Mini-course on Brownian geometry*
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1 Introduction

1.1 Planar maps

Planar maps are graphs embedded in the two-dimensional sphere. They have been used by physicists as discrete models of random geometry, especially in the setting of (two-dimensional) quantum gravity. The main goal of these lectures is to show that a large planar map chosen at random in an appropriate class, such as the class of all triangulations of the sphere, is close in a certain sense to a random compact metric space called the Brownian sphere (or the Brownian map). The Brownian sphere thus provides a universal model of random geometry, which gives insight into the properties of large planar maps chosen at random.

Definition 1. *A planar map is a proper embedding of a finite connected graph in the two-dimensional sphere* $\mathbf{S}^2$*, viewed up to orientation-preserving homeomorphisms of the sphere.*

We speak of a "proper" embedding to mean that there are no edge-crossings. In the preceding definition, we should have written "multigraph" instead of "graph", meaning that we allow self-loops and multiple edges. In these lectures, we will concentrate on quadrangulations, which have no self-loops, but multiple edges may occur as in the figure below.

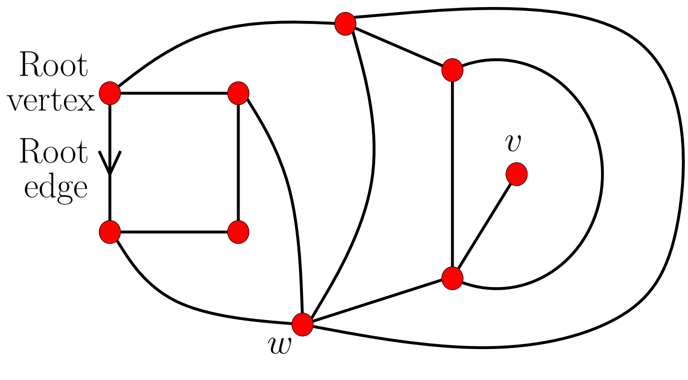

Figure 1: A (planar) quadrangulation

Definition 2. *A planar map is said to be rooted if there is a distinguished edge, and this distinguished edge called the root edge is oriented. The origin of the root edge is called the root vertex. The planar map is said to be rooted and pointed if in addition there is a distinguished vertex (which may or may not be the root vertex).*

When we identify two rooted (and pointed) planar maps modulo an orientation-preserving homeomorphism of the sphere, we of course require that this homeomorphism preserves the root edges (and the distinguished vertices in the pointed case).

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The faces of a planar map are the connected components of the complement of edges. The degree of a face counts the number of edges in the boundary of a face, with the convention that if both sides of an edge are incident to the same face, this edge is counted twice in the degree of the face (this is the case for the face incident to the vertex v in Fig. 1). A planar map is a *triangulation* if all its faces have degree 3, a *quadrangulation* if all faces have degree 4, and more generally a p-angulation if all faces have degree p. The planar map of Fig. 1 is a (rooted and pointed at v) quadrangulation with 7 faces.

It will also be important to introduce the notion of a *corner* of a vertex v of a planar map: a corner of v is an angular sector between two successive edges incident to v (which may be the same in the case when v is incident to a single edge). For instance in Fig. 1, the vertex v has only one corner, but w has 5 corners. Each face in a quadrangulation has exactly 4 corners (two of them may correspond to the same vertex, as in the case of the face containing v in Fig. 1).

We emphasize that planar maps are identified modulo orientation-preserving homeomorphisms of the sphere. Roughly speaking, this means that we can deform continuously the edges and move the vertices without changing the planar map in consideration.

1.2 Quadrangulations

Consider a planar map m, and let v, resp. f, resp. e, be the number of vertices, resp. of faces, resp. of edges, in m. By Euler's formula we have

$$v + f - e = 2.$$

Suppose in addition that m is a quadrangulation with n faces (thus $f = n$). Then, by counting the number of edge sides in two different manners, we have $4f = 2e$, and therefore $e = 2n$ and $v = n + 2$.

We denote the set of all rooted quadrangulations with n faces by $\mathcal{Q}_n$. The set $\mathcal{Q}_n$ is finite, and we have

$$\#\mathcal{Q}_n = \frac{2}{n+2}\, 3^n\, \mathbf{c}_n$$

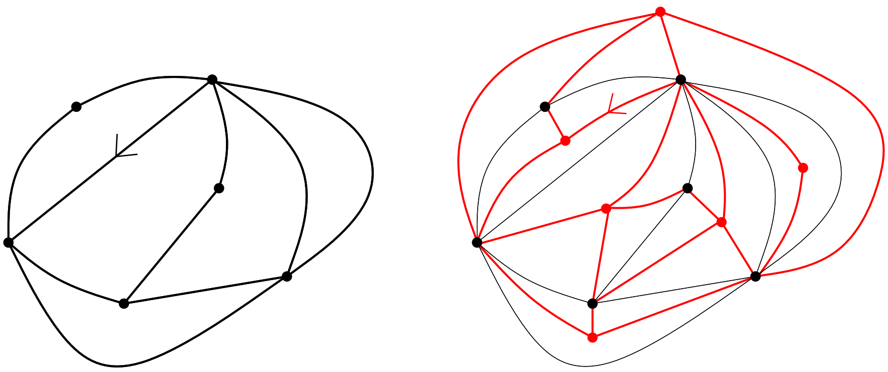
where $\mathbf{c}_n$ is the n-th Catalan number,

$$\mathbf{c}_n = \frac{1}{n+1}\binom{2n}{n}.$$

This formula (and many other similar formulas for the enumeration of planar maps) is due to Tutte [18]. Since for each rooted quadrangulation, there are exactly $n + 2$ manners of choosing an additional distinguished vertex, the set $\mathcal{Q}_n^\bullet$ of all rooted and pointed quadrangulations with n faces has cardinality

$$\#\mathcal{Q}_n^\bullet = (n + 2)\, \mathcal{Q}_n = 2 \cdot 3^n\, \mathbf{c}_n.$$

We will later give an explanation of the latter formula.



A general planar map The associated quadrangulation (red edges)

Figure 2: Tutte's bijection

A motivation for studying the special case of quadrangulations comes from the fact that there is a nice bijection called Tutte's bijection between the set of all rooted planar maps with n edges and the set $\mathcal{Q}_n$. Informally, starting from a general rooted planar map m, we add a vertex inside each face of m, and we then draw a red edge between this vertex and each corner of the face containing it. The collection of all red edges then forms a quadrangulation (some convention is needed to define the root edge of this quadrangulation from the root edge of m). See Fig. 2 for an illustration. As a consequence of this bijection, the previous formula for $\#\mathcal{Q}_n$ also gives the number of planar maps with n edges.

Let us now come to our main objective. For every $n \geq 1$, we choose a random quadrangulation Q_n uniformly at random in the set $\mathcal{Q}_n$ (or in $\mathcal{Q}_n^\bullet$, but this will make no difference for our purposes). Writing $V(Q_n)$ for the set of all vertices of $\mathcal{Q}_n$, we equip $V(Q_n)$ with the graph distance $\mathrm{d}_{\mathrm{gr}}^{Q_n}$: if $u, v \in V(Q_n)$, $\mathrm{d}_{\mathrm{gr}}^{Q_n}(u,v)$ is the minimal number of edges on a path connecting u and v in Q_n. Then $(V(Q_n), \mathrm{d}_{\mathrm{gr}}^{Q_n})$ is a (finite) random metric space, and we want to argue that when n is large this metric space is close (modulo an appropriate rescaling of the distance) to a certain random compact metric space. To make the preceding claim precise, we will need an appropriate notion of convergence of a sequence of metric spaces.

1.3 The main theorem

If (E, d) is a compact metric space, we use the notation d_{Haus}^E for the usual Hausdorff distance between compact subsets of E. If K and K' are two compact subsets of E,

$$d_{\mathrm{Haus}}^E(K, K') = \inf\{\varepsilon > 0 : K \subset K'_\varepsilon \text{ and } K' \subset K_\varepsilon\},$$

where K_ε stands for the ε-enlargment of K.

Then, if (E_1, d_1) and (E_2, d_2) are two compact metric spaces, the *Gromov-Hausdorff distance* $\mathrm{d}_{\mathrm{GH}}(E_1, E_2)$ is defined by

$$\mathrm{d}_{\mathrm{GH}}(E_1, E_2) = \inf\{d_{\mathrm{Haus}}^E(\psi_1(E_1), \psi_2(E_2))\},$$

where the infimum is over all choices of the compact metric space E and the isometric embeddings $\psi_1 : E_1 \longrightarrow E$ and $\psi_2 : E_2 \longrightarrow E$ of E_1 and E_2 into E.

It will be useful to have an alternative definition of $\mathrm{d}_{\mathrm{GH}}(E_1, E_2)$ in terms of correspondences. A correspondence between E_1 and E_2 is a subset $\mathcal{R}$ of $E_1 \times E_2$ such that, for every $x_1 \in E_1$, there exists at least one $x_2 \in E_2$ such that $(x_1, x_2) \in \mathcal{R}$ and, conversely, for every $y_2 \in E_2$, there exists at least one $y_1 \in E_1$ such that $(y_1, y_2) \in \mathcal{R}$. The distortion of the correspondence $\mathcal{R}$ is defined by

$$\mathrm{dis}(\mathcal{R}) = \sup\{|d_1(x_1, y_1) - d_2(x_2, y_2)| : (x_1, x_2), (y_1, y_2) \in \mathcal{R}\}.$$

We have then [3]

$$\mathrm{d}_{\mathrm{GH}}(E_1, E_2) = \frac{1}{2} \inf_{\mathcal{R} \in \mathcal{C}(E_1, E_2),\, (\rho_1, \rho_2) \in \mathcal{R}} \mathrm{dis}(\mathcal{R}), \tag{1}$$

where $\mathcal{C}(E_1, E_2)$ denotes the set of all correspondences between E_1 and E_2.

Let $\mathbb{K}$ denote the set of all compact metric spaces modulo isometries. Then one proves [3] that $(E_1, E_2) \longrightarrow \mathrm{d}_{\mathrm{GH}}(E_1, E_2)$ defines a distance on $\mathbb{K}$, and moreover the metric space $(\mathbb{K}, \mathrm{d}_{\mathrm{GH}})$ is separable and complete (separability can be deduced from the fact that finite metric spaces are dense in $\mathbb{K}$). Consequently, $(\mathbb{K}, \mathrm{d}_{\mathrm{GH}})$ is a Polish space. The convergence in distribution of a sequence of random compact metric spaces is then a special case of the familiar notion of convergence of random variables with values in a Polish space.

