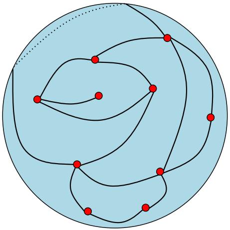
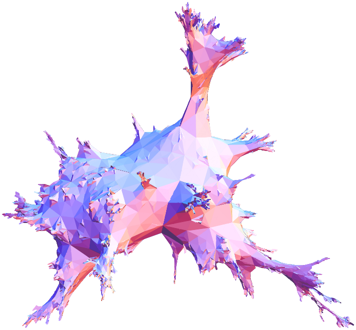
Random Geometry on the Sphere

Jean-François Le Gall

Université Paris-Sud Orsay and Institut universitaire de France





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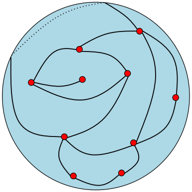

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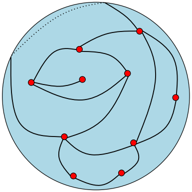

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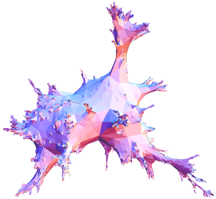

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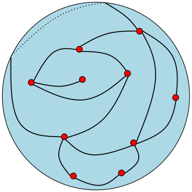

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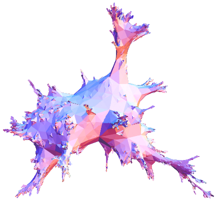

Strong analogy with Brownian motion, which is a canonical model for a random curve in space, obtained as the scaling limit of random walks on the lattice.

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Definition

A planar map is a proper embedding of a finite connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere).

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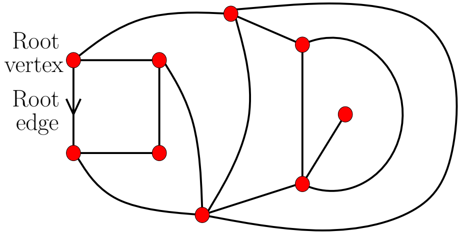

A rooted quadrangulation with 7 faces

Faces = connected components of the complement of edges

p-angulation:

- each face is bounded by p edges

$p = 3$: triangulation

$p = 4$: quadrangulation

Rooted map: distinguished oriented edge

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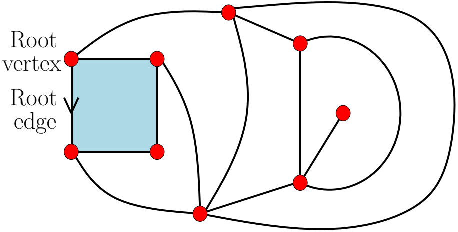

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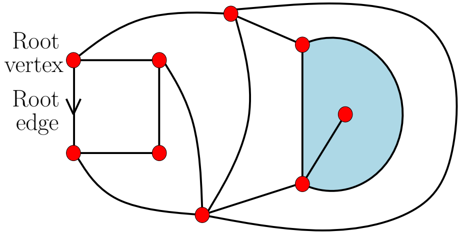

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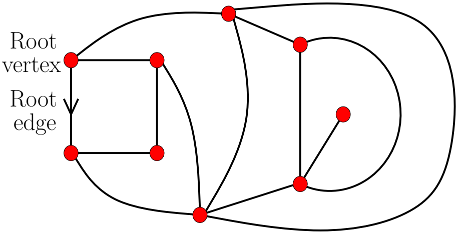

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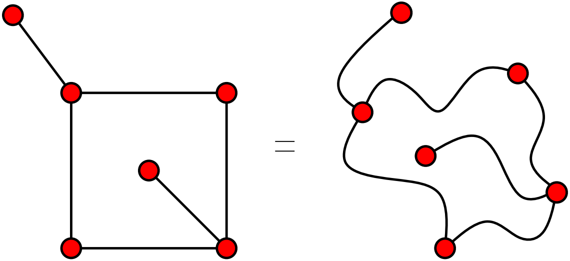

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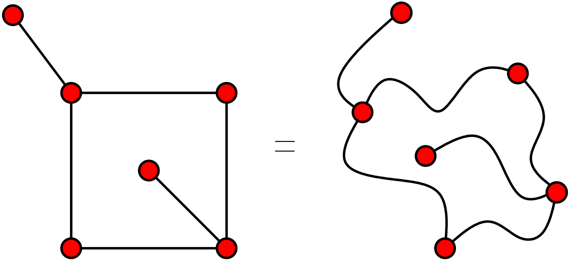
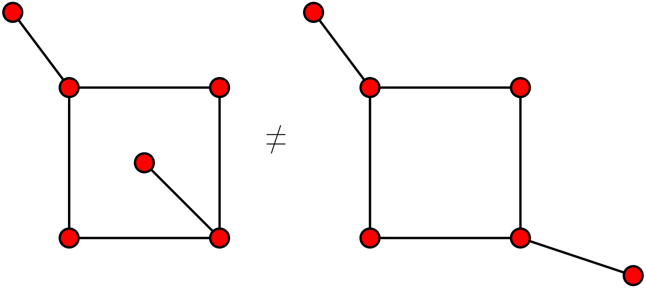
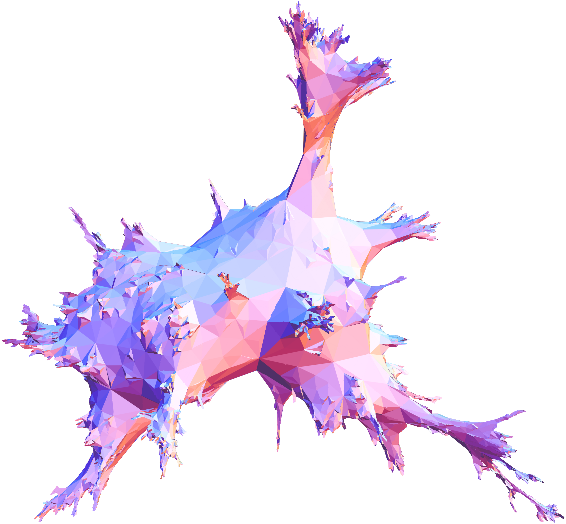


Two different planar maps:



A large triangulation of the sphere (simulation: N. Curien)

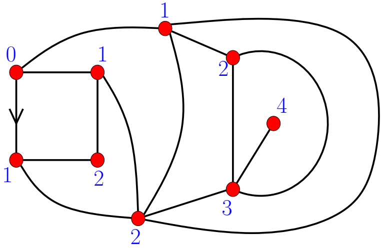
Can we get a continuous model out of this ?



Planar maps as metric spaces

M planar map

- $V(M)$ = set of vertices of M
- $d_{\rm gr}$ graph distance on $V(M)$
- $(V(M), d_{\rm gr})$ is a (finite) metric space



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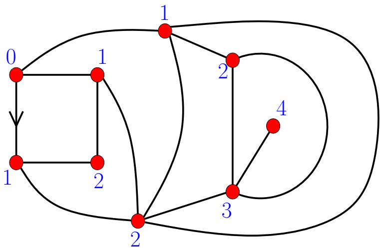

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$\mathbb{M}_n^p = \{\text{rooted } p-\text{angulations with } n \text{ faces}\}$

$\mathbb{M}_n^p$ is a finite set (*finite number of possible "shapes"*)

Choose M_n uniformly at random in $\mathbb{M}_n^p$.

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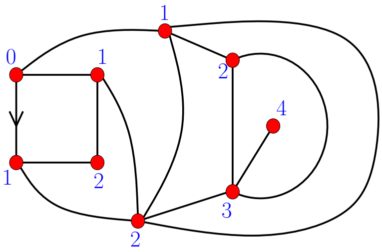

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View $(V(M_n), d_{\rm gr})$ as a random variable with values in

$$\mathbb{K} = \{\text{compact metric spaces, modulo isometries}\}$$

which is equipped with the Gromov-Hausdorff distance.

