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GEODESIC STARS IN RANDOM GEOMETRY

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A point of a metric space is called a geodesic star with m arms if it is the endpoint of m disjoint geodesics. For every $m \in \{1, 2, 3, 4\}$, we prove that the set of all geodesic stars with m arms in the Brownian sphere has dimension $5 - m$. This complements recent results of Miller and Qian, who proved that this dimension is smaller than or equal to $5 - m$.

1. Introduction. This work is concerned with the continuous models of random geometry that have been studied extensively in the recent years. In particular, we consider the Brownian sphere or Brownian map, which is the scaling limit in the Gromov-Hausdorff sense of triangulations or quadrangulations of the sphere with n faces chosen uniformly at random, and of much more general random planar maps (see in particular [1, 4, 7, 18, 24, 26]). We are primarily interested in the study of geodesics in the Brownian sphere, but our main result remains valid in the related models called the Brownian plane [8, 9] and the Brownian disk [6, 21].

Recall that a geodesic in a metric space (E, d) is a continuous path $(\gamma(t))_{t\in[0,\delta]}$, where $\delta > 0$, such that $d(\gamma(s), \gamma(t)) = |s - t|$ for every $s, t \in [0, \delta]$. For every $t \in (0, \delta)$, we say that $\gamma(t)$ is an interior point of the geodesic (whereas $\gamma(0)$ and $\gamma(\delta)$ are its endpoints). If $m \geq 1$ is an integer, we then say that a point x is a geodesic star with m arms (in short, an m-geodesic star) if there exist $\delta > 0$ and m geodesics $(\gamma_1(t))_{t\in[0,\delta]}, \ldots, (\gamma_m(t))_{t\in[0,\delta]}$ such that $\gamma_1(0) = \gamma_2(0) = \cdots = \gamma_m(0) = x$ and the sets $\{\gamma_j(t) : t \in (0, \delta]\}$, for $j \in \{1, \ldots, m\}$, are disjoint. If (E, d) is a geodesic space (having more than one point), any pair of distinct points is connected by a (possibly not unique) geodesic, and it is then immediate that every point is a 1-geodesic star. Our main result is the following theorem.

THEOREM 1. *Let $(\mathbf{m}_\infty, D)$ denote the Brownian sphere. For every integer $m \in \{1, 2, 3, 4\}$, let $\mathfrak{S}_m$ be the set of all m-geodesic stars in $(\mathbf{m}_\infty, D)$. Then the Hausdorff dimension of $\mathfrak{S}_m$ is a.s. equal to $5 - m$.*

The upper bound $\dim(\mathfrak{S}_m) \leq 5 - m$ has been obtained by Miller and Qian in [27, Theorem 1.4]. So the contribution of the present work is to prove the corresponding lower bound. We note that m-geodesic stars in the Brownian sphere were first discussed by Miermont [26, Definition 7], who conjectured that they exist for $m \leq 4$ but not for $m \geq 6$ (see the concluding remarks of [26]). In fact the non-existence of m-geodesic stars when $m \geq 6$ has been proved by Miller and Qian [27, Theorem 1.4].

Let us briefly comment on Theorem 1. The Brownian sphere is a geodesic space, and thus $\mathfrak{S}_1 = \mathbf{m}_\infty$, so that in the case $m = 1$ the result follows from the known fact [16] that $\dim(\mathbf{m}_\infty) = 4$. Next we may observe that any interior point of a geodesic is a 2-geodesic star, and therefore $\mathfrak{S}_2$ contains the set of all interior points of all geodesics. However, Miller and Qian [27, Corollary 1.3] proved that the Hausdorff dimension of the latter set is 1 (it is obviously greater than or equal to 1), thus confirming a conjecture of Angel, Kolesnik and

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Miermont [5]. Since $\dim(\mathfrak{S}_2) = 3$, this implies, at least informally, that typical 2-geodesic stars are not interior points of geodesics. Let us then consider $m = 3$. It is relatively easy to construct 3-geodesic stars in the Brownian sphere. Indeed, write x_* for the (first) distinguished point of $\mathbf{m}_\infty$ (see Section 2.5 below), and suppose that x and y are two points chosen independently according to the volume measure on $\mathbf{m}_\infty$. From the results of [17], the (a.s. unique) geodesics from x to x_* and from y to x_* coalesce before hitting x_*, and the point at which they coalesce is a 3-geodesic star. Again one expects that such points are not typical 3-geodesic stars. Finally, to the best of our knowledge, the existence of 4-geodesic stars had not been established before. We state an open question in the case $m = 5$.

Open problem. *Prove or disprove the existence of* 5*-geodesic stars in the Brownian map.*

If 5-geodesic stars do exist, [27, Theorem 1.4] implies that $\dim(\mathfrak{S}_5) = 0$.