Recall that Q_n is uniformly distributed on the set $\mathcal{Q}_n$.

Theorem 1. [9, 14] *We have*

$$\Big(V(Q_n), \Big(\frac{9}{8}\Big)^{1/4} n^{-1/4} \mathrm{d}_{\mathrm{gr}}^{Q_n}\Big) \xrightarrow[n\to\infty]{\text{(d)}} (\mathbf{m}_\infty, D^*),$$

where the convergence holds in distribution in $(\mathbb{K}, \mathrm{d}_{\mathrm{GH}})$*, and the limit* $(\mathbf{m}_\infty, D^*)$ *is a random compact metric space called the Brownian sphere.*

The fact that $n^{-1/4}$ is the correct rescaling factor will be explained later using the coding of quadrangulations by labeled trees. The constant $(9/8)^{1/4}$ is needed because, as we will see below, there is a "canonical" way of defining the limiting space $(\mathbf{m}_\infty, D^*)$ (otherwise we could just replace D^* by $(9/8)^{-1/4}D^*$ ...).

The preceding theorem can be extended to much more general classes of random planar maps (e.g. triangulations, p-angulations, general planar maps with a given number of vertices, planar maps with prescribed face degrees, etc.). The normalizing factor $n^{-1/4}$ is still the same in these extensions, but the constant $(9/8)^{1/4}$ has to be replaced by another constant depending on the class in consideration. These results show that the Brownian sphere is a "universal" model of random geometry, in the sense that $(\mathbf{m}_\infty, D^*)$ appears as the scaling limit of many different discrete models.

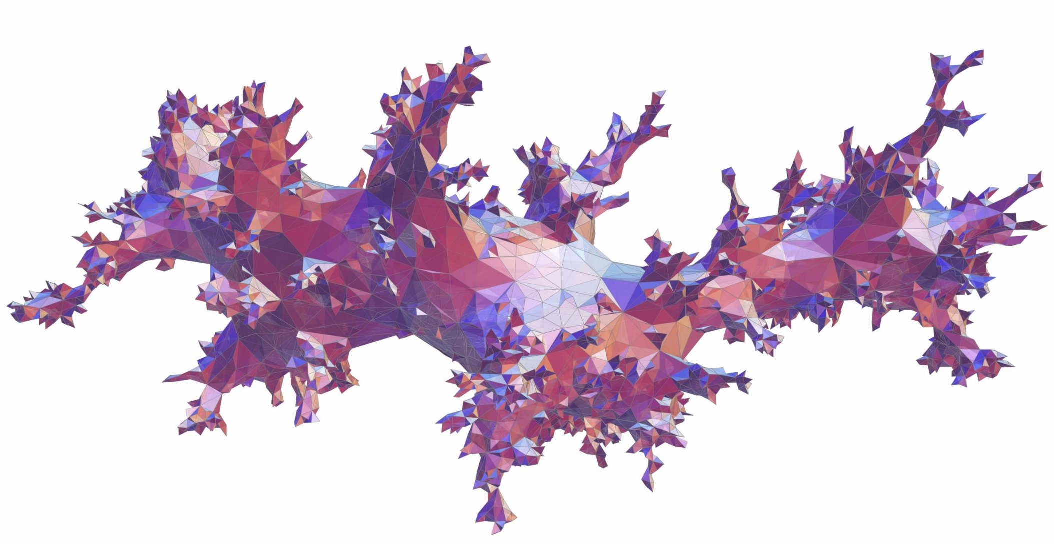


Figure 3: Simulation of a large triangulation (by N. Curien)

On the other hand, it is possible to get different limiting objects by considering random planar maps under distributions that favor the appearance of very large faces [10]. In particular, this leads to the so-called stable maps, which have aroused some interest in theoretical physics.

Among the recent developments around the Brownian sphere, we mention the work of Miller and Sheffield, who have used the Gaussian free field (and the Quantum Löwner Evolution) to provide another construction making it possible to equip the Brownian sphere with a conformal structure and involving deep connections with Liouville quantum gravity.

In order to explain the main ideas of the proof of Theorem 1, we mention the following key steps of the argument:

- the coding of quadrangulations by discrete labeled trees;
- scaling limits for random labeled trees, with convergence to the Continuum Random Tree (CRT) equipped with Brownian labels;
- the construction of the Brownian sphere from the CRT equipped with Brownian labels;
- the convergence of rescaled quadrangulations to the Brownian sphere from the convergence of random labeled trees.

The preceding steps are summarized in the following diagram.

$$\begin{tikzcd}
\text{(random) quadrangulations} \arrow[r, leftrightarrow, "\text{coding}"] \arrow[d, dashed, "\text{Theorem 3}"', "\text{scaling limit}"] & \text{(random) labeled trees} \arrow[d, "\text{scaling limit}"] \\
\text{Brownian sphere} & \text{CRT with Brownian labels} \arrow[l, "\text{coding}"']
\end{tikzcd}$$

We emphasize that the coding of quadrangulations by trees (which is explained in detail in the next section) remains valid in some form for much more general planar maps (see e.g. [2]), even if the technical details become more intricate. The principle saying that the geometry of a planar map can be described by an appropriately chosen labeled tree seems to hold in a very general setting.

2 Coding quadrangulations by labeled trees

We will consider rooted ordered trees, which are called plane trees in combinatorics (see e.g. [17]). We use the notation $\mathbb{N} = \{1, 2, \ldots\}$ and by convention $\mathbb{N}^0 = \{\varnothing\}$. We introduce the set

$$\mathcal{U} = \bigcup_{n=0}^{\infty} \mathbb{N}^n.$$

An element of $\mathcal{U}$ is thus a sequence $u = (u^1, \ldots, u^n)$ of elements of $\mathbb{N}$, and we set $|u| = n$, so that $|u|$ represents the "generation" of u. If $u = (u^1, \ldots u^m) \in \mathbb{U}$ and $j \in \mathbb{N}$, we write $uj = (u^1, \ldots u^m, j)$ (uj is a "child" of u). The mapping $\pi : \mathcal{U}\backslash\{\varnothing\} \longrightarrow \mathcal{U}$ is defined by $\pi((u^1, \ldots, u^n)) = (u^1, \ldots, u^{n-1})$ ($\pi(u)$ is the "parent" of u).

Definition 3. *A plane tree τ is a finite subset of $\mathcal{U}$ such that:*

(i) $\varnothing \in \tau$.

(ii) $u \in \tau\backslash\{\varnothing\} \Rightarrow \pi(u) \in \tau$.

(iii) *For every $u \in \tau$, there exists an integer $k_u(\tau) \geq 0$ such that, for every $j \in \mathbb{N}$, $uj \in \tau$ if and only if $1 \leq j \leq k_u(\tau)$*

The number $k_u(\tau)$ is interpreted as the "number of children" of u in τ.

For our purposes, it will be important to view a plane tree as a planar map (with only one face!) in the way suggested by the left side of Fig. 5: points of τ correspond to vertices of the graph and edges connect each vertex $u \in \tau\backslash\{\varnothing\}$ to its parent $\pi(u)$. Note that the children $u1, u2, \ldots$ of a vertex u are enumerated from left to right in the representation of Fig. 5.

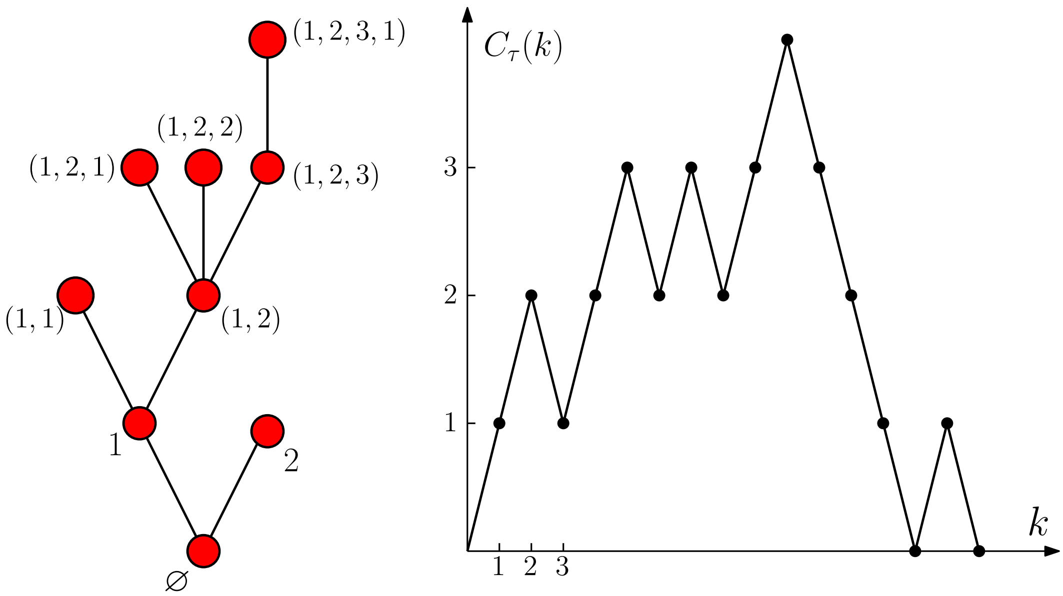


Figure 4: A plane tree and its contour function.

We denote the set of all plane trees by $\mathbf{A}$. In what follows, we see each vertex of the tree τ as an individual of a population whose τ is the family tree. By definition, the size $|\tau|$ of τ is the number of edges of τ, $|\tau| = \#\tau - 1$. For every integer $n \geq 0$, we put

$$\mathbf{A}_n = \{\tau \in \mathbf{A} : |\tau| = n\}.$$

We leave it as an exercise for the reader to check that the cardinality of $\mathbf{A}_n$ is the n-th Catalan number $\mathbf{c}_n$.

In the representation of Fig. 5, we can make sense of corners of each vertex u of the tree, exactly in the same way as we defined corners for a planar map (indeed we already said that we can view a tree as a planar map), and it is easy to verify that a tree $\tau \in \mathbf{A}_n$ has $2n$ corners. Corners of a tree τ can be listed in cyclic order by moving around the tree in clockwise order, starting and ending at the corner "below" the root $\varnothing$ (see the left side of Fig. 6 below for an example). The *contour sequence* of the tree is then the sequence of vertices attached to the corners in this enumeration. To make this more precise, if $\tau \in \mathbf{A}_n$, the contour sequence $(v_0, v_1, \dots, v_{2n})$ is defined inductively as follows:

- $v_0 = \varnothing$.
- For every $i = 0, 1, \dots, 2n-1$, v_{i+1} is either the first child of v_i that does not appear in $\{v_0, \dots, v_i\}$, or, if there is no such child, the parent of v_i.

