\omega)) \mid \mathcal{Z}_0 = z) \\ &= 2 \int_0^\infty dz z g(z) \int_0^\infty da h_a(z) f(a) \mathbb{N}_a^{(z)}(G). \end{aligned}$$

On the other hand, the left-hand side is also equal to

$$\sqrt{\frac{3}{2\pi}} \int_0^\infty dz z^{-5/2} g(z) \mathbb{N}^{*,z} \left( \int_0^\sigma dt G(W^{[t]}) f(\widehat{W}_t) \right).$$

Since this holds for any function  $g$ , we must have,  $dz$  a.e.,

$$z^{-2} \mathbb{N}^{*,z} \left( \int_0^\sigma dt G(W^{[t]}) f(\widehat{W}_t) \right) = \sqrt{\frac{8\pi}{3}} z^{3/2} \int_0^\infty da h_a(z) f(a) \mathbb{N}_a^{(z)}(G).$$

This is the identity of the proposition, except that we get it only  $dz$  a.e. However, a scaling argument shows that both sides of the preceding display are continuous functions of  $z$ , which gives the desired result for every  $z > 0$ .  $\square$



Recall from Section 2.3 the definition of the probability measure  $\bar{\mathbb{N}}^{*,z}(d\omega dt)$  on  $\mathcal{S} \times \mathbb{R}_+$ , and, for  $(\omega, t) \in \mathcal{S} \times \mathbb{R}_+$ , write  $U(\omega, t) = t$ . We can then rewrite the identity of Proposition 12 in the form

$$\bar{\mathbb{N}}^{*,z}(f(\widehat{\omega}_U)G(\omega^{[U]})) = \int_0^\infty da p_z(a) f(a) \mathbb{N}_a^{(z)}(G). \quad (31)$$

Let us now come to Brownian disks. We write  $\mathbb{K}$ , resp.  $\mathbb{K}^\bullet$ , for the space of all compact measure metric spaces, resp. pointed compact measure metric spaces, equipped with the Gromov-Hausdorff-Prokhorov topology. Theorem 1 in [28] provides a measurable mapping  $\Xi : \mathcal{S} \rightarrow \mathbb{K}$  such that the distribution of  $\Xi(\omega)$  under  $\mathbb{N}^{*,z}(d\omega)$  is the law of the free Brownian disk with perimeter  $z$ . Let us briefly recall the construction of this mapping, which is essentially an adaptation of the procedure of Section 4.1 to the finite spine case. Under  $\mathbb{N}^{*,z}(d\omega)$ , each vertex  $u$  of the the genealogical tree  $\mathcal{T}_{(\omega)}$  receives a nonnegative label  $\ell_u(\omega)$ , and we define the ‘‘boundary’’  $\partial\mathcal{T}_{(\omega)} := \{u \in \mathcal{T}_{(\omega)} : \ell_u(\omega) = 0\}$ . We also set  $\mathcal{T}_{(\omega)}^\circ := \mathcal{T}_{(\omega)} \setminus \partial\mathcal{T}_{(\omega)}$ , and for every  $u, v \in \mathcal{T}_{(\omega)}^\circ$ , we define  $\Delta^{d,\circ}(u, v)$  and  $\Delta^d(u, v)$  by the exact analogs of formulas (28) and (29), where  $\mathcal{T}$  and  $\mathcal{T}^\circ$  are replaced by  $\mathcal{T}_{(\omega)}$  and  $\mathcal{T}_{(\omega)}^\circ$  respectively, and the labels  $(\Lambda_u)_{u \in \mathcal{T}}$  are replaced by  $(\ell_u)_{u \in \mathcal{T}_{(\omega)}}$  (recall the definition of intervals on  $\mathcal{T}_{(\omega)}$  in Section 2.1).

A key technical point (Proposition 31 in [28]) is to verify that the mapping  $(u, v) \mapsto \Delta^d(u, v)$  can be extended continuously (in a unique way) to  $\mathcal{T}_{(\omega)} \times \mathcal{T}_{(\omega)}$ ,  $\mathbb{N}^{*,z}(d\omega)$  a.s. We then define  $\Xi(\omega)$  as the quotient space  $\mathcal{T}_{(\omega)}/\{\Delta^d = 0\}$ , which is equipped with the metric induced by  $\Delta^d$  and with a volume measure which is the pushforward of the volume measure on  $\mathcal{T}_{(\omega)}$  under the canonical projection. In the next definition, we use the notation  $\mathbf{p}_\omega$  for the composition of the canonical projection  $p_{(\omega)}$  from  $[0, \sigma(\omega)]$  onto  $\mathcal{T}_{(\omega)}$  with the canonical projection from  $\mathcal{T}_{(\omega)}$  onto  $\Xi(\omega)$ .

**Definition 13.** *The distribution of the free pointed Brownian disk with perimeter  $z$  is the law of the random measure metric space  $(\Xi(\omega), \Delta^d)$  pointed at the point  $\mathbf{p}_\omega(t)$ , under the probability measure  $\bar{\mathbb{N}}^{*,z}(d\omega dt)$ .*

The consistency of this definition of the free pointed Brownian disk with the one in [11] follows from the results in [28] (to be specific, [28, Theorem 1] identifies the law of  $\Xi(\omega)$  under  $\mathbb{N}^{*,z}$  as the distribution of the free Brownian disk of perimeter  $z$ , and one can then use formula (42) in [29] to see that Definition 13 is consistent with the definition of the free pointed Brownian disk in [11]). From [9, 11] one knows that,  $\mathbb{N}^{*,z}(d\omega)$  a.s.,  $\Xi(\omega)$  is homeomorphic to the unit disk. This makes it possible to define the boundary  $\partial\Xi(\omega)$  of  $\Xi(\omega)$ , and this boundary is identified in [28] with the image of  $\partial\mathcal{T}_{(\omega)}$  under the canonical projection. Furthermore, a.s. for every  $u \in \mathcal{T}_{(\omega)}$ , the label  $\ell_u(\omega)$  is equal to the distance in  $\Xi(\omega)$  from (the equivalence class of)  $u$  to the boundary  $\partial\Xi(\omega)$ .

We note that the random measure metric space  $\Xi(\omega)$  pointed at  $\mathbf{p}_\omega(t)$  is in fact a function of the re-rooted snake trajectory  $\omega^{[t]}$ , because  $\Xi(\omega)$  is canonically identified to  $\Xi(\omega^{[t]})$ , and  $\mathbf{p}_\omega(t)$  is mapped to  $\mathbf{p}_{\omega^{[t]}}(0)$  in this identification. To simplify notation, we may thus write  $\Xi^\bullet(\omega^{[t]})$  for the random metric space  $\Xi(\omega)$  pointed at  $\mathbf{p}_\omega(t)$ , or equivalently for  $\Xi(\omega^{[t]})$  pointed at  $\mathbf{p}_{\omega^{[t]}}(0)$ .

In the following developments, it will be convenient to write  $\mathbb{D}_z^\bullet$  for a free pointed Brownian disk with perimeter  $z$  and  $\partial\mathbb{D}_z^\bullet$  for the boundary of  $\mathbb{D}_z^\bullet$ . With a slight abuse of notation, we keep the notation  $\Delta^d$  for the distance on  $\mathbb{D}_z^\bullet$ . By definition, the height  $H_z$  of  $\mathbb{D}_z^\bullet$  is the distance from the distinguished point to the boundary. From the preceding interpretation of  $\Xi(\omega)$  pointed at  $\mathbf{p}_\omega(t)$  as a function of the re-rooted snake trajectory  $\omega^{[t]}$ , we get that, for any nonnegative measurable function  $\Phi$  on  $\mathbb{K}^\bullet$ , for any nonnegative measurable function  $h$  on  $\mathbb{R}_+$ ,

$$\mathbb{E}[\Phi(\mathbb{D}_z^\bullet) h(H_z)] = \bar{\mathbb{N}}^{*,z}\left(\Phi(\Xi^\bullet(\omega^{[U]})) h(\widehat{\omega}_U)\right)$$

using the same notation as in (31) and noting that  $\ell_{p_{(\omega)}(U)}(\omega) = \widehat{\omega}_U$  by definition. By (31), we can rewrite this as

$$\mathbb{E}[\Phi(\mathbb{D}_z^\bullet) h(H_z)] = \int_0^\infty da p_z(a) h(a) \mathbb{N}_a^{(z)}(\Phi \circ \Xi^\bullet). \quad (32)$$

The next proposition readily follows from (32).

**Proposition 14.** *The height  $H_z$  of  $\mathbb{D}_z^\bullet$  is distributed according to the density  $p_z(a)$ . Furthermore, the conditional distribution of  $\mathbb{D}_z^\bullet$  knowing that  $H_z = a$  is the law of  $\Xi^\bullet(\omega)$  under  $\mathbb{N}_a^{(z)}$ . By definition, this is the distribution of the free pointed Brownian disk with perimeter  $z$  and height  $a$ .*

At this point, we should note that the definition of  $\Xi^\bullet(\omega)$  requires the continuous extension of the mapping  $(u, v) \mapsto \Delta^d(u, v)$  from  $\mathcal{T}_{(\omega)}^\circ \times \mathcal{T}_{(\omega)}^\circ$  to  $\mathcal{T}_{(\omega)} \times \mathcal{T}_{(\omega)}$ . Proposition 31 in [28] and Proposition 11 above give the existence of this continuous extension  $\mathbb{N}_a^{(z)}(d\omega)$  a.s. for a.a.  $z > 0$ , for every fixed  $a > 0$ , and then one can use scaling arguments and Corollary 10 to get the same result for *every*  $z > 0$  and  $a > 0$  (this is used in Proposition 14). Similar considerations allow us to deduce the following property from Proposition 32 (iii) in [28]:  $\mathbb{N}_a^{(z)}(d\omega)$  a.s., for every  $u, v \in \partial\mathcal{T}_{(\omega)}$ , we have  $\Delta^d(u, v) = 0$  if and only if  $\ell_w(\omega) > 0$  for every  $w \in ]u, v[$ , or for every  $w \in ]v, u[$ . Finally, from [28, Proposition 30 (iv)], we also get that,  $\mathbb{N}_a^{(z)}(d\omega)$  a.s. for every  $u, v \in \mathcal{T}_{(\omega)}^\circ$ , we have  $\Delta^d(u, v) = 0$  if and only if  $\Delta^{d,\circ}(u, v) = 0$ .

It will be useful to introduce the uniform measure on the boundary  $\partial\mathbb{D}_z^\bullet$ . There exists a measure  $\mu_z$  on  $\partial\mathbb{D}_z^\bullet$  with total mass equal to  $z$ , such that, a.s. for any continuous function  $\varphi$  on  $\mathbb{D}_z^\bullet$ , we have

$$\langle \mu_z, \varphi \rangle = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \int_{\mathbb{D}_z^\bullet} \text{Vol}(dx) \mathbf{1}_{\{\Delta^d(x, \partial\mathbb{D}_z^\bullet) < \varepsilon\}} \varphi(x)$$

where  $\text{Vol}(dx)$  denotes the volume measure on  $\partial\mathbb{D}_z^\bullet$  (see [28, Corollary 37]). The preceding approximation and the definition of  $\mu_z$  are also valid for the free pointed Brownian disk with perimeter  $z$  and height  $a$ : In fact, if this Brownian disk is constructed as  $\Xi^\bullet(\omega)$  under  $\mathbb{N}_a^{(z)}$  (as in Proposition 14), we define  $\mu_z$  by setting

$$\langle \mu_z, \varphi \rangle = \int_0^\infty d\tilde{L}_s^0 \varphi(\mathbf{p}_\omega(s)), \quad (33)$$

where the exit local time  $\tilde{L}_s^0$  is defined under  $\mathbb{N}_a^{(z)}$  as explained at the end of Section 3.4. Note that the approximation formula for  $\mu_z$  then reduces to formula (27), thanks to the interpretation of labels as distances to the boundary.

For our purposes in the next sections, it will be important to consider a free Brownian disk (with perimeter  $z$  and height  $a$ ) equipped with a distinguished point chosen uniformly on the boundary. To this end, we proceed as follows. We start from a triple  $(Y^{(z)}, \tilde{\mathcal{M}}^{(z)}, \tilde{\mathcal{M}}'^{(z)})$  distributed according to  $\check{\Theta}_z^{(a)}$ . As explained before Proposition 11, the random snake trajectory  $\Omega(Y^{(z)}, \tilde{\mathcal{M}}^{(z)}, \tilde{\mathcal{M}}'^{(z)})$  is then distributed according to  $\mathbb{N}_a^{(z)}$ , and (Proposition 14) we obtain a free pointed Brownian disk  $\mathbb{D}_z^{\bullet,a}$  with perimeter  $z$  and height  $a$  by setting

$$\mathbb{D}_z^{\bullet,a} := \Xi^\bullet\left(\Omega(Y^{(z)}, \tilde{\mathcal{M}}^{(z)}, \tilde{\mathcal{M}}'^{(z)})\right). \quad (34)$$

In this construction,  $\mathbb{D}_z^{\bullet,a}$  comes with a distinguished vertex of its boundary, namely the one corresponding to the top of the spine of the tree coded by  $(Y^{(z)}, \tilde{\mathcal{M}}^{(z)}, \tilde{\mathcal{M}}'^{(z)})$ . We denote this special point by  $\alpha$ .

In the next proposition, we verify that  $\alpha$  is (in a certain sense) uniformly distributed over  $\partial\mathbb{D}_z^{\bullet,a}$ . To give a precise statement of this property, it is convenient to introduce the doubly pointed measure metric space  $\mathbb{D}_z^{\bullet\bullet,a}$  which is obtained by viewing  $\alpha$  as a second distinguished point of  $\mathbb{D}_z^{\bullet,a}$ .

**Proposition 15.** *Let  $F$  be a nonnegative measurable function on the space of all doubly pointed compact measure metric spaces, which is equipped with the Gromov-Hausdorff-Prokhorov topology. Then,*

$$\mathbb{E}[F(\mathbb{D}_z^{\bullet\bullet,a})] = \frac{1}{z} \mathbb{E}\left[\int \mu_z(dx) F([\mathbb{D}_z^{\bullet,a}, x])\right],$$

where  $\mu_z$  is the uniform measure on  $\partial\mathbb{D}_z^{\bullet,a}$ , and we use the notation  $[\mathbb{D}_z^{\bullet,a}, x]$  for the doubly pointed space obtained by equipping  $\mathbb{D}_z^{\bullet,a}$  with the second distinguished point  $x$ .

*Proof.* Let  $(Y, \mathcal{M}, \mathcal{M}')$  be distributed as in Proposition 2 (or in Section 3.2). As previously, let  $\tilde{\mathcal{M}}$  and  $\tilde{\mathcal{M}}'$  be obtained by truncating the atoms of  $\mathcal{M}$  and  $\mathcal{M}'$  at level 0. Let us introduce the notation  $\|\tilde{\mathcal{M}}\|$  for the sum of the quantities  $\sigma(\omega_i)$  over all atoms  $(t_i, \omega_i)$  of  $\tilde{\mathcal{M}}$ . Note that  $\|\tilde{\mathcal{M}}\|$  is the time at which the clockwise exploration of the tree associated with the triple  $(Y, \tilde{\mathcal{M}}, \tilde{\mathcal{M}}')$  (which coincides with the genealogical tree of  $\Omega(Y, \tilde{\mathcal{M}}, \tilde{\mathcal{M}}')$ ) visits the top of the spine. We start by observing that, for any

nonnegative measurable function  $G$  on  $\mathcal{S} \times \mathbb{R}_+$ , for any nonnegative measurable function  $h$  on  $\mathbb{R}_+$ ,

$$\begin{aligned} \mathbb{E}\left[G\left(\Omega(Y, \widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}'), \|\widetilde{\mathcal{M}}\|\right) h(\mathcal{Z}^Y)\right] &= \mathbb{N}_a\left(\int_0^\infty dL_r^0 G\left(\text{tr}_0(\omega), \int_0^r ds \mathbf{1}_{\{\zeta(\omega_s) \leq \tau_0(\omega_s)\}}\right) h(\mathcal{Z}_0)\right) \\ &= \mathbb{N}_a\left(\int_0^\infty d\widetilde{L}_r^0 G(\text{tr}_0(\omega), r) h(\mathcal{Z}_0)\right) \end{aligned}$$

where  $\widetilde{L}_r^0$  is defined under  $\mathbb{N}_a$  as in formula (4). The first equality follows from Proposition 2 as in the derivation of (25) and (26) above, and the second equality is just a time change formula. By conditioning on  $\mathcal{Z}^Y = z$  in the left-hand side and on  $\mathcal{Z}_0 = z$  in the right-hand side, we get

$$\mathbb{E}\left[G\left(\Omega(Y^{(z)}, \widetilde{\mathcal{M}}^{(z)}, \widetilde{\mathcal{M}}'^{(z)}), \|\widetilde{\mathcal{M}}^{(z)}\|\right)\right] = \frac{1}{z} \mathbb{N}_a^{(z)}\left(\int_0^\infty d\widetilde{L}_r^0 G(\omega, r)\right).$$

Now recall that  $\mathbb{D}_z^{\bullet, a} = \Xi^\bullet(\Omega(Y^{(z)}, \widetilde{\mathcal{M}}^{(z)}, \widetilde{\mathcal{M}}'^{(z)}))$  and that  $\mathbb{D}_z^{\bullet\bullet, a}$  is obtained by assigning to  $\mathbb{D}_z^{\bullet, a}$  a second distinguished point equal to  $\alpha = \mathbf{p}_\Omega(\|\widetilde{\mathcal{M}}^{(z)}\|)$ , where  $\mathbf{p}_\Omega$  denotes the composition of the canonical projection onto the genealogical tree of  $\Omega(Y, \widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}')$  with the projection from this genealogical tree onto  $\Xi^\bullet(\Omega(Y^{(z)}, \widetilde{\mathcal{M}}^{(z)}, \widetilde{\mathcal{M}}'^{(z)}))$ . Thanks to these observations, we obtain that

$$\mathbb{E}[F(\mathbb{D}_z^{\bullet\bullet, a})] = \frac{1}{z} \mathbb{N}_a^{(z)}\left(\int_0^\infty d\widetilde{L}_r^0 F\left([\Xi^\bullet(\omega), \mathbf{p}_\omega(r)]\right)\right),$$

and the desired result follows since (33) shows that  $\mu_z$  is the pushforward of the measure  $d\widetilde{L}_r^0$  under the mapping  $r \mapsto \mathbf{p}_\omega(r)$ .  $\square$

In view of forthcoming limit results where the distinguished point of  $\mathbb{D}_z^{\bullet, a}$  is “sent to infinity”, it will be convenient to introduce the random pointed measure metric space  $\overline{\mathbb{D}}_z^{\bullet, a}$  defined from the doubly pointed space  $\mathbb{D}_z^{\bullet\bullet, a}$  by forgetting the first distinguished point. So  $\overline{\mathbb{D}}_z^{\bullet, a}$  is pointed at a point which is uniformly distributed over its boundary.

#### 4.4 Infinite-volume Brownian disks

For every  $z > 0$  and  $a > 0$ , we keep the notation  $\mathbb{D}_z^{\bullet, a}$  for a pointed Brownian disk with perimeter  $z$  and height  $a$ . We may and will assume that  $\mathbb{D}_z^{\bullet, a}$  is constructed from a coding triple distributed according to  $\check{\Theta}_z^{(a)}$  as in formula (34), or equivalently, by time-reversal, from a coding triple distributed according to  $\Theta_z^{(a)}$ . The idea is now to let  $a \rightarrow \infty$  and to use the “convergence” of  $\Theta_z^{(a)}$  to  $\Theta_z$  in order to get the convergence of  $\mathbb{D}_z^{\bullet, a}$  as  $a \rightarrow \infty$ . As we already mentioned, for a precise statement of this convergence, it will be more convenient to replace  $\mathbb{D}_z^{\bullet, a}$  by  $\overline{\mathbb{D}}_z^{\bullet, a}$ . The limit, which will be denoted by  $\mathbb{D}_z^\infty$ , is a random pointed locally compact measure metric space, which we call the infinite-volume Brownian disk with perimeter  $z$ .

Let us start by explaining the construction of  $\mathbb{D}_z^\infty$ , which follows the general pattern of Section 4.1. We consider a coding triple  $((\rho_t)_{t \geq 0}, \mathcal{Q}, \mathcal{Q}')$  with distribution  $\Theta_z$ . From  $((\rho_t)_{t \geq 0}, \mathcal{Q}, \mathcal{Q}')$ , Section 2.4 allows us to construct an infinite tree  $\mathcal{T}_\infty^i$  equipped with nonnegative labels  $(\Lambda_v)_{v \in \mathcal{T}_\infty^i}$ , such that labels on the spine are given by the process  $(\rho_t)_{t \geq 0}$ . Note that the assumptions in Section 4.1 hold in particular thanks to Corollary 8.

We set  $\mathcal{T}_\infty^{i, \circ} := \{v \in \mathcal{T}_\infty^i : \Lambda_v > 0\}$  and  $\partial\mathcal{T}_\infty^i := \mathcal{T}_\infty^i \setminus \mathcal{T}_\infty^{i, \circ} = \{v \in \mathcal{T}_\infty^i : \Lambda_v = 0\}$ . We define the pseudo-metric functions  $\Delta^{i, \circ}(u, v)$  and  $\Delta^i(u, v)$  on  $\mathcal{T}_\infty^{i, \circ} \times \mathcal{T}_\infty^{i, \circ}$  as explained in Section 4.1.

- Lemma 16.** (i) *The mapping  $(u, v) \mapsto \Delta^i(u, v)$  has a.s. a continuous extension to  $\mathcal{T}_\infty^i \times \mathcal{T}_\infty^i$ .*  
(ii) *A.s., for every  $u, v \in \mathcal{T}_\infty^{i, \circ}$ , the property  $\Delta^i(u, v) = 0$  holds if and only if  $\Delta^{i, \circ}(u, v) = 0$ .*  
(iii) *A.s., for every  $u, v \in \partial\mathcal{T}_\infty^i$ , the property  $\Delta^i(u, v) = 0$  holds if and only if  $\Lambda_w > 0$  for every  $w \in ]u, v[$ , or for every  $w \in ]v, u[$ .*

*Proof.* Thanks to Proposition 9 and Corollary 8, it is enough to verify that properties analogous to (i),(ii),(iii) hold when  $\mathcal{T}_\infty^i$  is replaced by the labeled tree associated with a coding triple distributed according to  $\Theta_z^{(a)}$ , for some fixed  $a > 0$ . But then this is a consequence of the similar results in [28], as it was explained in the discussion after the statement of Proposition 14.  $\square$

We let  $\mathbb{D}_z^\infty$  denote the quotient space  $\mathcal{T}_\infty^i / \{\Delta^i = 0\}$ , which is equipped with the metric induced by  $\Delta^i$ . The volume measure on  $\mathbb{D}_z^\infty$  is (as usual) the pushforward of the volume measure on  $\mathcal{T}_\infty^i$ . We also distinguish a special point  $\alpha_\infty$  of  $\mathbb{D}_z^\infty$ , which is the equivalence class of the root of  $\mathcal{T}_\infty^i$ .

**Definition 17.** *The random pointed measure metric space  $(\mathbb{D}_z^\infty, \Delta^i)$  is the infinite-volume Brownian disk with perimeter  $z$ .*

As in Section 4.1, labels  $\Lambda_x$  make sense for  $x \in \mathbb{D}_z^\infty$ , and  $\Lambda_x$  is equal to the distance from  $x$  to the “boundary”  $\partial\mathbb{D}_z^\infty$ , which is defined as the set of all points of  $\mathbb{D}_z^\infty$  with zero label. From the scaling properties of the collection  $(\Theta_z)_{z>0}$ , one also gets that  $\lambda \cdot \mathbb{D}_z^\infty$  is distributed as  $\mathbb{D}_{\lambda^2 z}^\infty$ , for every  $\lambda > 0$ .