The Gromov-Hausdorff distance

The Hausdorff distance. K_1, K_2 compact subsets of a metric space

$$d_{\rm Haus}(K_1, K_2) = \inf\{\varepsilon > 0 : K_1 \subset U_\varepsilon(K_2) \text{ and } K_2 \subset U_\varepsilon(K_1)\}$$

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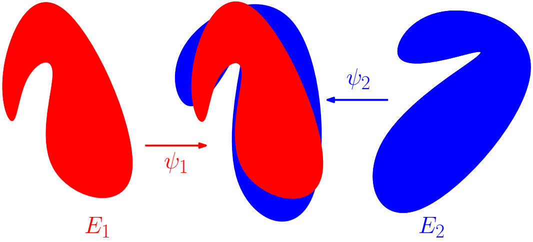
Definition (Gromov-Hausdorff distance)

If (E_1,d_1) and (E_2,d_2) are two compact metric spaces,

$$d_{\rm GH}(E_1,E_2) = \inf\{d_{\rm Haus}(\psi_1(E_1),\psi_2(E_2))\}$$

the infimum is over all isometric embeddings $\psi_1 : E_1 \to E$ and $\psi_2 : E_2 \to E$ of E_1 and E_2 into the same metric space E.



Gromov-Hausdorff convergence of rescaled maps

Fact

If $\mathbb{K} = \{$*isometry classes of compact metric spaces*$\}$*, then*

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$\rightarrow$ If M_n is uniformly distributed over $\{p-$ angulations with n faces$\}$, it makes sense to study the convergence in distribution of

$$(V(M_n), n^{-a} d_{\mathrm{gr}})$$

as random variables with values in $\mathbb{K}$.

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Choice of the rescaling factor n^{-a} **:** $a > 0$ is chosen so that $\mathrm{diam}(V(M_n)) \approx n^a$.

$\Rightarrow a = \frac{1}{4}$ [cf Chassaing-Schaeffer PTRF 2004 for quadrangulations]

Main result: The Brownian map

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M_n uniform over $\mathbb{M}_n^p$, $V(M_n)$ vertex set of M_n, $d_{\rm gr}$ graph distance

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Theorem (The scaling limit of p-angulations)

Suppose that either $p = 3$ (triangulations) or $p \geq 4$ is even. Set

$$c_3 = 6^{1/4} \quad , \quad c_p = \left(\frac{9}{p(p-2)}\right)^{1/4} \quad \text{if } p \text{ is even.}$$

Then,

$$(V(M_n), c_p\, n^{-1/4}\, d_{\rm gr}) \xrightarrow[n\to\infty]{\rm (d)} (\mathbf{m}_\infty, D^*)$$

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Remarks. The case $p = 4$ was obtained independently by Miermont.
The case $p = 3$ solves Schramm's problem (2006)
Expect the result to be also true for any odd value of p

Two properties of the Brownian map

Theorem (Hausdorff dimension)

$$\dim(\mathbf{m}_\infty, D^*) = 4 \qquad \textit{a.s.}$$

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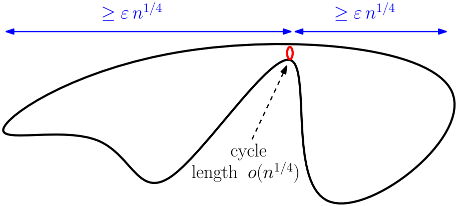
Consequence: for a typical planar map M_n with n faces, diameter $\approx n^{1/4}$ but: no cycle of size $o(n^{1/4})$ in M_n, such that both sides have diameter $\geq \varepsilon n^{1/4}$



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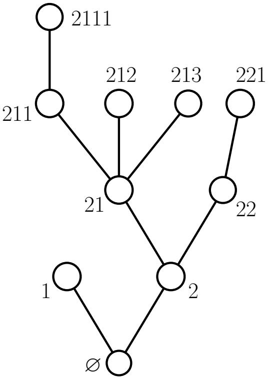

A plane tree τ with vertex set $V(\tau) = \{\varnothing, 1, 2, 21, 22, 212, \ldots\}$ (rooted **ordered** tree)

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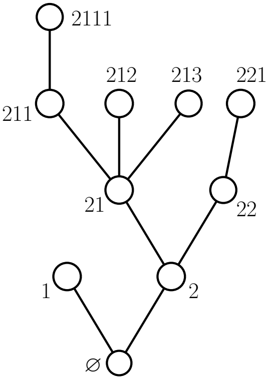

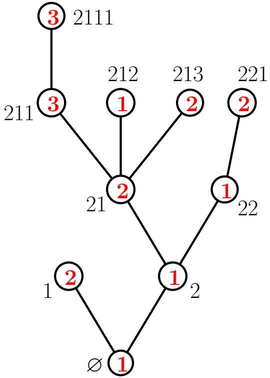

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A well-labeled tree $(\tau, (\ell_v)_{v \in V(\tau)})$
Properties of labels:

- $\ell_\varnothing = 1$
- $\ell_v \in \{1, 2, 3, \ldots\}, \forall v$
- $|\ell_v - \ell_{v'}| \leq 1$, if v, v' neighbors

Coding maps with trees, the case of quadrangulations

$\mathbb{T}_n = \{$well-labeled trees with n edges$\}$

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There is a bijection $\Phi : \mathbb{T}_n \longrightarrow \mathbb{M}_n^4$ such that, if $M = \Phi(\tau, (\ell_v)_{v \in V(\tau)})$, then

$$V(M) = V(\tau) \cup \{\partial\} \qquad (\partial \text{ is the root vertex of } M)$$
$$d_{\mathrm{gr}}(\partial, v) = \ell_v \quad , \forall v \in V(\tau)$$

Key properties.

- Vertices of τ become vertices of M
- The label in the tree becomes the distance from the root in the map.

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Coding of more general maps: Bouttier, Di Francesco, Guitter (2004)

Schaeffer's bijection between quadrangulations and well-labeled trees

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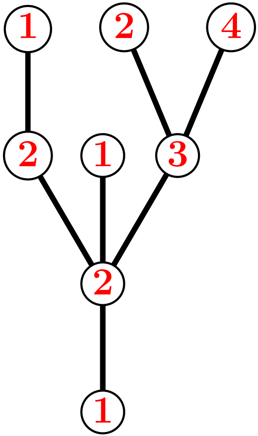

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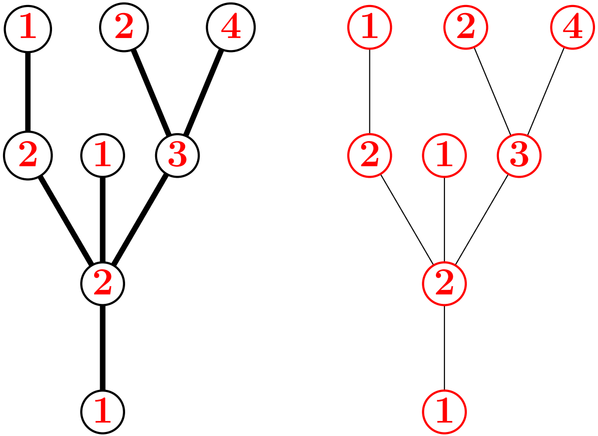

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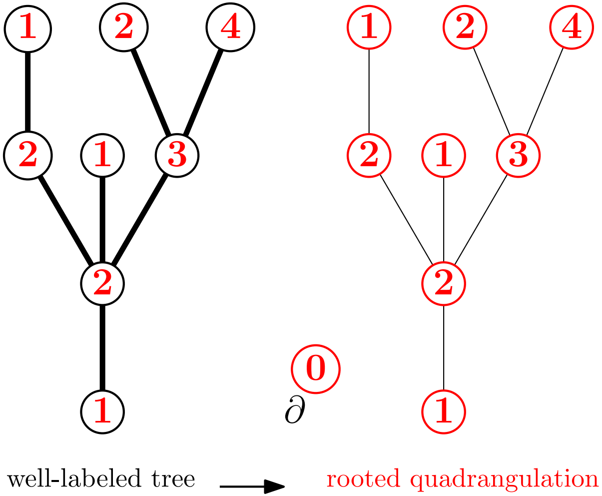

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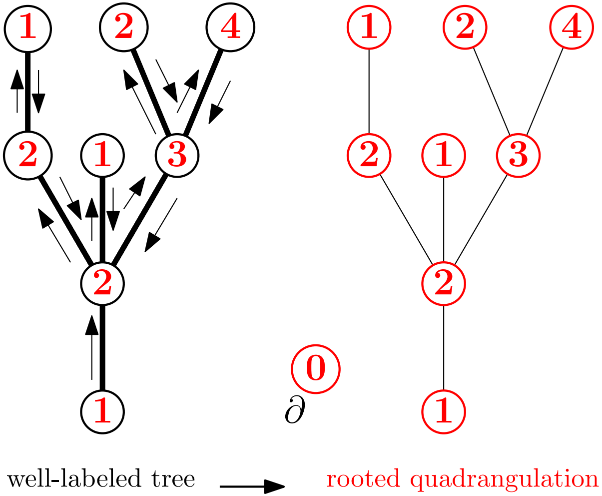