Note that $v_{2n} = \varnothing$. For instance, the contour sequence of the tree of of Fig. 4 starts with

$$\varnothing, 1, (1,1), 1, (1,2), (1,2,1), (1,2), (1,2,2), \dots$$

We define the *contour function* of τ by $C_\tau(i) = |v_i|$, for every $0 \le i \le 2n$. See Fig. 4 for an example.

Definition 4. *A labeled tree is a pair $(\tau, (\ell_v)_{v\in\tau})$ that consists of a plane tree τ a collection $(\ell_v)_{v\in\tau}$ of labels assigned to the vertices of τ such that the following properties hold:*

(i) *for every $v \in \tau$, $\ell_v \in \mathbb{Z}$;*

(ii) *$\ell_\varnothing = 0$;*

(iii) *for every $v \in \tau \backslash \{\varnothing\}$, $\ell_v - \ell_{\pi(v)} = 1, 0,$ or -1.*

Condition (iii) just means that when crossing an edge of τ the label can change by at most 1 in absolute value. See Fig. 5 for an example.

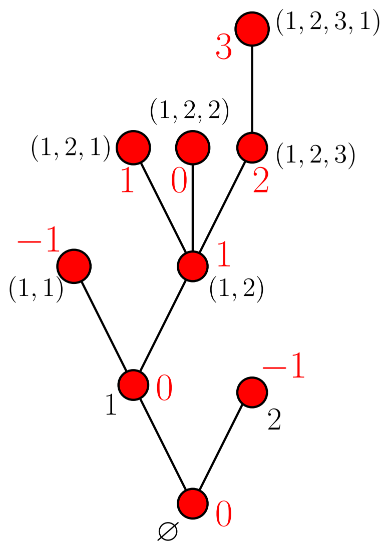


Figure 5: An admissible assignment of labels (in red) to the tree of Fig. 4.

If $\theta = (\tau, (\ell_v)_{v\in\tau})$ is a labeled tree, and $(v_0, v_1, \dots, v_{2n})$ is the contour sequence of τ, we define the *label function* of θ by setting $L_\theta(i) = \ell_{v_i}$ for every $0 \le i \le 2n$.

Let us write $\mathbb{T}_n$ for the set of all labeled trees with n edges. Since increments of labels along each edge can be chosen arbitrarily in $\{-1, 0, 1\}$, we get

$$\#\mathbb{T}_n = 3^n \#\mathbf{A}_n = 3^n \mathbf{c}_n.$$

The reason for introducing labeled trees comes from the following key theorem, where we recall the notation $\mathrm{d}_{\mathrm{gr}}^Q$ for the graph distance on the vertex set $V(Q)$ of a quadrangulation Q.

Theorem 2 (Cori-Vauquelin [5], Schaeffer [16]). *One can construct a bijection* $\Theta : \mathbb{T}_n \times \{-1, 1\} \longrightarrow \mathcal{Q}_n^\bullet$ *in such a way that the following properties hold. If* $(\tau, (\ell_v)_{v\in\tau}) \in \mathbb{T}_n$, $\eta \in \{-1, 1\}$, *and* $Q = \Theta((\tau, (\ell_v)_{v\in\tau}), \eta)$,

(i) *the vertex set of* τ *is canonically identified with* $V(Q)\backslash\{v_*\}$, *where* v_* *denotes the distinguished vertex of* Q;

(ii) *modulo the preceding identification, we have for every* $v \in V(Q)\backslash\{v_*\}$,

$$\mathrm{d}_{\mathrm{gr}}^Q(v_*, v) = \ell_v - \min\{\ell_w : w \in \tau\} + 1. \tag{2}$$

The preceding statement is somewhat informal, and a more precise version would require the construction of the bijection Θ, which will give additional properties. Before explaining the construction of Θ, let us emphasize that property (ii) will be of particular importance as it relates graph distances on Q to labels on the associated tree. The bijection of Theorem 2 is often called the CVS bijection (for Cori-Vauquelin and Schaeffer).

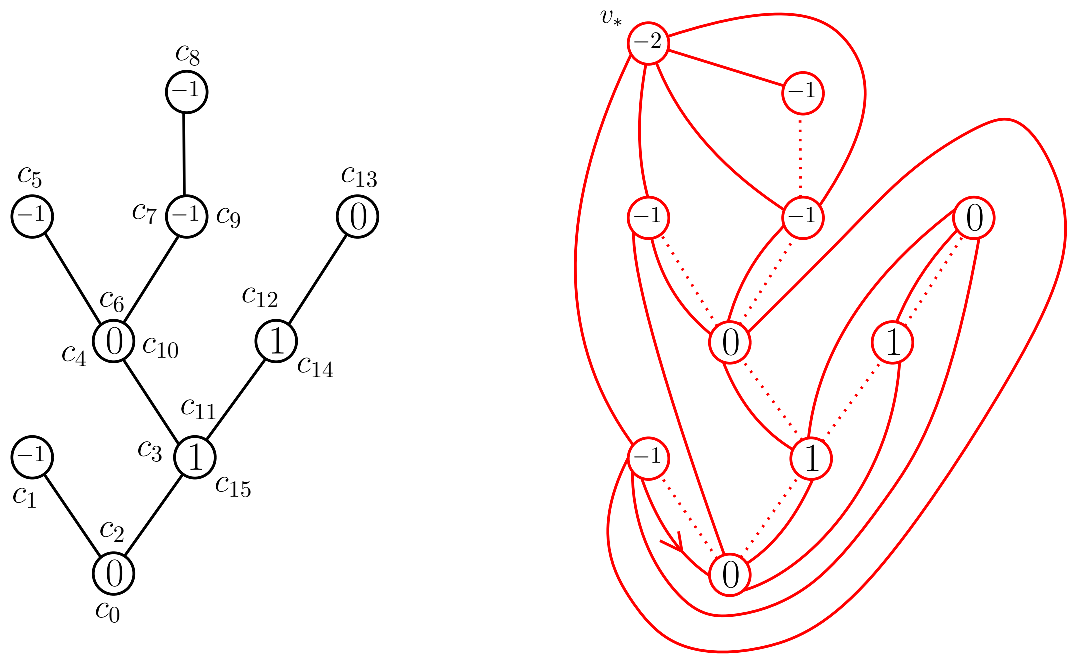


Figure 6: Illustration of the CVS bijection (case $\eta = -1$). On the left side, $c_0, c_1, \ldots$ are the corners of τ enumerated in cyclic order. On the right side, construction of the quadrangulation by drawing a red edge from each corner.

Construction of the bijection Θ. We start by drawing the (labeled) tree $(\tau, (\ell_v)_{v\in\tau})$ in the plane as in Fig. 4. On the example shown in Fig. 6, vertices are represented by small circles and their labels are displayed inside the circles (moreover in the right part of this figure, we have redrawn the tree τ with edges in dotted lines, because edges of the tree will not be edges of the associated quadrangulation, and we give the construction of the quadrangulation from this redrawn tree). Then the construction proceeds as follows. We first add an extra vertex "outside" τ, which is denoted by v_*, and we assign the label $\ell_{v*} = \min\{\ell_w : w \in \tau\} - 1$ to this vertex. We let $c_0, c_1, \ldots, c_{2n-1}$ be the corners of τ enumerated in cyclic ordering as explained previously (see the left side of Fig. 6). For every corner c_i, $0 \le i \le 2n-1$, we draw a red edge that connects this corner to the next corner in cyclic ordering with label $\ell_{c_i} - 1$ (here the label ℓ_c of a corner is obviously the label of the associated vertex). This is possible unless ℓ_{c_i} is equal to the minimal label on τ, in which case we connect the corner c_i to v_*. The reader will verify that it possible to draw all red edges so that they do not cross and do not cross the edges of the tree. See Fig. 6 for an example. In this example, the corner c_0 is connected to the next label with label $\ell_{c_0} - 1 = -1$, which is c_1, the corner c_1 has minimal label and is thus connected to v_*, the corner c_2 is connected to the next corner with label -1, which is now c_5, the corner c_3 is connected to the next corner with label $\ell_{c_3} - 1 = 0$, which is c_4, and so on.

The collection of the $2n$ red edges that we have drawn forms a quadrangulation Q — this is relatively easy to check but we omit the details — whose vertex set is the vertex set of τ to which we have added the extra vertex v_*. This quadrangulation has $2n$ edges and thus n faces (the reader may also observe that each face contains exactly one edge of τ). As we want to obtain a rooted and pointed quadrangulation, we still need to specify the root edge and the distinguished vertex of this quadrangulation: the distinguished vertex is v_* and the (unoriented) root edge is the red edge constructed from the corner c_0. To specify the orientation of the root edge, we use the parameter η: we decide that the root vertex is the root of the tree if and only $\eta = 1$.

It is not difficult to verify that the previous construction provides a mapping from $\mathbb{T}_n \times \{-1, 1\}$ into $\mathcal{Q}_n^\bullet$. The nontrivial part is to prove that this mapping is a bijection, which can be done by constructing the inverse mapping (see e.g. [4]).

Let us finally discuss properties (i) and (ii) in the theorem. Property (i) follows from our construction. As for (ii), we note that the labels of two vertices connected by an edge of Q differ by $+1$ or -1, and it readily follows that a path (in the quadrangulation) connecting a vertex v to v_* must have length at least $\ell_v - \ell_{v_*} = \ell_v - \min\{\ell_w : w \in \tau\} + 1$. The other inequality is also easy. From a vertex v we can construct a path from v to v_* in Q, with length $\ell_v - \ell_{v_*}$, as follows. We start from any corner of v in τ and consider the edge of Q from this corner to the next corner with label $\ell_v - 1$. From the latter corner, we also have an edge of Q to a corner with label $\ell_v - 2$. We iterate this procedure until we arrive at the unique vertex v_* with minimal label in $V(Q)$. Clearly, we have obtained a path from v to v_* with length $\ell_v - \ell_{v_*}$. This path is a geodesic in Q and is called the *simple geodesic* starting from the corner of v considered initially.

Remark. It follows from the CVS bijection that $\#\mathcal{Q}_n^\bullet = 2\#\mathbb{T}_n = 2 \cdot 3^n\, \mathbf{c}_n$, and we recover an enumeration formula given above.

3 Scaling limits of trees

3.1 Convergence of contour functions

For every $n \geq 1$, consider a random labeled tree $(\tau_n, (\ell_v^n)_{v \in \tau_n})$ which is uniformly distributed over $\mathbb{T}_n$, and note that τ_n is then uniformly distributed over $\mathbf{A}_n$. Our goal is to derive a scaling limit for $(\tau_n, (\ell_v^n)_{v \in \tau_n})$, or more precisely for the contour and label functions that code this labeled tree.