Let us discuss the earlier work about geodesics in the Brownian sphere. The paper [17] provides a complete description of geodesics ending at the distinguished point x_* , by showing that all such geodesics must be "simple geodesics" (see Section 3.2 for the definition of a simple geodesic in a slightly different model). It follows from this description that any two geodesics ending at x_* must coalesce before hitting x_*. This is the so-called confluence of geodesics phenomenon, which implies that x_* is (a.s.) not a 2-geodesic star. These results still hold if x_* is replaced by a point chosen according to the volume measure of $\mathbf{m}_\infty$, by the symmetry properties of the Brownian sphere (see Section 2.6 below), and they imply the uniqueness of the geodesic between two points chosen independently according to the volume measure. A different approach to the latter property was given by Miermont [25], in the more general setting of scaling limits of random planar maps in arbitrary genus.

Miermont's approach [26] to the uniqueness of the Brownian sphere as the scaling limit of random quadrangulations makes heavy use of the notion of geodesic stars. We also note that both [26] and the alternative approach to the uniqueness of the Brownian sphere developed in [18] strongly rely on the characterization of geodesics to x_*.

The paper [5] by Angel, Kolesnik and Miermont goes further in the study of geodesics in the Brownian sphere. In particular, it is proved in [5] that 2-geodesic stars form a set of first Baire category. Moreover, [5] contains a thorough discussion of the so-called geodesic networks: for each pair (x, y) of distinct points in the Brownian map, the geodesic network between x and y is the union of all geodesics from x to y. If $y = x_*$ (more generally, if y is a "typical" point) the results of [17] show that the geodesic network consists of the union of at most 3 geodesics, but things may be much more complicated if x and y are both exceptional points. The paper [5] studies the possible "normal" geodesic networks (normality implies that there is a common point other than x and y to all geodesics between x and y), and obtains in particular that there is a dense set of pairs $(x, y) \in \mathbf{m}_\infty \times \mathbf{m}_\infty$ such that the geodesic network between x and y consists of 9 distinct geodesics — in that case, both x and y must be 3-geodesic stars. Even deeper results (without the normality assumption) are derived in the recent paper [27] of Miller and Qian, which shows that 9 is indeed the maximal number of geodesics between two points of $\mathbf{m}_\infty$, and moreover computes the Hausdorff dimension of the set of pairs (x, y) such that there are exactly j geodesics between x and y (see [27, Theorem 1.6]). As already mentioned, [27] also gives the upper bound $\dim(\mathfrak{S}_m) \leq 5 - m$. Both [5] and [27] make heavy use of strong forms of the confluence of geodesics phenomenon, see in particular [5, Proposition 12] and [27, Theorem 1.1]. It is worth pointing that certain analogs of the results of [5] and [27] have been derived in the related setting of Liouville quantum gravity surfaces in the very recent papers [10] and [14] (the confluence of geodesics phenomenon in that setting [12] played a major role in the proof of the uniqueness of the Liouville quantum gravity metric, see [13]).

Let us outline the main steps of the proof of the lower bound $\dim(\mathfrak{S}_m) \geq 5 - m$ when $m \geq 2$. The key ideas are very similar to those that have been used in the study of exceptional points of Brownian motion. It is convenient to deal with the so-called free Brownian sphere, which means that $\mathbf{m}_\infty$ is defined under the infinite Brownian snake excursion measure $\mathbb{N}_0$ (see Section 2.5 below). For every $\varepsilon \in (0,1)$ we introduce a set $\mathfrak{S}_m^\varepsilon$ of "ε-approximate" m-geodesic stars. A point x of $\mathbf{m}_\infty$ belongs to $\mathfrak{S}_m^\varepsilon$ if $1 < D(x_*, x) < 2$ and if there exist m geodesics to x that start at distance 1 from x and are disjoint up to the time when they arrive at distance ε from x. More precisely, we require for technical reasons that these geodesics start from the boundary of the hull of radius 1 centered at x relative to x_* (roughly speaking, this hull is obtained by filling in the holes of the ball of radius 1 centered at x, except for the one containing x_*, see Section 2.5). Write $\mathrm{Vol}(\cdot)$ for the volume measure on $\mathbf{m}_\infty$. Using the symmetry properties of the Brownian sphere, it is not hard to verify that

$$\mathbb{N}_0\Big(\mathrm{Vol}(\mathfrak{S}_m^\varepsilon)\Big) \geq c_m\, \varepsilon^{m-1}, \tag{1}$$

with a positive constant c_m independent of ε. Then, if $\delta \in (0,1)$, we rely on a two-point estimate to get the bound

$$\mathbb{N}_0\Bigg(\iint \mathbf{1}_{\mathfrak{S}_m^\varepsilon \times \mathfrak{S}_m^\varepsilon}(x,y)\, D(x,y)^{-(5-m-\delta)}\, \mathrm{Vol}(\mathrm{d}x)\mathrm{Vol}(\mathrm{d}y)\Bigg) \leq c_{\delta,m}\, \varepsilon^{2(m-1)}, \tag{2}$$