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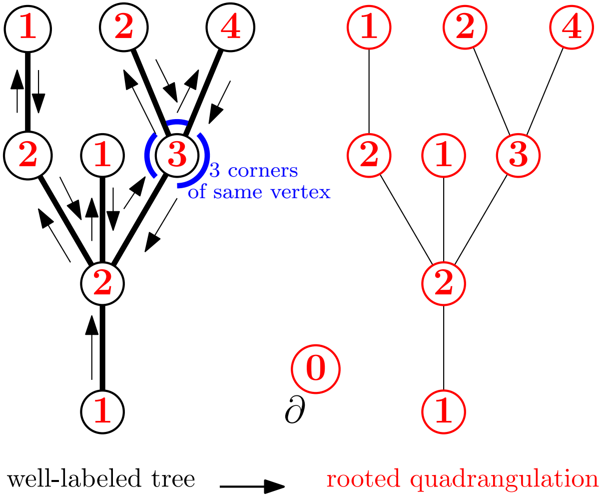

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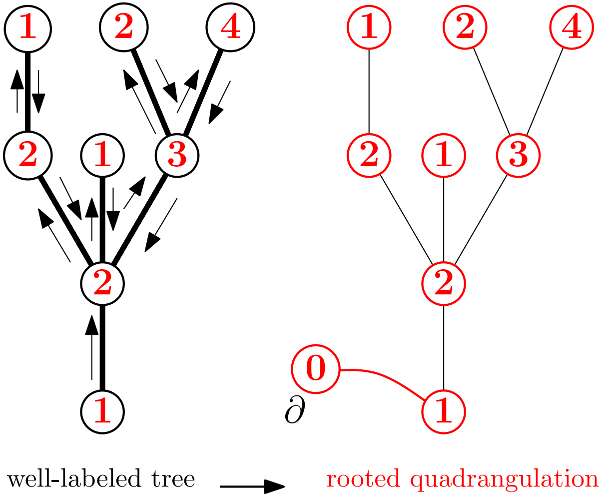

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- Draw a red edge between each corner and the last visited corner with smaller label (a corner with label 1 is connected to ∂)

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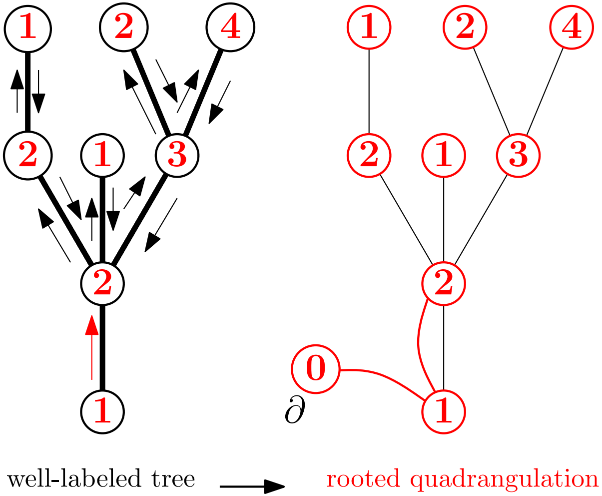

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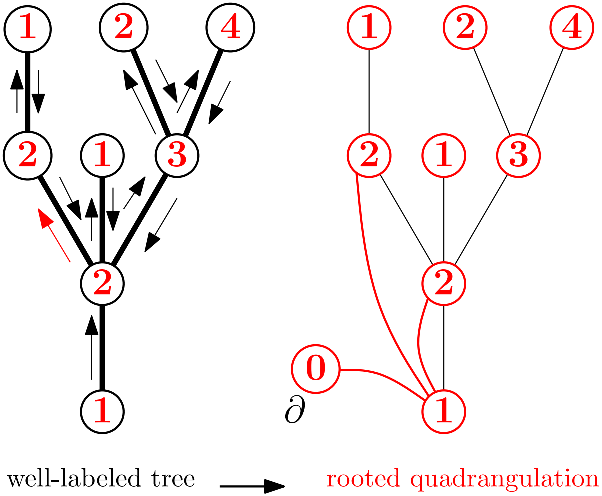

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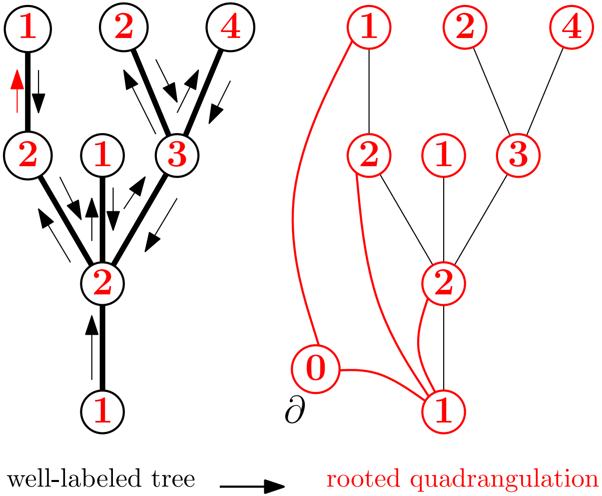

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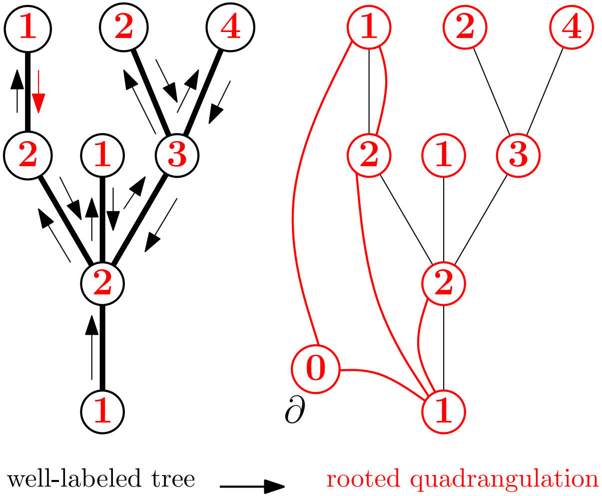

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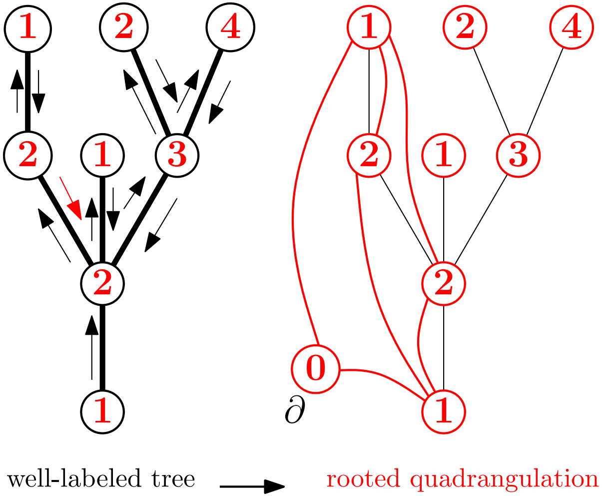

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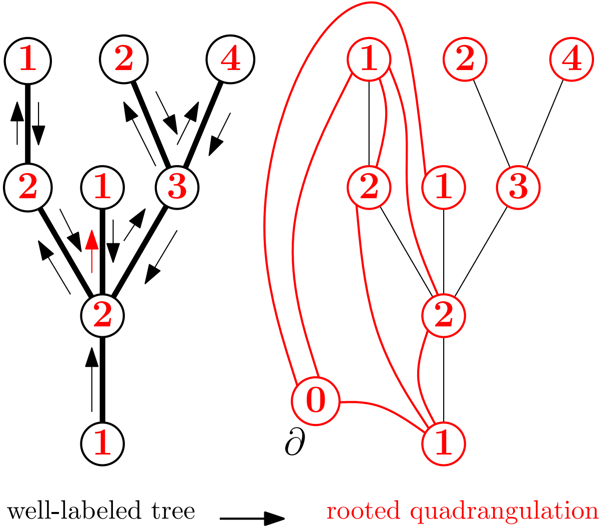