Let us start by discussing the tree τ_n. Recall that a Dyck path of length $2n$ is a sequence $(x_0, x_1, x_2, \ldots, x_{2n})$ of nonnegative integers such that $x_0 = x_{2n} = 0$, and $|x_i - x_{i-1}| = 1$ for every $i = 1, \ldots, 2n$. Recall the notation C_τ for the contour function of a plane tree τ (see Fig. 4). It is straightforward to verify that the mapping

$$\mathbf{A}_n \ni \tau \mapsto (C_\tau(i), 0 \leq i \leq n)$$

is a bijection from $\mathbf{A}_n$ onto the set of all Dyck paths of length $2n$.

On the other hand, let $(S_k)_{k \geq 0}$ be simple random walk (coin-tossing) on $\mathbb{Z}$, with $S_0 = 0$, and set

$$T = \min\{k \geq 0 : S_k = -1\}.$$

Then the conditional distribution of $(S_0, S_1, \ldots, S_{2n})$ given that $T = 2n + 1$ is the uniform distribution on the set of all Dyck paths of length $2n$.

The preceding discussion shows that, for the random tree τ_n which is uniformly distributed over $\mathbf{A}_n$, the contour function $(C_{\tau_n}(i), 0 \leq i \leq 2n)$ is distributed as $(S_i, 0 \leq i \leq 2n)$ under $\mathbb{P}(\cdot \,|\, T = 2n + 1)$. In the following proposition, it is convenient to agree that the contour function C_{τ_n} is extended to the real interval $[0, 2n]$ by linear interpolation between integer times (as in Fig. 4).

Proposition 3. *We have*

$$\Big(\frac{1}{\sqrt{2n}} C_{\tau_n}(2n\,t)\Big)_{0 \leq t \leq 1} \xrightarrow[n\to\infty]{\text{(d)}} (\mathbf{e}_t)_{0 \leq t \leq 1}$$

where $\mathbf{e}$ *is a normalized Brownian excursion and the convergence holds in the sense of weak convergence of the laws on the space* $C([0,1], \mathbb{R}_+)$ *of continuous functions from* $[0,1]$ *into* $\mathbb{R}_+$ *(equipped with the supremum norm).*

The law of the process $(\mathbf{e}_t)_{0\leq t\leq 1}$ is characterized by its finite-dimensional marginals. Write

$$p_t(x) = \frac{1}{\sqrt{2\pi t}}\,\exp(-\frac{|x|^2}{2t}), \qquad t>0,\, x\in\mathbb{R},$$

for the Brownian transition density. Then, for every integer $p\geq 1$, and every choice of $0<t_1<t_2<\cdots<t_p<1$, the distribution of $(\mathbf{e}(t_1),\ldots,\mathbf{e}(t_p))$ has density

$$(0,\infty)^p \ni (x_1,\ldots,x_p)\mapsto 2\sqrt{2\pi}\,q_{t_1}(x_1)\,p^*_{t_2-t_1}(x_1,x_2)\,p^*_{t_3-t_2}(x_2,x_3)\cdots p^*_{t_p-t_1}(x_{p-1},x_p)\,q_{1-t_p}(x_p) \tag{3}$$

where

$$q_t(x) = \frac{x}{t}\,p_t(x), \qquad t>0, x>0,$$

and

$$p^*_t(x,y) = p_t(y-x) - p_t(y+x)\ , t>0\ ,\ \ x,y>0$$

is the transition density of Brownian motion killed when it hits 0.

Let us discuss the convergence of finite-dimensional marginals in Proposition 3. It is convenient to write $\mathbb{P}_\ell$ for a probability measure under which the simple random walk S starts from $\ell\in\mathbb{Z}$. A useful ingredient is Kemperman's formula (see e.g. [15, Section 6.1]), which states that, for every integers $\ell\geq 0$ and $m\geq \ell+1$,

$$\mathbb{P}_\ell(T=n) = \frac{\ell+1}{n}\,\mathbb{P}_\ell(S_n=-1). \tag{4}$$

On the other hand, the classical local limit theorem applied to the random walk S gives, for every $\varepsilon>0$,

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}}\sup_{s\geq\varepsilon}\Big|\sqrt{n}\mathbb{P}\Big(S_{\lfloor ns\rfloor}=\lfloor x\sqrt{n}\rfloor \text{ or } \lfloor x\sqrt{n}\rfloor+1\Big) - 2p_s(0,x)\Big| = 0. \tag{5}$$

Fix $t\in(0,1)$. We use both (4) and (5) to verify that

$$\lim_{n\to\infty}\sqrt{2n}\,\mathbb{P}\Big(S_{\lfloor 2nt\rfloor}=\lfloor x\sqrt{2n}\rfloor \text{ or } \lfloor x\sqrt{2n}\rfloor+1\,\Big|\,T=2n+1\Big) = 4\sqrt{2\pi}\,q_t(x)\,q_{1-t}(x), \tag{6}$$

which gives the convergence of one-dimensional marginals in Proposition 3 (even in a strong form). Let us explain the derivation of (6). We first write for $i\in\{1,\ldots,2k\}$ and $\ell\geq 0$,

$$\mathbb{P}(S_i=\ell\mid T=2n+1) = \frac{\mathbb{P}(\{S_i=\ell\}\cap\{T=2n+1\})}{\mathbb{P}(T=2n+1)}.$$

By an application of the Markov property of S, we have

$$\mathbb{P}(\{S_i=\ell\}\cap\{T=2n+1\}) = \mathbb{P}(S_i=\ell, T>i)\,\mathbb{P}_\ell(T=2n+1-i).$$

Furthermore, a simple time-reversal argument shows that

$$\mathbb{P}(S_i=\ell, T>i) = \mathbb{P}_\ell(S_i=0, T>i) = 2\,\mathbb{P}_\ell(T=i+1).$$

Summarizing, we have obtained

$$\begin{aligned}\mathbb{P}(S_i=\ell\mid T=2n+1) &= \frac{2\mathbb{P}_\ell(T=i+1)\mathbb{P}_\ell(T=2n+1-i)}{\mathbb{P}(T=2n+1)}\\ &= \frac{2(2n+1)(\ell+1)^2}{(i+1)(2n+1-i)}\,\frac{\mathbb{P}_\ell(S_{i+1}=-1)\mathbb{P}_\ell(S_{2n+1-i}=-1)}{\mathbb{P}(S_{2n+1}=-1)}\end{aligned} \tag{7}$$

using (4) in the second equality.

We apply this identity with $i=\lfloor 2nt\rfloor$ and $\ell=\lfloor x\sqrt{2n}\rfloor$ or $\ell=\lfloor x\sqrt{2n}\rfloor+1$. Using (5), we have first

$$\frac{2(2n+1)(\lfloor x\sqrt{2n}\rfloor+1)^2}{(\lfloor 2nt\rfloor+1)(2n+1-\lfloor 2nt\rfloor)}\times\frac{1}{\mathbb{P}(S_{2n+1}=-1)} \approx 2\sqrt{2\pi}\,(2n)^{1/2}\,\frac{x^2}{t(1-t)}$$

and, using (5) once again,

$$\begin{aligned}
&\mathbb{P}_{\lfloor x\sqrt{2n}\rfloor}(S_{\lfloor 2nt\rfloor+1}=-1)\mathbb{P}_{\lfloor x\sqrt{2n}\rfloor}(S_{2n+1-\lfloor 2nt\rfloor}=-1)\\
&+\mathbb{P}_{\lfloor x\sqrt{2n}\rfloor+1}(S_{\lfloor 2nt\rfloor+1}=-1)\mathbb{P}_{\lfloor x\sqrt{2n}\rfloor+1}(S_{2n+1-\lfloor 2nt\rfloor}=-1)\approx 2\,(2n)^{-1}\,p_t(0,x)p_{1-t}(0,x).
\end{aligned}$$

Putting these estimates together, and recalling that $q_t(x)=(x/t)p_t(0,x)$, we arrive at (6).

Higher order marginals can be treated in a similar way. Let us sketch the case of two-dimensional marginals. We observe that, if $0<i<j<2n$ and if $\ell,m\in\mathbb{Z}_+$, we have, by the same arguments as above,

$$\mathbb{P}(S_i=\ell,S_j=m,T=2n+1)=2\,\mathbb{P}_\ell(T=i+1)\,\mathbb{P}_\ell(S_{j-i}=m,T>j-i)\,\mathbb{P}_m(T=2n+1-j).$$

Only the middle term $\mathbb{P}_\ell(S_{j-i}=m,T>j-i)$ requires a different treatment than in the case of one-dimensional marginals. However, by an application of the reflection principle, one has

$$\mathbb{P}_\ell(S_{j-i}=m,T>j-i)=\mathbb{P}_\ell(S_{j-i}=m)-\mathbb{P}_\ell(S_{j-i}=-m).$$

Hence, using (5), we easily obtain that for $x,y>0$ and $0<s<t<1$,

$$\mathbb{P}_{\lfloor x\sqrt{2n}\rfloor}(S_{\lfloor 2nt\rfloor-\lfloor 2ns\rfloor}=\lfloor y\sqrt{2n}\rfloor)+\mathbb{P}_{\lfloor x\sqrt{2n}\rfloor+1}(S_{\lfloor 2nt\rfloor-\lfloor 2ns\rfloor}=\lfloor y\sqrt{2n}\rfloor+1)\approx 2\,(2n)^{-1/2}\,p^*_{t-s}(x,y),$$

and the result for two-dimensional marginals follows in a straightforward way.

The convergence of finite-dimensional marginals is not sufficient to get the statement of Proposition 3: one also needs a tightness argument, which we omit here (see [11] for a detailed proof).

3.2 Interpretation in terms of convergence of trees

Our goal is to explain why Proposition 3 implies a convergence result for random trees. We first explain how the Brownian excursion $\mathbf{e}$ can be viewed as the "contour function" of a certain "continuous tree".