with a constant $c_{\delta,m}$ independent of ε. From (1) and (2), standard arguments show that, at least on a set of positive $\mathbb{N}_0$-measure, the volume measure restricted to $\mathfrak{S}_m^\varepsilon$ and scaled by the factor $\varepsilon^{-(m-1)}$ converges when ε tends to 0, along a suitable subsequence, to a limiting random measure μ satisfying

$$\iint D(x,y)^{-(5-m-\delta)}\, \mu(\mathrm{d}x)\mu(\mathrm{d}y) < \infty.$$

If we know that μ is supported on $\mathfrak{S}_m$, the classical Frostman lemma gives $\dim(\mathfrak{S}_m) \geq 5 - m - \delta$. However, it is not obvious that μ is supported on $\mathfrak{S}_m$, because, even if a sequence $(x_n)_{n\in\mathbb{N}}$ of ε_n-approximate m-geodesic stars (with $\varepsilon_n \to 0$) converges, it does not necessarily follow that the limit belongs to $\mathfrak{S}_m$. To overcome this difficulty, we need to modify the definition of $\mathfrak{S}_m^\varepsilon$ by imposing that the geodesics to x in this definition are not only disjoint but sufficiently far apart from each other. Another delicate point is to prove that the desired property holds $\mathbb{N}_0$-a.e. and not only on a set of positive $\mathbb{N}_0$-measure. As usual, we rely on a kind of zero-one law, which requires considering first the (scale invariant) Brownian plane and then using a strong coupling between the Brownian plane and the Brownian sphere.

The paper is organized as follows. Section 2 is devoted to a number of preliminaries, including the Brownian snake construction of the Brownian sphere as a measure metric space with two distinguished points denoted by x_* and x_0, and a discussion of the symmetry properties of the Brownian sphere, which roughly speaking say that x_* and x_0 play the same role as two points chosen independently according to the (normalized) volume measure. Section 3 starts with the construction of the random metric space corresponding to the hull of radius $r > 0$ centered at x_* relative to x_0, under $\mathbb{N}_0(\cdot \,|\, D(x_*, x_0) > r)$. This construction yields an explicit calculation of the probability that there are m geodesics from the boundary of the hull to x_* that stay disjoint until they hit the ball of radius ε centered at x_*. Then Theorem 8, which is a result of independent interest, shows that the hull of radius $r > 0$ centered at x_* and relative to x_0 is independent of its complement conditionally on its boundary size, and the complement itself is a Brownian disk — this is in fact an analog of a result proved in [22] for the Brownian plane. One then derives a two point-version saying that, under the conditional probability measure $\mathbb{N}_0(\cdot \,|\, D(x_*, x_0) > 2r)$, the hull of radius r centered at x_*

(relative to x_0) and the hull of radius r centered at x_0 (relative to x_*) are independent conditionally on their boundary sizes (Corollary 9). Section 4 is devoted to the proof of the version of (1) where the definition of $\mathfrak{S}^{\varepsilon}_m$ is modified as explained above to ensure that geodesics stay "sufficiently far apart" from each other. An important ingredient here is the notion of a slice, which roughly speaking separates two successive disjoint geodesics from the hull boundary to the ball of radius ε (slices also played a key role in the characterization of the distribution of Brownian disks in [6]). Section 5, which is the most technical part of the paper, uses the results of Section 3 to derive the key estimate (Lemma 15) that eventually leads to the bound (2). Section 6 then gives the proof of Theorem 1 along the lines of the preceding discussion. The Appendix contains the proofs of a couple of technical lemmas, including the strong coupling between the Brownian plane and the Brownian sphere that is used to justify the zero-one law argument.

We finally mention that Jason Miller and Wei Qian [28] have independently developed a different approach to Theorem 1. We believe that our general strategy and the intermediate steps of our proof are of independent interest and should prove useful to investigate other sets of exceptional points in the Brownian sphere. We hope to pursue this matter in the future.

2. Preliminaries.

2.1. *Measure metric spaces.* A (compact) measure metric space is a compact metric space (X,d) equipped with a finite Borel measure μ which is often called the volume measure. We write $\mathbb{M}$ for the set of all measure metric spaces, where two such spaces (X,d,μ) and (X',d',μ') are identified if there exists an isometry ϕ from X onto X' such that $\phi_*\mu=\mu'$.

For our purposes, it will be important to consider measure metric spaces given together with two distinguished closed subsets that we call the boundaries for reasons that will become clear later. We say that (X,d,μ,F_1,F_2) is a two-boundary measure metric space if (X,d,μ) is a measure metric space and if F_1 and F_2 are two closed subsets of X (the order between F_1 and F_2 is important). We write $\mathbb{M}^{bb}$ for the set of all two-boundary measure metric spaces modulo isometries (of course we now consider only isometries that preserve both the volume measures and the "boundaries").