In Section 6 below, we will verify that this definition of the infinite-volume Brownian disk is consistent with [8]. It then follows from [8, Corollary 3.13] that  $\mathbb{D}_z^\infty$  is homeomorphic to the complement of the open unit disk of the plane, so that the boundary  $\partial\mathbb{D}_z^\infty$  can be understood in a topological sense. We will not use this result, which may also be derived from our interpretation of complements of hulls of the Brownian plane as infinite-volume Brownian disks in Section 5.1.

**Remark.** We could have defined the infinite-volume Brownian disk without distinguishing a special point of the boundary. The reason for distinguishing  $\alpha_\infty$  comes from the use of the local Gromov-Hausdorff convergence in Theorem 21 below, which requires dealing with pointed spaces.

For the convergence result to follow, it is convenient to deal with the following definition of “balls”: for every  $h > 0$ ,

$$\mathcal{B}_h(\mathbb{D}_z^\infty) = \{v \in \mathbb{D}_z^\infty : \Delta^i(v, \partial\mathbb{D}_z^\infty) \leq h\},$$

and

$$\mathcal{B}_h(\bar{\mathbb{D}}_z^{\bullet,a}) = \{v \in \bar{\mathbb{D}}_z^{\bullet,a} : \Delta^{(a)}(v, \partial\bar{\mathbb{D}}_z^{\bullet,a}) \leq h\},$$

where we use the notation  $\Delta^{(a)}$  for the metric on  $\bar{\mathbb{D}}_z^{\bullet,a}$ . We view both  $\mathcal{B}_h(\mathbb{D}_z^\infty)$  and  $\mathcal{B}_h(\bar{\mathbb{D}}_z^{\bullet,a})$  as compact measure metric spaces, which are pointed at  $\alpha_\infty$  and  $\alpha$  respectively. The compactness of  $\mathcal{B}_h(\mathbb{D}_z^\infty)$  is a consequence of the fact that the set  $\{u \in \mathcal{T}_\infty^i : \Lambda_u \leq h\}$  is compact, by Corollary 8.

**Proposition 18.** *Let  $z > 0$  and  $h > 0$ . There exists a function  $(\varepsilon(a), a > 0)$  with  $\varepsilon(a) \rightarrow 0$  as  $a \rightarrow \infty$ , such that, for every  $a > 0$ , we can define on the same probability space both the infinite-volume Brownian disk  $\mathbb{D}_z^\infty$  and the pointed Brownian disk  $\bar{\mathbb{D}}_z^{\bullet,a}$ , in such a way that*

$$\mathbb{P}(\mathcal{B}_{h'}(\bar{\mathbb{D}}_z^{\bullet,a}) = \mathcal{B}_{h'}(\mathbb{D}_z^\infty), \text{ for every } 0 \leq h' \leq h) \geq 1 - \varepsilon(a).$$

In other words, when  $a$  large, one can couple the spaces  $\bar{\mathbb{D}}_z^{\bullet,a}$  and  $\mathbb{D}_z^\infty$  so that their tubular neighborhoods of the boundary (of any fixed radius  $h$ ) are isometric except on a set of small probability.

*Proof.* Let  $a > 0$ , and consider a triple  $((\rho_t^{(a)})_{0 \leq t \leq L^{(a)}}, \mathcal{Q}^{(a)}, \mathcal{Q}'^{(a)})$  with distribution  $\Theta_z^{(a)}$ . As it was explained before Proposition 15, this triple allows us to construct the Brownian disk  $\bar{\mathbb{D}}_z^{\bullet,a}$  of perimeter  $z$  and height  $a$  pointed at a boundary point. To be specific, the triple  $((\rho_t^{(a)})_{0 \leq t \leq L^{(a)}}, \mathcal{Q}^{(a)}, \mathcal{Q}'^{(a)})$  codes a random compact tree  $\mathcal{T}^{(a)}$  equipped with labels  $(\Lambda_v^{(a)})_{v \in \mathcal{T}^{(a)}}$ . The pseudo-metric functions  $\Delta^{(a),\circ}(u, v)$  and  $\Delta^{(a)}(u, v)$  are then defined as in Section 4.1 for  $u, v \in \mathcal{T}^{(a)}$  such that  $\Lambda_u^{(a)} > 0$  and  $\Lambda_v^{(a)} > 0$ . The function  $(u, v) \mapsto \Delta^{(a)}$  is extended by continuity to  $\mathcal{T}^{(a)} \times \mathcal{T}^{(a)}$ , and the resulting quotient metric space pointed at the root of  $\mathcal{T}^{(a)}$  is the pointed Brownian disk  $\bar{\mathbb{D}}_z^{\bullet,a}$  — here we observe that the distinguished point  $\alpha$  corresponds to the root and not to the top of the spine of  $\mathcal{T}^{(a)}$ , because the effect of dealing with the triple  $((\rho_t^{(a)})_{0 \leq t \leq L^{(a)}}, \mathcal{Q}^{(a)}, \mathcal{Q}'^{(a)})$  instead of its image under the time-reversal transformation (8) interchanges the roles of the root and the top of the spine (see the comments after (8)).

Recall the restriction operator  $\mathfrak{R}_a$  in (23).

**Lemma 19.** *We can couple the triple  $((\rho_t^{(a)})_{0 \leq t \leq L^{(a)}}, \mathcal{Q}^{(a)}, \mathcal{Q}'^{(a)})$  distributed according to  $\Theta_z^{(a)}$  and the triple  $((\rho_t)_{t \geq 0}, \mathcal{Q}, \mathcal{Q}')$  distributed according to  $\Theta_z$  so that the property*

$$\left( (\rho_t^{(a)})_{0 \leq t \leq L^{(a)}}, \mathcal{Q}^{(a)}, \mathcal{Q}'^{(a)} \right) = \mathfrak{R}_a \left( (\rho_t)_{t \geq 0}, \mathcal{Q}, \mathcal{Q}' \right) \quad (35)$$

*holds with probability tending to 1 as  $a \rightarrow \infty$ .*

*Proof.* It suffices to verify that the variation distance between  $\Theta_z^{(a)}$  and  $\mathfrak{R}_a(\Theta_z)$  tends to 0 as  $a \rightarrow \infty$ . This is an easy consequence of Proposition 9. Indeed, let  $A$  be a measurable subset of  $\mathcal{W} \times M_p(\mathbb{R}_+ \times \mathcal{S})^2$ . Then,

$$\begin{aligned}\Theta_z(\mathfrak{R}_a^{-1}(A)) &= \Theta_z(\mathfrak{R}_a^{-1}(A) \cap \{W_{*,(a)} > 0\}) + \Theta_z(\mathfrak{R}_a^{-1}(A) \cap \{W_{*,(a)} = 0\}) \\ &= \Theta_z(W_{*,(a)} > 0) \Theta_z^{(a)}(A) + \Theta_z(\mathfrak{R}_a^{-1}(A) \cap \{W_{*,(a)} = 0\}),\end{aligned}$$

by Proposition 9. It follows that the variation distance between  $\Theta_z^{(a)}$  and  $\mathfrak{R}_a(\Theta_z)$  is bounded above by  $1 - \Theta_z(W_{*,(a)} > 0)$ , which tends to 0 as  $a \rightarrow \infty$ .  $\square$

It will be convenient to write  $\mathcal{T}_\infty^{i,(a)}$  for the (labeled) compact tree derived from  $\mathcal{T}_\infty^i$  by removing the part of the spine above height  $\lambda_\infty^{(a)} := \sup\{t \geq 0 : \rho_t = a\}$  (and of course the subtrees branching off this part of the spine). We can also view  $\mathcal{T}_\infty^{i,(a)}$  as the labeled tree coded by  $\mathfrak{R}_a((\rho_t)_{t \geq 0}, \mathcal{Q}, \mathcal{Q}')$ . On the event where (35) holds, we can therefore also identify  $\mathcal{T}_\infty^{i,(a)}$  with the labeled tree  $\mathcal{T}^{(a)}$  coded by  $((\rho_t^{(a)})_{0 \leq t \leq L^{(a)}}, \mathcal{Q}^{(a)}, \mathcal{Q}'^{(a)})$ , and this identification is used in the next lemma and its proof.

From now on, we assume that the triples  $((\rho_t^{(a)})_{0 \leq t \leq L^{(a)}}, \mathcal{Q}^{(a)}, \mathcal{Q}'^{(a)})$  and  $((\rho_t)_{t \geq 0}, \mathcal{Q}, \mathcal{Q}')$  are coupled as in Lemma 19, and that  $\mathbb{D}_z^{\bullet,a}$  and  $\mathbb{D}_z^\infty$  are constructed from these triples as explained above.

**Lemma 20.** *Let  $h > 0$ . Set*

$$A = \max\{\Delta^i(x, y) : x, y \in \mathbb{D}_z^\infty, \Lambda_x \leq h, \Lambda_y \leq h\}.$$

*On the intersection of the event where (35) holds with the event where*

$$\inf\{\Lambda_v : v \in \mathcal{T}_\infty^i \setminus \mathcal{T}_\infty^{i,(a)}\} \geq A + h + 1 \quad (36)$$

*we have*

$$\Delta^i(v, w) = \Delta^{(a)}(v, w)$$

*for every  $v, w \in \mathcal{T}^{(a)}$  such that  $\Lambda_v \leq h$  and  $\Lambda_w \leq h$ .*

**Remark.** The statement of the lemma makes sense because on the event where (35) holds, the trees  $\mathcal{T}^{(a)}$  and  $\mathcal{T}_\infty^{i,(a)}$  are identified (as explained before the statement of the lemma), and so  $\Delta^{(a)}(v, w)$  and  $\Delta^i(v, w)$  both make sense when  $v, w \in \mathcal{T}^{(a)} = \mathcal{T}_\infty^{i,(a)}$ .

The statement of the proposition follows from Lemma 20. Indeed, by Corollary 8,

$$\inf\{\Lambda_u : u \in \mathcal{T}_\infty^i \setminus \mathcal{T}_\infty^{i,(a)}\} \xrightarrow{a \rightarrow \infty} +\infty, \text{ a.s.}$$

and so, when  $a$  is large, the property (36) will hold except on a set of small probability. Also, by Lemma 19, we know that the property (35) holds outside an event of small probability. Note that, when (36) holds, labels do not vanish on  $\mathcal{T}_\infty^i \setminus \mathcal{T}_\infty^{i,(a)}$ , and so the ‘‘boundary’’ (set of points with zero label) of  $\mathcal{T}_\infty^i$  is identified with the boundary of  $\mathcal{T}^{(a)}$ . When (35) and (36) both hold, the conclusion of Lemma 20 shows that  $\mathcal{B}_{h'}(\mathbb{D}_z^\infty)$  and  $\mathcal{B}_{h'}(\mathbb{D}_z^{\bullet,a})$  are isometric, for every  $0 \leq h' \leq h$ .

*Proof of Lemma 20.* Throughout the proof we assume that both (35) and (36) hold, so that  $\mathcal{T}^{(a)}$  and  $\mathcal{T}_\infty^{i,(a)}$  are identified. If  $u, v \in \mathcal{T}_\infty^{i,(a)} = \mathcal{T}^{(a)}$ , we use the notation  $[u, v]_{\mathcal{T}_\infty^i}$  for the interval from  $u$  to  $v$  in  $\mathcal{T}_\infty^i$ , and similarly  $[u, v]_{\mathcal{T}^{(a)}}$  for the same interval in  $\mathcal{T}^{(a)}$ . We note that either  $[u, v]_{\mathcal{T}_\infty^i} \subset \mathcal{T}_\infty^{i,(a)}$ , and then  $[u, v]_{\mathcal{T}_\infty^i} = [u, v]_{\mathcal{T}^{(a)}}$ , or  $[u, v]_{\mathcal{T}_\infty^i} \not\subset \mathcal{T}_\infty^{i,(a)}$ , and then  $[u, v]_{\mathcal{T}_\infty^i}$  is the union of  $[u, v]_{\mathcal{T}^{(a)}}$  and  $\mathcal{T}_\infty^i \setminus \mathcal{T}_\infty^{i,(a)}$ .

We use the notation  $\mathcal{T}_\infty^{i,(a),\circ} = \{u \in \mathcal{T}_\infty^{i,(a)} : \Lambda_u > 0\}$ . We first observe that, if  $u, v \in \mathcal{T}_\infty^{i,(a),\circ}$ , we have

$$\Delta^{(a),\circ}(u, v) \leq \Delta^{i,\circ}(u, v). \quad (37)$$

Let us explain this bound. Since labels do not vanish on  $\mathcal{T}_\infty^i \setminus \mathcal{T}_\infty^{i,(a)}$ , it is immediate that  $\Delta^{(a),\circ}(u, v) = \infty$  if and only if  $\Delta^{i,\circ}(u, v) = \infty$ . So we may assume that both are finite and then (37) directly follows

from the definition of these quantities and the fact that we have always  $[u, v]_{\mathcal{T}^{(a)}} \subset [u, v]_{\mathcal{T}_\infty^i}$  (and labels are the same on  $\mathcal{T}^{(a)}$  and  $\mathcal{T}_\infty^{i,(a)}$ ).

Next suppose that  $u, v \in \mathcal{T}_\infty^{i,(a),\circ}$  satisfy also  $\Lambda_u \leq h$  and  $\Lambda_v \leq h$ . Recall the definition (29) of  $\Delta^i(u, v)$  as an infimum involving all possible choices of  $u_0, u_1, \dots, u_p$  in  $\mathcal{T}_\infty^{i,\circ}$  such that  $u_0 = u$  and  $u_p = v$ . We claim that in this definition we can restrict our attention to the case when  $u_0, u_1, \dots, u_p$  satisfy  $\Lambda_{u_j} < A + h + 1$  for every  $1 \leq j \leq p$ , and therefore  $u_0, u_1, \dots, u_p \in \mathcal{T}_\infty^{i,(a)}$ , by (36). To see this, suppose that  $\Lambda_{u_k} \geq A + h + 1$ , for some  $k \in \{1, \dots, p-1\}$ . Then, from the definition of  $A$ , we have also  $\Lambda_{u_k} \geq \Delta^i(u, v) + h + 1$ . Hence, using the bound (30), we get

$$\sum_{j=1}^p \Delta^{i,\circ}(u_{j-1}, u_j) \geq |\Lambda_{u_k} - \Lambda_u| \geq (\Delta^i(u, v) + h + 1) - h = \Delta^i(u, v) + 1,$$

so that we may disregard the sequence  $u_0, u_1, \dots, u_p$  in the infimum defining  $\Delta^i(u, v)$ .

By the previous considerations and (37), we get, for  $u, v \in \mathcal{T}_\infty^{i,(a),\circ}$  such that  $\Lambda_u \leq h$  and  $\Lambda_v \leq h$ ,

$$\Delta^i(u, v) = \inf_{\substack{u_0=u, u_1, \dots, u_p=v \\ u_1, \dots, u_{p-1} \in \mathcal{T}_\infty^{i,(a)}}} \sum_{j=1}^p \Delta^{i,\circ}(u_{j-1}, u_j) \geq \inf_{\substack{u_0=u, u_1, \dots, u_p=v \\ u_1, \dots, u_{p-1} \in \mathcal{T}^{(a)}}} \sum_{j=1}^p \Delta^{(a),\circ}(u_{j-1}, u_j) = \Delta^{(a)}(u, v). \quad (38)$$

We now want to argue that we have indeed the equality  $\Delta^{(a)}(u, v) = \Delta^i(u, v)$ . To this end it is enough to show that, for any sequence  $u_0 = u, u_1, \dots, u_p = v$  in  $\mathcal{T}^{(a)}$  such that

$$\sum_{j=1}^p \Delta^{(a),\circ}(u_{j-1}, u_j) < \Delta^{(a)}(u, v) + 1, \quad (39)$$

we have in fact  $\Delta^{(a),\circ}(u_{j-1}, u_j) = \Delta^{i,\circ}(u_{j-1}, u_j)$  for every  $j \in \{1, \dots, p\}$  (this will entail that the two infima in (38) are equal). We argue by contradiction and suppose that  $\Delta^{(a),\circ}(u_{j-1}, u_j) < \Delta^{i,\circ}(u_{j-1}, u_j)$  for some  $j \in \{1, \dots, p\}$ . This means that we have

$$\inf_{w \in [u_{j-1}, u_j]_{\mathcal{T}_\infty^i}} \Lambda_w < \inf_{w \in [u_{j-1}, u_j]_{\mathcal{T}^{(a)}}} \Lambda_w,$$

or the same with  $[u_{j-1}, u_j]$  replaced by  $[u_j, u_{j-1}]$ . However,  $[u_{j-1}, u_j]_{\mathcal{T}_\infty^i}$  can be different from  $[u_{j-1}, u_j]_{\mathcal{T}^{(a)}}$  only if  $[u_{j-1}, u_j]_{\mathcal{T}_\infty^i}$  is the union of  $[u_{j-1}, u_j]_{\mathcal{T}^{(a)}}$  and  $\mathcal{T}_\infty^i \setminus \mathcal{T}_\infty^{i,(a)}$ , and we get

$$\inf_{w \in [u_{j-1}, u_j]_{\mathcal{T}^{(a)}}} \Lambda_w \geq \inf_{w \in \mathcal{T}_\infty^i \setminus \mathcal{T}_\infty^{i,(a)}} \Lambda_w \geq A + h + 1.$$

This implies in particular that  $\Lambda_{u_j} \geq A + h + 1$ , and by the same argument as above this gives a contradiction with (39). This completes the proof of the lemma.  $\square$

In the next statement, we use the local Gromov-Hausdorff-Prokhorov convergence for pointed locally compact measure length spaces, as defined in [2].

**Theorem 21.** *We have*

$$\mathbb{D}_z^{\bullet, a} \xrightarrow[a \rightarrow \infty]{(d)} \mathbb{D}_z^\infty$$

*in distribution in the sense of the local Gromov-Hausdorff-Prokhorov convergence.*

*Proof.* The statement of the theorem is an immediate consequence of Proposition 18. In fact, it is enough to verify that, for every  $h > 0$ , the closed ball of radius  $h$  centered at the distinguished point  $\alpha$  of  $\mathbb{D}_z^{\bullet, a}$  converges in distribution to the corresponding ball in  $\mathbb{D}_z^\infty$  as  $a \rightarrow \infty$ , in the sense of the Gromov-Hausdorff-Prokhorov convergence for pointed compact measure metric spaces. However, this readily follows from the coupling obtained in Proposition 18, since the closed ball of radius  $h$  centered at  $\alpha$  is obviously contained in  $\mathcal{B}_h(\mathbb{D}_z^{\bullet, a})$  and similarly in the limiting space.  $\square$

We conclude this section with a couple of almost sure properties of the infinite-volume Brownian disk  $\mathbb{D}_z^\infty$  that can be derived from our approach. First, from the analogous result for the disk  $\mathbb{D}_z^{\bullet,a}$  (see the remarks after Proposition 14) and the coupling in Proposition 18, one easily obtains the existence of the uniform measure  $\mu_z^\infty$  on  $\partial\mathbb{D}_z^\infty$ , which is a measure of total mass  $z$  satisfying

$$\langle \mu_z^\infty, \varphi \rangle = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \int_{\mathbb{D}_z^\infty} \text{Vol}(dx) \mathbf{1}_{\{\Delta^i(x, \partial\mathbb{D}_z^\infty) < \varepsilon\}} \varphi(x),$$

for any continuous function  $\varphi$  on  $\mathbb{D}_z^\infty$ , a.s. In particular the volume of the tubular neighborhood of radius  $\varepsilon$  of  $\partial\mathbb{D}_z^\infty$  behaves like  $z\varepsilon^2$  when  $\varepsilon \rightarrow 0$ .

Our construction of  $\mathbb{D}_z^\infty$  is well suited to the analysis of geodesics to the boundary. Write  $(\mathcal{E}_s^i)_{s \in \mathbb{R}}$  for the clockwise exploration of the tree  $\mathcal{T}_\infty^i$ , and set

$$s_0 := \min\{s \in \mathbb{R} : \Lambda_{\mathcal{E}_s^i} = 0\}, \quad s_1 := \max\{s \in \mathbb{R} : \Lambda_{\mathcal{E}_s^i} = 0\}.$$

Also set  $x_0 := \Pi^i(\mathcal{E}_{s_0}) = \Pi^i(\mathcal{E}_{s_1})$ .

**Proposition 22.** *Almost surely, there exists  $h_0 > 0$  such that, for every  $x \in \mathbb{D}_z^\infty$  with  $\Delta^i(x, \partial\mathbb{D}_z^\infty) > h_0$ , any geodesic from  $x$  to  $\partial\mathbb{D}_z^\infty$  hits  $\partial\mathbb{D}_z^\infty$  at  $x_0$ .*

The proof shows more precisely that all geodesics to  $\partial\mathbb{D}_z^\infty$  starting outside a sufficiently large ball coalesce before hitting the boundary.

*Proof.* Recall the notation  $\mathcal{T}_\infty^{i,(a)}$  in the proof of Proposition 18. By Corollary 8, we may choose  $a$  large enough so that labels  $\Lambda_v$  do not vanish on  $\mathcal{T}_\infty^i \setminus \mathcal{T}_\infty^{i,(a)}$ . Then we may take  $h_0 = \max\{\Lambda_v : v \in \mathcal{T}_\infty^{i,(a)}\}$ . To verify this, fix  $x \in \mathbb{D}_z^\infty$  such that  $\Delta^i(x, \partial\mathbb{D}_z^\infty) > h_0$ , then we may write  $x = \Pi^i(v)$  with  $v \in \mathcal{T}_\infty^i \setminus \mathcal{T}_\infty^{i,(a)}$ , and we have  $\Delta^i(x, \partial\mathbb{D}_z^\infty) = \Lambda_v$ . Consider a simple geodesic  $\gamma = (\gamma_r)_{0 \leq r \leq \Lambda_v}$  from  $x$  to  $\partial\mathbb{D}_z^\infty$  constructed as in Section 4.1. Then it is straightforward to verify that  $\gamma_{\Lambda_v} = x_0$ . To complete the proof, we just need the fact that any geodesic from  $x$  to  $\partial\mathbb{D}_z^\infty$  is a simple geodesic. This follows via Theorem 29 below from the analogous result in the Brownian plane, which is itself a consequence of the study of geodesics in the Brownian map [24]. We omit the details.  $\square$

One may also consider geodesic rays in the infinite Brownian disk (a geodesic ray  $\gamma = (\gamma_t)_{t \geq 0}$  is an infinite geodesic path). In a way analogous to the case of the Brownian plane (see [14, Theorem 18]) one obtains that any two geodesic rays in  $\mathbb{D}_z^\infty$  coalesce in finite time. Again this can be deduced from the Brownian plane result via Theorem 29, but this also follows, with some more work, from the alternative construction of the infinite Brownian disk presented in Section 6 below.

## 4.5 The Brownian half-plane

In this section, we define the Brownian half-plane and show that it is the tangent cone in distribution of the free pointed Brownian disk at a point chosen uniformly on its boundary. Let us start with the definition. We consider a coding triple  $(R, \mathcal{P}, \mathcal{P}')$ , where  $R = (R_t)_{t \in [0, \infty)}$  is a three-dimensional Bessel process started from 0, and, conditionally on  $R$ ,  $\mathcal{P}$  and  $\mathcal{P}'$  are independent Poisson point measures on  $\mathbb{R}_+ \times \mathcal{S}$  with intensity  $2 dt \mathbb{N}_{R_t}(d\omega)$ . For every  $r > 0$ , we set  $\mathbf{L}_r := \sup\{t \geq 0 : R_t = r\}$  (as in Section 3.2), and we let  $\tilde{\mathcal{P}}$  and  $\tilde{\mathcal{P}}'$  stand for the point measures  $\mathcal{P}$  and  $\mathcal{P}'$  truncated at level 0.