We consider a (deterministic) continuous function $h:[0,1]\longrightarrow\mathbb{R}_+$ such that $h(0)=h(1)=0$. To avoid trivialities, we will also assume that h is not identically zero. For every $s,t\in[0,1]$, we set

$$m_h(s,t)=\inf_{r\in[s\wedge t,s\vee t]}h(r),$$

and

$$d_h(s,t)=h(s)+h(t)-2m_h(s,t).$$

Clearly $d_h(s,t)=d_h(t,s)$ and it is also easy to verify the triangle inequality

$$d_h(s,u)\leq d_h(s,t)+d_h(t,u)$$

for every $s,t,u\in[0,1]$. We then introduce the equivalence relation $s\sim t$ iff $d_h(s,t)=0$ (or equivalently iff $h(s)=h(t)=m_h(s,t)$). Let $\mathcal{T}_h$ be the quotient space

$$\mathcal{T}_h=[0,1]/\sim.$$

Obviously the function d_h induces a distance on $\mathcal{T}_h$, and we keep the notation d_h for this distance. We denote by $p_h:[0,1]\longrightarrow\mathcal{T}_h$ the canonical projection. Clearly p_h is continuous (when $[0,1]$ is equipped with the Euclidean metric and $\mathcal{T}_h$ with the metric d_h), and the metric space $(\mathcal{T}_h,d_h)$ is thus compact.

Proposition 4. *The metric space $(\mathcal{T}_h,d_h)$ is a compact $\mathbb{R}$-tree.*

This means that $(\mathcal{T}_h,d_h)$ is a compact metric space and that, for every $a,b\in\mathcal{T}_h$ there is a unique continuous injective path going from a to b, up to reparameterization, and the range of this path is isometric to the line segment $[0,d_h(a,b)]$. Informally, a compact $\mathbb{R}$-tree should be viewed as a connected union of line segments in the plane, which is a tree in the sense that there is no cycle.

We will say that $(\mathcal{T}_h,d_h)$ is the tree *coded* by h. By definition, the root ρ_h of the tree $\mathcal{T}_h$ is $\rho_h=p_h(0)=p_h(1)$, and the volume measure on $\mathcal{T}_h$ is the pushforward of Lebesgue measure on $[0,1]$ under p_h.

It will be important for us to define "intervals" in the tree $\mathcal{T}_h$. Roughly speaking, the mapping $[0,1] \ni t \mapsto p_h(t)$ corresponds to a cyclic exploration of the tree (recall that $p_h(0) = p_h(1)$), and, if $a, b \in \mathcal{T}_h$, the interval $[a, b]$ corresponds to the set of points of $\mathcal{T}_h$ that are visited when going from a to b in this exploration. In view of a more precise definition, let us make the special convention that, for $s, t \in [0,1]$ such that $s > t$, the "interval" $[s, t]$ is defined by $[s, t] = [s, 1] \cup [1, t]$ (of course, if $s \le t$, $[s, t]$ is the usual interval). Then, for $a, b \in \mathcal{T}_h$, there is a smallest interval $[s, t]$ with $p_h(s) = a$ and $p_h(t) = b$, and we set $[a, b] = p_h([s, t])$. It is important to observe that $[a, b] \neq [b, a]$ in general.

Definition 5. *The Continuum Random Tree (CRT) is the random $\mathbb{R}$-tree $\mathcal{T}_{\mathbf{e}}$ coded by a normalized Brownian excursion* $\mathbf{e}$.

The CRT was first introduced by Aldous [1] with a different presentation. See [7] for the presentation that is used here.

We view $\mathcal{T}_{\mathbf{e}}$ as a random variable with values in the space $\mathbb{K}$ introduced in Section 1. This makes sense because the mapping $h \mapsto \mathcal{T}_h$, from $\{h \in C([0,1], \mathbb{R}_+) : h(0) = h(1) = 0\}$ into $\mathbb{K}$ is measurable: the next lemma even gives the continuity of this mapping.

Lemma 5. *Let $h_1, h_2 \in C([0,1], \mathbb{R}_+)$ such that $h_1(0) = h_1(1) = 0$ and $h_2(0) = h_2(1) = 0$. Then,*

$$\mathrm{d_{GH}}(\mathcal{T}_{h_1}, \mathcal{T}_{h_2}) \le 2 \sup\{|h_1(t) - h_2(t)| : t \in [0,1]\}.$$

Proof. We use the definition of $\mathrm{d_{GH}}$ in terms of correspondences (1). To this end, we consider the correspondence between $\mathcal{T}_{h_1}$ and $\mathcal{T}_{h_2}$ defined by

$$\mathcal{R} = \{(p_{h_1}(t), p_{h_2}(t)) : t \in [0,1]\}.$$

The distortion of $\mathcal{R}$ is bounded by noting that, if $s, t \in [0,1]$ with $s \le t$,

$$\begin{aligned} d_{h_1}(s,t) &= h_1(s) + h_1(t) - 2\min\{h_1(r) : r \in [s,t]\}, \\ d_{h_2}(s,t) &= h_2(s) + h_2(t) - 2\min\{h_2(r) : r \in [s,t]\}; \end{aligned}$$

so that

$$|d_{h_1}(p_{h_1}(s), p_{h_1}(t)) - d_{h_2}(p_{h_2}(s), p_{h_2}(t))| = |d_{h_1}(s,t) - d_{h_2}(s,t)| \le 4 \,\sup\{|h_1(r) - h_2(r)| : r \in [0,1]\}$$

and therefore

$$\mathrm{dis}(\mathcal{R}) \le 4 \,\sup\{|h_1(r) - h_2(r)| : r \in [0,1]\}$$

and we just have to use (1). □

Recall that τ_n is a random plane tree uniformly distributed over $\mathbf{A}_n$. As previously, we view τ_n as a graph (in the way suggested by Fig. 4) and write $\mathrm{d}_{\mathrm{gr}}^{\tau_n}$ for the graph distance on τ_n.

Theorem 6. *We have*

$$\Big(\tau_n, \frac{1}{\sqrt{2n}} \mathrm{d}_{\mathrm{gr}}^{\tau_n}\Big) \xrightarrow[n\to\infty]{\mathrm{(d)}} (\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}}),$$

in distribution in the space $\mathbb{K}$.

Proof. By Proposition 3 and the Skorokhod representation theorem, we may assume that we have almost surely

$$\sup_{t \in [0,1]} \Big| \frac{1}{\sqrt{2n}} C_{\tau_n}(2n\,t) - \mathbf{e}(t) \Big| \xrightarrow[n\to\infty]{} 0.$$

Set $C^{(n)}(t) = \frac{1}{\sqrt{2n}} C_{\tau_n}(2n\,t)$ for every $t \in [0,1]$. From the last display and Lemma 5, we get

$$\mathcal{T}_{C^{(n)}} \xrightarrow[n\to\infty]{\mathrm{a.s.}} \mathcal{T}_{\mathbf{e}}$$

where the convergence holds in $\mathbb{K}$. Finally, it is straightforward to verify that

$$\mathrm{d_{GH}}\Big((\mathcal{T}_{C^{(n)}}, d_{C^{(n)}}), (\tau_n, \frac{1}{\sqrt{2n}} \mathrm{d}_{\mathrm{gr}}^{\tau_n})\Big) \le \frac{1}{\sqrt{2n}},$$

and the statement of the theorem follows. □

4 The scaling limit of labels

Recall our notation $(\tau_n, (\ell_v^n)_{v\in\tau_n})$ for a random labeled tree uniformly distributed over $\mathbb{T}_n$. We are now interested in deriving a scaling limit for the labels $(\ell_v^n)_{v\in\tau_n}$.

Let $h : [0,1] \longrightarrow \mathbb{R}_+$ be a continuous function such that $h(0) = h(1) = 0$ (as in subsection 3.2). We also assume that h is Hölder continuous: There exist two positive constants K and γ such that, for every $s,t \in [0,1]$,

$$|h(s) - h(t)| \leq K\,|s-t|^\gamma.$$

We use the same notation $m_h(s,t) = \min\{h(r) : r \in [s\wedge t, s\vee t]\}$ as above.

Lemma 7. *The function $(m_h(s,t))_{s,t\in[0,1]}$ is nonnegative definite in the sense that, for every integer $p \geq 1$, for every $s_1,\dots,s_p \in [0,1]$ and every $\lambda_1,\dots,\lambda_p \in \mathbb{R}$, we have*

$$\sum_{i=1}^{p}\sum_{j=1}^{p} \lambda_i\lambda_j\, m_h(s_i,s_j) \geq 0.$$

Proof. Fix $s_1,\dots,s_p \in [0,1]$, and $t \geq 0$. For $i,j \in \{1,\dots,p\}$, put $i \approx j$ if $m_p(s_i,s_j) \geq t$. Then $\approx$ is an equivalence relation on $\{i : h(s_i) \geq t\} \subset \{1,\dots,p\}$. By summing over the different classes of this equivalence relation, we get that

$$\sum_{i=1}^{p}\sum_{j=1}^{p} \lambda_i\lambda_j \mathbf{1}_{\{t\leq m_h(s_i,s_j)\}} = \sum_{\mathcal{C}\text{ equiv. class of }\approx} \Big(\sum_{i\in\mathcal{C}} \lambda_i\Big)^2 \geq 0.$$

Now integrate with respect to dt to get the desired result. □

By Lemma 7 and a standard application of the Kolmogorov extension theorem, there exists a centered Gaussian process $(Z_s^h)_{s\in[0,1]}$ whose covariance is

$$\mathbb{E}[Z_s^h Z_t^h] = m_h(s,t)$$

for every $s,t \in [0,1]$. Consequently we have

$$\mathbb{E}[(Z_s^h - Z_t^h)^2] = \mathbb{E}[(Z_s^h)^2] + \mathbb{E}[(Z_t^h)^2] - 2\mathbb{E}[Z_s^h Z_t^h] = h(s) + h(t) - 2m_h(s,t) \leq 2K\,|s-t|^\gamma,$$

where the last bound follows from our Hölder continuity assumption on h (this calculation also shows that $\mathbb{E}[(Z_s^h - Z_t^h)^2] = d_h(s,t)$, in the notation of subsection 3.2). From the previous bound and an application of the Kolmogorov continuity criterion, the process $(Z_s^h)_{s\in[0,1]}$ has a modification with continuous sample paths. This leads us to the following definition.

Definition 6. *The Brownian snake driven by the function h is the centered Gaussian process $(Z_s^h)_{s\in[0,1]}$ with continuous sample paths and covariance*

$$\mathbb{E}[Z_s^h Z_t^h] = m_h(s,t)\ , \quad s,t \in [0,1].$$

Notice that we have in particular $Z_0^h = Z_1^h = 0$. More generally, for every $t \in [0,1]$, Z_t^h is normal with mean 0 and variance $h(t)$. We observe that our terminology is different from the one in [6], where the Brownian snake is defined as a path-valued process: what we call the Brownian snake is the "head" of the Brownian snake considered in [6].