The Gromov-Hausdorff-Prokhorov distance on $\mathbb{M}^{bb}$ is then defined by

$$\begin{aligned}&d_{\mathrm{GHP}}((X,d,\mu,F_1,F_2),(X',d',\mu',F_1',F_2'))\\&=\inf\Big\{d_{\mathrm{H}}^E(\phi(X),\phi'(X))\vee d_{\mathrm{H}}^E(\phi(F_1),\phi'(F_1'))\vee d_{\mathrm{H}}^E(\phi(F_2),\phi'(F_2'))\vee d_{\mathrm{P}}^E(\phi_*\mu,\phi'_*\mu')\Big\},\end{aligned}$$

where the infimum is over all isometric embeddings ϕ and ϕ' of X and X' into a compact metric space (E,d^E), d_{H}^E stands for the Hausdorff distance between compact subsets of E and d_{P}^E is the Prokhorov distance on the space of finite measures on E. Then, by an easy generalization of [2, Theorem 2.5], one verifies that d_{GHP} is a distance on $\mathbb{M}^{bb}$, and $(\mathbb{M}^{bb},d_{\mathrm{GHP}})$ is a Polish space.

We also let $\mathbb{M}^{\bullet b}$, resp. $\mathbb{M}^{\bullet\bullet}$, denote the closed subset of $\mathbb{M}^{bb}$ that consists of all (isometry classes of) two-boundary measure metric spaces (X,d,μ,F_1,F_2) such that F_1 is a singleton, resp. both F_1 and F_2 are singletons. Note that $\mathbb{M}^{\bullet\bullet}$ is just the space of two-pointed measure metric spaces as considered in [21, Section 2.1].

Remark. Gwynne and Miller [11] consider the closely related notion of a curve-decorated measure metric space.

2.2. *Snake trajectories.* We will use the formalism of snake trajectories as developed in [3]. First recall that a finite path w is a continuous mapping $\mathrm{w} : [0,\zeta] \longrightarrow \mathbb{R}$, where the number $\zeta = \zeta_{(\mathrm{w})} \geq 0$ is called the lifetime of w. We let $\mathcal{W}$ denote the space of all finite paths, which is a Polish space when equipped with the distance

$$d_{\mathcal{W}}(\mathrm{w},\mathrm{w}') = |\zeta_{(\mathrm{w})} - \zeta_{(\mathrm{w}')}| + \sup_{t\geq 0} |\mathrm{w}(t\wedge\zeta_{(\mathrm{w})}) - \mathrm{w}'(t\wedge\zeta_{(\mathrm{w}')})|.$$

The endpoint or tip of the path w is denoted by $\widehat{\mathrm{w}} = \mathrm{w}(\zeta_{(\mathrm{w})})$. For $x \in \mathbb{R}$, we set $\mathcal{W}_x = \{\mathrm{w} \in \mathcal{W} : \mathrm{w}(0) = x\}$. The trivial element of $\mathcal{W}_x$ with zero lifetime is identified with the point x of $\mathbb{R}$.

DEFINITION 2. *Let $x \in \mathbb{R}$. A snake trajectory with initial point x is a continuous mapping $s \mapsto \omega_s$ from $\mathbb{R}_+$ into $\mathcal{W}_x$ which satisfies the following two properties:*

(i) *We have $\omega_0 = x$ and the number $\sigma(\omega) := \sup\{s \geq 0 : \omega_s \neq x\}$, called the duration of the snake trajectory ω, is finite (by convention $\sigma(\omega) = 0$ if $\omega_s = x$ for every $s \geq 0$).*
(ii) (Snake property) *For every $0 \leq s \leq s'$, $\omega_s(t) = \omega_{s'}(t)$ for every $t \in [0, \min_{s\leq r\leq s'} \zeta_{(\omega_r)}]$.*

We will write $\mathcal{S}_x$ for the set of all snake trajectories with initial point x and $\mathcal{S} = \bigcup_{x\in\mathbb{R}} \mathcal{S}_x$ for the set of all snake trajectories. If $\omega \in \mathcal{S}$, we often write $W_s(\omega) = \omega_s$ and $\zeta_s(\omega) = \zeta_{(\omega_s)}$ for every $s \geq 0$. The set $\mathcal{S}$ is a Polish space for the distance $d_{\mathcal{S}}(\omega,\omega') = |\sigma(\omega) - \sigma(\omega')| + \sup_{s\geq 0} d_{\mathcal{W}}(W_s(\omega), W_s(\omega'))$. A snake trajectory ω is completely determined by the knowledge of the lifetime function $s \mapsto \zeta_s(\omega)$ and of the tip function $s \mapsto \widehat{W}_s(\omega)$: See [3, Proposition 8].