Following Section 2.4, we can use the coding triple  $(R, \tilde{\mathcal{P}}, \tilde{\mathcal{P}}')$  to construct a tree  $\mathcal{T}_\infty^{hp}$  equipped with nonnegative labels  $(\Lambda_v)_{v \in \mathcal{T}_\infty^{hp}}$ . In contrast with the measures  $\Theta_z$  used to define the infinite-volume Brownian disk, there is no conditioning on the total exit measure at 0, which is here infinite a.s., as it can be seen from a scaling argument. There are subtrees carrying zero labels that branch off the right side or the left side of the spine at arbitrary high levels, so that labels along the clockwise exploration of  $\mathcal{T}_\infty^{hp}$  vanish in both intervals  $(-\infty, -K)$  and  $[K, \infty)$ , for any  $K > 0$ .

We then follow the general procedure of Section 4.1. We set  $\mathcal{T}_\infty^{hp, \circ} := \{v \in \mathcal{T}_\infty^{hp} : \Lambda_v > 0\}$  and  $\partial\mathcal{T}_\infty^{hp} := \mathcal{T}_\infty^{hp} \setminus \mathcal{T}_\infty^{hp, \circ}$ , and we let  $\Delta^{hp, \circ}(u, v)$  and  $\Delta^{hp}(u, v)$ , for  $u, v \in \mathcal{T}_\infty^{hp, \circ}$ , be the pseudo-metric functions associated with the triple  $(R, \tilde{\mathcal{P}}, \tilde{\mathcal{P}}')$  as in Section 4.1.

- Lemma 23.** (i) *The mapping  $(u, v) \mapsto \Delta^{hp}(u, v)$  has a.s. a continuous extension to  $\mathcal{T}_\infty^{hp} \times \mathcal{T}_\infty^{hp}$ .*  
(ii) *A.s., for every  $u, v \in \mathcal{T}_\infty^{hp, \circ}$ , the property  $\Delta^{hp}(u, v) = 0$  holds if and only if  $\Delta^{hp, \circ}(u, v) = 0$ .*  
(iii) *A.s., for every  $u, v \in \partial\mathcal{T}_\infty^{hp}$ , the property  $\Delta^{hp}(u, v) = 0$  holds if and only if  $\Lambda_w > 0$  for every  $w \in ]u, v[$ , or for every  $w \in ]v, u[$ .*

*Proof.* Property (i) can be derived by minor modifications of the proof of [28, Proposition 31], noting that we may restrict our attention to the bounded subtree obtained by truncating  $\mathcal{T}_\infty^{hp}$  at height  $\mathbf{L}_r$  for some  $r > 0$ . We omit the details. As for (ii) and (iii), there is an additional complication due to the fact that it is not immediately clear why we can restrict our attention to a bounded subtree. Let us explain the argument for (iii), which is the property we use below. The fact that  $\Lambda_w > 0$  for every  $w \in ]u, v[$  implies  $\Delta^{hp}(u, v) = 0$  is easy and left to the reader. Suppose then that  $u, v$  are distinct points of  $\partial\mathcal{T}_\infty^{hp}$  are such that  $\Delta^{hp}(u, v) = 0$ . Without loss of generality we can assume that  $[u, v]$  is compact, and we then have to check that  $\Lambda_w > 0$  for every  $w \in ]u, v[$ . Recall the notation  $[[u, \infty[[$  for the unique geodesic ray from  $u$  in the tree  $\mathcal{T}_\infty^{hp}$ , and  $]]u, \infty]] = [[u, \infty[[ \setminus \{u\}$ . We claim that, for every  $\delta > 0$ , we can find points  $u' \in ]u, \infty[[$  and  $v' \in ]v, \infty[[$  such that  $\Lambda_{u'} < \delta$  and  $\Lambda_{v'} < \delta$  and there exist  $w_0 = u', w_1, \dots, w_p = v' \in [u, v]$  such that

$$\sum_{i=1}^p \Delta^{hp, \circ}(w_{i-1}, w_i) < \delta. \quad (40)$$

If the claim holds, we can use [28, Proposition 32 (ii)] to see that necessarily  $\Lambda_w > 0$  for every  $w \in ]u, v[$  (the point is the fact that all  $w_i$ 's belong to  $[u, v]$ , and thus we are dealing with a compact subtree of  $\mathcal{T}_\infty^{hp}$ ). So it remains to prove our claim. First note that we can find  $u' \in ]u, \infty[[$  and  $v' \in ]v, \infty[[$  such that  $\Lambda_{u'} < \delta/2$ ,  $\Lambda_{v'} < \delta/2$  and  $\Delta^{hp}(u', v') < \delta/2$  and in particular there exist  $w_0, w_1, \dots, w_p \in \mathcal{T}_\infty^{hp, \circ}$  such that (40) holds with  $\delta$  replaced by  $\delta/2$ . It may happen that some of the  $w_i$ 's do not belong to  $[u, \infty)$ , but then we can replace  $u'$  by  $w_{j+1}$ , where  $j = \max\{i : w_i \notin [u, \infty)\}$ , noting that necessarily  $w_{j+1} \in ]u, \infty[[$  (otherwise  $\Delta^{hp, \circ}(w_j, w_{j+1})$  would be  $\infty$ ) and  $\Lambda_{w_{j+1}} < \delta$  by the bound  $|\Lambda_{w_i} - \Lambda_{w_{i-1}}| \leq \Delta^{hp, \circ}(w_{i-1}, w_i)$  for  $1 \leq i \leq j+1$ . Therefore we can assume that all  $w_i$ 's belong to  $[u, \infty)$ , and then a symmetric argument shows that we can assume that they all belong to  $[u, v]$  as desired.  $\square$

We set  $\mathbb{H}_\infty = \mathcal{T}_\infty^{hp} / \{\Delta^{hp} = 0\}$ , and we let  $\Pi^{hp}$  denote the canonical projection from  $\mathcal{T}_\infty^{hp}$  onto  $\mathbb{H}_\infty$ . We equip  $\mathbb{H}_\infty$  with the distance induced by  $\Delta^{hp}$  and the volume measure which is the pushforward of the volume measure on  $\mathcal{T}_\infty^{hp}$  under the canonical projection. We observe that  $\mathbb{H}_\infty$  has a distinguished vertex, namely the root  $\rho$  of  $\mathcal{T}_\infty^{hp}$  (or bottom of the spine). By Lemma 23 (iii), the equivalence class of  $\rho$  in the quotient  $\mathcal{T}_\infty^{hp} / \{\Delta^{hp} = 0\}$  must be a singleton, since there are points of  $\mathcal{T}_\infty^{hp}$  with zero label arbitrarily close to  $\rho$ , both on the left side and on the right side of the spine.

**Definition 24.** *The random pointed locally compact measure metric space  $\mathbb{H}_\infty$  is called the Brownian half-plane.*

At the end of Section 6, we will explain why this definition is consistent with the one found in [8] or in [19]. The Brownian half-plane enjoys the same scale invariance property as the Brownian plane: Recalling the notation  $\lambda \cdot E$  introduced in Section 4.2,  $\lambda \cdot \mathbb{H}_\infty$  has the same distribution as  $\mathbb{H}_\infty$ , for every  $\lambda > 0$ . The boundary  $\partial\mathbb{H}_\infty$  is defined by  $\partial\mathbb{H}_\infty := \Pi^{hp}(\partial\mathcal{T}_\infty^{hp})$  (one can prove that  $\mathbb{H}_\infty$  is homeomorphic to the usual half-plane and then  $\partial\mathbb{H}_\infty$  is also the set of all points of  $\mathbb{H}_\infty$  that have no neighborhood homeomorphic to an open disk, but we do not need these facts here). As noted in Section 4.1, for any  $v \in \mathcal{T}_\infty^{hp}$ ,  $\Lambda_v$  is equal to the distance from  $\Pi^{hp}(v)$  to the boundary  $\partial\mathbb{H}_\infty$ .

Let  $r > 0$  and let  $\mathcal{T}_{\infty, r}^{hp}$  be the closed subset of  $\mathcal{T}_\infty^{hp}$  consisting of the part  $[0, \mathbf{L}_r]$  of the spine and the subtrees branching off  $[0, \mathbf{L}_r]$ .

**Lemma 25.** *We have*

$$\inf_{v \notin \mathcal{T}_{\infty, r}^{hp}} \Delta^{hp}(\rho, v) > 0, \quad \text{a.s.}$$

*Proof.* We argue by contradiction and assume that there is a sequence  $(u_n)_{n \geq 1}$  in the complement of  $\mathcal{T}_{\infty, r}^{hp}$  such that  $\Delta^{hp}(\rho, u_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Suppose that infinitely many points of this sequence belong to  $[\rho, \infty)$ . Let  $v_{(r)}$  be the last point of  $\mathcal{T}_{\infty, r}^{hp} \cap \partial\mathcal{T}_\infty^{hp}$  visited by the exploration of  $\mathcal{T}_\infty^{hp}$ , and note



that  $\Pi^{hp}(\rho) \neq \Pi^{hp}(v_{(r)})$ , by Lemma 23 (iii). Then an argument very similar to the proof of Lemma 23 (iii) shows that we can find another sequence  $(v_n)_{n \geq 1}$  with  $v_n \in \llbracket v_{(r)}, \infty \llbracket$  and such that we still have  $\Delta^{hp}(\rho, v_n) \rightarrow 0$  as  $n \rightarrow \infty$ . In particular,  $\Lambda_{v_n} \rightarrow 0$ , and this implies that  $v_n \rightarrow v_{(r)}$  in  $\mathcal{T}_\infty^{hp}$ , and thus  $\Delta^{hp}(v_{(r)}, v_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Finally we get  $\Delta^{hp}(\rho, v_{(r)}) = 0$ , which is a contradiction. The case when infinitely many points of this sequence belong to  $(\infty, \rho]$  is treated in a symmetric manner.  $\square$

It follows from Lemma 25 that  $\Pi^{hp}(\mathcal{T}_{\infty, r}^{hp})$  contains a ball of positive radius centered at  $\rho$  in  $\mathbb{H}_\infty$ . Then, by scale invariance, we have a.s.

$$\lim_{r \rightarrow \infty} \left( \inf_{v \notin \mathcal{T}_{\infty, r}^{hp}} \Delta^{hp}(\rho, v) \right) = +\infty.$$

This implies in particular that  $\mathbb{H}_\infty$  is boundedly compact (any ball centered at  $\rho$  is contained in the image of a compact subtree of  $\mathcal{T}_\infty^{hp}$  under  $\Pi^{hp}$ ).

Our next goal is to prove that  $\mathbb{H}_\infty$  is the tangent cone in distribution of the pointed Brownian disk at a point chosen uniformly on its boundary — this will eventually allow us to make the connection with previous definitions of the Brownian half-plane. Recall from the end of Section 4.3 the notation  $\bar{\mathbb{D}}_z^{\bullet, a}$  for the pointed measure metric space obtained from  $\mathbb{D}_z^{\bullet, a}$  by “forgetting” the first distinguished point (so  $\bar{\mathbb{D}}_z^{\bullet, a}$  is pointed at a point chosen uniformly on its boundary).

**Theorem 26.** *Let  $z > 0$  and  $a > 0$ . We have*

$$\lambda \cdot \bar{\mathbb{D}}_z^{\bullet, a} \xrightarrow[\lambda \rightarrow \infty]{(d)} \mathbb{H}_\infty,$$

*in distribution in the sense of the local Gromov-Hausdorff-Prokhorov convergence.*

We give below the proof of Theorem 26 for  $a = 1$ , but a scaling argument yields the general case. Before we proceed to the proof of Theorem 26, we start with some preliminary estimates. We consider again a triple  $(Y, \widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}')$  distributed as explained at the beginning of Section 3.2 with  $a = 1$ . Recall that the random path  $Y$  is defined on the interval  $[0, T^Y]$ , that  $Y_0 = 1$  and  $T^Y = \inf\{t \geq 0 : Y_t = 0\}$ . For every  $\varepsilon \in (0, 1)$ , we also set

$$T_\varepsilon := \inf\{t \geq 0 : Y_t = \varepsilon\}.$$

We let  $\widetilde{\mathcal{M}}^\varepsilon(dtd\omega)$ , resp.  $\widetilde{\mathcal{M}}'^\varepsilon(dtd\omega)$ , be the image of  $\mathbf{1}_{[T_\varepsilon, T^Y]}(t) \widetilde{\mathcal{M}}(dtd\omega)$ , resp. of  $\mathbf{1}_{[T_\varepsilon, T^Y]}(t) \widetilde{\mathcal{M}}'(dtd\omega)$ , under the mapping  $(t, \omega) \mapsto (t - T_\varepsilon, \omega)$ . We also set  $Y_t^\varepsilon := Y_{T_\varepsilon + t}$  for  $0 \leq t \leq T^Y - T_\varepsilon$ . Recall that

$$\mathcal{Z}^Y = \int \widetilde{\mathcal{M}}(dtd\omega) \mathcal{Z}_0(\omega) + \int \widetilde{\mathcal{M}}'(dtd\omega) \mathcal{Z}_0(\omega),$$

and also set

$$\mathcal{Z}^{Y, \varepsilon} := \int \widetilde{\mathcal{M}}^\varepsilon(dtd\omega) \mathcal{Z}_0(\omega) + \int \widetilde{\mathcal{M}}'^\varepsilon(dtd\omega) \mathcal{Z}_0(\omega).$$

Set  $\Gamma^\varepsilon := (Y^\varepsilon, \widetilde{\mathcal{M}}^\varepsilon, \widetilde{\mathcal{M}}'^\varepsilon)$ , and observe that  $\Gamma^\varepsilon$  is a coding triple in the sense of Section 2.4. Moreover, the conditional distribution of  $\Gamma^\varepsilon$  knowing  $\mathcal{Z}^{Y, \varepsilon} = z$  is  $\check{\Theta}_z^{(\varepsilon)}$ . Our first goal is to show that the conditional distribution of  $\Gamma^\varepsilon$  given  $\mathcal{Z}^Y = z$  is close to its unconditional distribution when  $\varepsilon \rightarrow 0$ .

From (13), we have

$$\mathbb{E}[e^{-\lambda \mathcal{Z}^Y}] = \left(1 + \sqrt{2\lambda/3}\right)^{-3}, \quad \mathbb{E}[e^{-\lambda \mathcal{Z}^{Y, \varepsilon}}] = \left(1 + \varepsilon \sqrt{2\lambda/3}\right)^{-3}.$$

Furthermore, we may write  $\mathcal{Z}^Y = \mathcal{Z}^{Y, \varepsilon} + \widehat{\mathcal{Z}}^{Y, \varepsilon}$ , where  $\mathcal{Z}^{Y, \varepsilon}$  and  $\widehat{\mathcal{Z}}^{Y, \varepsilon}$  are independent (more precisely,  $\widehat{\mathcal{Z}}^{Y, \varepsilon}$  is independent of  $\Gamma^\varepsilon$ ). Hence

$$\mathbb{E}[e^{-\lambda \widehat{\mathcal{Z}}^{Y, \varepsilon}}] = \left( \frac{1 + \varepsilon \sqrt{2\lambda/3}}{1 + \sqrt{2\lambda/3}} \right)^3. \tag{41}$$

The distribution of  $\widehat{\mathcal{Z}}^{Y, \varepsilon}$  can be written in the form

$$\varepsilon^3 \delta_0(dy) + \widehat{\Upsilon}_\varepsilon(dy),$$

where the measure  $\widehat{\Upsilon}_\varepsilon(dy)$  is supported on  $(0, \infty)$ . To simplify notation, we also write  $\varphi(y) = y h_1(y)$  for the density of  $\mathcal{Z}^Y$  and  $\varphi_\varepsilon(y) = y h_\varepsilon(y)$  for the density of  $\mathcal{Z}^{Y, \varepsilon}$ .

**Lemma 27.** We have  $\widehat{\Upsilon}_\varepsilon(dy) = \widehat{\varphi}_\varepsilon(y) dy$ , where the functions  $\widehat{\varphi}_\varepsilon(y)$  satisfy

$$\lim_{\varepsilon \rightarrow 0} \widehat{\varphi}_\varepsilon(y) = \varphi(y)$$

uniformly on every interval of the form  $[\delta, \infty)$ ,  $\delta > 0$ .

*Proof.* From (41), we have, for every  $\lambda > 0$ ,

$$\int_0^\infty \widehat{\Upsilon}_\varepsilon(dy) e^{-\lambda y} = \left( \frac{1 + \varepsilon\sqrt{2\lambda/3}}{1 + \sqrt{2\lambda/3}} \right)^3 - \varepsilon^3.$$

Now observe that

$$\left( \frac{1 + \varepsilon\sqrt{\lambda}}{1 + \sqrt{\lambda}} \right)^3 - \varepsilon^3 = \frac{(1 - \varepsilon)^3 + 3\varepsilon(1 - \varepsilon)^2(1 + \sqrt{\lambda}) + 3\varepsilon^2(1 - \varepsilon)(1 + \sqrt{\lambda})^2}{(1 + \sqrt{\lambda})^3} \quad (42)$$

where we have expanded  $(1 + \varepsilon\sqrt{\lambda})^3 = ((1 - \varepsilon) + \varepsilon(1 + \sqrt{\lambda}))^3$ . It follows from formulas (A.1), (A.2), (A.3) in the Appendix that the Laplace transform of the function  $\chi_{(\varepsilon)}$  defined by

$$\chi_{(\varepsilon)}(y) = (1 - \varepsilon)^3 \chi_3(y) + 3\varepsilon(1 - \varepsilon)^2 \chi_2(y) + 3\varepsilon^2(1 - \varepsilon) \chi_1(y).$$

is the quantity in (42). Consequently, we have  $\widehat{\Upsilon}_\varepsilon(dy) = \widehat{\varphi}_\varepsilon(y) dy$  with  $\widehat{\varphi}_\varepsilon(y) = \frac{3}{2} \chi_{(\varepsilon)}(\frac{3y}{2})$ . Furthermore, the explicit formulas for  $\chi_1, \chi_2, \chi_3$  show that  $\chi_{(\varepsilon)}(y)$  converge to  $\chi_3(y)$  as  $\varepsilon \rightarrow 0$ , uniformly on every interval of the form  $[\delta, \infty)$ ,  $\delta > 0$ . The result of the proposition follows since  $\varphi(y) = \frac{3}{2} \chi_3(\frac{3y}{2})$  by definition.  $\square$

**Lemma 28.** Let  $z > 0$ . The total variation distance between the conditional distribution of  $\Gamma^\varepsilon$  knowing that  $\mathcal{Z}^Y = z$  and the unconditional distribution of  $\Gamma^\varepsilon$  converges to 0 as  $\varepsilon \rightarrow 0$ .

**Remark.** We have made a canonical choice for the conditional distribution  $\check{\Theta}_z^{(1)}$  of  $(Y, \widetilde{M}, \widetilde{M}')$  knowing  $\mathcal{Z}^Y = z$ , and so the conditional distribution of  $\Gamma^\varepsilon$  knowing that  $\mathcal{Z}^Y = z$  is also well defined for every  $z$ .

*Proof.* The equality  $\mathcal{Z}^Y = \mathcal{Z}^{Y,\varepsilon} + \widehat{\mathcal{Z}}^{Y,\varepsilon}$  gives

$$\varphi(x) = \varepsilon^3 \varphi_\varepsilon(x) + \int_0^x \varphi_\varepsilon(y) \widehat{\varphi}_\varepsilon(x - y) dy \quad (43)$$

for every  $x > 0$ . Let  $G$  and  $g$  be measurable functions defined respectively on  $\mathcal{W} \times M_p(\mathbb{R}_+ \times \mathcal{S})^2$  and on  $\mathbb{R}_+$ , such that  $0 \leq G \leq 1$  and  $0 \leq g \leq 1$ . Then,

$$\begin{aligned} \mathbb{E}[G(\Gamma^\varepsilon)g(\mathcal{Z}^Y)] &= \mathbb{E}[G(\Gamma^\varepsilon)g(\mathcal{Z}^{Y,\varepsilon} + \widehat{\mathcal{Z}}^{Y,\varepsilon})] \\ &= \int dz \widehat{\varphi}_\varepsilon(z) \mathbb{E}[G(\Gamma^\varepsilon)g(\mathcal{Z}^{Y,\varepsilon} + z)] + \varepsilon^3 \mathbb{E}[G(\Gamma^\varepsilon)g(\mathcal{Z}^{Y,\varepsilon})] \\ &= \mathbb{E}\left[G(\Gamma^\varepsilon) \int_{\mathcal{Z}^{Y,\varepsilon}}^\infty dz g(z) \widehat{\varphi}_\varepsilon(z - \mathcal{Z}^{Y,\varepsilon})\right] + \varepsilon^3 \int dz g(z) \varphi_\varepsilon(z) \mathbb{E}[G(\Gamma^\varepsilon) | \mathcal{Z}^{Y,\varepsilon} = z] \\ &= \int dz g(z) \left( \mathbb{E}[G(\Gamma^\varepsilon) \mathbf{1}_{\{\mathcal{Z}^{Y,\varepsilon} < z\}} \widehat{\varphi}_\varepsilon(z - \mathcal{Z}^{Y,\varepsilon})] + \varepsilon^3 \varphi_\varepsilon(z) \mathbb{E}[G(\Gamma^\varepsilon) | \mathcal{Z}^{Y,\varepsilon} = z] \right). \end{aligned}$$

Recalling that the density of  $\mathcal{Z}^Y$  is  $\varphi$ , it follows that we have  $dz$  a.e.,

$$\mathbb{E}[G(\Gamma^\varepsilon) | \mathcal{Z}^Y = z] = \mathbb{E}\left[G(\Gamma^\varepsilon) \mathbf{1}_{\{\mathcal{Z}^{Y,\varepsilon} < z\}} \frac{\widehat{\varphi}_\varepsilon(z - \mathcal{Z}^{Y,\varepsilon})}{\varphi(z)}\right] + \varepsilon^3 \mathbb{E}[G(\Gamma^\varepsilon) | \mathcal{Z}^{Y,\varepsilon} = z] \frac{\varphi_\varepsilon(z)}{\varphi(z)}, \quad (44)$$

where we observe that  $\mathbb{E}[G(\Gamma^\varepsilon) | \mathcal{Z}^{Y,\varepsilon} = z] = \check{\Theta}_z^{(\varepsilon)}(G)$  is well defined for every  $z$ . We now want to argue that (44) holds for every  $z > 0$  and not only  $dz$  a.e. To this end, it is enough to consider the special case  $G(w, \mu, \mu') = \exp(-f(w) - \langle \mu, h \rangle - \langle \mu', h' \rangle)$  where,  $f, h, h'$  are nonnegative functions,  $f$  is bounded and continuous on  $\mathcal{W}$ ,  $h$  and  $h'$  are bounded and continuous on  $\mathbb{R}_+ \times \mathcal{S}$  and both  $h$  and  $h'$  vanish on

$\{(t, \omega) : \sigma(\omega) \leq \delta\}$  for some  $\delta > 0$ . In that case, using a scaling argument and Corollary 10, one checks that both sides of (44) are continuous functions of  $z$ , so that they must be equal for every  $z > 0$ .