Remark. Recall from subsection 3.2 the definition of the equivalence relation $\sim$ associated with h: $s \sim t$ iff $d_h(s,t) = 0$. If $s,t \in [0,1]$ are fixed and such that $s \sim t$ we have $Z_s^h = Z_t^h$ a.s. (this is obvious since $\mathbb{E}[(Z_s^h - Z_t^h)^2] = d_h(s,t)$). Via a continuity argument, one can get the stronger fact that, almost surely for every $s,t \in [0,1]$, the condition $s \sim t$ implies that $Z_s^h = Z_t^h$. In other words we may view Z^h as a process indexed by the quotient $[0,1]\,/\!\sim$, that is by the tree $\mathcal{T}_h$. Indeed, it is then very natural to interpret Z^h as Brownian motion indexed by the tree $\mathcal{T}_h$: In the particular case when $\mathcal{T}_h$ is a finite union of segments (which holds if h is piecewise monotone), Z^h can be constructed by running independent Brownian motions along the branches of $\mathcal{T}_h$. It is however more convenient to view Z^h as a process indexed by $[0,1]$ because later the function h (and thus the tree $\mathcal{T}_h$) will be random and we avoid considering a random process indexed by a random set.

Recall that we have defined the *label function* L_θ of a labeled tree $\theta = (\tau, (\ell_v)_{v\in\tau})$ by setting $L_\theta(i) = \ell_{v_i}$ for every $0 \le i \le 2n$, where $(v_0, \dots, v_{2n})$ is the contour sequence of τ. We extend the definition of L_θ to the real interval $[0, 2n]$ by linear interpolation.

Theorem 8. *Let $\theta_n = (\tau_n, (\ell^n_v)_{v\in\tau_n})$ be uniformly distributed over $\mathbb{T}_n$. Then*

$$\Big(\frac{1}{\sqrt{2n}} C_{\tau_n}(2n\,t), \Big(\frac{9}{8}\Big)^{1/4} n^{-1/4}\, L_{\theta_n}(2n\,t)\Big)_{0\le t\le 1} \xrightarrow[n\to\infty]{(d)} (\mathbf{e}_t, Z_t)_{0\le t\le 1}$$

where, conditionally on $\mathbf{e}$*, Z is distributed as the Brownian snake driven by* $\mathbf{e}$*, and the convergence holds in distribution in the space $C([0,1],\mathbb{R}_+)^2$.*

Proof. To simplify notation, we write $C_n(s) = C_{\tau_n}(s)$ and $L_n(s) = L_{\theta_n}(s)$ in this proof. As in the proof of Theorem 6, we may assume that

$$\sup_{t\in[0,1]} \Big|\frac{1}{\sqrt{2n}} C_n(2n\,t) - \mathbf{e}(t)\Big| \xrightarrow[n\to\infty]{\text{a.s.}} 0. \qquad (8)$$

Let us discuss the convergence of finite-dimensional marginals in Theorem 8. We aim to prove that for every choice of $0 \le t_1 < t_2 < \cdots < t_p \le 1$, we have

$$\Big(\frac{1}{\sqrt{2n}} C_n(2n\,t_i), \Big(\frac{9}{8n}\Big)^{1/4} L_n(2n\,t_i)\Big)_{1\le i\le p} \xrightarrow[n\to\infty]{(d)} (\mathbf{e}_{t_i}, Z_{t_i})_{1\le i\le p}. \qquad (9)$$

Since for every $i \in \{1, \dots, p\}$,

$$|C_n(2nt_i) - C_n(\lfloor 2nt_i\rfloor)| \le 1\ , \quad |L_n(2nt_i) - L_n(\lfloor 2nt_i\rfloor)| \le 1$$

we may replace $2nt_i$ by its integer part $\lfloor 2nt_i\rfloor$ in (9).

Consider the case $p = 1$. We may assume that $0 < t_1 < 1$, because otherwise the result is trivial. It is immediate that conditionally on τ_n, the label increments $\ell^n_v - \ell^n_{\pi(v)}$, $v \in \tau_n\backslash\{\emptyset\}$, are i.i.d. with uniform distribution on $\{-1, 0, 1\}$. Consequently, we may write

$$(C_n(\lfloor 2nt_1\rfloor), L_n(\lfloor 2nt_1\rfloor)) \overset{(d)}{=} \Big(C_n(\lfloor 2nt_1\rfloor), \sum_{i=1}^{C_n(\lfloor 2nt_1\rfloor)} \eta_i\Big)$$

where the variables $\eta_1, \eta_2, \dots$ are i.i.d. with uniform distribution on $\{-1, 0, 1\}$, and are also independent of the trees τ_n. By the central limit theorem,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \eta_i \xrightarrow[n\to\infty]{(d)} \Big(\frac{2}{3}\Big)^{1/2} N$$

where N is a standard normal variable. Thus if we set for $\lambda \in \mathbb{R}$,

$$\Phi(n, \lambda) = \mathbb{E}\Big[\exp\Big(\mathrm{i}\frac{\lambda}{\sqrt{n}} \sum_{i=1}^{n} \eta_i\Big)\Big]$$

we have $\Phi(n, \lambda) \longrightarrow \exp(-\lambda^2/3)$ as $n \to \infty$.

Then, for every $\lambda, \lambda' \in \mathbb{R}$, we get by conditioning on τ_n,

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big(\mathrm{i}\frac{\lambda}{\sqrt{2n}} C_n(\lfloor 2nt_1\rfloor) + \mathrm{i}\frac{\lambda'}{\sqrt{C_n(\lfloor 2nt_1\rfloor)}} \sum_{i=1}^{C_n(\lfloor 2nt_1\rfloor)} \eta_i\Big)\Big] \\
&\quad = \mathbb{E}\Big[\exp\Big(\mathrm{i}\frac{\lambda}{\sqrt{2n}} C_n(\lfloor 2nt_1\rfloor)\Big) \times \Phi(C_n(\lfloor 2nt_1\rfloor), \lambda')\Big] \\
&\quad \xrightarrow[n\to\infty]{} \mathbb{E}[\exp(\mathrm{i}\lambda \mathbf{e}_{t_1})] \times \exp(-\lambda'^2/3)
\end{aligned}$$

using the (almost sure) convergence of $(2n)^{-1/2}C_n(\lfloor 2nt_1\rfloor)$ towards $\mathbf{e}_{t_1} > 0$. In other words we have obtained the joint convergence in distribution

$$\Big(\frac{C_n(\lfloor 2nt_1\rfloor)}{\sqrt{2n}}, \frac{1}{\sqrt{C_n(\lfloor 2nt_1\rfloor)}} \sum_{i=1}^{C_n(\lfloor 2nt_1\rfloor)} \eta_i\Big) \xrightarrow[n\to\infty]{\text{(d)}} (\mathbf{e}_{t_1}, (2/3)^{1/2}N), \tag{10}$$

where the normal variable N is independent of $\mathbf{e}$.

From preceding observations, we have

$$\Big(\frac{C_n(\lfloor 2nt_1\rfloor)}{\sqrt{2n}}, \Big(\frac{9}{8n}\Big)^{1/4} L_n(\lfloor 2nt_1\rfloor)\Big) \overset{\text{(d)}}{=} \Big(\frac{C_n(\lfloor 2nt_1\rfloor)}{\sqrt{2n}}, \Big(\frac{3}{2}\Big)^{1/2} \Big(\frac{C_n(\lfloor 2nt_1\rfloor)}{\sqrt{2n}}\Big)^{1/2} \frac{1}{\sqrt{C_n(\lfloor 2nt_1\rfloor)}} \sum_{i=1}^{C_n(\lfloor 2nt_1\rfloor)} \eta_i\Big)$$

and from (10) we get

$$\Big(\frac{C_n(\lfloor 2nt_1\rfloor)}{\sqrt{2n}}, \Big(\frac{9}{8n}\Big)^{1/4} L_n(\lfloor 2nt_1\rfloor)\Big) \xrightarrow[n\to\infty]{\text{(d)}} (\mathbf{e}_{t_1}, \sqrt{\mathbf{e}_{t_1}}\, N).$$

This gives (9) in the case $p = 1$, since by construction $(\mathbf{e}_{t_1}, Z_{t_1}) \overset{\text{(d)}}{=} (\mathbf{e}_{t_1}, \sqrt{\mathbf{e}_{t_1}}\, N)$.

Let us discuss the case $p = 2$ of (9). We fix t_1 and t_2 with $0 < t_1 < t_2 < 1$. Let us set

$$\check{C}_n^{i,j} = \min_{i\wedge j \le k \le i\vee j} C_n(k)\ , \qquad \text{for } i,j \in \{0,1,\dots,2n\}.$$

Write $v_0^n = \varnothing, v_1^n, \dots, v_{2n}^n = \varnothing$ for the contour sequence of the tree τ_n. Then we know that

$$C_n(\lfloor 2nt_1\rfloor) = |v^n_{\lfloor 2nt_1\rfloor}|,\ C_n(\lfloor 2nt_2\rfloor) = |v^n_{\lfloor 2nt_2\rfloor}|,\ L_n(\lfloor 2nt_1\rfloor) = \ell^n_{v^n_{\lfloor 2nt_1\rfloor}},\ L_n(\lfloor 2nt_2\rfloor) = \ell^n_{v^n_{\lfloor 2nt_2\rfloor}},$$

and furthermore it follows from the construction of the contour function that $\check{C}_n^{\lfloor 2nt_1\rfloor,\lfloor 2nt_2\rfloor}$ is the generation of the last common ancestor to $v^n_{\lfloor 2nt_1\rfloor}$ and $v^n_{\lfloor 2nt_2\rfloor}$ in τ_n. From the properties of labels in θ_n, we now see that, conditionally on τ_n,

$$(L_n(\lfloor 2nt_1\rfloor), L_n(\lfloor 2nt_2\rfloor)) \overset{\text{(d)}}{=} \Big(\sum_{i=1}^{\check{C}_n^{\lfloor 2nt_1\rfloor,\lfloor 2nt_2\rfloor}} \eta_i + \sum_{i=\check{C}_n^{\lfloor 2nt_1\rfloor,\lfloor 2nt_2\rfloor}+1}^{C_n(\lfloor 2nt_1\rfloor)} \eta'_i\ ,\ \sum_{i=1}^{\check{C}_n^{\lfloor 2nt_1\rfloor,\lfloor 2nt_2\rfloor}} \eta_i + \sum_{i=\check{C}_n^{\lfloor 2nt_1\rfloor,\lfloor 2nt_2\rfloor}+1}^{C_n(\lfloor 2nt_2\rfloor)} \eta''_i\Big) \tag{11}$$

where the variables $\eta_i, \eta'_i, \eta''_i$ are independent and uniformly distributed over $\{-1,0,1\}$.