Let $\omega \in \mathcal{S}$ be a snake trajectory and $\sigma = \sigma(\omega)$. The lifetime function $s \mapsto \zeta_s(\omega)$ codes a compact $\mathbb{R}$-tree, which will be denoted by $\mathcal{T}_{(\omega)}$ and called the *genealogical tree* of the snake trajectory. This $\mathbb{R}$-tree is the quotient space $\mathcal{T}_{(\omega)} := [0,\sigma]/\sim$ of the interval $[0,\sigma]$ for the equivalence relation

$$s \sim s' \text{ if and only if } \zeta_s(\omega) = \zeta_{s'}(\omega) = \min_{s\wedge s'\leq r\leq s\vee s'} \zeta_r(\omega),$$

and $\mathcal{T}_{(\omega)}$ is equipped with the distance induced by

$$d_{(\omega)}(s,s') = \zeta_s(\omega) + \zeta_{s'}(\omega) - 2\min_{s\wedge s'\leq r\leq s\vee s'} \zeta_r(\omega).$$

(notice that $d_{(\omega)}(s,s') = 0$ if and only if $s \sim s'$). We write $p_{(\omega)} : [0,\sigma] \longrightarrow \mathcal{T}_{(\omega)}$ for the canonical projection. By convention, $\mathcal{T}_{(\omega)}$ is rooted at the point $\rho_{(\omega)} := p_{(\omega)}(0)$, and the volume measure on $\mathcal{T}_{(\omega)}$ is defined as the pushforward of Lebesgue measure on $[0,\sigma]$ under $p_{(\omega)}$. The mapping $s \mapsto p_{(\omega)}(s)$ may be interpreted as a (clockwise) *cyclic exploration* of $\mathcal{T}_{(\omega)}$.

It will be useful to define also intervals on the tree $\mathcal{T}_{(\omega)}$. For $s, s' \in [0,\sigma]$, we use the convention $[s,s'] = [s,\sigma] \cup [0,s']$ if $s > s'$ (and of course, $[s,s']$ is the usual interval if $s \leq s'$). If $a, b \in \mathcal{T}_{(\omega)}$ are distinct, we can find $s, s' \in [0,\sigma]$ in a unique way so that $p_{(\omega)}(s) = a$ and $p_{(\omega)}(s') = b$ and the interval $[s,s']$ is as small as possible, and we define $[a,b] := p_{(\omega)}([s,s'])$. Informally, $[a,b]$ is the set of all points that are visited when going from a to b in "clockwise order" around the tree. We also take $[a,a] = \{a\}$.

By property (ii) in the definition of a snake trajectory, the condition $p_{(\omega)}(s) = p_{(\omega)}(s')$ implies that $W_s(\omega) = W_{s'}(\omega)$. So the mapping $s \mapsto W_s(\omega)$ could be viewed as defined on the quotient space $\mathcal{T}_{(\omega)}$. For $a \in \mathcal{T}_{(\omega)}$, we set $\ell_a(\omega) := \widehat{W}_s(\omega)$ for any $s \in [0,\sigma]$ such that $a = p_{(\omega)}(s)$ (by the previous observation, this does not depend on the choice of s). We then interpret $\ell_a(\omega)$ as a label assigned to the point a of $\mathcal{T}_{(\omega)}$, and we observe that, if $p_{(\omega)}(s) = a$, the path $[0,\zeta_s]\ni t\mapsto W_s(t)$ records the labels along the line segment from $\rho_{(\omega)}$ to a in $\mathcal{T}_{(\omega)}$. We also note that the mapping $a\mapsto \ell_a(\omega)$ is continuous on $\mathcal{T}_{(\omega)}$. We will use the notation

$$W_*(\omega):=\min\{\ell_a(\omega): a\in\mathcal{T}_{(\omega)}\}=\min\{\widehat{W}_s(\omega):0\leq s\leq\sigma\}.$$

Let us introduce a truncation operation on snake trajectories. Let $x,y\in\mathbb{R}$ with $y<x$. If $\mathrm{w}\in\mathcal{W}_x$, we set $\tau_y(\mathrm{w}):=\inf\{t\geq 0:\mathrm{w}(t)=y\}$, with the usual convention $\inf\varnothing=\infty$. Then, if $\omega\in\mathcal{S}_x$, we set, for every $s\geq 0$,

$$\eta_s(\omega)=\inf\Big\{t\geq 0:\int_0^t \mathrm{d}u\,\mathbf{1}_{\{\zeta_{(\omega_u)}\leq\tau_y(\omega_u)\}}>s\Big\}.$$

Note that the condition $\zeta_{(\omega_u)}\leq\tau_y(\omega_u)$ holds if and only if $\tau_y(\omega_u)=\infty$ or $\tau_y(\omega_u)=\zeta_{(\omega_u)}$. Then, setting $\omega'_s=\omega_{\eta_s(\omega)}$ for every $s\geq 0$ defines an element ω' of $\mathcal{S}_x$, which will be denoted by $\mathrm{tr}_y(\omega)$ and called the truncation of ω at y (see [3, Proposition 10]). The effect of the time change $\eta_s(\omega)$ is to “eliminate” those paths ω_s that hit y and then survive for a positive amount of time. We can then also define the excursions of ω “below” level y. To this end, we let (α_j,β_j), $j\in J$, be the connected components of the open set

$$\{s\in[0,\sigma]:\tau_y(\omega_s)<\zeta_{(\omega_s)}\},$$

and notice that $\omega_{\alpha_j}=\omega_{\beta_j}$ for every $j\in J$. For every $j\in J$ we define a snake trajectory $\omega^j\in\mathcal{S}_0$ by setting

$$\omega^j_s(t):=\omega_{(\alpha_j+s)\wedge\beta_j}(\zeta_{(\omega_{\alpha_j})}+t)-y\;,\text{ for }0\leq t\leq\zeta_{(\omega^j_s)}:=\zeta_{(\omega_{(\alpha_j+s)\wedge\beta_j})}-\zeta_{(\omega_{\alpha_j})}\text{ and }s\geq 0.$$

We say that ω_j, $j\in J$ are the excursions of ω below level y.