From (12), we have  $\varphi_\varepsilon(z) = O(\varepsilon)$  as  $\varepsilon \rightarrow 0$ , hence, for every fixed  $z > 0$ ,

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^3 \mathbb{E}[G(\Gamma^\varepsilon) \mid \mathcal{Z}^{Y, \varepsilon} = z] \frac{\varphi_\varepsilon(z)}{\varphi(z)} = 0, \quad (45)$$

uniformly in the choice of  $G$ . On the other hand, using Lemma 27 and the fact that  $\mathcal{Z}^{Y, \varepsilon} \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , we have

$$\lim_{\varepsilon \rightarrow 0} \mathbf{1}_{\{\mathcal{Z}^{Y, \varepsilon} < z\}} \frac{\widehat{\varphi}_\varepsilon(z - \mathcal{Z}^{Y, \varepsilon})}{\varphi(z)} = 1 \quad (46)$$

almost surely. Moreover, using (43), we have

$$\mathbb{E} \left[ \mathbf{1}_{\{\mathcal{Z}^{Y, \varepsilon} < z\}} \frac{\widehat{\varphi}_\varepsilon(z - \mathcal{Z}^{Y, \varepsilon})}{\varphi(z)} \right] = \frac{1}{\varphi(z)} \int_0^z dy \varphi_\varepsilon(y) \widehat{\varphi}_\varepsilon(z - y) = \frac{1}{\varphi(z)} (\varphi(z) - \varepsilon^3 \varphi_\varepsilon(z)),$$

which tends to 1 as  $\varepsilon \rightarrow 0$ . By Scheffé's lemma, the convergence (46) also holds in  $L^1$ . The statement of the lemma then follows from (44) and (45).  $\square$

*Proof of Theorem 26.* The proof is based on a coupling argument relying on Lemma 28. If  $E$  is a pointed metric space, we use the notation  $B_r(E)$  for the closed ball of radius  $r$  centered at the distinguished point. The theorem will follow if we can prove that, for every  $K > 0$  and every  $\delta > 0$ , if  $\lambda$  is large enough we can couple  $\mathbb{H}_\infty$  and  $\mathbb{D}_z^{\bullet, 1}$  in such a way that the balls  $B_K(\lambda \cdot \mathbb{D}_z^{\bullet, 1})$  and  $B_K(\mathbb{H}_\infty)$  are isometric with probability at least  $1 - \delta$  (with an isometry preserving the volume measure and the distinguished point). Equivalently, recalling that  $\lambda \cdot \mathbb{H}_\infty$  has the same distribution as  $\mathbb{H}_\infty$ , it suffices to prove that, for  $\eta > 0$  small enough,  $\mathbb{H}_\infty$  and  $\mathbb{D}_z^{\bullet, 1}$  can be coupled so that  $B_\eta(\mathbb{D}_z^{\bullet, 1})$  and  $B_\eta(\mathbb{H}_\infty)$  are isometric with probability at least  $1 - \delta$  (again with an isometry preserving the volume measure and the distinguished point).

As explained at the end of Section 4.3, we may and will assume that  $\mathbb{D}_z^{\bullet, 1}$  is constructed from a coding triple  $(Y^{(z)}, \widetilde{\mathcal{M}}^{(z)}, \widetilde{\mathcal{M}}'^{(z)})$  distributed according to  $\check{\Theta}_z^{(1)}$ . The labeled tree associated with  $(Y^{(z)}, \widetilde{\mathcal{M}}^{(z)}, \widetilde{\mathcal{M}}'^{(z)})$  is denoted by  $\mathcal{T}^{(z)}$ , and we write  $\Delta^{(z), \circ}$  and  $\Delta^{(z)}$  for the pseudo-distance functions on  $\mathcal{T}^{(z)}$ , so that  $\Delta^{(z)}$  induces the metric on  $\mathbb{D}_z^{\bullet, 1}$ . The set of all points of  $\mathcal{T}^{(z)}$  with positive label is denoted by  $\mathcal{T}^{(z), \circ}$ .

For  $\varepsilon \in (0, 1)$ , let  $\Gamma^{(z), \varepsilon}$  be defined as  $\Gamma^\varepsilon$  but replacing the triple  $(Y, \widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}')$  by  $(Y^{(z)}, \widetilde{\mathcal{M}}^{(z)}, \widetilde{\mathcal{M}}'^{(z)})$  (so  $\Gamma^{(z), \varepsilon}$  is distributed as  $\Gamma^\varepsilon$  conditioned on  $\mathcal{Z}^Y = z$ ). Let  $\check{\Gamma}^{(z), \varepsilon}$ , resp.  $\check{\Gamma}^\varepsilon$ , denote the image of  $\Gamma^{(z), \varepsilon}$ , resp.  $\Gamma^\varepsilon$ , under the time reversal operation  $\mathbf{TR}$  in (8). We fix  $\delta > 0$  and claim that:

1. For  $\varepsilon \in (0, 1)$  small enough, the triples  $(Y^{(z)}, \widetilde{\mathcal{M}}^{(z)}, \widetilde{\mathcal{M}}'^{(z)})$  and  $(R, \mathcal{P}, \mathcal{P}')$  can be coupled in such a way that the equality

$$\check{\Gamma}^{(z), \varepsilon} = \left( (R_t)_{0 \leq t \leq \mathbf{L}_\varepsilon}, \mathbf{1}_{[0, \mathbf{L}_\varepsilon]}(t) \widetilde{\mathcal{P}}(dtd\omega), \mathbf{1}_{[0, \mathbf{L}_\varepsilon]}(t) \widetilde{\mathcal{P}}'(dtd\omega) \right) \quad (47)$$

holds with probability at least  $1 - \frac{\delta}{2}$ .

2. For  $\varepsilon \in (0, 1)$  small enough, we can choose  $\eta_0 > 0$  so that for every  $0 < \eta \leq \eta_0$ , we have

$$B_\eta(\mathbb{D}_z^{\bullet, 1}) = B_\eta(\mathbb{H}_\infty)$$

on the event where (47) holds, except possibly on an event of probability at most  $\frac{\delta}{2}$ .

Clearly the theorem follows from Properties 1 and 2. Property 1 is a consequence of Lemma 28: just note that the distribution of the coding triple in the right-hand side of (47) is the (unconditional) distribution of  $\check{\Gamma}^\varepsilon$ .

It remains to verify Property 2. We fix  $\varepsilon > 0$  small enough so that we can apply Property 1. We then assume that the triples  $(Y^{(z)}, \widetilde{\mathcal{M}}^{(z)}, \widetilde{\mathcal{M}}'^{(z)})$  and  $(R, \mathcal{P}, \mathcal{P}')$  have been coupled in such a way that the event where (47) holds has probability greater than  $1 - \frac{\delta}{2}$ , and we denote the latter event by  $\mathcal{F}$ .

We argue on the intersection  $\mathcal{F} \cap \mathcal{F}'$ , where  $\mathcal{F}'$  denotes the event where  $W_*(\omega_i) = 0$  for at least one atom  $(t_i, \omega_i)$  of  $\widetilde{\mathcal{M}}^{(z)}$  or of  $\widetilde{\mathcal{M}}'^{(z)}$  such that  $t_i < T_\varepsilon^{(z)} := \inf\{t \geq 0 : Y_t^{(z)} = \varepsilon\}$ . Clearly we can also assume that  $\mathcal{F}'$  has probability greater than  $1 - \frac{\delta}{6}$  by choosing  $\varepsilon$  even smaller if necessary.

Recall the notation  $\mathcal{T}_{\infty, r}^{hp}$  introduced before Lemma 25. From this lemma, we know that, for  $\eta > 0$  small enough, the set  $\{v \in \mathcal{T}_\infty^{hp} : \Delta^{hp}(\rho, v) \leq 4\eta\}$  will be contained in  $\mathcal{T}_{\infty, \varepsilon}^{hp}$ , except on an event of probability at most  $\frac{\delta}{6}$ . Moreover, if the latter property holds, we claim that we have also, for every  $u, v \in \mathcal{T}_\infty^{hp, \circ}$  such that  $\Delta^{hp}(\rho, u) \leq \eta$  and  $\Delta^{hp}(\rho, v) \leq \eta$ ,

$$\Delta^{hp}(u, v) = \inf_{\substack{u_0=u, u_1, \dots, u_p=v \\ u_1, \dots, u_{p-1} \in \mathcal{T}_{\infty, \varepsilon}^{hp} \cap \mathcal{T}_\infty^{hp, \circ}}} \sum_{i=1}^p \Delta^{hp, \circ}(u_{i-1}, u_i). \quad (48)$$

In other words, in formula (29) applied to  $\Delta^{hp}(u, v)$ , we may restrict the infimum to the case where all  $u_i$ 's belong to  $\mathcal{T}_{\infty, \varepsilon}^{hp}$ . Let us justify (48). Assume that  $\Delta^{hp}(\rho, u) \leq \eta$  and  $\Delta^{hp}(\rho, v) \leq \eta$  (so that in particular  $\Delta^{hp}(u, v) \leq 2\eta$ ) and  $u_0 = u, u_1, \dots, u_q \in \mathcal{T}_\infty^{hp, \circ}$  are such that

$$\sum_{i=1}^q \Delta^{hp, \circ}(u_{i-1}, u_i) < \Delta^{hp}(u, v) + \eta.$$

It then follows that  $\Delta^{hp}(u, u_q) < 3\eta$  and  $\Delta^{hp}(\rho, u_q) < 4\eta$  which implies  $u_q \in \mathcal{T}_{\infty, \varepsilon}^{hp}$ .

Furthermore, when applying formula (28) to compute the quantities  $\Delta^{hp, \circ}(u_{i-1}, u_i)$  in the right-hand side of (48), it is enough to consider the case when the interval  $[u_{i-1}, u_i]$  (resp.  $[u_i, u_{i-1}]$ ) is contained in  $\mathcal{T}_{\infty, \varepsilon}^{hp}$ , because otherwise this interval contains  $\mathcal{T}_\infty^{hp} \setminus \mathcal{T}_{\infty, \varepsilon}^{hp}$  and then the infimum of labels on  $[u_{i-1}, u_i]$  is 0. To summarize, on the event where (48) holds for every  $u, v \in \mathcal{T}_\infty^{hp, \circ}$  such that  $\Delta^{hp}(\rho, u) \leq \eta$  and  $\Delta^{hp}(\rho, v) \leq \eta$ , we get that the value of  $\Delta^{hp}(u, v)$  for such points  $u$  and  $v$  is determined by the tree  $\mathcal{T}_{\infty, \varepsilon}^{hp}$  and the labels on this tree.

On the event where (48) holds, we thus get that the ball  $B_\eta(\mathbb{H}_\infty)$  can be written as a function of the coding triple

$$\left( (R_t)_{0 \leq t \leq L_\varepsilon}, \mathbf{1}_{[0, L_\varepsilon]}(t) \widetilde{\mathcal{P}}(dtd\omega), \mathbf{1}_{[0, L_\varepsilon]}(t) \widetilde{\mathcal{P}}'(dtd\omega) \right)$$

since the tree  $\mathcal{T}_{\infty, \varepsilon}^{hp}$  and the labels on this tree are functions of this triple (and also the distinguished point of  $B_\eta(\mathbb{H}_\infty)$  corresponds to the root of this coding triple). To complete the argument (recalling that we assume (47)), we need to justify that  $B_\eta(\overline{\mathbb{D}}_z^{\bullet, 1})$  is given by the same function applied to the triple  $\check{\Gamma}^{(z), \varepsilon}$ , except possibly on a set of small probability. To get this, recall that  $\mathbb{D}_z^{\bullet, 1}$  is obtained by applying  $\Xi^\bullet$  to the snake trajectory  $\Omega(Y^{(z)}, \widetilde{\mathcal{M}}^{(z)}, \widetilde{\mathcal{M}}'^{(z)})$ . With the coding triple  $\check{\Gamma}^{(z), \varepsilon}$  we associate a labeled tree  $\mathcal{T}_\varepsilon^{(z)}$ , which is identified to a subtree of the labeled tree  $\mathcal{T}^{(z)}$ , and, modulo this identification,  $\mathcal{T}_\varepsilon^{(z)}$  is rooted at the top of the spine of the tree  $\mathcal{T}^{(z)}$ , which corresponds to the distinguished point  $\alpha$  of  $\overline{\mathbb{D}}_z^{\bullet, 1}$ . We claim that the image of  $\mathcal{T}_\varepsilon^{(z)}$  (viewed as a subset of  $\mathcal{T}^{(z)}$ ) under the canonical projection from  $\mathcal{T}^{(z)}$  onto  $\overline{\mathbb{D}}_z^{\bullet, 1}$  must contain a neighborhood of  $\alpha$ . As in the proof of Lemma 25, this property holds because the equivalence class of  $\alpha$  in  $\mathcal{T}^{(z)} / \{\Delta^{(z)} = 0\}$  is a singleton, which is a consequence of the fact that two points  $u$  and  $v$  of  $\mathcal{T}^{(z)}$  with zero label are identified in  $\overline{\mathbb{D}}_z^{\bullet, 1}$  if and only if labels stay positive on the interval  $]u, v[$ , or on the interval  $]v, u[$  (see the discussion after Proposition 14).

It follows from the preceding claim that, for  $\eta$  small enough, we have  $\Delta^{(z)}(\alpha, v) > 4\eta$  whenever  $v \notin \mathcal{T}_\varepsilon^{(z)}$ , except on an event of probability at most  $\frac{\delta}{6}$ . Discarding the latter event of small probability, the same argument as above shows that the analog of (48) holds for every  $u, v \in \mathcal{T}^{(z), \circ}$  such that  $\Delta^{(z)}(\alpha, u) < \eta$  and  $\Delta^{(z)}(\alpha, v) < \eta$ , provided we replace  $\Delta^{hp}$  by  $\Delta^{(z)}$ ,  $\mathcal{T}_\infty^{hp, \circ}$  by  $\mathcal{T}^{(z), \circ}$ , and  $\mathcal{T}_{\infty, \varepsilon}^{hp}$  by  $\mathcal{T}_\varepsilon^{(z)}$ . Furthermore, the quantities  $\Delta^{(z), \circ}(u_{i-1}, u_i)$  appearing in this analog can be computed from the labeled tree  $\mathcal{T}_\varepsilon^{(z)}$  (here we use our definition of  $\mathcal{F}'$ , which implies that  $\mathcal{T}^{(z)} \setminus \mathcal{T}_\varepsilon^{(z)}$  contains points with zero label).

It follows from the preceding discussion that, on the event  $\mathcal{F}$  that has probability at least  $1 - \frac{\delta}{2}$ , and except on an event of probability at most  $\frac{\delta}{2}$ , the ball  $B_\eta(\overline{\mathbb{D}}_z^{\bullet, 1})$  is obtained from the triple  $\check{\Gamma}^{(z), \varepsilon}$  by applying the same function that can be used to get the ball  $B_\eta(\mathbb{H}_\infty)$  from the triple in the right-hand side of (47). The desired result follows.  $\square$

## 5 Applications

### 5.1 Infinite Brownian disks in the Brownian plane

Recall the construction of the Brownian plane  $(\mathbb{B}\mathbb{P}_\infty, \Delta^p)$  from the coding triple  $(X, \mathcal{L}, \mathcal{R})$  in Section 4.2 and note that the same triple was also considered in Section 3.3. We use the notation  $(\mathcal{T}_\infty^p, (\Lambda_v)_{v \in \mathcal{T}_\infty^p})$  for the labeled tree associated with the triple  $(X, \mathcal{L}, \mathcal{R})$ , and we write  $\Pi^p$  for the canonical projection from  $\mathcal{T}_\infty^p$  onto  $\mathbb{B}\mathbb{P}_\infty$ . The distinguished point  $\rho$  of  $\mathbb{B}\mathbb{P}_\infty$  is the image of the root of  $\mathcal{T}_\infty^p$  under  $\Pi^p$ .

To simplify notation, for every  $r > 0$ , we write  $B(r) = B_r(\mathbb{B}\mathbb{P}_\infty)$  for the closed ball of radius  $r$  centered at  $\rho$  in  $\mathbb{B}\mathbb{P}_\infty$ . The hull  $B^\bullet(r)$  is then the subset of  $\mathbb{B}\mathbb{P}_\infty$  defined by saying that  $\mathbb{B}\mathbb{P}_\infty \setminus B^\bullet(r)$  is the unique unbounded connected component of  $\mathbb{B}\mathbb{P}_\infty \setminus B(r)$  (this component is unique since  $\mathbb{B}\mathbb{P}_\infty$  is homeomorphic to the plane [14]). Informally,  $B^\bullet(r)$  is obtained by filling in the (bounded) holes in  $B(r)$ . As in the introduction, it will be convenient to use the notation

$$\check{B}^\bullet(r) = \mathbb{B}\mathbb{P}_\infty \setminus B^\bullet(r).$$

One can give an explicit description of  $\check{B}^\bullet(r)$  in terms of the labeled tree  $(\mathcal{T}_\infty^p, (\Lambda_v)_{v \in \mathcal{T}_\infty^p})$ . For  $v \in \mathcal{T}_\infty^p$ , we recall that  $\llbracket v, \infty \llbracket$  is the geodesic ray from  $v$  in  $\mathcal{T}_\infty^p$ . Then,  $\check{B}^\bullet(r) = \Pi^p(F_r)$ , where

$$F_r := \{v \in \mathcal{T}_\infty^p : \Lambda_w > r \text{ for every } w \in \llbracket v, \infty \llbracket\}. \quad (49)$$

Similarly, the topological boundary of  $\check{B}^\bullet(r)$  (or of  $B^\bullet(r)$ ) is  $\partial\check{B}^\bullet(r) = \partial B^\bullet(r) = \Pi^p(\partial F_r)$ , with

$$\partial F_r = \{v \in \mathcal{T}_\infty^p : \Lambda_w = r \text{ and } \Lambda_w > r \text{ for every } w \in \llbracket v, \infty \llbracket\}, \quad (50)$$

with the obvious notation  $\llbracket v, \infty \llbracket$ . See formulas (16) and (17) in [15]. We note that the intersection of the set  $F_r$  with the spine of  $\mathcal{T}_\infty^p$  is just the interval  $(L_r, \infty)$ , where we recall the notation  $L_r$  in (18) (as in Section 2.4, the spine is identified to  $\mathbb{R}_+$ ). Following [15], we define the boundary size of  $B^\bullet(r)$  to be  $|\partial B^\bullet(r)| = \mathcal{Z}^{(r)}$ , where the quantity  $\mathcal{Z}^{(r)}$  is defined in (19):  $\mathcal{Z}^{(r)}$  is the sum over all atoms  $(t, \omega)$  of  $\mathcal{L}$  and  $\mathcal{R}$  such that  $t > L_r$  of the exit measures  $\mathcal{Z}_r(\omega)$  at level  $r$  — see formula (18) in [15]. We write  $\text{cl}(\check{B}^\bullet(r)) = \check{B}^\bullet(r) \cup \partial\check{B}^\bullet(r)$  for the closure of  $\check{B}^\bullet(r)$ , and similarly  $\text{cl}(F_r) = F_r \cup \partial F_r$ .

Recall that the intrinsic metric on an open subset  $O$  of  $\mathbb{B}\mathbb{P}_\infty$  is defined by declaring that the distance between two points  $x$  and  $y$  of  $O$  is the infimum of the lengths of all continuous curves  $\gamma : [0, 1] \rightarrow O$  such that  $\gamma(0) = x$  and  $\gamma(1) = y$ . Here the lengths are of course computed with respect to the metric  $\Delta^p$  of  $\mathbb{B}\mathbb{P}_\infty$ .

**Theorem 29.** *Let  $r > 0$ . Then a.s. the intrinsic metric on  $\check{B}^\bullet(r)$  has a unique continuous extension to  $\text{cl}(\check{B}^\bullet(r))$ , which is a metric on this set. We write  $(\mathbb{D}^{\infty, (r)}, \Delta^{\infty, (r)})$  for the resulting random locally compact metric space, which is equipped with the restriction of the volume measure on  $\mathbb{B}\mathbb{P}_\infty$  and pointed at  $\Pi^p(L_r)$ . Then, conditionally on  $|\partial B^\bullet(r)|$ ,  $(\mathbb{D}^{\infty, (r)}, \Delta^{\infty, (r)})$  is an infinite-volume Brownian disk with perimeter  $|\partial B^\bullet(r)|$ .*

*Proof.* Recall the notation  $(X^{(r)}, \tilde{\mathcal{L}}^{(r)}, \tilde{\mathcal{R}}^{(r)})$  introduced in Section 3.3, and the fact that, conditionally on  $\mathcal{Z}^{(r)} = z$ , this coding triple is distributed according to  $\Theta_z$  (Proposition 6). The construction of Section 4.4 produces, from the triple  $(X^{(r)}, \tilde{\mathcal{L}}^{(r)}, \tilde{\mathcal{R}}^{(r)})$ , a random measure metric space  $(\mathbb{D}^{\infty, (r)}, \Delta^{\infty, (r)})$  such that, conditionally on  $\mathcal{Z}^{(r)} = z$ ,  $(\mathbb{D}^{\infty, (r)}, \Delta^{\infty, (r)})$  is an infinite-volume Brownian disk with perimeter  $z$ . Furthermore,  $\mathbb{D}^{\infty, (r)}$  is obtained as a quotient space of the labeled tree  $\mathcal{T}_\infty^i$  coded by the triple  $(X^{(r)}, \tilde{\mathcal{L}}^{(r)}, \tilde{\mathcal{R}}^{(r)})$ . Here we use the same notation  $\mathcal{T}_\infty^i$  as in Section 4.4, where we were dealing with a different triple distributed according to  $\Theta_z$ , but this should create no confusion. We write  $\Pi^i$  for the canonical projection from  $\mathcal{T}_\infty^i$  onto  $\mathbb{D}^{\infty, (r)}$ .

It is easy to verify that the tree  $\mathcal{T}_\infty^i$  can be identified with  $\text{cl}(F_r)$ . The spine of  $\mathcal{T}_\infty^i$  is identified with the part  $[L_r, \infty)$  of the spine of  $\mathcal{T}_\infty^p$ , and we observe that, for each atom  $(t_i, \omega_i)$  of  $\mathcal{L}$  or  $\mathcal{R}$  such that  $t_i > L_r$  (so that  $\text{tr}_r(\omega_i)$  shifted by  $-r$  corresponds to an atom of  $\tilde{\mathcal{L}}^{(r)}$  or  $\tilde{\mathcal{R}}^{(r)}$ ), the genealogical tree  $\mathcal{T}_{(\text{tr}_r(\omega_i))}$  is identified with  $\{v \in \mathcal{T}_{(\omega_i)} : \Lambda_w > r \text{ for every } w \in \llbracket \rho(\omega_i), v \llbracket\}$  (see the end of Section 2.1). The identification of  $\mathcal{T}_\infty^i$  with  $\text{cl}(F_r)$  preserves labels, provided labels on  $\text{cl}(F_r)$  are shifted by  $-r$ . With a slight abuse of notation, if  $u \in \mathcal{T}_\infty^i$ , we will also write  $\Lambda_u$  for the label of the point of  $\text{cl}(F_r)$  corresponding to  $u$  in the identification of  $\mathcal{T}_\infty^i$  with  $\text{cl}(F_r)$  (so the label of  $u$  in  $\mathcal{T}_\infty^i$  is  $\Lambda_u - r$ ).

Furthermore, two vertices of  $\text{cl}(F_r)$  are identified in the quotient space  $\mathbb{B}\mathbb{P}_\infty$  if and only if the corresponding vertices of  $\mathcal{T}_\infty^i$  are identified in the quotient  $\mathbb{D}^{\infty,(r)}$ : to check this property in the case where the vertices belong to the boundary (the other case is immediate) we use the fact that two vertices  $u$  and  $v$  of  $\mathcal{T}_\infty^i$  with zero label are identified if and only if labels remain positive on one of the two intervals  $]u, v[$  and  $]v, u[$  of the tree  $\mathcal{T}_\infty^i$  (Lemma 16 (iii)). Thus we can identify  $\mathbb{D}^{\infty,(r)}$  with the set  $\Pi^p(\text{cl}(F_r)) = \text{cl}(\check{B}^\bullet(r))$ , in such a way that  $\partial\mathbb{D}^{\infty,(r)}$  is identified with  $\partial\check{B}^\bullet(r)$ , and this identification preserves the volume measures.