From (8), we have

$$\Big((2n)^{-1/2}C_n(\lfloor 2nt_1\rfloor), (2n)^{-1/2}C_n(\lfloor 2nt_2\rfloor), (2n)^{-1/2}\check{C}_n^{\lfloor 2nt_1\rfloor,\lfloor 2nt_2\rfloor}\Big) \xrightarrow[n\to\infty]{\text{a.s.}} (\mathbf{e}_{t_1}, \mathbf{e}_{t_2}, m_{\mathbf{e}}(t_1,t_2)).$$

By arguing as in the case $p = 1$, we now deduce from (11) that

$$\begin{aligned}&\Big(\frac{C_n(\lfloor 2nt_1\rfloor)}{\sqrt{2n}}, \frac{C_n(\lfloor 2nt_2\rfloor)}{\sqrt{2n}}, \Big(\frac{9}{8n}\Big)^{1/4} L_n(\lfloor 2nt_1\rfloor), \Big(\frac{9}{8n}\Big)^{1/4} L_n(\lfloor 2nt_2\rfloor)\Big)\\ &\quad\xrightarrow[n\to\infty]{\text{(d)}} (\mathbf{e}_{t_1}, \mathbf{e}_{t_2}, \sqrt{m_{\mathbf{e}}(t_1,t_2)}\, N + \sqrt{\mathbf{e}_{t_1} - m_{\mathbf{e}}(t_1,t_2)}\, N'\ ,\ \sqrt{m_{\mathbf{e}}(t_1,t_2)}\, N + \sqrt{\mathbf{e}_{t_2} - m_{\mathbf{e}}(t_1,t_2)}\, N'')\end{aligned}$$

where N, N', N'' are three independent standard normal variables, which are also independent of $\mathbf{e}$. The limiting distribution in the last display is easily identified with that of $(\mathbf{e}_{t_1}, \mathbf{e}_{t_2}, Z_{t_1}, Z_{t_2})$, and this gives the case $p = 2$ in (9). The general case of (9) is proved by similar arguments. As was already the case for Proposition 3, the proof is completed by a tightness argument which we omit. Again, details can be found in [11]. □

5 The definition of the Brownian sphere

Recall our notation $\mathcal{T}_{\mathbf{e}}$ for the CRT, and the definition of intervals $[a,b]$ on $\mathcal{T}_{\mathbf{e}}$. We let $Z = (Z_t)_{0\le t\le 1}$ be as in Theorem 8. Recall from the remark following Definition 6 that Z can be viewed as indexed by the tree $\mathcal{T}_{\mathbf{e}}$ (so Z_a makes sense for $a \in \mathcal{T}_{\mathbf{e}}$). For every $a, b \in \mathcal{T}_{\mathbf{e}}$, we set

$$D^\circ(a,b) = Z_a + Z_b - 2\max\left(\min_{c\in[a,b]} Z_c, \min_{c\in[b,a]} Z_c\right).$$

The motivation for this definition comes from the following discrete observation.

Lemma 9. *Let $\theta = (\tau, (\ell_v)_{v\in\tau})$ be a labeled tree in $\mathbb{T}_n$, and let Q be the quadrangulation associated with θ by the CVS bijection. Let $v_0, v_1, \dots, v_{2n}$ be the contour sequence of τ_n and recall that $v_0, v_1, \dots, v_{2n}$ can be viewed as vertices of Q. Then, for every $i, j \in \{0, 1, \dots, n\}$,*

$$\mathrm{d}_{\mathrm{gr}}^{Q}(v_i, v_j) \le \ell_{v_i} + \ell_{v_j} - 2\max\left(\min_{k\in[\![i,j]\!]} \ell_{v_k}, \min_{k\in[\![j,i]\!]} \ell_{v_k}\right) + 2,$$

where $[\![i,j]\!] = \{i, i+1, \dots, j\}$ if $i \le j$, and $[\![i,j]\!] = \{i, i+1, \dots, 2n\} \cup \{0, 1, \dots, j\}$ if $i > j$.

Proof. In the enumeration of corners of τ the index i corresponds a corner of v_i, and we may consider the simple geodesic γ from v_i to v_* starting from this corner (cf. the end of Section 2). Similarly, we can consider the simple geodesic γ' starting from the corner indexed by j. It follows from the construction of simple geodesics that γ and γ' coalesce at a vertex of Q with label

$$\max\left(\min_{k\in[\![i,j]\!]} \ell_{v_k}, \min_{k\in[\![j,i]\!]} \ell_{v_k}\right) + 1.$$

The stated bound for $\mathrm{d}_{\mathrm{gr}}^{Q}(v_i, v_j)$ then follows by considering the path made of γ and γ' up to the point where they coalesce. $\square$

We then set, for every $a, b \in \mathcal{T}_{\mathbf{e}}$,

$$D^*(a,b) = \inf_{k\ge1, a=a_0,a_1,\dots,a_{k-1},a_k=b} \sum_{i=1}^{k} D^\circ(a_{i-1}, a_i),$$

where the infimum is over all choices of the integer $k \ge 1$ and the points $a_1, \dots, a_{k-1}$ of $\mathcal{T}_{\mathbf{e}}$. Note that D^* is the largest symmetric function of $(a,b) \in \mathcal{T}_{\mathbf{e}} \times \mathcal{T}_{\mathbf{e}}$ that is bounded above by D° and satisfies the triangle inequality.

Lemma 10. *Almost surely, for every $a, b \in \mathcal{T}_{\mathbf{e}}$, we have $D^*(a,b) = 0$ if and only if $D^\circ(a,b) = 0$.*

The "if" part is trivial since $D^* \le D^\circ$. The "only if" part is more delicate, and we omit the proof (see [8]). We note that $D^\circ(a,b) = 0$ holds if and only if

$$Z_a = Z_b = \max\left(\min_{c\in[a,b]} Z_c, \min_{c\in[b,a]} Z_c\right).$$

We then introduce an equivalence relation on $\mathcal{T}_{\mathbf{e}}$ by setting

$$a \simeq b \text{ if and only if } D^*(a,b) = 0.$$

Definition 7. *The Brownian sphere $\mathbf{m}_\infty$ is the quotient space $\mathcal{T}_{\mathbf{e}}/\simeq$, which is equipped with the distance induced by D^*.*

The Brownian sphere was first introduced by Marckert and Mokadem [13] (they use the name Brownian map), with a somewhat different presentation.

Remark. In the discrete setting, the CVS bijection identifies the vertex set of the quadrangulation Q (up to the exception of the distinguished vertex v_*) with the vertex set of the associated tree. In the definition of the Brownian sphere, there is also an associated tree $\mathcal{T}_{\mathbf{e}}$, but we need to identify certain pairs of points of $\mathcal{T}_{\mathbf{e}}$. The reason why these identifications are needed can be explained as follows. If Q is a "large" quadrangulation and $(\tau, (\ell_u)_{u\in\tau})$ is the associated labeled tree, there will exist vertices u and v of τ such that v is "far" from u at a macroscopic scale, and

- $\ell_v = \ell_u - 1$,
- when exploring the tree from (a corner of) u to (a corner of) v in the cyclic ordering of corners of τ, one encounters only vertices with label at least equal to ℓ_u.

These two properties imply that there is an edge of Q between u and v. In the scaling limit (distances are rescaled by a factor tending to 0), this means that two "distant" points of $\mathcal{T}_{\mathbf{e}}$ have to be identified. Notice that the property $D^\circ(a,b) = 0$ is a continuous analog of the two discrete properties listed above.

Let us state without proof a few important properties.

- Equivalence classes for $\simeq$ may contain $1, 2$ or 3 points (but not more)
- The metric space $(\mathbf{m}_\infty, D^*)$ is a.s. homeomorphic to the sphere $\mathbf{S}^2$ [12].
- The Hausdorff dimension of $(\mathbf{m}_\infty, D^*)$ is a.s. equal to 4 [8].

6 Sketch of proof of Theorem 1

Let $(\tau_n, (\ell^n_v)_{v\in\tau_n})$ be the labeled tree associated with Q_n via the CVS bijection. By Theorem 8,

$$\Big(\frac{1}{\sqrt{2n}} C_{\tau_n}(2n\,t), \Big(\frac{9}{8}\Big)^{1/4} n^{-1/4}\, L_{\theta_n}(2n\,t)\Big)_{0\le t\le 1} \xrightarrow[n\to\infty]{\text{(d)}} (\mathbf{e}_t, Z_t)_{0\le t\le 1} \tag{12}$$

where we recall that $C_{\tau_n}(i) = |u^n_i|$ and $L_{\tau_n}(i) = \ell^n_{u^n_i}$, if $u^n_0, u^n_1, \dots, u^n_{2n}$ is the contour sequence of τ_n. We also know that, for every $0 \le i \le n$,

$$\mathrm{d}^{Q_n}_{\mathrm{gr}}(v^n_*, u^n_i) = \ell^n_{u^n_i} - \ell^n_* + 1 \tag{13}$$

where v^n_* is the distinguished vertex of Q_n, and we use the notation $\ell^n_* = \min\{\ell^n_u : u \in \tau_n\}$. Furthermore, by Lemma 9, we have, for every $0 \le i \le j \le n$,

$$\mathrm{d}^{Q_n}_{\mathrm{gr}}(u^n_i, u^n_j) \le d^\circ_n(i,j),$$

where we have set

$$d^\circ_n(i,j) = \ell_{u^n_i} + \ell_{u^n_j} - 2 \min_{k\in[i,j]\cap\mathbb{Z}} \ell^n_{u^n_k} + 2.$$

By convention, we also take $d^\circ_n(j,i) = d^\circ_n(i,j)$.

We now set, for every $i, j \in \{0, 1, \dots, 2n\}$,

$$d_n(i,j) = \mathrm{d}^{Q_n}_{\mathrm{gr}}(u^n_i, u^n_j).$$

The Gromov-Hausdorff convergence stated in Theorem 1 will essentially follow from the convergence in distribution of the processes

$$\Big(n^{-1/4} d_n(\lfloor 2ns \rfloor, \lfloor 2nt \rfloor)\Big)_{0\le s,t\le 1}.$$

We start by extending d_n and d°_n to real values of $s, t \in [0, 2n]$. We set

$$\begin{aligned} d_n(s,t) &= (s - \lfloor s \rfloor)(t - \lfloor t \rfloor) d_n(\lceil s \rceil, \lceil t \rceil) + (s - \lceil s \rceil)(t - \lfloor t \rfloor) d_n(\lfloor s \rfloor, \lceil t \rceil) \\ &\quad + (s - \lfloor s \rfloor)(t - \lceil t \rceil) d_n(\lceil s \rceil, \lfloor t \rfloor) + (s - \lceil s \rceil)(t - \lceil t \rceil) d_n(\lfloor s \rfloor, \lfloor t \rfloor), \end{aligned}$$

and we similarly define $d_n^\circ(s,t)$. Clearly, the bound $d_n^\circ \le d_n$ and the triangle inequality still hold for these extended versions.