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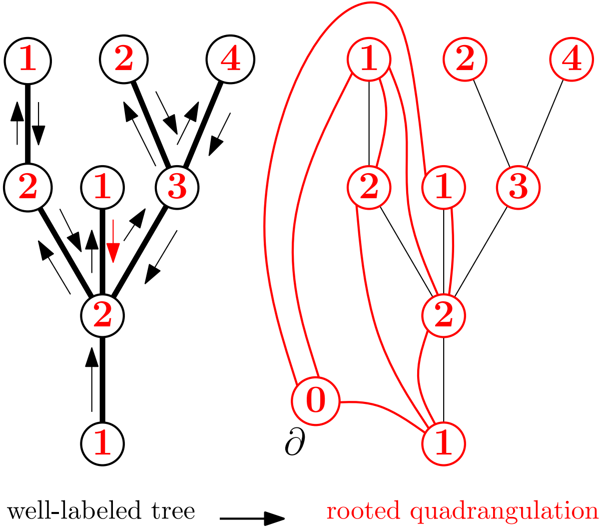

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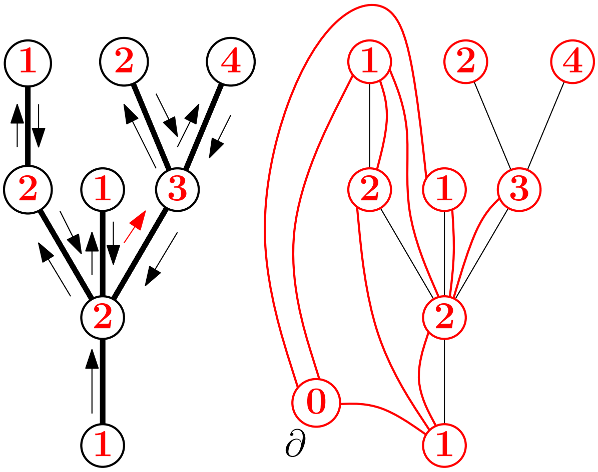

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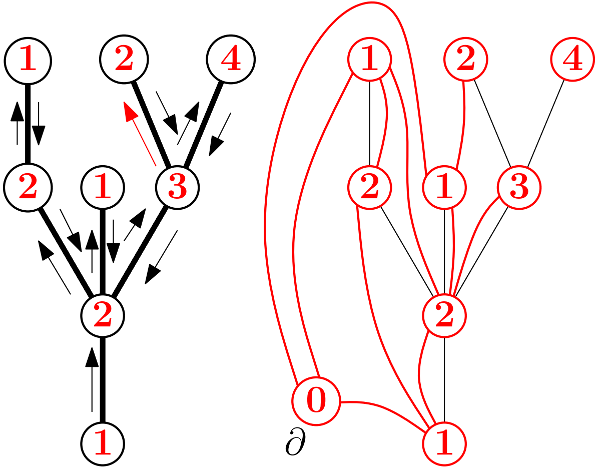

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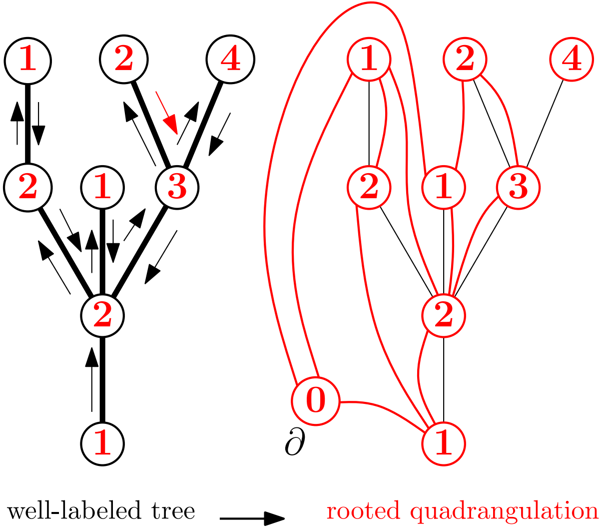

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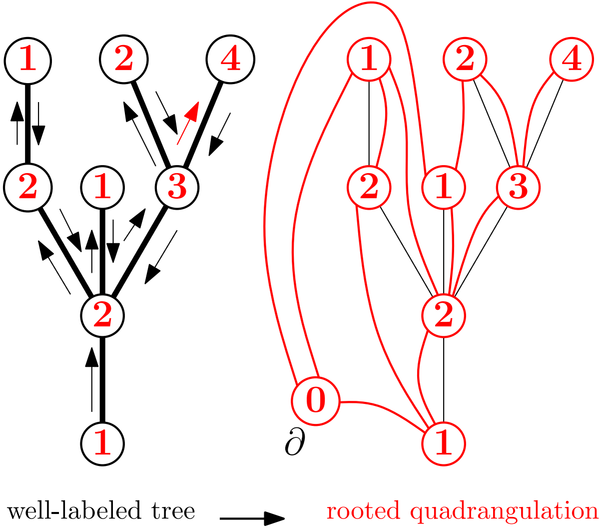

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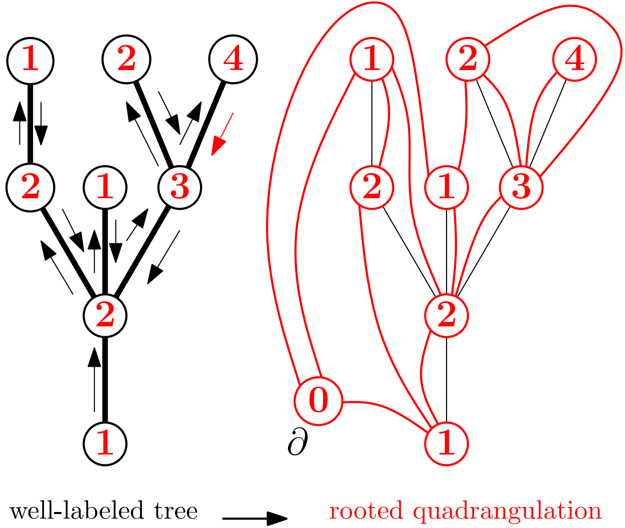

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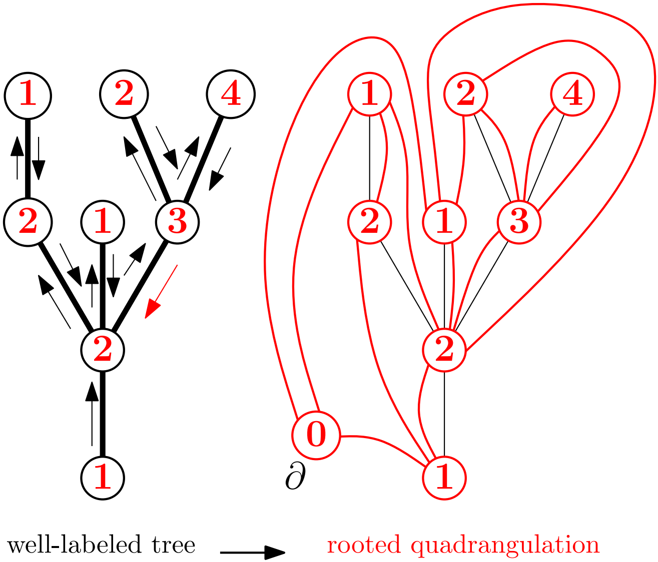

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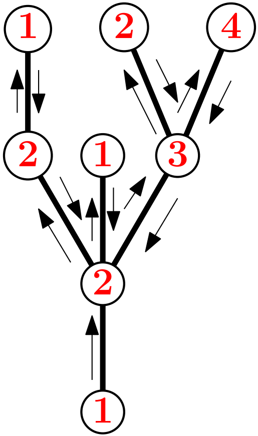