We finally introduce the re-rooting operation on snake trajectories (see [3, Section 2.2]). Let $\omega\in\mathcal{S}_0$ and $r\in[0,\sigma(\omega)]$. Then $\omega^{[r]}$ is the snake trajectory in $\mathcal{S}_0$ such that $\sigma(\omega^{[r]})=\sigma(\omega)$ and for every $s\in[0,\sigma(\omega)]$,

$$\zeta_s(\omega^{[r]})=d_{(\omega)}(r,r\oplus s),$$
$$\widehat{W}_s(\omega^{[r]})=\widehat{W}_{r\oplus s}(\omega)-\widehat{W}_r(\omega),$$

where we use the notation $r\oplus s=r+s$ if $r+s\leq\sigma(\omega)$, and $r\oplus s=r+s-\sigma(\omega)$ otherwise. These prescriptions completely determine $\omega^{[r]}$. The genealogical tree $\mathcal{T}_{(\omega^{[r]})}$ may be interpreted as the tree $\mathcal{T}_{(\omega)}$ re-rooted at the vertex $p_{(\omega)}(r)$, and vertices of $\mathcal{T}_{(\omega^{[r]})}$ receive the same labels as in $\mathcal{T}_{(\omega)}$, shifted so that the label of the (new) root is still 0.

2.3. *The Brownian snake excursion measure on snake trajectories.* Let $x\in\mathbb{R}$. The Brownian snake excursion measure $\mathbb{N}_x$ is the σ-finite measure on $\mathcal{S}_x$ that satisfies the following two properties: Under $\mathbb{N}_x$,

(i) the distribution of the lifetime function $(\zeta_s)_{s\geq 0}$ is the Itô measure of positive excursions of linear Brownian motion, normalized so that, for every $\varepsilon>0$,

$$\mathbb{N}_x\Big(\sup_{s\geq 0}\zeta_s>\varepsilon\Big)=\frac{1}{2\varepsilon};$$

(ii) conditionally on $(\zeta_s)_{s\geq 0}$, the tip function $(\widehat{W}_s)_{s\geq 0}$ is a Gaussian process with mean x and covariance function

$$K(s,s'):=\min_{s\wedge s'\leq r\leq s\vee s'}\zeta_r.$$

Informally, the lifetime process $(\zeta_s)_{s\geq 0}$ evolves under $\mathbb{N}_x$ like a Brownian excursion, and conditionally on $(\zeta_s)_{s\geq 0}$, each path W_s is a linear Brownian path started from x with lifetime ζ_s, which is "erased" from its tip when ζ_s decreases and is "extended" when ζ_s increases. We note that the density of σ under $\mathbb{N}_0$ is $(2\sqrt{2\pi s^3})^{-1}$.

For every $y < x$, we have

$$\mathbb{N}_x(W_* \leq y) = \frac{3}{2(x-y)^2}. \tag{3}$$

See e.g. [15, Section VI.1] for a proof. Additionally, one can prove that $\mathbb{N}_x(\mathrm{d}\omega)$ a.e. there is a unique $s_* \in (0,\sigma)$ such that $\widehat{W}_{s_*} = W_*$ (see [23, Proposition 2.5]) and we set $a_* = p_{(\omega)}(s_*)$ so that $\ell_{a_*} = W_*$.

The following scaling property is often useful. For $\lambda > 0$, for every $\omega \in \mathcal{S}_x$, we define $\theta_\lambda(\omega) \in \mathcal{S}_{x\sqrt{\lambda}}$ by $\theta_\lambda(\omega) = \omega'$, with

$$\omega'_s(t) := \sqrt{\lambda}\,\omega_{s/\lambda^2}(t/\lambda)\,, \quad \text{for } s \geq 0 \text{ and } 0 \leq t \leq \zeta'_s := \lambda\zeta_{s/\lambda^2}.$$

Then it is a simple exercise to verify that the pushforward of $\mathbb{N}_x$ under θ_λ is $\lambda\,\mathbb{N}_{x\sqrt{\lambda}}$.

For every $t > 0$, we define the conditional probability measure $\mathbb{N}_0^{(t)} := \mathbb{N}_0(\cdot \,|\, \sigma = t)$. If $s \in [0,t]$, $\mathbb{N}_0^{(t)}$ is invariant under the re-rooting operation $\omega \mapsto \omega^{[s]}$ (see e.g. [23, Theorem 2.3]).