Modulo the preceding identification, both assertions of the theorem follow from the next lemma.

**Lemma 30.** *Let  $x$  and  $y$  be two points of  $\mathbb{D}^{\infty,(r)} \setminus \partial\mathbb{D}^{\infty,(r)}$ , and let  $x'$  and  $y'$  be the corresponding points in  $\check{B}^\bullet(r)$ . Then the intrinsic distance (relative to the open set  $\check{B}^\bullet(r)$ ) between  $x'$  and  $y'$  coincides with  $\Delta^{\infty,(r)}(x, y)$ .*

*Proof.* Let  $\Delta^{\infty,(r),\circ}(v, w)$  be defined as in (28) for the labeled tree  $(\mathcal{T}_\infty^i, (\Lambda_u - r)_{u \in \mathcal{T}_\infty^i})$  (recall that the label in  $\mathcal{T}_\infty^i$  of a point  $u \in \mathcal{T}_\infty^i$  is equal to  $\Lambda_u - r$ ), so that  $\Delta^{\infty,(r)}(v, w)$  is then given from  $\Delta^{\infty,(r),\circ}(v, w)$  by formula (29).

We first prove that the intrinsic distance between  $x'$  and  $y'$  is bounded above by  $\Delta^{\infty,(r)}(x, y)$ . To this end, let  $v$  and  $w$  be points of  $\mathcal{T}_\infty^i$  such that  $\Pi^i(v) = x$  and  $\Pi^i(w) = y$ . We claim that, if  $\Delta^{\infty,(r),\circ}(v, w) < \infty$ , then  $\Delta^{\infty,(r),\circ}(v, w)$  is the length of a continuous curve in  $\check{B}^\bullet(r)$  that connects  $x'$  to  $y'$ . Let us explain this. Without loss of generality, we may assume that

$$\Delta^{\infty,(r),\circ}(v, w) = \Lambda_v + \Lambda_w - 2 \inf_{u \in [v, w]} \Lambda_u,$$

with  $\inf_{u \in [v, w]} \Lambda_u > r$ . We let  $v'$  and  $w'$  be the points of  $F_r$  corresponding to  $v$  and  $w$  in the identification of  $\mathcal{T}_\infty^i$  with  $\text{cl}(F_r)$  (in particular  $\Lambda_{v'} = \Lambda_v$  and  $\Lambda_{w'} = \Lambda_w$ ). We note that the condition  $\inf_{u \in [v, w]} \Lambda_u > r$  implies that the interval  $[v, w]$  of  $\mathcal{T}_\infty^i$  is also identified with the interval  $[v', w']$  of  $\mathcal{T}_\infty^p$  (in particular we have  $\Delta^{p,\circ}(v', w') = \Delta^{\infty,(r),\circ}(v, w)$ ), and furthermore  $\Pi^p([v', w'])$  is contained in  $\check{B}^\bullet(r)$ . By concatenating two simple geodesics starting from  $\Pi^p(v') = x'$  and  $\Pi^p(w') = y'$  respectively up to their merging time, as explained at the end of Section 4.1, we construct a path from  $x'$  to  $y'$  whose length is equal to  $\Delta^{p,\circ}(v', w')$ , and which stays in  $\Pi^p([v', w']) \subset \check{B}^\bullet(r)$ . This gives our claim.

From the definition of  $\Delta^{\infty,(r)}$  as an infimum, we now get that  $\Delta^{\infty,(r)}(x, y)$  is bounded below by the infimum of lengths of continuous curves connecting  $x'$  and  $y'$  that stay in  $\check{B}^\bullet(r)$ . We thus obtain that the intrinsic distance between  $x'$  and  $y'$  (with respect to the open set  $\check{B}^\bullet(r)$ ) is bounded above by  $\Delta^{\infty,(r)}(x, y)$ .

It remains to prove the reverse bound. To this end, we need to verify that, if  $\gamma : [0, 1] \rightarrow \check{B}^\bullet(r)$  is a continuous curve such that  $\gamma(0) = x'$  and  $\gamma(1) = y'$ , then the length of  $\gamma$  is bounded below by  $\Delta^{\infty,(r)}(x, y)$ . We write  $\bar{\gamma}(t)$  for the point of  $\mathbb{D}^{\infty,(r)}$  corresponding to  $\gamma(t)$  in the identification of  $\text{cl}(\check{B}^\bullet(r))$  with  $\mathbb{D}^{\infty,(r)}$ . We may find  $\delta > 0$  such that  $\Lambda_{\gamma(t)} > r + \delta$  for every  $t \in [0, 1]$  (recall that  $\Lambda_z = \Delta^p(\rho, z)$  for every  $z \in \mathbb{B}\mathbb{P}_\infty$ ). Then, we may choose  $n$  large enough so that  $\Delta^p(\gamma(\frac{i-1}{n}), \gamma(\frac{i}{n})) < \delta/2$  for every  $1 \leq i \leq n$ . The length of  $\gamma$  is bounded below by  $\sum_{i=1}^n \Delta^p(\gamma(\frac{i-1}{n}), \gamma(\frac{i}{n}))$ , and so to get the desired result it suffices to verify that, for every  $1 \leq i \leq n$ ,

$$\Delta^p\left(\gamma\left(\frac{i-1}{n}\right), \gamma\left(\frac{i}{n}\right)\right) \geq \Delta^{\infty,(r)}\left(\bar{\gamma}\left(\frac{i-1}{n}\right), \bar{\gamma}\left(\frac{i}{n}\right)\right).$$

Fix  $1 \leq i \leq n$ , and recall the definition (29) of  $\Delta^p(\gamma(\frac{i-1}{n}), \gamma(\frac{i}{n}))$  as an infimum over possible choices of  $u_0 = \gamma(\frac{i-1}{n}), u_1, \dots, u_p = \gamma(\frac{i}{n})$  in  $\mathcal{T}_\infty^p$ , where we may restrict our attention to choices of  $u_0, u_1, \dots, u_p$  such that  $\Lambda_{u_j} > r + \delta/2$  (use  $\Delta^p(u, v) \geq |\Lambda_v - \Lambda_u|$ ) and  $\Delta^{p,\circ}(u_{j-1}, u_j) < \delta/2$  for every  $1 \leq j \leq p$ . It suffices to consider one such choice and to prove that

$$\sum_{j=1}^p \Delta^{p,\circ}(u_{j-1}, u_j) \geq \Delta^{\infty,(r)}\left(\bar{\gamma}\left(\frac{i-1}{n}\right), \bar{\gamma}\left(\frac{i}{n}\right)\right). \quad (51)$$

For every  $1 \leq j \leq p$ , the properties  $\Lambda_{u_j} > r + \delta/2$  and  $\Delta^{p,\circ}(u_{j-1}, u_j) < \delta/2$  imply that the minimal label on  $[u_{j-1}, u_j]$  is greater than  $r$  (or the same holds with  $[u_{j-1}, u_j]$  replaced by  $[u_j, u_{j-1}]$ ). This

shows in particular that there is a continuous curve from  $\gamma(\frac{i}{n})$  to  $\Pi^p(u_j)$  that stays in the complement of  $B(r)$ , so that  $\Pi^p(u_j)$  belongs to  $\check{B}^\bullet(r)$  and  $u_j$  must belong to  $F_r$ , which allows us to define  $\bar{u}_j$  as the point of  $\mathcal{T}_\infty^i$  corresponding to  $u_j$ . Furthermore the fact that the minimal label on  $[u_{j-1}, u_j]$  is greater than  $r$  also implies that the interval  $[u_{j-1}, u_j]$  in  $\mathcal{T}_\infty^p$  is identified to the interval  $[\bar{u}_{j-1}, \bar{u}_j]$  in  $\mathcal{T}_\infty^i$ , and then that  $\Delta^{p,\circ}(u_{j-1}, u_j) = \Delta^{\infty,(r),\circ}(\bar{u}_{j-1}, \bar{u}_j)$ . The bound (51) follows, which completes the proof of the lemma and of Theorem 5.1.  $\square$

In view of applications to isoperimetric inequalities in the Brownian plane [36], we state another result which complements Theorem 29 by showing that, in some sense, the exterior of the hull  $B^\bullet(r)$  is independent of this hull, conditionally on its boundary size. We keep the notation introduced at the beginning of this section, and in particular, we recall that the Brownian plane  $\mathbb{B}\mathbb{P}_\infty$  is constructed from the labeled tree  $(\mathcal{T}_\infty^p, (\Lambda_v)_{v \in \mathcal{T}_\infty^p})$  associated with the coding triple  $(X, \mathcal{L}, \mathcal{R})$ . We fix  $r > 0$  and write  $K_r$  for the complement of the set  $F_r$  defined in (49),

$$K_r := \{v \in \mathcal{T}_\infty^p : \Lambda_w \leq r \text{ for some } w \in \llbracket v, \infty \llbracket\}.$$

We have then  $B^\bullet(r) = \Pi^p(K_r)$  (cf. formulas (16) and (17) in [15]). Recall that, for every  $u, v \in \mathbb{B}\mathbb{P}_\infty$ ,

$$\Delta^{p,\circ}(u, v) = \Lambda_u + \Lambda_v - 2 \max \left( \inf_{w \in [u, v]} \Lambda_w, \inf_{w \in [v, u]} \Lambda_w \right). \quad (52)$$

We then set, for every  $u, v \in K_r$ ,

$$\Delta^{p,(r)}(u, v) = \inf_{\substack{u_0 = u, u_1, \dots, u_p = v \\ u_0, u_1, \dots, u_p \in K_r}} \sum_{i=1}^p \Delta^{p,\circ}(u_{i-1}, u_i) \quad (53)$$

where the infimum is over all choices of the integer  $p \geq 1$  and of the finite sequence  $u_0, u_1, \dots, u_p$  in  $K_r$  such that  $u_0 = u$  and  $u_p = v$ . For every  $u, v \in K_r$ , we have  $\Delta^p(u, v) \leq \Delta^{p,(r)}(u, v)$  (just note that  $\Delta^p(u, v)$  is defined by the same formula (53) without the restriction to  $u_0, \dots, u_p \in K_r$ ) and we also know that  $\Delta^p(u, v) = 0$  implies  $\Delta^{p,\circ}(u, v) = 0$  and a fortiori  $\Delta^{p,(r)}(u, v) = 0$ . It follows that  $\Delta^{p,(r)}$  induces a metric on  $\Pi^p(K_r) = B^\bullet(r)$ , and we keep the notation  $\Delta^{p,(r)}$  for this metric. For future use, we also observe that, in the right-hand side of formula (52) applied to  $u, v \in K_r$ , we may replace the infimum over  $w \in [u, v]$  by an infimum over  $w \in [u, v] \cap K_r$ : The point is that, if the clockwise exploration going from  $u$  to  $v$  (or from  $v$  to  $u$ ) intersects  $F_r$ , then it necessarily visits a point with label at most  $r$ , because otherwise  $u$  and  $v$  would have to be in  $F_r$ .

**Theorem 31.** *Conditionally on  $|\partial B^\bullet(r)|$ , the random compact measure metric space  $(B^\bullet(r), \Delta^{p,(r)})$  and the space  $(\mathbb{D}^{\infty,(r)}, \Delta^{\infty,(r)})$  in Theorem 29 are independent. Furthermore, the restriction of the metric  $\Delta^{p,(r)}$  to the interior of  $B^\bullet(r)$  coincides with the intrinsic metric induced by  $\Delta^p$  on this open set.*

*Proof.* The general idea is to show that the space  $(B^\bullet(r), \Delta^{p,(r)})$  can be constructed from random quantities that are independent of  $(\mathbb{D}^{\infty,(r)}, \Delta^{\infty,(r)})$  conditionally on  $|\partial B^\bullet(r)|$ . We start by introducing the labeled tree  $\mathcal{T}^{p,(r)}$  which consists of the part  $[0, L_r]$  of the spine of  $\mathcal{T}_\infty^p$ , and of the subtrees branching off  $[0, L_r]$ . Equivalently,  $\mathcal{T}^{p,(r)}$  is associated with the finite spine coding triple

$$\mathfrak{X}^{(r)} := \left( (X_t)_{0 \leq t \leq L_r}, \mathbf{1}_{[0, L_r]}(t) \mathcal{L}(dtd\omega), \mathbf{1}_{[0, L_r]}(t) \mathcal{R}(dtd\omega) \right). \quad (54)$$

Clearly,  $\mathcal{T}^{p,(r)}$  viewed as a subset of  $\mathcal{T}_\infty^p$  is contained in  $K_r$ . If  $(\mathcal{E}_s)_{s \in \mathbb{R}}$  is the clockwise exploration of  $\mathcal{T}_\infty^p$ ,  $\mathcal{T}^{p,(r)}$  corresponds to the points visited by  $(\mathcal{E}_s)_{s \in \mathbb{R}}$  during an interval of the form  $[-\sigma_r, \sigma'_r]$ , with  $\sigma_r, \sigma'_r > 0$ . We also let  $\check{\mathfrak{X}}^{(r)}$  be the image of  $\mathfrak{X}^{(r)}$  under the time-reversal operation (8) and denote the associated labeled tree by  $\check{\mathcal{T}}^{p,(r)}$  (replacing  $\mathcal{T}^{p,(r)}$  by  $\check{\mathcal{T}}^{p,(r)}$  just amounts to interchanging the roles of the root and the top of the spine).

We then consider all subtrees branching off the spine of  $\mathcal{T}_\infty^p$  at a level higher than  $L_r$ , and, for each such subtree whose labels hit  $[0, r]$ , the ‘‘excursions outside’’  $(r, \infty)$ . To make this precise, write

$$\mathcal{L} = \sum_{i \in I} \delta_{(t_i, \omega^i)}, \quad \mathcal{R} = \sum_{i \in J} \delta_{(t_i, \omega^i)},$$

where the indexing sets  $I$  and  $J$  are disjoint. In the time scale of the clockwise exploration, each  $\omega^i$  corresponds to an interval  $[\alpha_i, \beta_i]$  contained in  $(-\infty, 0)$  if  $i \in J$ , or in  $(0, \infty)$  if  $i \in I$ , and  $\sigma(\omega^i) = \beta_i - \alpha_i$ . Set  $I_r := \{i \in I : t_i > L_r \text{ and } W_*(\omega^i) \leq r\}$  and  $J_r := \{i \in J : t_i > L_r \text{ and } W_*(\omega^i) \leq r\}$ . For each  $i \in I_r \cup J_r$ , we can make sense of the exit local time of  $\omega^i$  at level  $r$ , as defined in Section 2.2, and we denote this local time by  $(L_t^{i,r})_{t \in [0, \sigma(\omega^i)]}$ . We then set, for every  $t \in \mathbb{R}$ ,

$$L_t^{*,r} = \sum_{i \in I \cup J} L_{t \wedge \beta_i - t \wedge \alpha_i}^{i,r},$$

so that, in some sense,  $L_t^{*,r}$  represents the total exit local time accumulated at  $r$  by the clockwise exploration up to time  $t$ . We note that

$$L_\infty^{*,r} = \sum_{i \in I \cup J} L_{\sigma(\omega^i)}^{i,r} = \mathcal{Z}^{(r)},$$

and  $|\partial B^\bullet(r)| = \mathcal{Z}^{(r)}$  by definition.

Then, for every  $i \in I_r \cup J_r$ , we consider the excursions  $(\omega^{i,k})_{k \in \mathbb{N}}$  of  $\omega^i$  outside  $(r, \infty)$  (we refer to [3, Section 2.4] for more information about such excursions). These excursions  $\omega^{i,k}$ ,  $k \in \mathbb{N}$  are in one-to-one correspondence with the connected components  $(a_{i,k}, b_{i,k})$ ,  $k \in \mathbb{N}$ , of the open set  $\{s \in [0, \sigma(\omega^i)] : \tau_r(\omega_s^i) < \zeta_s(\omega^i)\}$ , in such a way that, for every  $s \geq 0$ ,

$$\omega_s^{i,k}(t) := \omega_{(a_{i,k}+s) \wedge b_{i,k}}^i(\zeta_{a_{i,k}}(\omega^i) + t), \text{ for } 0 \leq t \leq \zeta_s(\omega^{i,k}) := \zeta_{(a_{i,k}+s) \wedge b_{i,k}}(\omega^i) - \zeta_{a_{i,k}}(\omega^i).$$

In the time scale of the clockwise exploration,  $\omega^{i,k}$  corresponds to the interval  $[\alpha_{i,k}, \beta_{i,k}]$ , where  $\alpha_{i,k} = \alpha_i + a_{i,k}$  and  $\beta_{i,k} = \alpha_i + b_{i,k}$ . In particular, the (labeled) tree  $\mathcal{T}_{(\omega^{i,k})}$  coincides with the subtree of  $\mathcal{T}_\infty^p$  consisting of the descendants of  $\mathcal{E}_{\alpha_{i,k}} = \mathcal{E}_{\beta_{i,k}}$  (this set of descendants is  $\{\mathcal{E}_s : s \in [\alpha_{i,k}, \beta_{i,k}]\}$ ).

Recall the coding triple  $(X^{(r)}, \tilde{\mathcal{L}}^{(r)}, \tilde{\mathcal{R}}^{(r)})$  which is used to construct the space  $(\mathbb{D}^{\infty, (r)}, \Delta^{\infty, (r)})$ . An application of the special Markov property, in the form given in the appendix of [27], shows that, conditionally on  $\mathcal{Z}^{(r)}$ , the point measure

$$\mathcal{N}_{(r)}(dtd\omega) := \sum_{i \in I_r \cup J_r} \sum_{k \in \mathbb{N}} \delta_{(L_{\alpha_{i,k}}^{*,r}, \omega^{i,k})}(dtd\omega)$$

is Poisson with intensity  $\mathbf{1}_{[0, \mathcal{Z}^{(r)}]}(t) dt \mathbb{N}_r(d\omega)$ , and is independent of  $(X^{(r)}, \tilde{\mathcal{L}}^{(r)}, \tilde{\mathcal{R}}^{(r)})$ . Note in particular that  $\mathcal{Z}^{(r)}$  is a measurable function of  $\mathcal{N}_{(r)}$ . On the other hand, the coding triple  $\mathfrak{X}^{(r)}$  in (54) is clearly independent of the pair  $(\mathcal{N}_{(r)}, (X^{(r)}, \tilde{\mathcal{L}}^{(r)}, \tilde{\mathcal{R}}^{(r)}))$ . So the first assertion of the theorem would follow if we could prove that the space  $(B^\bullet(r), \Delta^{p, (r)})$  is a function of  $\mathcal{N}_{(r)}$  and  $\mathfrak{X}^{(r)}$ . This is not correct, but we will see that  $(B^\bullet(r), \Delta^{p, (r)})$  is a function of  $(\mathcal{N}_{(r)}, L_0^{*,r}, \mathfrak{X}^{(r)})$ , whose conditional distribution given  $(\mathcal{Z}^{(r)}, L_0^{*,r})$  only depends on  $\mathcal{Z}^{(r)}$ . This will suffice to get the first assertion of Theorem 31.

Let us explain how the space  $(B^\bullet(r), \Delta^{p, (r)})$  can be written as a function of  $(\mathcal{N}_{(r)}, L_0^{*,r})$  and  $\mathfrak{X}^{(r)}$ . To begin with, we introduce the right-continuous inverse of the process  $(L_t^{*,r})_{t \in \mathbb{R}}$ : for every  $s \in [0, \mathcal{Z}^{(r)})$ ,

$$\tau_s^{*,r} := \inf\{t \in \mathbb{R} : L_t^{*,r} > s\},$$

and we also make the convention that  $\tau_{\mathcal{Z}^{(r)}}^{*,r}$  is the left limit of  $s \mapsto \tau_s^{*,r}$  at  $s = \mathcal{Z}^{(r)}$ . Then one verifies that  $(\Pi^p(\mathcal{E}_{\tau_s^{*,r}}))_{0 \leq t \leq \mathcal{Z}^{(r)}}$  is an injective loop whose range is precisely  $\partial B^\bullet(r)$ . Let us briefly justify this. Recall that  $\partial B^\bullet(r) = \Pi^p(\partial F_r)$ , with  $\partial F_r$  given by (50). We first observe that the mapping  $s \mapsto \Pi^p(\mathcal{E}_{\tau_s^{*,r}})$  is continuous. Indeed, we already know that the function  $s \mapsto \Pi^p(\mathcal{E}_s)$  is continuous. Furthermore, if  $\tau_{s-}^{*,r} < \tau_s^{*,r}$ , the support property of the exit local time implies that either all points of the form  $\mathcal{E}_u$  with  $u \in (\tau_{s-}^{*,r}, \tau_s^{*,r})$  are descendants of  $\mathcal{E}_{\tau_{s-}^{*,r}}$  and necessarily  $\mathcal{E}_{\tau_{s-}^{*,r}} = \mathcal{E}_{\tau_s^{*,r}}$ , or the labels of all such points  $\mathcal{E}_u$  are greater than  $r$ . In both cases, we have  $\Pi^p(\mathcal{E}_{\tau_{s-}^{*,r}}) = \Pi^p(\mathcal{E}_{\tau_s^{*,r}})$ . Then one easily deduces from the same support property that any point of the form  $\Pi^p(\mathcal{E}_{\tau_s^{*,r}})$  belongs to  $\partial B^\bullet(r)$ . Conversely, using (50), any point  $x$  of  $\partial B^\bullet(r)$ , with the exception of the point  $L_r$  of the spine, must be of the form  $\Pi^p(v)$  where  $v$  belongs to a subtree  $\mathcal{T}_{(\omega^i)}$  with  $i \in I \cup J$ , and labels along the line segment between  $v$  and the root of  $\mathcal{T}_{(\omega^i)}$  are greater than  $r$  except at  $v$ . From the support property of the exit



local time, it follows that  $v = \mathcal{E}_{\tau_s^{*,r}}$  for some  $s \in [\alpha_i, \beta_i]$ . The formula  $v = \mathcal{E}_{\tau_s^{*,r}}$  also holds for  $v = L_r$  with  $s = L_0^{*,r}$ . Finally, from the description of the distribution of  $\mathcal{N}_{(r)}$  and the fact that  $\Delta^p(u, v) = 0$  holds if and only if  $\Delta^{p,\circ}(u, v) = 0$ , one checks that, for every  $0 \leq s < s' \leq \mathcal{Z}^{(r)}$ , the points  $\Pi^p(\mathcal{E}_{\tau_s^{*,r}})$  and  $\Pi^p(\mathcal{E}_{\tau_{s'}^{*,r}})$  are distinct, except in the case  $s = 0$  and  $s' = \mathcal{Z}^{(r)}$ .