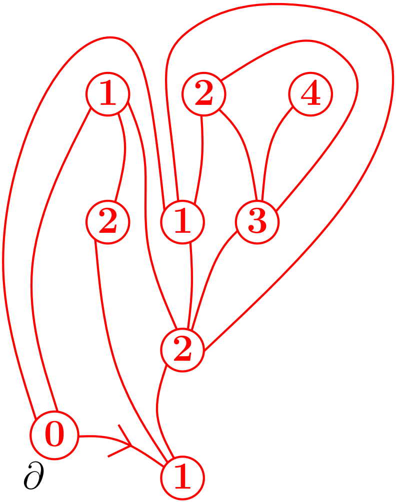

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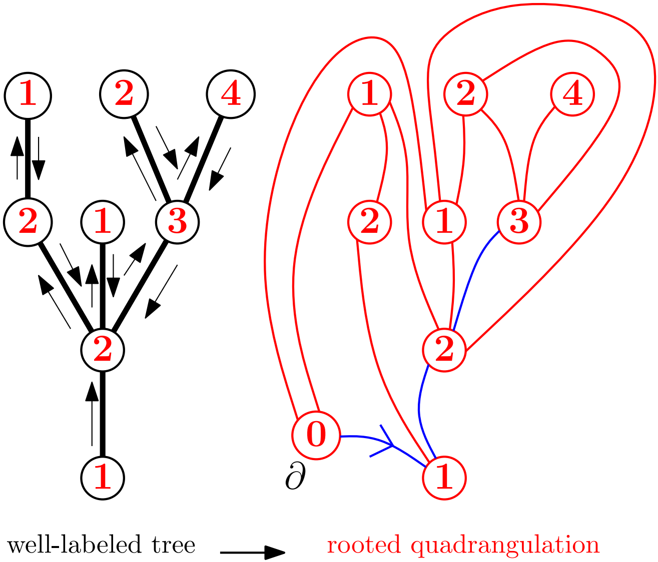

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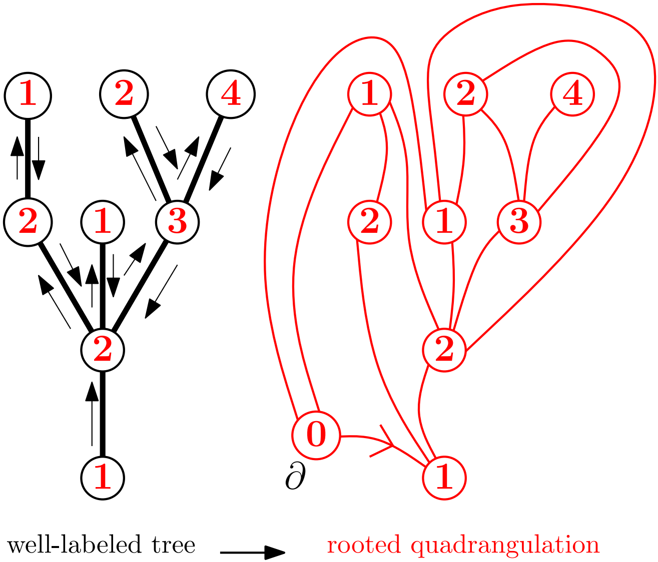

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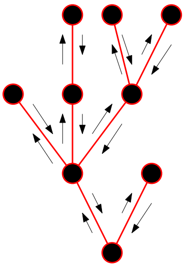

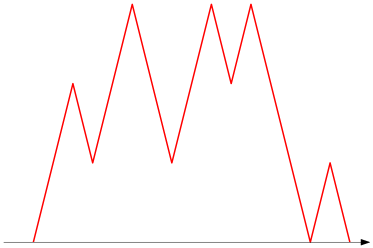

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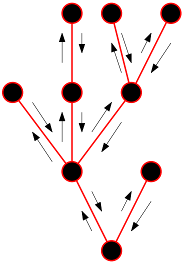

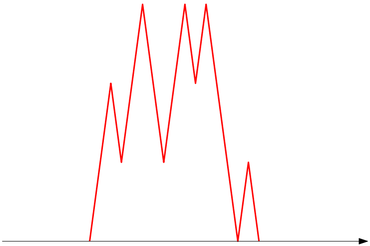

From the contour back to the tree (by gluing certain pairs of segments of the graph)

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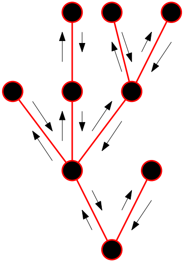

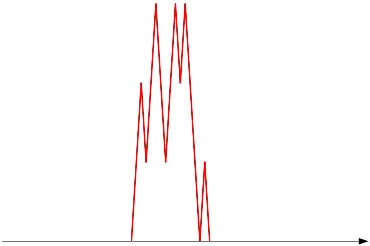

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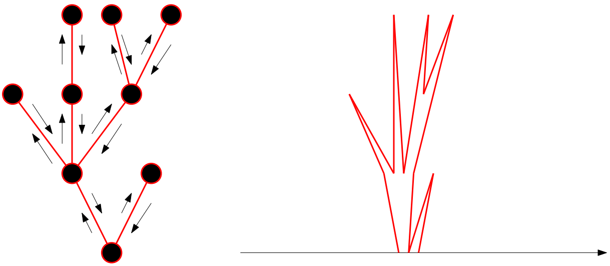

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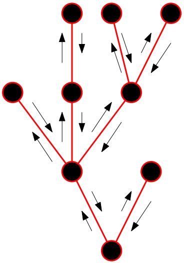
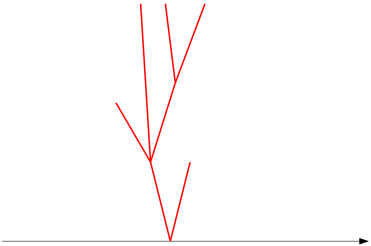




From the contour back to the tree (by gluing certain pairs of segments of the graph)

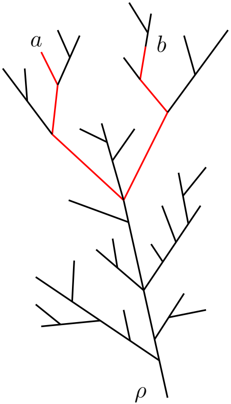
The notion of a real tree

Definition

A real tree, or $\mathbb{R}$-tree, is a (compact) metric space $\mathcal{T}$ such that:

- any two points $a, b \in \mathcal{T}$ are joined by a unique continuous and injective path (up to re-parametrization)
- this path is isometric to a line segment

$\mathcal{T}$ is a rooted real tree if there is a distinguished point ρ, called the root.



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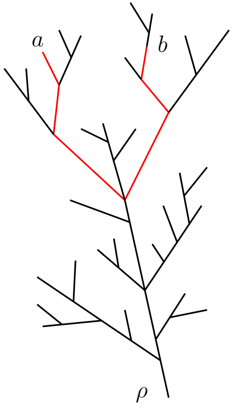

Remark. A real tree can have

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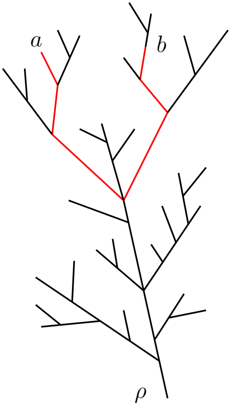

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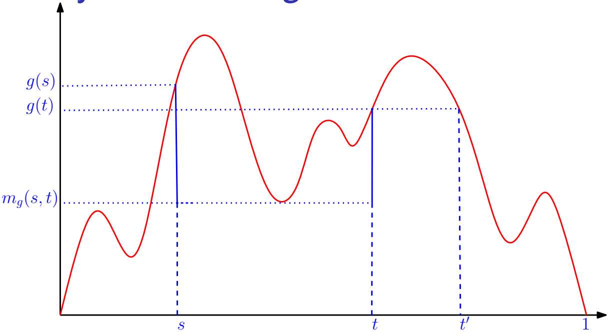
Fact. The coding of discrete trees by contour functions can be extended to real trees: also gives a cyclic ordering on the tree.