Exit measures. Let $x, y \in \mathbb{R}$, with $y < x$. Under the measure $\mathbb{N}_x$, one can make sense of the "quantity" of paths W_s that hit level y. One shows [21, Proposition 34] that the limit

$$L_t^y := \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^2}\int_0^t \mathrm{d}s\,\mathbf{1}_{\{\tau_y(W_s)=\infty,\,\widehat{W}_s<y+\varepsilon\}} \tag{4}$$

exists uniformly for $t \geq 0$, $\mathbb{N}_x$ a.e., and defines a continuous nondecreasing function, which is obviously constant on $[\sigma,\infty)$. The process $(L_t^y)_{t\geq 0}$ is called the exit local time at level y, and the exit measure $\mathcal{Z}_y$ is defined by $\mathcal{Z}_y = L_\infty^y = L_\sigma^y$. Then, $\mathbb{N}_x$ a.e., the topological support of the measure $\mathrm{d}L_t^y$ is exactly the set $\{s \in [0,\sigma] : \tau_y(W_s) = \zeta_s\}$, and, in particular, $\mathcal{Z}_y > 0$ if and only if one of the paths W_s hits y. The definition of $\mathcal{Z}_y$ is a special case of the theory of exit measures (see [15, Chapter V] for this general theory). Notice that the quantities in the right-hand side of (4) are functions of $\mathrm{tr}_y(\omega)$.

The special Markov property of the Brownian snake states that, under $\mathbb{N}_x(\mathrm{d}\omega)$ and conditionally on the truncation $\mathrm{tr}_y(\omega)$, the excursions of ω below y form a Poisson measure with intensity $\mathcal{Z}_y\mathbb{N}_0$ (see the appendix of [20] for a more precise statement).

2.4. *Decomposing the Brownian snake at its minimum.* Let $u > 0$. We will use the description of the conditional measure $\mathbb{N}_0(\cdot \,|\, W_* = -u)$, which can be found in [19]. Let (α_i, β_i), $i \in I$, be the connected components of $\{s \in [0, s_*] : \zeta_s > \min_{[s,s_*]} \zeta_r\}$, and similarly let (α_i, β_i), $i \in J$, be the connected components of $\{s \in [s_*, \sigma] : \zeta_s > \min_{[s_*,s]} \zeta_r\}$ (the indexing sets I and J are disjoint). Notice that $\zeta_{\alpha_i} = \zeta_{\beta_i}$ and $\widehat{W}_{\alpha_i} = \widehat{W}_{\beta_i} = W_{s_*}(\zeta_{\alpha_i})$ by the snake property. For every $i \in I \cup J$, we write $\omega^{(i)}$ for the unique snake trajectory such that

$$\zeta_s(\omega^{(i)}) = \zeta_{(\alpha_i+s)\wedge\beta_i} - \zeta_{\alpha_i}\,, \quad \widehat{W}_s(\omega^{(i)}) = \widehat{W}_{(\alpha_i+s)\wedge\beta_i}.$$

Then, under $\mathbb{N}_0(\cdot \,|\, W_* = -u)$, the finite path $(u + W_{s_*}(\zeta_{s_*} - t))_{0\leq t\leq\zeta_{s_*}}$ is distributed as a nine-dimensional Bessel process started at 0 and stopped at its last passage time at u, and, conditionally on W_{s_*}, the point measures

$$\sum_{i\in I} \delta_{(\zeta_{\alpha_i},\omega^{(i)})}(\mathrm{d}t\mathrm{d}\omega') \text{ and } \sum_{i\in J} \delta_{(\zeta_{\alpha_i},\omega^{(i)})}(\mathrm{d}t\mathrm{d}\omega')$$

are independent Poisson measures with intensity

$$2\,\mathbf{1}_{[0,\zeta_{s_*}]}(t)\,\mathbf{1}_{\{W_*(\omega')>-u\}}\,\mathrm{d}t\,\mathbb{N}_{W_{s_*}(t)}(\mathrm{d}\omega').$$

Informally, this corresponds to a spine decomposition of the labeled tree $\mathcal{T}_{(\omega)}$ under $\mathbb{N}_0(\cdot\,|\,W_*=-u)$: W_{s_*} records the labels along a spine isometric to $[0,\zeta_{s_*}]$, and each (labeled) tree $\mathcal{T}_{(\omega^{(i)})}$ for $i\in I$, resp. for $i\in J$, is grafted to the left side of the spine, resp. to the right side of the spine, at level ζ_{α_i}.