We let  $\mathfrak{H}$  be derived from the disjoint union

$$[0, \mathcal{Z}^{(r)}] \cup \left( \bigcup_{\substack{i \in I_r \cup J_r \\ k \in \mathbb{N}}} \mathcal{T}_{(\omega^{i,k})} \right) \cup \check{\mathcal{T}}^{p,(r)}$$

by identifying 0 with  $\mathcal{Z}^{(r)}$ , the root of  $\check{\mathcal{T}}^{p,(r)}$  with the point  $L_0^{*,r}$  of  $[0, \mathcal{Z}^{(r)}]$ , and, for every  $i \in I_r \cup J_r$  and  $k \in \mathbb{N}$ , the root of  $\mathcal{T}_{(\omega^{i,k})}$  with the point  $L_{\alpha_{i,k}}^{*,r}$  of  $[0, \mathcal{Z}^{(r)}]$ . We assign labels  $(\Lambda_x^{(r)})_{x \in \mathfrak{H}}$  to the points of  $\mathfrak{H}$ : the label of any point of  $[0, \mathcal{Z}^{(r)}]$  is equal to  $r$ , and points of the labeled trees  $\mathcal{T}_{(\omega^{i,k})}$  and  $\check{\mathcal{T}}^{p,(r)}$  keep their labels. We also define a volume measure on  $\mathfrak{H}$  by summing the volume measures of the trees  $\mathcal{T}_{(\omega^{i,k})}$  and of  $\check{\mathcal{T}}^{p,(r)}$ . The total volume of  $\mathfrak{H}$  is

$$\Sigma^{(r)} := |\check{\mathcal{T}}^{p,(r)}| + \sum_{i \in I_r \cup J_r} \sum_{k \in \mathbb{N}} \sigma(\omega^{i,k}),$$

using the notation  $|\check{\mathcal{T}}^{p,(r)}|$  for the total volume of  $\check{\mathcal{T}}^{p,(r)}$ .

We need to define a cyclic clockwise exploration of  $\mathfrak{H}$ , which will be denoted by  $(\mathcal{E}_s^{(r)})_{s \in [0, \Sigma^{(r)})}$ . Roughly speaking this exploration corresponds to concatenating the clockwise explorations of the trees  $\mathcal{T}_{(\omega^{i,k})}$  and  $\check{\mathcal{T}}^{p,(r)}$  in the order prescribed by the exploration of  $\mathcal{T}_\infty^p$ . To give a more precise definition, we first observe that we can write  $K_r = K_r^\circ \cup \partial F_r$ , where  $\partial F_r$  is as in (50), and

$$K_r^\circ := \{v \in \mathcal{T}_\infty^p : \Lambda_w \leq r \text{ for some } w \in \llbracket v, \infty \rrbracket\}.$$

If  $v \in \partial F_r$ , we know that  $\Pi^p(v) \in \partial B^\bullet(r)$ , so that, by previous observations, there is a unique  $s \in [0, \mathcal{Z}^{(r)})$  such that  $v = \Pi^p(\mathcal{E}_{\tau_s^{*,r}})$ , and we set  $\Phi_{(r)}(v) := s$ .

We then define, for every  $s \in \mathbb{R}$ ,

$$A_s^{(r)} := \int_{-\infty}^s dt \mathbf{1}_{\{\mathcal{E}_t \in K_r\}}.$$

Note that  $A_\infty^{(r)} = \Sigma^{(r)}$ . We set  $\eta_t^{(r)} := \inf\{s \in \mathbb{R} : A_s^{(r)} > t\}$  for every  $t \in [0, \Sigma^{(r)})$ . Then, for every  $t \in [0, \Sigma^{(r)})$ , either  $\mathcal{E}_{\eta_t^{(r)}}$  belongs to  $K_r^\circ$ , which implies that  $\mathcal{E}_{\eta_t^{(r)}}$  is a point of one of the trees  $\mathcal{T}_{(\omega^{i,k})}$  or of  $\check{\mathcal{T}}^{p,(r)}$ , and we let  $\mathcal{E}_t^{(r)}$  be the ‘‘same’’ point in  $\mathfrak{H}$ , or  $\mathcal{E}_{\eta_t^{(r)}}$  belongs to  $\partial F_r$ , and we set  $\mathcal{E}_t^{(r)} = \Phi_{(r)}(\mathcal{E}_{\eta_t^{(r)}}) \in [0, \mathcal{Z}^{(r)})$ . Finally we take  $\mathcal{E}_{\Sigma^{(r)}}^{(r)} = \mathcal{E}_0^{(r)} = 0$ . Although this is not apparent in the preceding presentation, the reader will easily check that this exploration process  $\mathcal{E}^{(r)}$  only depends on  $(\mathcal{N}_{(r)}, L_0^{*,r})$  and  $\mathfrak{X}^{(r)}$  (the reason why we need  $L_0^{*,r}$  is because we have to rank the tree  $\mathcal{T}^{p,(r)}$  among the trees  $\mathcal{T}_{(\omega^{i,k})}$  — of course the order between the different trees  $\mathcal{T}_{(\omega^{i,k})}$  is prescribed by the point measure  $\mathcal{N}_{(r)}$ ).

The clockwise exploration of  $\mathfrak{H}$  allows us to make sense of intervals on  $\mathfrak{H}$ . In turn, we can then define the function  $D'^{(r),\circ}(u, v)$ , for  $u, v \in \mathfrak{H}$ , by the right-hand side of (52), where we simply replace  $\Lambda$  by  $\Lambda^{(r)}$ . Similarly, we define  $D'^{(r)}(u, v)$ , for  $u, v \in \mathfrak{H}$ , by replacing  $\Delta^{p,\circ}$  with  $D'^{(r),\circ}$  in the right-hand side of (53) (and of course replacing  $u_0, u_1, \dots, u_p \in K_r$  by  $u_0, u_1, \dots, u_p \in \mathfrak{H}$ ). We now claim that the quotient space  $\mathfrak{H}/\{D'^{(r)} = 0\}$ , equipped with the metric induced by  $D'^{(r)}$  and with the volume measure which is the pushforward of the volume measure on  $\mathfrak{H}$ , coincides with  $(B^\bullet(r), \Delta^{p,(r)})$ . This is a straightforward consequence of our construction (using the fact that one can replace  $[u, v]$  by  $[u, v] \cap K_r$  in the right-hand side of (52) when  $u, v \in K_r$ ), and we omit the details.

We also observe that the conditional distribution of the space  $\mathfrak{H}/\{D'^{(r)} = 0\}$  given  $(\mathcal{Z}^{(r)}, L_0^{*,r})$  does not depend on  $L_0^{*,r}$ . This follows from the fact that the law of a Poisson point measure on  $[0, z] \times \mathbb{S}$  with intensity  $dt \mathbb{N}_r(d\omega)$  is invariant under the shift  $t \mapsto t + a \bmod z$ , for any fixed  $a \in [0, z]$ .

Finally, we can write  $(B^\bullet(r), \Delta^{p,(r)}) = (\mathfrak{H}/\{D^{(r)} = 0\}, D^{(r)}) = \Psi(\mathcal{N}_{(r)}, L_0^{*,r}, \mathfrak{X}^{(r)})$  with a  $\mathbb{K}$ -valued function  $\Psi$ , and we have for every nonnegative measurable function  $F$  on  $\mathbb{K}$ ,

$$\begin{aligned} \mathbb{E}[F(\Psi(\mathcal{N}_{(r)}, L_0^{*,r}, \mathfrak{X}^{(r)})) \mid (X^{(r)}, \tilde{\mathcal{L}}^{(r)}, \tilde{\mathcal{R}}^{(r)})] &= \mathbb{E}[F(\Psi(\mathcal{N}_{(r)}, L_0^{*,r}, \mathfrak{X}^{(r)})) \mid (\mathcal{Z}^{(r)}, L_0^{*,r})] \\ &= \mathbb{E}[F(\Psi(\mathcal{N}_{(r)}, L_0^{*,r}, \mathfrak{X}^{(r)})) \mid \mathcal{Z}^{(r)}]. \end{aligned}$$

The second equality follows from the preceding observation, and the first one holds because  $\mathcal{N}_{(r)}$  and  $(X^{(r)}, \tilde{\mathcal{L}}^{(r)}, \tilde{\mathcal{R}}^{(r)})$  are conditionally independent given  $\mathcal{Z}^{(r)}$  (and  $L_0^{*,r}$  is a measurable function of  $(X^{(r)}, \tilde{\mathcal{L}}^{(r)}, \tilde{\mathcal{R}}^{(r)})$ ). Since  $(\mathbb{D}^{\infty,(r)}, \Delta^{\infty,(r)})$  is a function of  $(X^{(r)}, \tilde{\mathcal{L}}^{(r)}, \tilde{\mathcal{R}}^{(r)})$ , this gives the first assertion of Theorem 31.

The proof of the second assertion of Theorem 31 is very similar to the proof of Lemma 30, and we leave the details to the reader.  $\square$

**Remark.** The preceding proof gives a description of the distribution of the hull  $B^\bullet(r)$  equipped with its intrinsic metric in terms of the space  $\mathfrak{H}$ . We note that the labeled tree  $\check{\mathcal{T}}^{p,(r)}$  has the same distribution as the tree  $\mathcal{T}_{(\omega)}$  under  $\mathbb{N}_r(d\omega \mid W_*(\omega) = 0)$  (see [26]). So the conditional distribution of  $B^\bullet(r)$  knowing  $\mathcal{Z}^{(r)} = z$  could as well be defined from a Poisson point measure  $\sum_{i \in I} \delta_{(t_i, \omega_i)}$  with intensity  $\mathbf{1}_{[0,z]}(t) dt \mathbb{N}_r(d\omega)$  conditioned on the event  $\{\inf_{i \in I} W_*(\omega_i) = 0\}$ . In this form, there is a striking analogy with the construction of the (free) Brownian disk with perimeter  $z$  found in [9] or [11] — see Section 6 below for a presentation within the formalism of the present work. The essential difference comes from the fact that the construction of the hull assigns constant labels equal to  $r$  to points of  $\mathfrak{H}$  that belong to  $[0, z]$ , whereas, in the construction of the Brownian disk, labels along  $[0, z]$  evolve like a scaled Brownian bridge.

## 5.2 Horohulls in the Brownian plane

In this section, we explain how pointed Brownian disks with a given height appear as horohulls in the Brownian plane. Let us first recall the definition of these horohulls. We consider the Brownian plane  $(\mathbb{B}\mathbb{P}_\infty, \Delta^p)$ , with the distinguished point  $\rho$ . One can prove [14] that, a.s. for every  $a, b \in \mathbb{B}\mathbb{P}_\infty$ , the limit

$$\lim_{x \rightarrow \infty} (\Delta^p(a, x) - \Delta^p(b, x))$$

exists in  $\mathbb{R}$ . Here the limit as  $x \rightarrow \infty$  means that  $x$  tends to the point at infinity in the Alexandroff compactification of  $\mathbb{B}\mathbb{P}_\infty$ . Clearly, the limit in the preceding display can be written in the form  $\mathcal{H}_a - \mathcal{H}_b$ , where the “horofunction”  $a \mapsto \mathcal{H}_a$  is uniquely defined if we impose  $\mathcal{H}_\rho = 0$ . We interpret  $\mathcal{H}_a$  as a (relative) distance from  $a$  to  $\infty$ , and call  $\mathcal{H}_a$  the horodistance from  $a$ . Note the bound  $|\mathcal{H}_a - \mathcal{H}_b| \leq \Delta^p(a, b)$ .

For every  $r > 0$ , let  $\mathfrak{B}^\circ(r)$  be the connected component of the open set  $\{x \in \mathbb{B}\mathbb{P}_\infty : \mathcal{H}_x > -r\}$  that contains  $\rho$ . So a point  $x$  belongs to  $\mathfrak{B}^\circ(r)$  if and only if there is a continuous path from  $\rho$  to  $x$  that stays at horodistance greater than  $-r$ . The horohull  $\mathfrak{B}^\bullet(r)$  is defined as the closure of  $\mathfrak{B}^\circ(r)$ . We view  $\mathfrak{B}^\bullet(r)$  as a pointed compact measure metric space with distinguished point  $\rho$ . Note that the compactness of  $\mathfrak{B}^\bullet(r)$  is not obvious a priori, but will follow from the description that we give in the proof of the next statements.

We write  $\text{Vol}(\cdot)$  for the volume measure on  $\mathbb{B}\mathbb{P}_\infty$ . In the following two statements, we fix  $r > 0$ .

**Proposition 32.** *The limit*

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-2} \text{Vol}(\{x \in \mathfrak{B}^\bullet(r) : \mathcal{H}_x < -r + \varepsilon\})$$

*exists a.s. This limit is called the boundary size of  $\mathfrak{B}^\bullet(r)$  and denoted by  $|\partial\mathfrak{B}^\bullet(r)|$ .*

**Theorem 33.** *The intrinsic metric on  $\mathfrak{B}^\circ(r)$  has a.s. a continuous extension to  $\mathfrak{B}^\bullet(r)$ , which is denoted by  $\Delta_\infty^{\text{horo},r}$ . Then conditionally on  $|\partial\mathfrak{B}^\bullet(r)| = z$ , the pointed measure metric space  $(\mathfrak{B}^\bullet(r), \Delta_\infty^{\text{horo},r})$  is a pointed Brownian disk with perimeter  $z$  and height  $r$ .*

The proof of both Proposition 32 and Theorem 33 relies on the construction of the Brownian plane found in [14], which is different from the one given in Section 4.2. Let us recall the construction of [14] using our formalism of coding triples (the presentation therefore seems to differ from the one in [14], but the relevant random objects are the same). We consider a coding triple  $(B, \mathcal{P}, \mathcal{P}')$ , such that:

- $B = (B_t)_{t \geq 0}$  is a linear Brownian motion started from 0.
- Conditionally on  $B$ ,  $\mathcal{P}$  and  $\mathcal{P}'$  are independent Poisson point measures on  $\mathbb{R}_+ \times \mathcal{S}$  with intensity  $2 dt \mathbb{N}_{B_t}(d\omega)$ .

Following Section 2.4, we then consider the infinite labeled tree  $(\mathcal{T}_\infty^p, (\Lambda'_v)_{v \in \mathcal{T}_\infty^p})$  associated with this coding triple. We define the functions  $D^{\infty, \circ}(u, v)$  and  $D^\infty(u, v)$ , for  $u, v \in \mathcal{T}_\infty^p$ , in a way similar to Section 4.1 (note however that labels are here of arbitrary sign):

$$D^{\infty, \circ}(u, v) = \Lambda'_u + \Lambda'_v - 2 \max \left( \inf_{w \in [u, v]} \Lambda'_w, \inf_{w \in [v, u]} \Lambda'_w \right), \quad (55)$$

and

$$D^\infty(u, v) = \inf_{u_0=u, u_1, \dots, u_p=v} \sum_{i=1}^p D^{\infty, \circ}(u_{i-1}, u_i) \quad (56)$$

where the infimum is over all choices of the integer  $p \geq 1$  and of the finite sequence  $u_0, u_1, \dots, u_p$  in  $\mathcal{T}_\infty^p$  such that  $u_0 = u$  and  $u_p = v$ . We have  $D^\infty(u, v) = 0$  if and only if  $D^{\infty, \circ}(u, v) = 0$  [14, Proposition 11].

We let  $\mathbb{B}\mathbb{P}'_\infty$  be the quotient space  $\mathcal{T}_\infty^p / \{D^\infty = 0\}$ , which is equipped with the distance induced by  $D^\infty(u, v)$  and the volume measure which is the pushforward of the volume measure on  $\mathcal{T}_\infty^p$ , and with the distinguished point which is the equivalence class of the root of  $\mathcal{T}_\infty^p$ . We also let  $\Pi^p$  stand for the canonical projection from  $\mathcal{T}_\infty^p$  onto  $\mathbb{B}\mathbb{P}'_\infty$ .

Then the pointed measure metric space  $\mathbb{B}\mathbb{P}'_\infty$  is a Brownian plane, that is, it has the same distribution as  $\mathbb{B}\mathbb{P}_\infty$  (see [15, Theorem 3.4]). Therefore, we can replace  $\mathbb{B}\mathbb{P}_\infty$  by  $\mathbb{B}\mathbb{P}'_\infty$  in the proof of both Proposition 32 and Theorem 33. The point of this replacement is the fact that the horodistance from a point  $a$  of  $\mathbb{B}\mathbb{P}'_\infty$  is now equal to its label  $\Lambda'_a$  [15, Proposition 17]. Indeed, we can summarize the difference between the two constructions of the Brownian plane by saying that labels correspond to distances from the distinguished point in the first construction, and to horodistances in the second one. In the proofs below, we assume that  $\mathfrak{B}^\circ(r)$  and  $\mathfrak{B}^\bullet(r)$  are defined in  $\mathbb{B}\mathbb{P}'_\infty$ , and without risk of confusion we use the notation  $\rho$  both for the root of  $\mathcal{T}_\infty^p$  and for the distinguished point of  $\mathbb{B}\mathbb{P}'_\infty$ .

*Proof of Proposition 32 and Theorem 33.* The first step is to observe that we have  $\mathfrak{B}^\circ(r) = \Pi^p(G_r)$ , where

$$G_r := \{v \in \mathcal{T}_\infty^p : \Lambda'_w > -r \text{ for every } w \in \llbracket \rho, v \rrbracket\}, \quad (57)$$

and  $\mathfrak{B}^\bullet(r) = \mathfrak{B}^\circ(r) \cup \partial \mathfrak{B}^\circ(r)$ , with  $\partial \mathfrak{B}^\circ(r) = \Pi^p(\partial G_r)$ , and

$$\partial G_r = \{v \in \mathcal{T}_\infty^p : \Lambda'_v = -r \text{ and } \Lambda'_w > -r \text{ for every } w \in \llbracket \rho, v \rrbracket\}. \quad (58)$$

Notice the similarity with (49) and (50). Let us justify the equality  $\mathfrak{B}^\circ(r) = \Pi^p(G_r)$ . The inclusion  $\mathfrak{B}^\circ(r) \supset \Pi^p(G_r)$  is easy, because, if  $v \in G_r$ , the image under  $\Pi^p$  of the geodesic segment  $\llbracket \rho, v \rrbracket$  yields a continuous path from  $v$  to  $\rho$  along which labels (horodistances) stay greater than  $-r$ . The reverse inclusion comes from the so-called ‘‘cactus bound’’ which says that any continuous path between  $\rho$  and  $\Pi^p(v)$  must visit a point whose label is smaller than or equal to  $\min_{u \in \llbracket \rho, v \rrbracket} \Lambda'_u$  (see formula (4) in [14] for a short proof in the case of the Brownian map, which is immediately extended to the present setting). Once the equality  $\mathfrak{B}^\circ(r) = \Pi^p(G_r)$  is established, the property  $\partial \mathfrak{B}^\circ(r) = \Pi^p(\partial G_r)$  is easy and we omit the details.

Write  $\text{cl}(G_r) = G_r \cup \partial G_r$ , which we can view as a (compact) subtree of the tree  $\mathcal{T}_\infty^p$ . In a way very similar to the proof of Theorem 29, we may interpret  $\text{cl}(G_r)$  as the (labeled) tree associated with a coding triple derived from the triple  $(B, \mathcal{P}, \mathcal{P}')$ . To this end, we set

$$T_r := \inf\{t \geq 0 : B_t = -r\},$$

and we note that  $\text{cl}(G_r)$  consists of the union of the part  $[0, T_r]$  of the spine of  $\mathcal{T}_\infty^p$  with the subtrees branching off the spine between levels 0 and  $T_r$  and truncated at label  $-r$ . To make this more precise, if

$$\mathcal{P} = \sum_{i \in I} \delta_{(t_i, \omega_i)},$$

we define

$$\mathcal{P}^{(r)} = \sum_{i \in I, t_i < T_r} \delta_{(t_i, \text{tr}_{-r}(\omega_i))}$$

and we similarly define  $\mathcal{P}'^{(r)}$  from  $\mathcal{P}'$ . Let  $B^{(r)}$  stand for the stopped path  $(B_t)_{0 \leq t \leq T_r}$ . Then  $\text{cl}(G_r)$  is canonically and isometrically identified with the (labeled) tree coded by the triple  $(B^{(r)}, \mathcal{P}^{(r)}, \mathcal{P}'^{(r)})$ . This identification preserves the labels and the volume measures. The fact that the limit in Proposition 32 exists, and is in fact given by

$$|\partial \mathfrak{B}^\bullet(r)| = \int \mathcal{Z}_{-r}(\omega) \mathcal{P}^{(r)}(d\omega) + \int \mathcal{Z}_{-r}(\omega) \mathcal{P}'^{(r)}(d\omega)$$

now follows from the approximation formula (2) for exit measures, using also Proposition 2 and (9).

Recall the notation  $\vartheta_r$  introduced before Proposition 4. In order to derive the statement of Theorem 33, we now notice that the triple  $\mathfrak{T}^{(r)} := (B^{(r)} + r, \vartheta_{-r} \mathcal{P}^{(r)}, \vartheta_{-r} \mathcal{P}'^{(r)})$  has the same distribution as the coding triple  $(Y, \widetilde{\mathcal{M}}, \widetilde{\mathcal{M}'})$  considered at the beginning of Section 3.2, provided we take  $a = r$ . It follows that the conditional distribution of  $\mathfrak{T}^{(r)}$  knowing that  $|\partial \mathfrak{B}^\bullet(r)| = z$  is  $\check{\Theta}_z^r$ . Recall the mapping  $\Omega$  defined in Section 2.4. Then  $\Omega(\mathfrak{T}^{(r)})$  is a random snake trajectory which, conditionally on  $|\partial \mathfrak{B}^\bullet(r)| = z$ , is distributed according to  $\mathbb{N}_r^{(z)}$ . Furthermore, by Proposition 14, the random metric space  $\mathbb{D}^{\bullet, r} := \Xi^\bullet(\Omega(\mathfrak{T}^{(r)}))$  is a pointed Brownian disk with perimeter  $z$  and height  $r$ , conditionally on  $|\partial \mathfrak{B}^\bullet(r)| = z$ . To complete the proof, we just need to identify  $\mathfrak{B}^\bullet(r)$  (equipped with the intrinsic distance) with  $\mathbb{D}^{\bullet, r}$ .

By a preceding observation,  $\text{cl}(G_r)$  is identified to the genealogical tree of  $\Omega(\mathfrak{T}^{(r)})$  (which is the labeled tree associated with  $\mathfrak{T}^{(r)}$ ) and this identification preserves labels, provided labels on  $\text{cl}(G_r)$  are shifted by  $r$ . One then verifies that two points of  $\text{cl}(G_r)$  are identified in  $\Pi^p(\text{cl}(G_r)) = \mathfrak{B}^\bullet(r)$  if and only if the corresponding points of the genealogical tree of  $\Omega(\mathfrak{T}^{(r)})$  are identified in  $\mathbb{D}^{\bullet, r}$ . It follows that  $\mathfrak{B}^\bullet(r)$  and  $\mathbb{D}^{\bullet, r}$  can be identified as sets. To complete the proof of Theorem 33, it then remains to show that the intrinsic distance between two points of  $\mathfrak{B}^\bullet(r)$  coincides with the distance between the corresponding points of the interior of  $\mathbb{D}^{\bullet, r}$  (from the discussion in Section 4.3, this will imply first that the intrinsic distance on  $\mathfrak{B}^\bullet(r)$  can be extended to the boundary, and then that  $\mathfrak{B}^\bullet(r)$  is isometric to  $\mathbb{D}^{\bullet, r}$  as desired). This is derived by arguments very similar to the end of the proof of Theorem 29, and we omit the details.  $\square$

We conclude this section with some explicit distributional properties of the process of horohulls. It will be convenient to use the Skorokhod space  $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$  of càdlàg functions from  $\mathbb{R}_+$  into  $\mathbb{R}$ . We write  $(Z_t)_{t \geq 0}$  for the canonical process on  $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$ , and  $(\mathcal{F}_t)_{t \geq 0}$  for the canonical filtration. We then introduce, for every  $x \geq 0$ , the probability measure  $P_x$  which is the law of the continuous-state branching process with branching mechanism  $\Phi$  (in short, the  $\Phi$ -CSBP), where  $\Phi(\lambda) = \sqrt{8/3} \lambda^{3/2}$ . We refer to [15, Section 2.1] for the definition and some properties of the  $\Phi$ -CSBP.