The real tree coded by a function g

$g : [0,1] \longrightarrow [0,\infty)$
continuous,

$g(0) = g(1) = 0$

$m_g(s,t) = \min_{[s \wedge t, s \vee t]} g$



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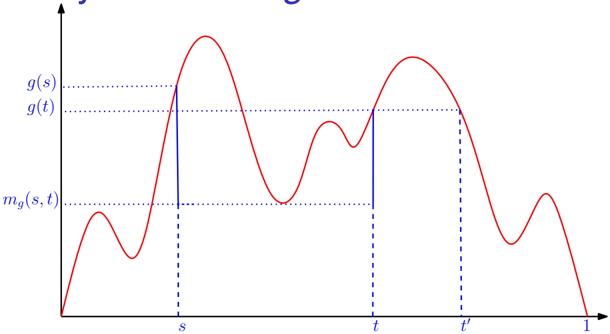

$d_g(s,t) = g(s) + g(t) - 2\, m_g(s,t)$ pseudo-metric on $[0,1]$

$t \sim t'$ iff $d_g(t,t') = 0$ (or equivalently $g(t) = g(t') = m_g(t,t')$)

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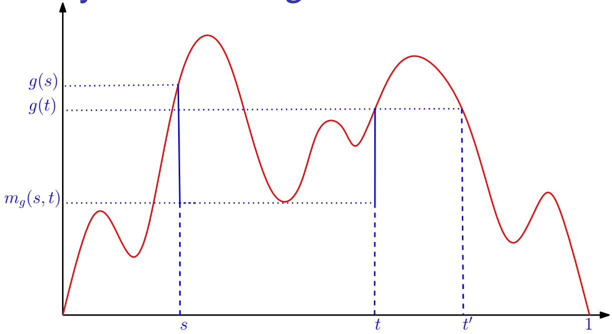

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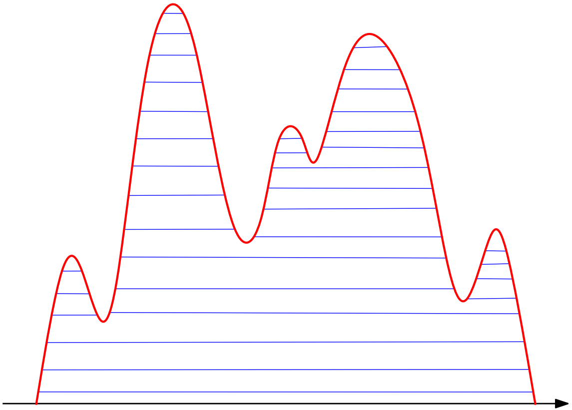
Proposition

$\mathcal{T}_g := [0,1]/\sim$ equipped with d_g is a real tree, called the tree coded by g. It is rooted at $\rho = 0$.

The canonical projection $[0,1] \to \mathcal{T}_g$ induces a cyclic ordering on $\mathcal{T}_g$

Coding a tree by a function



Every horizontal blue line segment below the curve is identified to a single point.

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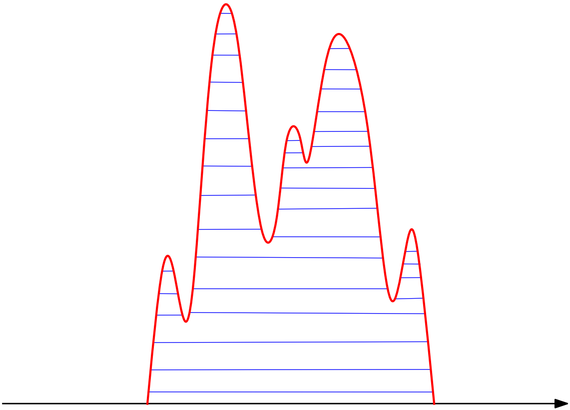

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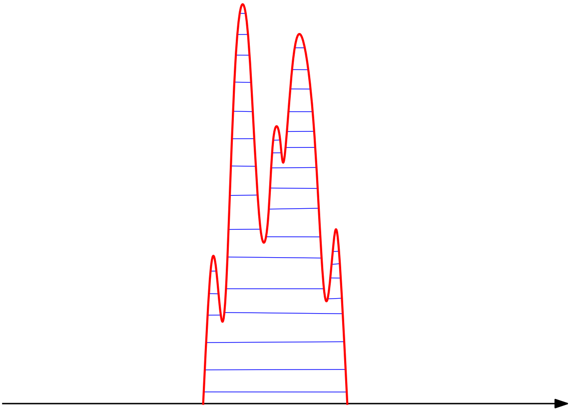

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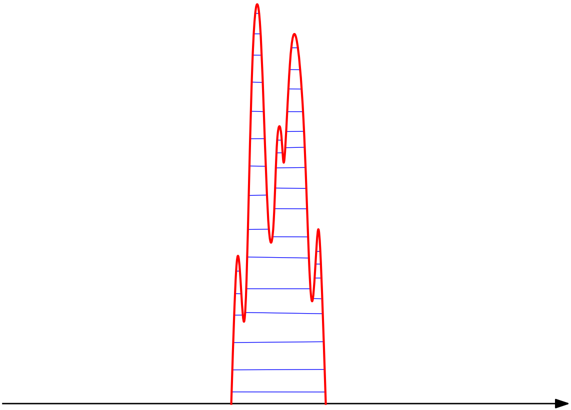

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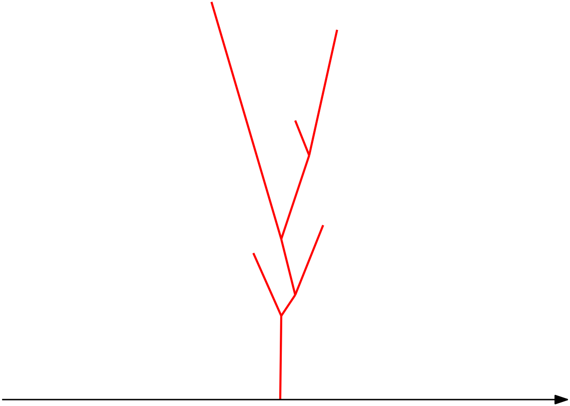

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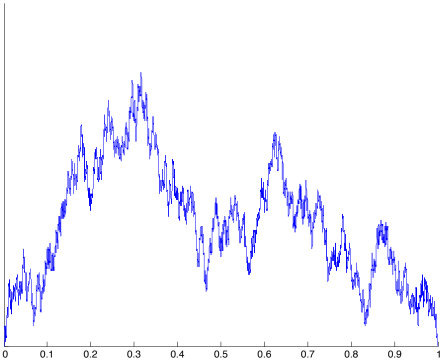
Definition of the CRT

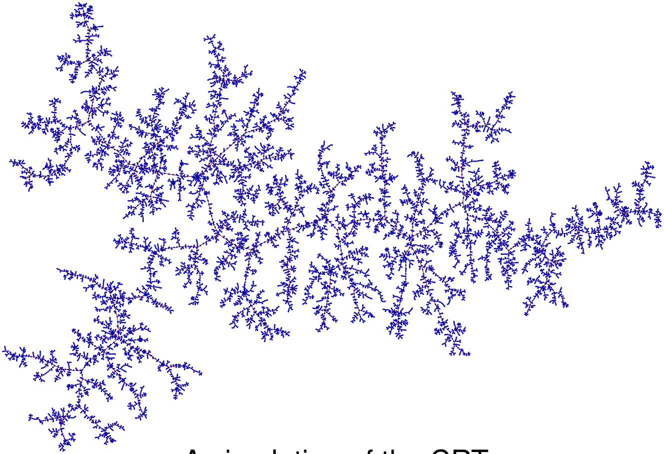
Let $(\mathbf{e}_t)_{0\leq t\leq 1}$ be a Brownian excursion with duration 1 (= Brownian motion started from 0 conditioned to be at 0 at time 1 and to stay ≥ 0)

Definition

The CRT $(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$ is the (random) real tree coded by the Brownian excursion $\mathbf{e}$.



Simulation of a Brownian excursion



A simulation of the CRT
(simulation: I. Kortchemski)

Assigning Brownian labels to a real tree

Let $(\mathcal{T}, d)$ be a real tree with root ρ.

$(Z_a)_{a\in\mathcal{T}}$: Brownian motion indexed by $(\mathcal{T}, d)$
= centered Gaussian process such that

- $Z_\rho = 0$
- $E[(Z_a - Z_b)^2] = d(a, b), \qquad a, b \in \mathcal{T}$

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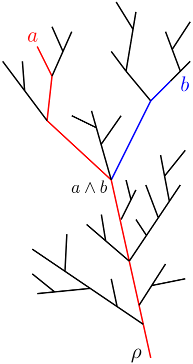

Labels evolve like Brownian motion along the branches of the tree:

- The label Z_a is the value at time $d(\rho, a)$ of a standard Brownian motion
- Similar property for Z_b, but one uses
  - the same BM between 0 and $d(\rho, a \wedge b)$
  - an independent BM between $d(\rho, a \wedge b)$ and $d(\rho, b)$

The definition of the Brownian map

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$$D^0(a,b) = Z_a + Z_b - 2\max\Big(\min_{c\in[a,b]} Z_c, \min_{c\in[b,a]} Z_c\Big)$$

where $[a, b]$ is the “interval” from a to b corresponding to the cyclic ordering on $\mathcal{T}_{\mathbf{e}}$ (vertices visited when going from a to b in clockwise order around the tree).