By definition, if $0 \le s \le t \le 1$,

$$d_n^\circ(\lfloor 2ns, 2nt\rfloor) = L_{\tau_n}(\lfloor 2ns\rfloor) + L_{\tau_n}(\lfloor 2nt\rfloor) - 2 \min_{\lfloor 2ns\rfloor \le k \le \lfloor 2nt\rfloor} L_{\tau_n}(k) + 2.$$

It then follows from (12) that

$$\left(\left(\frac{9}{8}\right)^{1/4} n^{-1/4}\, d_n^\circ(2ns,2nt)\right)_{0\le s,t\le 1} \xrightarrow[n\to\infty]{\text{(d)}} \left(Z_s + Z_t - 2 \min_{s\wedge t\le r\le s\vee t} Z_r\right).$$

The limiting process is clearly continuous in the pair (s,t), and vanishes on the diagonal. From the convergence in the last display, we have, for every $\varepsilon > 0$ and $\delta > 0$,

$$\limsup_{n\to\infty} \mathbb{P}\left(\sup_{|s-t|\le\delta} \left(n^{-1/4}\, d_n^\circ(2ns,2nt)\right) \ge \varepsilon\right) \le \mathbb{P}\left(\sup_{|s-t|\le\delta} \left(Z_s + Z_t - 2\min_{s\wedge t\le r\le s\vee t} Z_r\right) \ge \varepsilon\right).$$

For a fixed value of $\varepsilon > 0$, we can choose $\delta > 0$ small enough so that the right-hand side of the last display is arbitrarily small. Hence given $\eta \in (0,1)$, for every integer $k \ge 1$ we can find $\delta_k > 0$ and an integer n_k such that, for every $n \ge n_k$, we have

$$\mathbb{P}\left(\sup_{[s-t|\le\delta_k} (n^{-1/4} d_n^\circ(2ns,2nt)) \ge 2^{-k}\right) \le \eta 2^{-k}. \tag{14}$$

Then, by the triangle inequality, for every $s, s', t, t' \in [0,1]$,

$$\Big| d_n(2ns,2nt) - d_n(2ns',2nt')\Big| \le d_n(2ns,2ns') + d_n(2nt,2nt') \le d_n^\circ(2ns,2ns') + d_n^\circ(2nt,2nt'),$$

and it follows from (14) that, for every $k \ge 1$ and every $n \ge n_k$,

$$\mathbb{P}\left(\sup_{|s-s'|\le\delta_k, |t-t'|\le\delta_k} n^{-1/4}\Big|d_n(2ns,2nt) - d_n(2ns',2nt')\Big| > 2\cdot 2^{-k}\right) \le 2\eta 2^{-k}. \tag{15}$$

If $n \in \{1,\ldots,n_k - 1\}$ is fixed, the continuity of the mapping $(u,v] \mapsto d_n(2nu,2nv)$ shows that (15) still holds provided we take δ_k smaller if necessary. Finally, for every $k \ge 0$, we can find $\delta_k > 0$ so that (15) holds for every $n \ge 1$, and therefore, for every $n \ge 1$,

$$\mathbb{P}\left(\bigcap_{k\ge 1}\left\{\sup_{|s-s'|\le\delta_k, |t-t'|\le\delta_k} n^{-1/4}\Big|d_n(2ns,2nt) - d_n(2ns',2nt')\Big| \le 2\cdot 2^{-k}\right\}\right) \ge 1 - 2\eta. \tag{16}$$

The set of all continuous functions $\varphi : [0,1]^2 \longrightarrow \mathbb{R}_+$ such that $\varphi(0,0) = 0$ and, for every $k \ge 1$,

$$\sup_{|s-s'|\le\delta_k, |t-t'|\le\delta_k} |\varphi(s,t) - \varphi(s',t')| \le 2\cdot 2^{-k}$$

is compact by Ascoli's theorem. We thus obtain from (16) that the sequence of the laws of $(n^{-1/4} d_n(2ns,2nt))_{(s,t)\in[0,1]^2}$ is tight in the set of probability measures on $C([0,1]^2,\mathbb{R}_+)$. Thanks to this tightness property, we may assume that, along a suitable sequence of values of n, we have

$$\left(\left(\frac{9}{8}\right)^{1/4} n^{-1/4}\, d_n(2ns,2nt)\right)_{(s,t)\in[0,1]^2} \xrightarrow[n\to\infty]{\text{(d)}} \Big(D(s,t)\Big)_{(s,t)\in[0,1]^2}, \tag{17}$$

where the random process $(D(s,t))_{(s,t)\in[0,1]^2}$ has continuous sample paths. We may further assume that this convergence holds jointly with (12). Finally, thanks to the Skorokhod representation theorem, we may assume that both (12) and (17) hold almost surely (still along the chosen subsequence).

Let us state some properties of D that follows from the preceding convergences.

- If $s \sim t$ (meaning that $p_{\mathbf{e}}(s) = p_{\mathbf{e}}(t)$), we have $D(s,t) = 0$. The idea is to write $s = \lim k_n/(2n)$ and $t = \lim \ell_n/(2n)$, in such a way that $u^n_{k_n} = u^n_{\ell_n}$, forcing $d_n(k_n, \ell_n) = 0$ (observe that, for $s \le t$, the condition $\mathbf{e}_s = \mathbf{e}_t = \min\{\mathbf{e}_r : s \le r \le t\}$ allows us to find the sequences k_n and ℓ_n in such a way that $C_{\tau_n}(k_n) = C_{\tau_n}(\ell_n) = \min\{C_{\tau_n}(i) : k_n \le i \le \ell_n\}$, which implies $u^n_{k_n} = u^n_{\ell_n}$).

- The preceding observation allows us to define $D(a,b)$ for $a, b \in \mathcal{T}_{\mathbf{e}}$, in such a way that $D(a,b) = D(s,t)$ if $a = p_{\mathbf{e}}(s)$ and $b = p_{\mathbf{e}}(t)$. Note that D satisfies the triangle inequality, as a consequence of the (a.s.) convergence (17).

- For every $a, b \in \mathcal{T}_{\mathbf{e}}$, we have $D(a,b) \le D^\circ(a,b)$. This essentially follows from the bound $d_n(i,j) \le d^\circ_n(i,j)$, which implies after passage to the limit that, for $s \le t$,
$$D(s,t) \le Z_s + Z_t - 2\min\{Z_r : r \in [s,t]\}.$$

- If s_* is the (unique) element of $[0,1]$ such that $Z_{s_*} = \min\{Z_r : r \in [0,1]\}$, we have
$$D(s_*, t) = Z_s - Z_{s_*}.$$
This follows from (13) by passage to the limit $n \to \infty$.

Let us know explain how we get the Gromov-Hausdorff convergence of $(V(Q_n), (\frac{9}{8})^{1/4} n^{-1/4} \mathrm{d}^{Q_n}_{\mathrm{gr}})$ along the chosen subsequence of values of n. Thanks to the fact that (12) and (17) hold almost surely, we will in fact get an almost sure convergence. For $a, b \in \mathcal{T}_{\mathbf{e}}$, we set
$$a \approx b \text{ iff } D(a,b) = 0.$$
Then D induces a distance on the quotient space $\mathcal{T}_{\mathbf{e}}/\approx$. We claim that we have
$$\Big(V(Q_n), \Big(\frac{9}{8}\Big)^{1/4} n^{-1/4} \mathrm{d}^{Q_n}_{\mathrm{gr}}\Big) \xrightarrow[n\to\infty]{\text{a.s.}} (\mathcal{T}_{\mathbf{e}}/\approx, D) \tag{18}$$
along the sequence of values of n chosen previously so that (12) and (17) hold. To verify our claim, we use the formulation of the Gromov-Hausdorff distance in terms of correspondences. If Π denotes the canonical projection from $\mathcal{T}_{\mathbf{e}}$ onto $\mathcal{T}_{\mathbf{e}}/\approx$, we define a correspondence between $V(Q_n)$ and $\mathcal{T}_{\mathbf{e}}/\approx$ by setting
$$\mathcal{C}_n := \{(u^n_{\lfloor 2ns \rfloor}, \Pi(p_{\mathbf{e}}(s))) : s \in [0,1]\} \cup \{v^n_*, \Pi(p_{\mathbf{e}}(s_*))\},$$
where s_* is such that $Z_{s_*} = \min\{Z_s : 0 \le s \le 1\}$.

We then check that the distortion of $\mathcal{C}_n$ tends to 0. This is easy since, for $s, t \in [0,1]$,
$$\Big|\Big(\frac{9}{8}\Big)^{1/4} n^{-1/4} \mathrm{d}^{Q_n}_{\mathrm{gr}}(u^n_{\lfloor 2ns \rfloor}, u^n_{\lfloor 2nt \rfloor}) - D(\Pi(p_{\mathbf{e}}(s)), \Pi(p_{\mathbf{e}}(t)))\Big| = \Big|\Big(\frac{9}{8}\Big)^{1/4} n^{-1/4} d_n(\lfloor 2ns \rfloor, \lfloor 2nt \rfloor) - D(s,t)\Big|$$
tends to 0 as $n \to \infty$, uniformly in s, t, by (17), and on the other hand,
$$\Big(\frac{9}{8}\Big)^{1/4} n^{-1/4} \mathrm{d}^{Q_n}_{\mathrm{gr}}(v^n_*, u^n_{\lfloor 2ns \rfloor}) \xrightarrow[n\to\infty]{} Z_t - Z_{s_*} = D(\Pi(p_{\mathbf{e}}(s_*)), \Pi(p_{\mathbf{e}}(s))),$$
again uniformly in s, by (13) and (12). This completes the proof of our claim (18).

However, the proof of Theorem 1 is not yet complete, since on one hand we get only convergence on a sequence of values of n, and on the other hand we have not checked that $D(a,b) = D^*(a,b)$ for every $a, b \in \mathcal{T}_{\mathbf{e}}$. It is in fact enough to prove the latter assertion (because then, the limit on any subsequence will be the same, and a tightness argument gives the desired result). Note that the bound $D \le D^*$ is very easy since we already know that $D \le D^\circ$ and D satisfies the triangle inequality. Unfortunately, the other bound $D \ge D^*$ is much harder to obtain, and we refer to [9] and [14] for two different approaches.

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