The  $\Phi$ -CSBP is critical, meaning that  $E_x[Z_t] = x$  for every  $t \geq 0$  and  $x \geq 0$ . Then, for every  $x > 0$ , we can define the law  $P_x^\uparrow$  of the  $\Phi$ -CSBP started from  $x$  and conditioned to non-extinction via the  $h$ -transform

$$\frac{dP_x^\uparrow}{dP_x} \Big|_{\mathcal{F}_t} = \frac{Z_t}{x}. \quad (59)$$

See [22, Section 4.1] for a discussion of continuous-state branching processes conditioned on non-extinction. The preceding formula does not make sense for  $x = 0$ . However, [21, Theorem 2] shows that the laws  $P_x^\uparrow$  converge weakly as  $x \downarrow 0$  to a limiting law denoted by  $P_0^\uparrow$ , which is characterized by the following two properties:

(i) for every  $t > 0$ , the law of  $Z_t$  under  $P_0^\uparrow$  is given by

$$E_0^\uparrow[e^{-\lambda Z_t}] = \left(1 + t \sqrt{2\lambda/3}\right)^{-3}, \quad \lambda \geq 0,$$

so that in particular  $Z_t > 0$ ,  $P_0^\uparrow$  a.s.;

(ii) for every  $t > 0$ , under  $P_0^\uparrow$ , conditionally on  $(Z_u)_{0 \leq u \leq t}$ , the process  $(Z_{t+s})_{s \geq 0}$  is distributed according to  $P_{Z_t}^\uparrow$ .

From (13), property (i) is equivalent to saying that the density of  $Z_t$  is  $z h_t(z)$ .

In the next proposition, we take  $|\partial \mathfrak{B}^\bullet(0)| = 0$  by convention.

**Proposition 34.** *The process  $(|\partial \mathfrak{B}^\bullet(r)|)_{r \geq 0}$  has a càdlàg modification, which is distributed according to  $P_0^\uparrow$ .*

*Proof.* As a preliminary observation, we recall from [15, Section 2.2] that the exit measure process  $(Z_{-r})_{r > 0}$  is Markovian under  $\mathbb{N}_0$ , with the transition kernels of the  $\Phi$ -CSBP. In other words, we can find a càdlàg modification of  $(Z_{-r})_{r > 0}$  such that, for every  $t > 0$ , the conditional distribution of  $(Z_{-t-r})_{r \geq 0}$  under  $\mathbb{N}_0$  and knowing  $(Z_{-u})_{0 < u \leq t}$  is  $P_{Z_{-t}}$ .

In order to get the statement of the proposition, it suffices to verify that the finite-dimensional distributions of the process  $(|\partial \mathfrak{B}^\bullet(r)|)_{r \geq 0}$  coincide with the finite-dimensional marginals under  $\mathbb{P}_0^\uparrow$ . So we need to verify that, for every  $0 < t_1 < \dots < t_p$ , for every nonnegative measurable functions  $\varphi_1, \dots, \varphi_p$  on  $\mathbb{R}_+$ , we have

$$\mathbb{E}\left[\varphi_1(|\partial \mathfrak{B}^\bullet(r_1)|) \cdots \varphi_p(|\partial \mathfrak{B}^\bullet(r_p)|)\right] = E_0^\uparrow[\varphi_1(Z_{r_1}) \cdots \varphi_p(Z_{r_p})]. \quad (60)$$

Now recall from the preceding proof that, for every  $j = 1, \dots, p$ ,

$$|\partial \mathfrak{B}^\bullet(r_j)| = \int \mathbf{1}_{\{t < T_{r_j}\}} Z_{-r_j}(\omega) \mathcal{P}(dt d\omega) + \int \mathbf{1}_{\{t < T_{r_j}\}} Z_{-r_j}(\omega) \mathcal{P}'(dt d\omega).$$

It then follows from (10) that

$$\begin{aligned} \mathbb{E}\left[\varphi_1(|\partial \mathfrak{B}^\bullet(r_1)|) \cdots \varphi_p(|\partial \mathfrak{B}^\bullet(r_p)|)\right] &= \mathbb{N}_0\left(Z_{-r_p} \varphi_1(Z_{-r_1}) \cdots \varphi_p(Z_{-r_p})\right) \\ &= \mathbb{N}_0\left(\varphi_1(Z_{-r_1}) E_{Z_{-r_1}}\left[Z_{r_p-r_1} \varphi_2(Z_{r_2-r_1}) \cdots \varphi_p(Z_{r_p-r_1})\right]\right), \end{aligned}$$

where we use the first observation of the proof in the last equality. Thanks to the  $h$ -transform relation (59), the right-hand side is also equal to

$$\mathbb{N}_0\left(\varphi_1(Z_{-r_1}) Z_{-r_1} E_{Z_{-r_1}}^\uparrow\left[\varphi_2(Z_{r_2-r_1}) \cdots \varphi_p(Z_{r_p-r_1})\right]\right) = \int_0^\infty dz z h_{r_1}(z) \varphi_1(z) E_z^\uparrow\left[\varphi_2(Z_{r_2-r_1}) \cdots \varphi_p(Z_{r_p-r_1})\right]$$

since the density of  $Z_{-r_1}$  under  $\mathbb{N}_0(\cdot \cap \{Z_{-r_1} \neq 0\})$  is  $h_{r_1}$  (Proposition 3). Finally, properties (i) and (ii) above show that the right-hand side of the last display equals  $E_0^\uparrow[\varphi_1(Z_{r_1}) \cdots \varphi_p(Z_{r_p})]$ , which completes the proof.  $\square$

In the next proposition, we compute the joint distribution of the boundary size and the volume of the horohull  $\mathfrak{B}^\bullet(r)$ .

**Proposition 35.** *Let  $r > 0$ . We have, for every  $\lambda \geq 0$  and  $\mu > 0$ ,*

$$\mathbb{E}\left[\exp\left(-\lambda |\partial \mathfrak{B}^\bullet(r)| - \mu \text{Vol}(\mathfrak{B}^\bullet(r))\right)\right] = \frac{\left(\frac{2}{3} + \frac{\lambda}{3} \sqrt{2/\mu}\right)^{-1/2} \sinh((2\mu)^{1/4} r) + \cosh((2\mu)^{1/4} r)}{\left(\left(\frac{2}{3} + \frac{\lambda}{3} \sqrt{2/\mu}\right)^{1/2} \sinh((2\mu)^{1/4} r) + \cosh((2\mu)^{1/4} r)\right)^3}.$$

*Proof.* From the fact that  $\mathfrak{B}^\bullet(r) = \Pi^p(G_r \cup \partial G_r)$ , with  $G_r$  and  $\partial G_r$  given by (57) and (58), we easily obtain that

$$\text{Vol}(\mathfrak{B}^\bullet(r)) = \int \mathbf{1}_{\{t < T_r\}} \sigma(\text{tr}_{-r}(\omega)) \mathcal{P}(dt d\omega) + \int \mathbf{1}_{\{t < T_r\}} \sigma(\text{tr}_{-r}(\omega)) \mathcal{P}'(dt d\omega).$$

Thanks to the similar formula for  $|\partial\mathfrak{B}^\bullet(r)|$ , and to Proposition 2, we get

$$\mathbb{E}\left[\exp(-\lambda|\partial\mathfrak{B}^\bullet(r)| - \mu\text{Vol}(\mathfrak{B}^\bullet(r)))\right] = \mathbb{N}_0\left(\mathcal{Z}_{-r} \exp(-\lambda\mathcal{Z}_{-r} - \mu\mathcal{Y}_{-r})\right),$$

where  $\mathcal{Y}_{-r} = \int_0^\sigma ds \mathbf{1}_{\{\tau_{-r}(W_s) = \infty\}}$ . By Lemma 4.5 in [15], we have, for  $\lambda > \sqrt{2\mu}$ ,

$$\mathbb{N}_0\left(1 - \exp(-\lambda\mathcal{Z}_{-r} - \mu\mathcal{Y}_{-r})\right) = \sqrt{\frac{\mu}{2}} \left(3 \left(\coth\left((2\mu)^{1/4}r + \coth^{-1}\left(\sqrt{\frac{2}{3} + \frac{\lambda}{3}\sqrt{2/\mu}}\right)\right)\right)^2 - 2\right).$$

By differentiating with respect to  $\lambda$ , we get the formula of the proposition. The restriction to  $\lambda > \sqrt{2\mu}$  can be removed by an argument of analytic continuation.  $\square$

**Remark.** Up to unimportant scaling constants, the formula of Proposition 35 already appears in [16, Proposition 4], which deals with asymptotics for the boundary size and volume of the (discrete) horohulls in the UIPT. This should not come as a surprise since the Brownian plane is known to be the scaling limit of the UIPT [12]. Note however that it would not be easy to deduce Proposition 35 from the corresponding discrete result.

Our last proposition characterizes the distribution of the process  $(|\partial\mathfrak{B}^\bullet(r)|, \text{Vol}(\mathfrak{B}^\bullet(r)))_{r>0}$ . This is an analog of [15, Theorem 1.3], which is concerned with the usual hulls in the Brownian plane.

**Proposition 36.** *Let  $U = (U_t)_{t \geq 0}$  be a random process distributed according to  $P_0^\uparrow$ , and let  $s_1, s_2, \dots$  be a measurable enumeration of jump times of  $U$ . Let  $\xi_1, \xi_2, \dots$  be an independent sequence of positive random variables distributed according to the density  $(2\pi x^5)^{-1/2} \exp(-1/2x)$ . Assume that the sequence  $(\xi_1, \xi_2, \dots)$  is independent of the process  $U$ . Then,*

$$\left(|\partial\mathfrak{B}^\bullet(r)|, \text{Vol}(\mathfrak{B}^\bullet(r))\right)_{r>0} \stackrel{(d)}{=} \left(U_r, \sum_{i:s_i \leq r} \xi_i (\Delta U_{s_i})^2\right)_{r>0}.$$

From our presentation of the Brownian plane in terms of the triple  $(B, \mathcal{P}, \mathcal{P}')$ , and using Proposition 2 to relate this triple to the Brownian snake excursion measure, Proposition 36 follows as a straightforward application of the excursion theory developed in [3] (see in particular Theorem 40 and Proposition 32 in [3]). We omit the details of the proof.

### 5.3 Removing a strip from the Brownian half-plane

In this section, we give an analog of Theorem 29 showing that, if one removes a strip of width  $r$  from the Brownian half-plane, the resulting space equipped with its intrinsic metric is again a Brownian half-plane. We let  $(\mathbb{H}_\infty, \Delta^{hp})$  stand for the Brownian half-plane constructed from a coding triple  $(R, \tilde{\mathcal{P}}, \tilde{\mathcal{P}}')$  as explained in Section 4.5. Recall that  $\mathbb{H}_\infty$  is obtained as a quotient space of the labeled tree  $\mathcal{T}_\infty^{hp}$  associated with  $(R, \tilde{\mathcal{P}}, \tilde{\mathcal{P}}')$ , and that every  $x \in \mathbb{H}_\infty$  thus has a label  $\Lambda_x$ , which is equal to the distance from  $x$  to the boundary  $\partial\mathbb{H}_\infty$ .

We fix  $r > 0$  and set

$$\mathbb{H}_\infty^{(r)} := \{x \in \mathbb{H}_\infty : \Lambda_x \geq r\}.$$

The interior  $\mathbb{H}_\infty^{(r), \circ}$  is  $\{x \in \mathbb{H}_\infty : \Lambda_x > r\}$ . We distinguish a special point  $x_{(r)}$  of the boundary of  $\mathbb{H}_\infty^{(r)}$ , which corresponds to the point of the spine of  $\mathcal{T}_\infty^{hp}$  at height  $\mathbf{L}_r = \sup\{t \geq 0 : R_t = r\}$ .

**Theorem 37.** *The intrinsic metric on  $\mathbb{H}_\infty^{(r), \circ}$  has a unique continuous extension to  $\mathbb{H}_\infty^{(r)}$ , which is a metric on this space. Furthermore, the resulting random measure metric space pointed at  $x_{(r)}$  is a Brownian half-plane.*

*Proof.* This is very similar to the proof of Theorem 29, and we only sketch the arguments. We first introduce the process  $(R_t^{(r)})_{t \geq 0}$  defined by

$$R_t^{(r)} := R_{\mathbf{L}_r+t} - r,$$

and we note that  $(R_t^{(r)})_{t \geq 0}$  is also a three-dimensional Bessel process started at 0. Recalling the point measures  $\tilde{\mathcal{P}}$  and  $\tilde{\mathcal{P}}'$  used in the construction of  $\mathbb{H}_\infty$ , we define two other point measures  $\tilde{\mathcal{P}}^{(r)}$  and  $\tilde{\mathcal{P}}'^{(r)}$  on  $\mathbb{R}_+ \times \mathcal{S}$  by setting, for every nonnegative measurable function  $\Phi$  on  $\mathbb{R}_+ \times \mathcal{S}$ ,

$$\langle \tilde{\mathcal{P}}^{(r)}, \Phi \rangle = \int \mathcal{P}(dtd\omega) \mathbf{1}_{(\mathbf{L}_r, \infty)}(t) \Phi(t - \mathbf{L}_r, \vartheta_r(\mathrm{tr}_r \omega)),$$

and similarly for  $\tilde{\mathcal{P}}'$ , where we recall the notation  $\vartheta_r$  for the shift on snake trajectories. Then it is straightforward to verify that the coding triples  $(R, \tilde{\mathcal{P}}, \tilde{\mathcal{P}}')$  and  $(R^{(r)}, \tilde{\mathcal{P}}^{(r)}, \tilde{\mathcal{P}}'^{(r)})$  have the same distribution.

Consequently, the construction of Section 4.5 applied to the triple  $(R^{(r)}, \tilde{\mathcal{P}}^{(r)}, \tilde{\mathcal{P}}'^{(r)})$  yields a pointed measure metric space  $(\mathbb{H}_\infty^{(r)}, \Delta^{hp, (r)})$  which is a Brownian half-plane. To complete the proof we just have to identify  $(\mathbb{H}_\infty^{(r)}, \Delta^{hp, (r)})$  with the space  $\mathbb{H}_\infty^{(r)}$  equipped with its intrinsic metric. This is done in the same way as in the proof of Theorem 29 and we omit the details.  $\square$

**Remark.** We could also have derived an analog of Theorem 31 showing that the space  $\mathbb{H}_\infty^{(r)}$  in Theorem 37 is independent of the strip  $\mathbb{H}_\infty \setminus \mathbb{H}_\infty^{(r)}$  equipped with its intrinsic metric. We leave the precise formulation and proof of this result to the reader.

## 6 Consistency with previous definitions

In this section, we show that our definitions of the infinite-volume Brownian disk and of the Brownian half-plane are consistent with the previous definitions in [8] and [19]. This is relatively easy for the Brownian half-plane but somewhat more delicate for the infinite-volume Brownian disk.

We start by recalling the definition of the free pointed Brownian disk that can be found in [8, 9, 11]. Our presentation uses the notation introduced in the preceding sections and is therefore slightly different from the one in the previous papers.

We fix  $z > 0$  and consider a Poisson point measure  $\mathcal{N} = \sum_{i \in I} \delta_{(t_i, \omega_i)}$  on  $\mathbb{R}_+ \times \mathcal{S}$  with intensity

$$2 \mathbf{1}_{[0, z]}(t) dt \mathbb{N}_0(d\omega).$$

We then introduce the compact metric space  $\mathcal{T}'$ , which is obtained from the disjoint union

$$[0, z] \cup \left( \bigcup_{i \in I} \mathcal{T}_{(\omega_i)} \right) \tag{61}$$

by identifying 0 with  $z$  and, for every  $i \in I$ , the root  $\rho_{(\omega_i)}$  of  $\mathcal{T}_{(\omega_i)}$  with the point  $t_i$  of  $[0, z]$ . The metric on  $\mathcal{T}'$  is defined in a very similar manner to Section 2.4. For instance, if  $v \in \mathcal{T}_{(\omega_i)}$  and  $w \in \mathcal{T}_{(\omega_j)}$ , with  $j \neq i$ , the distance between  $v$  and  $w$  is

$$d_{(\omega_i)}(v, \rho_{(\omega_i)}) + \min\{(t_i \vee t_j) - (t_i \wedge t_j), z - (t_i \vee t_j) + (t_i \wedge t_j)\} + d_{(\omega_j)}(\rho_{(\omega_j)}, w),$$

and the reader will easily guess the formula in other cases. The volume measure on  $\mathcal{T}'$  is just the sum of the volume measures on the trees  $\mathcal{T}_{(\omega_i)}$ ,  $i \in I$ .

Set  $\sigma' := \sum_{i \in I} \sigma(\omega_i)$ . We can define a clockwise exploration  $(\mathcal{E}'_t)_{0 \leq t \leq \sigma'}$  of  $\mathcal{T}'$ , basically by concatenating the mappings  $p_{(\omega_i)} : [0, \sigma(\omega_i)] \rightarrow \mathcal{T}_{(\omega_i)}$  in the order prescribed by the  $t_i$ 's. Note that, as in the finite spine case of Section 2.4, this exploration is cyclic (because 0 and  $z$  have been identified in  $\mathcal{T}'$ ). The clockwise exploration allows us to define intervals in the space  $\mathcal{T}'$ , exactly as in Section 2.4.

We next assign real labels to the points of  $\mathcal{T}'$ . To this end we let  $(\beta_t)_{0 \leq t \leq z}$  be a standard Brownian bridge (starting and ending at 0) over the time interval  $[0, z]$ , which is independent of  $\mathcal{N}$ . For  $t \in [0, z]$ , we set  $\ell'_t = \sqrt{3} \beta_t$ , and for  $v \in \mathcal{T}_{(\omega_i)}$ ,  $i \in I$ ,

$$\ell'_v = \sqrt{3} \beta_{t_i} + \ell_v(\omega_i),$$

where  $\ell_v(\omega_i)$  denotes the label of  $v$  in  $\mathcal{T}_{(\omega_i)}$ , as in Section 2.1. Then,  $\min\{\ell'_v : v \in \mathcal{T}'\}$  is attained at a unique point  $v_*$  of  $\mathcal{T}'$ .

We may now define the pseudo-metric functions  $D'^{\circ}$  and  $D'$  exactly as in (55) and (56),

$$D'^{\circ}(u, v) = \ell'_u + \ell'_v - 2 \max \left( \inf_{w \in [u, v]} \ell'_w, \inf_{w \in [v, u]} \ell'_w \right), \quad (62)$$

and

$$D'(u, v) = \inf_{u_0=u, u_1, \dots, u_p=v} \sum_{i=1}^p D'^{\circ}(u_{i-1}, u_i) \quad (63)$$

where the infimum is over all choices of the integer  $p \geq 1$  and of the finite sequence  $u_0, u_1, \dots, u_p$  in  $\mathcal{T}'$  such that  $u_0 = u$  and  $u_p = v$ . It is immediate to verify that, for every  $u \in \mathcal{T}'$ ,  $D'^{\circ}(u, v_*) = D'(u, v_*) = \ell'_u - \ell'_{v_*}$ .

Let  $\mathbb{D}'_z$  denote the space  $\mathcal{T}'/\{D' = 0\}$ , which is equipped with the metric induced by  $D'$ , with the pushforward of the volume measure on  $\mathcal{T}'$ , and with the distinguished point which is the equivalence class of  $v_*$  (without risk of confusion, we will also write  $v_*$  for this equivalence class). Then  $\mathbb{D}'_z$  is a free pointed Brownian disk with perimeter  $z$  whose boundary  $\partial\mathbb{D}'_z$  is the image of  $[0, z]$  under the canonical projection. This construction is basically the one in [11, Section 2.3], and, as already noted, it is consistent with Definition 13. We set

$$H'_z = D'(v_*, \partial\mathbb{D}'_z) = \min\{\ell'_v : v \in [0, z]\} - \ell'_{v_*}. \quad (64)$$

A variant of the preceding construction yields the infinite-volume Brownian disk with perimeter  $z$  as considered<sup>2</sup> in [8]. We keep the same notation as before, and we introduce an infinite labeled tree  $\mathcal{T}'_{\infty}$  which has the same distribution as the tree  $\mathcal{T}'_{\infty}^p$  of Section 5.2 (so this is the labeled tree associated with a triple  $(B, \mathcal{P}, \mathcal{P}')$  whose distribution is specified in Section 5.2). We assume that  $\mathcal{N}$  and  $\mathcal{T}'_{\infty}$  are independent, and we also consider a random variable  $U$  uniformly distributed over  $[0, z]$  and independent of the pair  $(\mathcal{N}, \mathcal{T}'_{\infty})$ . Then we let  $\mathcal{T}'^{(\infty)}$  be derived from the disjoint union

$$[0, z] \cup \left( \bigcup_{i \in I} \mathcal{T}'_{(\omega_i)} \right) \cup \mathcal{T}'_{\infty} \quad (65)$$

by the same identifications as in (61), and furthermore by identifying the root of  $\mathcal{T}'_{\infty}$  with the point  $U$  of  $[0, z]$ . The metric on  $\mathcal{T}'^{(\infty)}$  is defined as in the case of  $\mathcal{T}'$ . The clockwise exploration  $(\mathcal{E}'_t^{(\infty)})_{t \in \mathbb{R}}$  of  $\mathcal{T}'^{(\infty)}$  is then defined in much the same way as in the infinite spine case of Section 2.4: We have  $\mathcal{E}'_0^{(\infty)} = 0 = z$ , and the points  $(\mathcal{E}'_t^{(\infty)})_{t < 0}$  now correspond to the right side of the tree  $\mathcal{T}'_{\infty}$ , to the trees  $\mathcal{T}'_{(\omega_i)}$  with  $t_i > U$  and to the interval  $[U, z]$ , and similarly for the points  $(\mathcal{E}'_t^{(\infty)})_{t > 0}$ . The labels  $\ell'_v^{(\infty)}$  on  $\mathcal{T}'^{(\infty)}$  are obtained exactly as in the case of  $\mathcal{T}'$ , using the same Brownian bridge  $\beta$  and taking  $\ell'_v^{(\infty)} = \beta_U + \Lambda'_v$  when  $v \in \mathcal{T}'_{\infty}$ , where  $\Lambda'_v$  stands for the label of  $v$  in  $\mathcal{T}'_{\infty}$ .

We may now define the pseudo-metric functions  $D'^{\circ(\infty)}(u, v)$  and  $D'^{(\infty)}(u, v)$  on  $\mathcal{T}'^{(\infty)}$  by the very same formulas as in (62) and (63), just replacing the labels  $\ell'_v$  by  $\ell'_v^{(\infty)}$  (and noting that the clockwise exploration  $(\mathcal{E}'_t^{(\infty)})_{t \in \mathbb{R}}$  allows us to define intervals on  $\mathcal{T}'^{(\infty)}$ , exactly as in Section 2.4).

We then define  $\mathbb{D}'_z^{(\infty)}$  as the quotient space  $\mathcal{T}'^{(\infty)}/\{D'^{(\infty)} = 0\}$ , which is equipped with the metric induced by  $D'^{(\infty)}$ , with the volume measure which is the pushforward of the volume measure on  $\mathcal{T}'^{(\infty)}$  and with the distinguished point which is the equivalence class of 0. In the terminology of [8],  $\mathbb{D}'_z^{(\infty)}$  is an infinite-volume Brownian disk with perimeter  $z$ . The next proposition shows that this is consistent with Definition 17.

**Proposition 38.** *The pointed locally compact measure metric spaces  $\mathbb{D}'_z^{\infty}$  and  $\mathbb{D}'_z^{(\infty)}$  have the same distribution.*

<sup>2</sup>Unfortunately, it seems that the definition given in [8] is slightly incorrect. We believe that the construction below is the correct way to define the infinite-volume Brownian disk as it appears in the limit theorems proved in [8].