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Then set

$$D^*(a,b) = \inf_{a_0=a,a_1,\ldots,a_{k-1},a_k=b} \sum_{i=1}^{k} D^0(a_{i-1}, a_i),$$

$$a \approx b \text{ if and only if } D^*(a,b) = 0 \quad (\text{equivalent to } D^0(a,b) = 0).$$

The definition of the Brownian map

$(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$ is the CRT, $(Z_a)_{a\in\mathcal{T}_{\mathbf{e}}}$ Brownian motion indexed by the CRT (Two levels of randomness!).

Set, for every $a, b \in \mathcal{T}_{\mathbf{e}}$,

$$D^0(a,b) = Z_a + Z_b - 2\max\Big(\min_{c\in[a,b]} Z_c, \min_{c\in[b,a]} Z_c\Big)$$

where $[a, b]$ is the "interval" from a to b corresponding to the cyclic ordering on $\mathcal{T}_{\mathbf{e}}$ (vertices visited when going from a to b in clockwise order around the tree).

Then set

$$D^*(a,b) = \inf_{a_0=a,a_1,\ldots,a_{k-1},a_k=b} \sum_{i=1}^{k} D^0(a_{i-1},a_i),$$

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Definition

The Brownian map $\mathbf{m}_\infty$ is the quotient space $\mathbf{m}_\infty := \mathcal{T}_{\mathbf{e}} / \approx$, which is equipped with the distance induced by D^*.

Summary and interpretation

Starting from the CRT $\mathcal{T}_{\mathbf{e}}$, with Brownian labels $Z_a, a \in \mathcal{T}_{\mathbf{e}}$,
$\rightarrow$ Identify two vertices $a, b \in \mathcal{T}_{\mathbf{e}}$ if:

- they have the same label $Z_a = Z_b$,
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Remark. Not many vertices are identified:

- A “typical” equivalence class is a singleton.
- Equivalence classes may contain at most 3 points.

Still these identifications drastically change the topology.

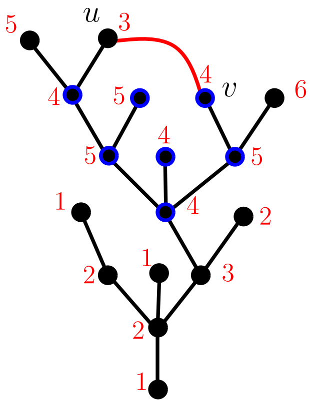
Interpretation of the equivalence relation $\approx$

In Schaeffer's bijection:
$\exists$ edge between u and v if

- $\ell_u = \ell_v - 1$
- $\ell_w \geq \ell_v$, $\forall w \in]u, v]$

Explains why in the continuous limit

$$Z_a = Z_b = \min_{c\in[a,b]} Z_c$$
$\Rightarrow$ *a* and *b* are identified



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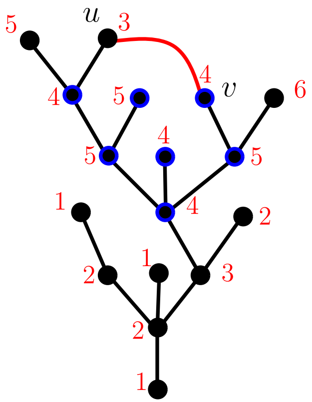

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Key points of the proof of the main theorem:

- Prove the converse (no other pair of points are identified)
- Obtain the formula for the limiting distance D^*

Properties of distances in the Brownian map

Let ρ_* be the (unique) vertex of $\mathcal{T}_{\mathbf{e}}$ such that

$$Z_{\rho_*} = \min_{c \in \mathcal{T}_{\mathbf{e}}} Z_c$$

Then, for every $a \in \mathcal{T}_{\mathbf{e}}$,

$$D^*(\rho_*, a) = Z_a - \min Z.$$

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No such simple expression for $D^*(a, b)$ in terms of labels, but

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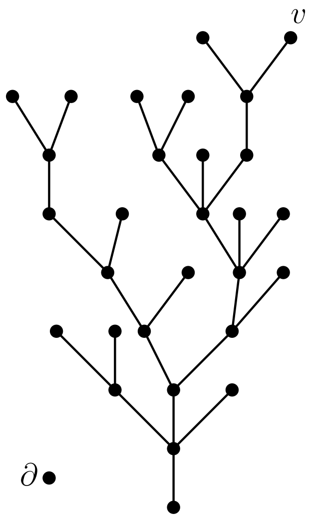
D^* is the maximal metric that satisfies this inequality

4. Geodesics in the Brownian map

Geodesics in quadrangulations

Use Schaeffer's bijection between quadrangulations and well-labeled trees.

To construct a geodesic from v to ∂:



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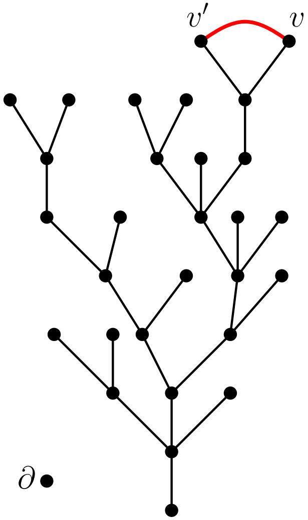

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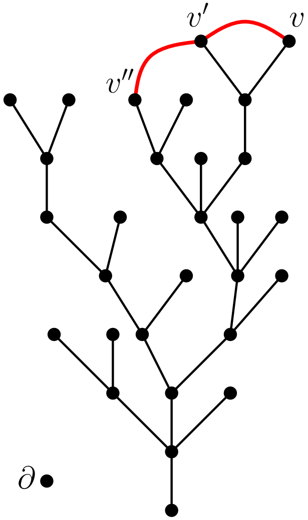

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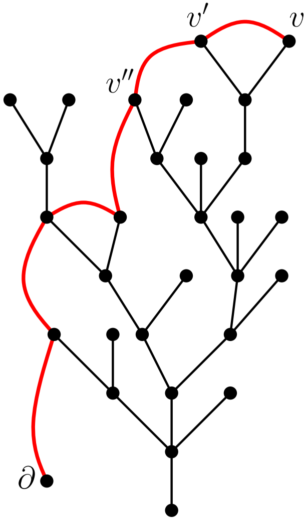
- And so on.
- Eventually one reaches ∂.



Geodesics to ρ_* in the Brownian map

Recall : ρ_* is the unique point of $\mathcal{T}_{\mathbf{e}}$ s.t.

$$Z_{\rho_*} = \min_{c \in \mathcal{T}_{\mathbf{e}}} Z_c$$

then, for every $b \in \mathcal{T}_{\mathbf{e}}$,

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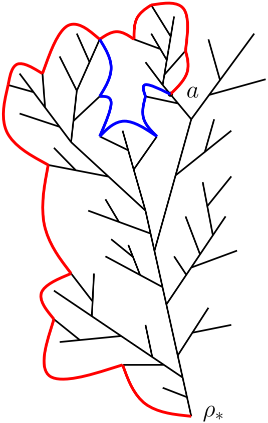
("last" refers to the cyclic order)



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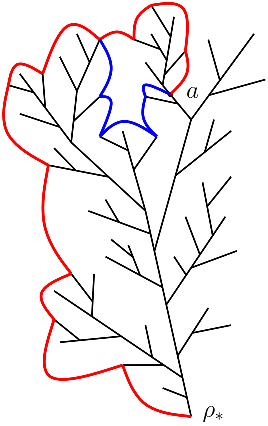

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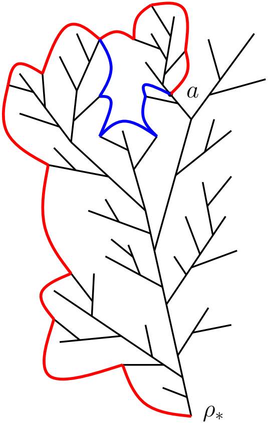

Fact

All geodesics to ρ_* are of this form.

If a is not a leaf, there are several possible choices, depending on which side of a one starts.

The main result about geodesics

Define the skeleton of $\mathcal{T}_{\mathbf{e}}$ by $\mathrm{Sk}(\mathcal{T}_{\mathbf{e}}) = \mathcal{T}_{\mathbf{e}} \backslash \{\text{leaves of } \mathcal{T}_{\mathbf{e}}\}$ and set

$\mathrm{Skel} = \pi(\mathrm{Sk}(\mathcal{T}_{\mathbf{e}}))$, where $\pi : \mathcal{T}_{\mathbf{e}} \to \mathcal{T}_{\mathbf{e}}/\!\approx\, = \mathbf{m}_\infty$ canonical projection

Then

- the restriction of π to $\mathrm{Sk}(\mathcal{T}_{\mathbf{e}})$ is a homeomorphism onto Skel
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Theorem (Geodesics from the root)

Let $x \in \mathbf{m}_\infty$. Then,

- *if $x \notin \mathrm{Skel}$, there is a unique geodesic from ρ_* to x*
- *if $x \in \mathrm{Skel}$, the number of distinct geodesics from ρ_* to x is the multiplicity $m(x)$ of x in Skel (note: $m(x) \leq 3$).*

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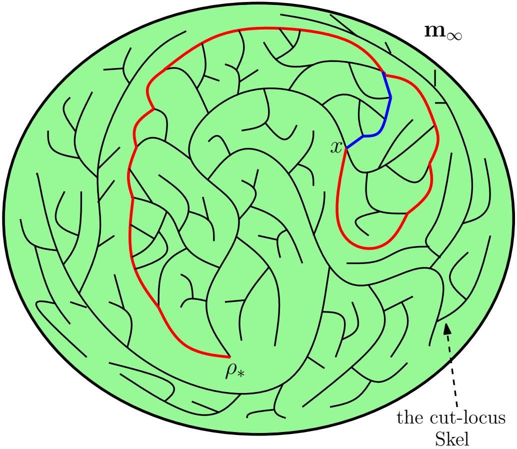
Remarks

- Skel is the cut-locus of $\mathbf{m}_\infty$ relative to ρ_*: cf classical Riemannian geometry [Poincaré, Myers, ...], where the cut-locus is a tree.
- same results if ρ_* replaced by a point chosen “at random” in $\mathbf{m}_\infty$.

Illustration of the cut-locus



The cut-locus Skel
is homeomorphic to
a non-compact real tree
and is dense in $\mathbf{m}_\infty$

Geodesics to ρ_*
do not visit Skel
(except possibly
at their starting point)
but “move around” Skel.

Confluence property of geodesics

Fact: Two geodesics to ρ_* coincide near ρ_*.
(easy from the definition)

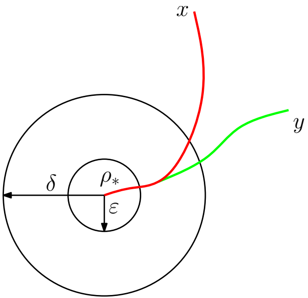
Corollary

Given $\delta > 0$, there exists $\varepsilon > 0$ s.t.

- *if $D^*(\rho_*, x) \geq \delta$, $D^*(\rho_*, y) \geq \delta$*
- *if γ is any geodesic from ρ_* to x*
- *if γ' is any geodesic from ρ_* to y*

then

$$\gamma(t) = \gamma'(t) \quad \text{for all } t \leq \varepsilon$$



"Only one way" of leaving ρ_* along a geodesic.
(also true if ρ_* is replaced by a typical point of $\mathbf{m}_\infty$)

Uniqueness of geodesics in discrete maps

M_n uniform distributed over $\mathbb{M}_n^p = \{p-\text{angulations with } n \text{ faces}\}$
$V(M_n)$ set of vertices of M_n, ∂ root vertex of M_n, $d_{\rm gr}$ graph distance

For $v \in V(M_n)$, set $\mathrm{Geo}(\partial \to v) = \{\text{geodesics from } \partial \text{ to } v\}$

If γ, γ' are two discrete paths in M_n (with the same length)

$$d(\gamma,\gamma') = \max_i d_{\rm gr}(\gamma(i),\gamma'(i))$$

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Corollary

Let $\delta > 0$. *Then,*

$$\frac{1}{n}\#\{v \in V(M_n) : \exists \gamma,\gamma' \in \mathrm{Geo}(\partial \to v),\ d(\gamma,\gamma') \geq \delta n^{1/4}\} \underset{n\to\infty}{\longrightarrow} 0$$

Two discrete geodesics (between two typical points) are within a distance $o(n^{-1/4})$

(Macroscopic uniqueness, also true for "approximate geodesics"= paths with length $d_{\rm gr}(\partial, v) + o(n^{1/4})$)

5. Canonical embeddings: Open problems

Recall that a planar map is defined up to (orientation-preserving) homeomorphisms of the sphere.

It is possible to choose a particular (canonical) embedding of the graph satisfying conformal invariance properties, and this choice is unique (at least up to the Möbius transformations, which are the conformal transformations of the sphere $\mathbb{S}^2$).

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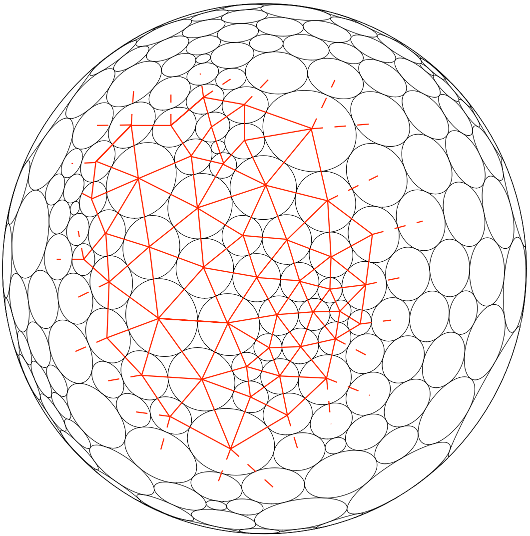
Question

Applying this canonical embedding to M_n (uniform over p-angulations with n faces), can one let n tend to infinity and get a random metric Δ on the sphere $\mathbb{S}^2$ satisfying conformal invariance properties, and such that

$$(\mathbb{S}^2, \Delta) \overset{(\mathrm{d})}{=} (\mathbf{m}_\infty, D^*)$$

Canonical embeddings via circle packings 1



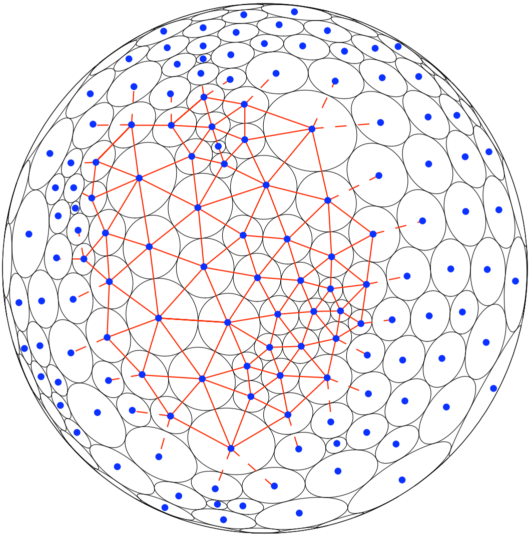
From a circle packing, construct a graph M :

- $V(M) = \{$centers of circles$\}$
- edge between a and b if the corresponding circles are tangent.

A triangulation (without loops or multiple edges) can always be represented in this way.

Representation unique up to Möbius transformations.

Canonical embeddings via circle packings 2



Apply to M_n uniform over {triangulations with n faces}. Let $n \to \infty$. Expect to get

- Random metric Δ on $\mathbb{S}^2$ (with conformal invariance properties) such that $(\mathbb{S}^2, \Delta) = (\mathbf{m}_\infty, D^*)$
- Random volume measure on $\mathbb{S}^2$

Connections with the Gaussian free field ?

Recent progress: Miller-Sheffield (Quantum Loewner Evolut.)

A few references

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