We will deduce Proposition 38 from Proposition 39 below, which shows that  $\mathbb{D}'_z^\infty$  is a limit of conditioned Brownian disks, in a way similar to Theorem 21 for  $\mathbb{D}_z^\infty$ . We note that [8] proves that the space  $\mathbb{D}'^\infty$  is the limit in distribution of Brownian disks with perimeter  $z$  conditioned to have a large volume, but it is not so easy to verify that this conditioning has the same effect as the one in Theorem 21, which involves the height of the distinguished point.

Let us start with some preliminary observations. Since  $\mathbb{D}'_z^\bullet$  has the same distribution as  $\mathbb{D}_z^\bullet$ , we know from the discussion after Proposition 14 that there exists a measure  $\mu'_z$  on  $\partial\mathbb{D}'_z^\bullet$  with total mass  $z$ , such that, a.s. for any continuous function  $\varphi$  on  $\mathbb{D}'_z^\bullet$ , we have

$$\langle \mu'_z, \varphi \rangle = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \int_{\mathbb{D}'_z^\bullet} \text{Vol}(dx) \mathbf{1}_{\{D(x, \partial\mathbb{D}'_z^\bullet) < \varepsilon\}} \varphi(x)$$

where  $\text{Vol}(dx)$  denotes the volume measure on  $\partial\mathbb{D}'_z^\bullet$ . For our purposes, it is important to know that  $\mu'_z$  is also the pushforward of Lebesgue measure on  $[0, z]$  under the canonical projection from  $\mathcal{T}'$  onto  $\mathbb{D}'_z^\bullet$ . This is proved in [29, Theorem 9].

We now note that, in addition to  $v_*$ ,  $\mathbb{D}'_z^\bullet$  has another distinguished point (belonging to its boundary) namely the point  $v_\partial$  which is the equivalence class of 0 in the quotient  $\mathcal{T}'/\{D' = 0\}$ . We note that  $v_\partial$  is uniformly distributed over  $\partial\mathbb{D}'_z^\bullet$ , in the following sense. Similarly as in Proposition 15, we introduce the doubly pointed measure metric space  $\mathbb{D}'_z^{\bullet\bullet}$  which is obtained by viewing  $v_\partial$  as a second distinguished point of  $\mathbb{D}'_z^\bullet$ . We have then, for any nonnegative measurable function  $F$  on the space of all doubly pointed compact measure metric spaces,

$$\mathbb{E}[F(\mathbb{D}'_z^{\bullet\bullet})] = \frac{1}{z} \mathbb{E}\left[\int \mu'_z(dx) F([\mathbb{D}'_z^\bullet, x])\right], \quad (66)$$

with the same notation as in Proposition 15. The proof of (66) is straightforward: For  $r, t \in [0, z]$  use the notation  $t \oplus r = t + r$  if  $t + r \leq z$  and  $t \oplus r = t + r - z$  if  $t + r > z$ , and note that, for every  $r \in [0, z]$ , the point measure  $\sum_{i \in I} \delta_{(t_i \oplus r, \omega_i)}$  has the same distribution as  $\mathcal{N}$ , whereas  $(\beta_{r \oplus t} - \beta_r)_{0 \leq t \leq z}$  has the same distribution as  $(\beta_t)_{0 \leq t \leq z}$ .

Consider a random doubly pointed space  $\mathbb{D}_z^{\bullet\bullet}$  whose distribution is obtained by integrating the distribution of  $\mathbb{D}_z^{\bullet\bullet, a}$  with respect to the probability density  $p_z(a)$  in Proposition 3. By integrating the formula of Proposition 15 with respect to  $p_z(a) da$ , we get

$$\mathbb{E}[F(\mathbb{D}_z^{\bullet\bullet})] = \frac{1}{z} \mathbb{E}\left[\int \mu_z(dx) F([\mathbb{D}_z^\bullet, x])\right]. \quad (67)$$

Since the pairs  $(\mathbb{D}'_z^\bullet, \mu'_z)$  and  $(\mathbb{D}_z^\bullet, \mu_z)$  have the same distribution, we obtain by comparing (66) and (67) that  $\mathbb{D}'_z^{\bullet\bullet}$  and  $\mathbb{D}_z^{\bullet\bullet}$  have the same distribution. Let  $\bar{\mathbb{D}}_z^\bullet$ , resp.  $\bar{\mathbb{D}}_z^{\bullet\bullet}$ , be the pointed space obtained from  $\mathbb{D}'_z^\bullet$ , resp. from  $\mathbb{D}'_z^{\bullet\bullet}$ , by forgetting the first distinguished point. Then  $(\bar{\mathbb{D}}_z^\bullet, H_z)$  and  $(\bar{\mathbb{D}}_z^{\bullet\bullet}, H'_z)$  also have the same distribution.

**Proposition 39.** *For every  $a > 0$ , let  $\bar{\mathbb{D}}_z^{\bullet\bullet, (a)}$  be distributed as  $\bar{\mathbb{D}}_z^{\bullet\bullet}$  conditioned on the event  $\{H'_z \geq a\}$ . Then*

$$\bar{\mathbb{D}}_z^{\bullet\bullet, (a)} \xrightarrow[a \rightarrow \infty]{(d)} \mathbb{D}'_z^\infty$$

*in distribution in the sense of the local Gromov-Hausdorff-Prokhorov convergence.*

Before proving Proposition 39, let us explain why the statement of Proposition 38 follows from this proposition. Recall from Section 4.3 that  $H_z$  denotes the distance from the distinguished point of  $\mathbb{D}_z^\bullet$  to the boundary. Since  $(\bar{\mathbb{D}}_z^\bullet, H_z)$  and  $(\bar{\mathbb{D}}_z^{\bullet\bullet}, H'_z)$  have the same distribution,  $\bar{\mathbb{D}}_z^{\bullet\bullet, (a)}$  has the same distribution as  $\bar{\mathbb{D}}_z^\bullet$  conditioned on  $H_z \geq a$ , whereas the pointed space  $\bar{\mathbb{D}}_z^{\bullet\bullet, a}$  in Theorem 21 has the distribution of  $\bar{\mathbb{D}}_z^\bullet$  conditioned on  $H_z = a$ . Hence, by comparing the convergences in Theorem 21 and in Proposition 39, we conclude that  $\mathbb{D}_z^\infty$  and  $\mathbb{D}'_z^\infty$  have the same distribution.

*Proof of Proposition 39.* Let  $E_a$  stand for the event  $\{H'_z \geq a\}$ . The idea of the proof is to study the effect on the pair  $(\beta, \mathcal{N})$  (which determines  $\mathbb{D}'_z^{\bullet\bullet}$ ) of conditioning on  $E_a$ . To this end, it will be useful

to replace  $E_a$  by another event for which the conditioning will be easier to study. We first note that, by (64) and the definition of labels on  $\mathcal{T}'$ , we have

$$H'_z = \min_{0 \leq t \leq z} (\sqrt{3} \beta_t) - \inf_{i \in I} (\sqrt{3} \beta_{t_i} + W_*(\omega_i)). \quad (68)$$

Set  $\|\beta\| = \sup\{|\beta_t| : 0 \leq t \leq z\}$ , and consider the events

$$\tilde{E}_a := \left\{ \inf_{i \in I} W_*(\omega_i) \leq -a \right\}, E'_a := \left\{ \inf_{i \in I} W_*(\omega_i) \leq -a - 2\sqrt{3}\|\beta\| \right\}, E''_a := \left\{ \inf_{i \in I} W_*(\omega_i) \leq -a + 2\sqrt{3}\|\beta\| \right\}.$$

From (68), we have  $E'_a \subset E_a \subset E''_a$ . On the other hand, it is an easy exercise to check that the ratio  $\mathbb{P}(E'_a)/\mathbb{P}(E''_a)$  tends to 1 as  $a \rightarrow \infty$  (in fact, it follows from (1) that both  $\mathbb{P}(E'_a)$  and  $\mathbb{P}(E''_a)$  are asymptotic to  $3z/a^2$ ). Since we have also  $E'_a \subset \tilde{E}_a \subset E''_a$ , we may condition on  $\tilde{E}_a$  instead of conditioning on  $E_a = \{H'_z \geq a\}$  in order to get the convergence of the proposition.

Conditioning on  $\tilde{E}_a$  does not affect  $\beta$ . On the other hand, when  $a$  is large, the conditional distribution of  $\mathcal{N}$  knowing  $\tilde{E}_a$  is close in total variation to the law of

$$\mathcal{N}'^{(a)} + \delta_{(\tilde{U}, \omega^{(a)})},$$

where  $\mathcal{N}'^{(a)}$  is a Poisson point measure with intensity  $2 \mathbf{1}_{[0, z]}(t) \mathbf{1}_{\{W_*(\omega) > -a\}} dt \mathbb{N}_0(d\omega)$ ,  $\omega^{(a)}$  is distributed according to  $\mathbb{N}_0(\cdot \mid W_* \leq -a)$ , and  $\tilde{U}$  is uniformly distributed over  $[0, z]$  (and  $\mathcal{N}'$ ,  $\omega^{(a)}$  and  $\tilde{U}$  are independent). When  $a$  is large,  $\mathcal{N}$  and  $\mathcal{N}'^{(a)}$  can be coupled so that they are equal with high probability.

We then want to argue that, when  $a$  is large, we can couple  $\omega^{(a)}$  and the labeled tree  $\mathcal{T}'_\infty$  used to define  $\mathbb{D}'_\infty$  so that  $\mathcal{T}_{(\omega^{(a)})}$  and  $\mathcal{T}'_\infty$ , both viewed as labeled trees, are close in some appropriate sense. Recall that  $\mathcal{T}'_\infty$  was constructed from a coding triple  $(B, \mathcal{P}, \mathcal{P}')$  such that  $B = (B_t)_{t \geq 0}$  is a linear Brownian motion started from 0 and, conditionally on  $B$ ,  $\mathcal{P}$  and  $\mathcal{P}'$  are independent Poisson point measures on  $\mathbb{R}_+ \times \mathcal{S}$  with intensity  $2 dt \mathbb{N}_{B_t}(d\omega)$ . On the other hand, the main results of [26] give the distribution of  $\omega^{(a)}$ . If  $b \in [a, \infty)$ , the conditional distribution of  $\omega^{(a)}$  knowing that  $W_*(\omega^{(a)}) = -b$  is that of the snake trajectory corresponding to a coding triple  $(V, \mathcal{M}, \mathcal{M}')$  such that  $V = (V_t - b)_{0 \leq t \leq T^{V'}}$ , where  $(V_t)_{0 \leq t \leq T^{V'}}$  is a Bessel process of dimension  $-5$  started from  $b$  and stopped when it hits 0, and, conditionally on  $V$ ,  $\mathcal{M}$  and  $\mathcal{M}'$  are independent Poisson measures on  $\mathbb{R}_+ \times \mathcal{S}$  with intensity

$$2 \mathbf{1}_{[0, T^{V'}]}(t) \mathbf{1}_{\{W_*(\omega) > -b\}} dt \mathbb{N}_{V_t}(d\omega).$$

From this description, we easily get that, for every  $h > 0$  and  $\varepsilon \in (0, 1)$ , we can for  $a$  large enough couple the coding triples  $(B, \mathcal{P}, \mathcal{P}')$  and  $(V, \mathcal{M}, \mathcal{M}')$  in such a way that the following two properties hold except on a set of probability smaller than  $\varepsilon$ :

- $V_t = B_t$  for  $0 \leq t \leq h$ ;
- the restriction of  $\mathcal{P}$ , resp. of  $\mathcal{P}'$ , to  $[0, h] \times \mathcal{S}$  coincides with the restriction of  $\mathcal{M}$ , resp. of  $\mathcal{M}'$ , to  $[0, h] \times \mathcal{S}$ .

Now recall that the construction of  $\mathbb{D}'_\infty$  relies on the 4-tuple  $(\beta, \mathcal{N}, U, \mathcal{T}'_\infty)$ , whereas, up to an event of small probability when  $a$  is large, the space  $\bar{\mathbb{D}}_z^{\bullet, (a)}$  (which is  $\bar{\mathbb{D}}_z^\bullet$  conditioned on  $\{H_z \geq a\}$ ) may be obtained from the 4-tuple  $(\beta, \mathcal{N}'^{(a)}, \tilde{U}, \mathcal{T}_{(\omega^{(a)})})$ . It follows from the preceding considerations that, up to a set of small probability when  $a$  is large, we can couple these two 4-tuples in such a way that their first three components coincide and moreover the labeled trees  $\mathcal{T}'_\infty$  and  $\mathcal{T}_{(\omega^{(a)})}$  with their spines “truncated at height  $h$ ” also coincide (in the case of  $\mathcal{T}_{(\omega^{(a)})}$ , the spine corresponds to the line segment between the root and the vertex with minimal label). Given  $r > 0$ , we deduce from the preceding observation (by choosing  $h$  large enough) that we can couple the spaces  $\mathbb{D}'_z^\infty$  and  $\bar{\mathbb{D}}_z^{\bullet, (a)}$  so that the balls of radius  $r$  centered at the distinguished point are the same in both spaces, except on an event of small probability when  $a$  is large. We omit the detailed verification of this last coupling, which is very similar to the proof of Proposition 18 above or Theorem 1 in [14]. The convergence in distribution stated in Proposition 39 follows.  $\square$

**Remark.** The quantities  $H_z$  and  $H'_z$  have the same distribution, and thus the density of the random variable in the right-hand side of (68) is equal to  $p_z(a)$ . The reader is invited to give a direct proof of this fact, as a verification of the consistency of our definition of the free pointed Brownian disk with the one in [11].

To conclude this section, we explain why our definition of the Brownian half-plane  $\mathbb{H}_\infty$  is consistent with the one given in [8] or [19]. We use the notation  $\mathbb{H}'_\infty$  for the Brownian half-plane as defined in [8]. Then, from [8, Corollary 3.9], we get that  $\mathbb{H}'_\infty$  is the tangent cone in distribution of the Brownian disk with perimeter  $z$  and volume  $r$  at a point uniformly distributed over its boundary — here “uniformly distributed” refers to the analog of the measure  $\mu'_z$  (the construction of  $\bar{\mathbb{D}}_z^\bullet$  given above also works for the Brownian disk with fixed volume  $r$ , just by conditioning  $\sum_{i \in I} \sigma(\omega_i)$  to be equal to  $r$ ). By randomizing the volume  $r$ , we infer that we have also

$$\lambda \cdot \bar{\mathbb{D}}_z^\bullet \xrightarrow[\lambda \rightarrow \infty]{(d)} \mathbb{H}'_\infty,$$

in distribution in the sense of the local Gromov-Hausdorff-Prokhorov convergence. On the other hand, it follows from Theorem 26 that we have

$$\lambda \cdot \bar{\mathbb{D}}_z^\bullet \xrightarrow[\lambda \rightarrow \infty]{(d)} \mathbb{H}_\infty.$$

Since  $\mathbb{D}_z^\bullet$  and  $\bar{\mathbb{D}}_z^\bullet$  have the same distribution, we conclude that  $\mathbb{H}_\infty$  and  $\mathbb{H}'_\infty$  also have the same distribution as desired.

## Appendix: Some Laplace transforms

Recall the standard notation

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt.$$

Then the function  $\chi_1$  defined for  $x > 0$  by

$$\chi_1(x) = \frac{1}{\sqrt{\pi}} x^{-1/2} - e^x \operatorname{erfc}(\sqrt{x}) = \frac{1}{\sqrt{\pi}} e^x \int_{\sqrt{x}}^\infty \frac{1}{t^2} e^{-t^2} dt, \quad (\text{A.0})$$

satisfies, for every  $\lambda > 0$ ,

$$\int_0^\infty dx e^{-\lambda x} \chi_1(x) = (1 + \sqrt{\lambda})^{-1}. \quad (\text{A.1})$$

This is easily verified via an integration by parts which gives for  $\lambda > 0$ ,

$$\int_0^\infty \operatorname{erfc}(\sqrt{x}) e^x e^{-\lambda x} dx = \frac{1}{\sqrt{\lambda}(1 + \sqrt{\lambda})}.$$

From the last two displays and an integration by parts, one checks that the function  $\chi_2 = \chi_1 * \chi_1$ , which satisfies

$$\int_0^\infty dx e^{-\lambda x} \chi_2(x) = (1 + \sqrt{\lambda})^{-2}, \quad (\text{A.2})$$

is given for  $x > 0$  by

$$\chi_2(x) = e^x \operatorname{erfc}(\sqrt{x}) - 2x \chi_1(x) = (2x + 1) e^x \operatorname{erfc}(\sqrt{x}) - \frac{2}{\sqrt{\pi}} x^{1/2}.$$

Similar manipulations show that the function  $\chi_3 = \chi_1 * \chi_1 * \chi_1$  satisfying

$$\int_0^\infty dx e^{-\lambda x} \chi_3(x) = (1 + \sqrt{\lambda})^{-3}. \quad (\text{A.3})$$

is given by

$$\chi_3(x) = \frac{2}{\sqrt{\pi}} (x^{3/2} + x^{1/2}) - 2x(x + \frac{3}{2}) e^x \operatorname{erfc}(\sqrt{x}).$$

We observe that  $\chi_1(x) > 0$  for every  $x > 0$  (this is obvious from (A.0)) and thus we have also  $\chi_3(x) > 0$  for every  $x > 0$ . Finally, we note that

$$\int_0^\infty \frac{1 - e^{-\lambda x}}{x} \chi_3(x) dx = \int_0^\lambda d\mu \int_0^\infty e^{-\mu x} \chi_3(x) dx = \int_0^\lambda d\mu (1 + \sqrt{\mu})^{-3} = (1 + \lambda^{-1/2})^{-2}. \quad (\text{A.4})$$

## References

- [1] C. Abraham, Rescaled bipartite planar maps converge to the Brownian map. *Ann. Inst. H. Poincaré Probab. Stat.* 52, 575-595 (2016)
- [2] R. Abraham, J.-F. Delmas, P. Hoscheit, A note on Gromov-Hausdorff-Prokhorov distance between (locally) compact measure spaces. *Electron. J. Probab.* 18, 1-21 (2013)
- [3] C. Abraham, J.-F. Le Gall, Excursion theory for Brownian motion indexed by the Brownian tree. *J. Eur. Math. Soc. (JEMS)* 20, 2951-3016 (2018)
- [4] L. Addario-Berry, M. Albenque, The scaling limit of random simple triangulations and random simple quadrangulations. *Ann. Probab.* 45, 2767-2825 (2017)
- [5] M. Albenque, N. Holden, X. Sun, Scaling limit of large triangulations of polygons. Preprint, arXiv:1910.04946
- [6] D. Aldous, The continuum random tree I. *Ann. Probab.*, 19, 1-28 (1991)
- [7] O. Angel, O. Schramm, Uniform infinite planar triangulations. *Comm. Math. Phys.*, 241, 191-213 (2003)
- [8] E. Baur, G. Miermont, G. Ray, Classification of scaling limits of uniform quadrangulations with a boundary. *Ann. Probab.* 47, 3397-3477 (2019)
- [9] J. Bettinelli, Scaling limit of random planar quadrangulations with a boundary. *Ann. Inst. H. Poincaré Probab. Stat.* 51, 432-477 (2015)
- [10] J. Bettinelli, E. Jacob, G. Miermont, The scaling limit of uniform random plane maps, via the Ambjørn-Budd bijection. *Electron. J. Probab.* 19, no. 74, 1-16 (2014)
- [11] J. Bettinelli, G. Miermont, Compact Brownian surfaces I. Brownian disks. *Probab. Theory Related Fields* 167, 555-614 (2017)
- [12] T. Budzinski, The hyperbolic Brownian plane. *Probab. Theory Related Fields* 171, 503-541 (2018)
- [13] A. Caraceni, N. Curien, Geometry of the Uniform Infinite Half-Planar Quadrangulation. *Random Struct. Alg.* 52, 454-494 (2018)
- [14] N. Curien, J.-F. Le Gall, The Brownian plane. *J. Theoret. Probab.* 27, 1240-1291 (2014)
- [15] N. Curien, J.-F. Le Gall, The hull process of the Brownian plane. *Probab. Theory Related Fields* 166, 187-231 (2016)
- [16] N. Curien, L. Ménard, The skeleton of the UIPT, seen from infinity. *Annales Henri Lebesgue* 1, 87-125 (2018)
- [17] N. Curien, G. Miermont, Uniform infinite planar quadrangulations with a boundary. *Random Struct. Alg.* 47, 30-58 (2015)
- [18] T. Duquesne, T., J.-F. Le Gall, Probabilistic and fractal aspects of Lévy trees. *Probab. Theory Related Fields* 131, 553-603 (2005)
- [19] E. Gwynne, J. Miller, Scaling limit of the uniform infinite half-plane quadrangulation in the Gromov-Hausdorff-Prokhorov-uniform topology. *Electron. J. Probab.* 22, paper no 84, 47pp. (2017)
- [20] E. Gwynne, J. Miller, Convergence of the free Boltzmann quadrangulation with simple boundary to the Brownian disk. *Ann. Inst. Henri Poincaré Probab. Stat.* 55, 1-60 (2019)
- [21] A.E. Kyprianou, J.C. Pardo, Continuous-state branching processes and self-similarity. *J. Appl. Probab.* 45, 1140-1160 (2008)

- [22] A. Lambert, Quasi-stationary distributions and the continuous-state branching process conditioned to be never extinct. *Electron. J. Probab.* 12, 420–446 (2007)
- [23] J.-F. Le Gall, *Spatial Branching Processes, Random Snakes and Partial Differential Equations*. Lectures in Mathematics ETH Zürich. Birkhäuser, Boston, 1999.
- [24] J.-F. Le Gall, Geodesics in large planar maps and in the Brownian map. *Acta Math.* 205, 287–360 (2010)
- [25] J.-F. Le Gall, Uniqueness and universality of the Brownian map. *Ann. Probab.* 41, 2880–2960 (2013)
- [26] J.-F. Le Gall, Bessel processes, the Brownian snake and super-Brownian motion. In: Séminaire de Probabilités XLVII - In Memoriam Marc Yor, Lecture Notes Math. 2137, 89–105. Springer 2015.
- [27] J.-F. Le Gall, Subordination of trees and the Brownian map. *Probab. Theory Related Fields* 171, 819–864 (2018)
- [28] J.-F. Le Gall, Brownian disks and the Brownian snake. *Ann. Inst. H. Poincaré Probab. Stat.* 55, 237–313 (2019)
- [29] J.-F. Le Gall, The Brownian disk viewed from a boundary point. Preprint, arXiv:2006.11050
- [30] J.-F. Le Gall, G. Miermont, Scaling limits of random trees and planar maps. In: Probability and Statistical Physics in Two and More Dimensions, Clay Mathematics Proceedings, vol.15, pp.155-211, AMS-CMI, 2012
- [31] C. Marzouk, Scaling limits of random bipartite planar maps with a prescribed degree sequence. *Random Struct. Alg.* 53, 448–503 (2018)
- [32] C. Marzouk, Brownian limits of planar maps with a prescribed degree sequence. Preprint, available at arXiv:1903.06138
- [33] G. Miermont, The Brownian map is the scaling limit of uniform random plane quadrangulations. *Acta Math.* 210, 319–401 (2013)
- [34] J. Miller, S. Sheffield, Liouville quantum gravity and the Brownian map II: geodesics and continuity of the embedding. Preprint, available at arXiv:1605.03563
- [35] D. Revuz, M. Yor, *Continuous Martingales and Brownian Motion*. Springer, Berlin, 1991.
- [36] A. Riera, Isoperimetric inequalities in the Brownian map and the Brownian plane. In preparation.
- [37] D. Williams, Path decomposition and continuity of local time for one-dimensional diffusions, I. *Proc. London Math. Soc., Ser. 3*, 28, 738–768 (1974)