Let $r>0$. The preceding decomposition can be used to make sense of the exit measure $\mathcal{Z}_{W_*+r}$ under $\mathbb{N}_0(\cdot\,|\,W_*<-r)$. Notice that the exit measure $\mathcal{Z}_y$ was defined in the previous section for a *deterministic* level y, whereas here W_*+r is random. Nonetheless, we may argue under $\mathbb{N}_0(\cdot\,|\,W_*=-u)$ for every fixed $u>r$, and then define

$$\mathcal{Z}_{W_*+r}:=\sum_{\{i\in I\cup J:\,\zeta_{\alpha_i}>\tau_{-u+r}(W_{s_*})\}}\mathcal{Z}_{-u+r}(\omega^{(i)}) \tag{5}$$

with the notation above. Moreover, the special Markov property implies that the distribution of $\mathrm{tr}_{-u+r}(\omega)$ under $\mathbb{N}_0(\cdot\,|\,W_*=-u)$ is absolutely continuous with respect to its distribution under $\mathbb{N}_0(\cdot\,|\,W_*<-u+r)$, so that we can apply (4) (with $y=-u+r$ and $x=0$) under $\mathbb{N}_0(\cdot\,|\,W_*=-u)$, and get that

$$\mathcal{Z}_{W_*+r}=\lim_{\varepsilon\to0}\varepsilon^{-2}\int_0^\sigma \mathrm{d}s\,\mathbf{1}_{\{\tau_{W_*+r}(W_s)=\infty,\widehat{W}_s<W_*+r+\varepsilon\}}\,, \tag{6}$$

$\mathbb{N}_0(\cdot\,|\,W_*=-u)$ a.e. and thus also $\mathbb{N}_0(\cdot\,|\,W_*<-r)$ a.e.

2.5. *The Brownian sphere.* Let us argue under the excursion measure $\mathbb{N}_0(\mathrm{d}\omega)$, and recall the notation $\ell_a=\ell_a(\omega)$ for the label assigned to $a\in\mathcal{T}_{(\omega)}$, and the definition of intervals on $\mathcal{T}_{(\omega)}$. We define, for every $a,b\in\mathcal{T}_{(\omega)}$,

$$D^\circ(a,b):=\ell_a+\ell_b-2\max\Big(\min_{c\in[a,b]}\ell_c,\,\min_{c\in[b,a]}\ell_c\Big). \tag{7}$$

We record two easy but important properties of D°. First, for every $a,b\in\mathcal{T}_\zeta$,

$$D^\circ(a,b)\geq|\ell_a-\ell_b|. \tag{8}$$

Then, recalling that a_* is the unique point such that $\ell_{a_*}=W_*$, we have for every $a\in\mathcal{T}_\zeta$,

$$D^\circ(a_*,a)=\ell_a-\ell_{a_*}. \tag{9}$$

We let $D(a,b)$ be the largest symmetric function of the pair (a,b) that is bounded above by $D^\circ(a,b)$ and satisfies the triangle inequality: For every $a,b\in\mathcal{T}_\zeta$,

$$D(a,b)=\inf\Big\{\sum_{i=1}^k D^\circ(a_{i-1},a_i)\Big\}, \tag{10}$$

where the infimum is over all choices of the integer $k\geq1$ and of the elements $a_0,a_1,\dots,a_k$ of $\mathcal{T}_{(\omega)}$ such that $a_0=a$ and $a_k=b$. We note that D is a pseudo-metric on $\mathcal{T}_{(\omega)}$, and we let $\mathbf{m}_\infty(\omega):=\mathcal{T}_{(\omega)}/\{D=0\}$ be the associated quotient space (that is, the quotient space of $\mathcal{T}_{(\omega)}$ for the equivalence relation $a\approx b$ if and only if $D(a,b)=0$), which is equipped with the distance induced by D, for which we keep the same notation D. Then $(\mathbf{m}_\infty,D)$ is a compact metric space and also a geodesic space. The canonical projection from $\mathcal{T}_\zeta$ onto $\mathbf{m}_\infty$ is denoted by Π, and the volume measure Vol on $\mathbf{m}_\infty$ is defined as the pushforward of the volume measure on $\mathcal{T}_{(\omega)}$ under Π. Note that the total mass of Vol is σ.

We view $(\mathbf{m}_\infty,D,\mathrm{Vol})$ as a random two-pointed measure metric space, or equivalently as a random variable with values in the space $\mathbb{M}^{\bullet\bullet}$ of Section 2.1: the first distinguished point of

$\mathbf{m}_\infty$ is $x_* = \Pi(a_*)$ and the second distinguished point is $x_0 := \Pi(\rho_{(\omega)})$. As a consequence of (9), we have

$$D(x_*, x_0) = -W_*.$$

The free Brownian sphere is the (two-pointed measure) metric space $\mathbf{m}_\infty$ under the measure $\mathbb{N}_0$. We note that $\mathbf{m}_\infty$ is a geodesic space. It makes sense, and is also of interest, to consider $\mathbf{m}_\infty$ under conditional measures. The space $\mathbf{m}_\infty$ under the probability measure $\mathbb{N}_0^{(1)} := \mathbb{N}_0(\cdot \,|\, \sigma = 1)$ is the standard Brownian sphere (or Brownian map). For every $r > 0$, we will also consider $\mathbf{m}_\infty$ under the conditional measure $\mathbb{N}_0^{[r]} := \mathbb{N}_0(\cdot \,|\, W_* < -r)$. This corresponds to conditioning the free Brownian sphere on the event that the distance between the two distinguished points is greater than r.

The metric space $\mathbf{m}_\infty$ is homeomorphic to the usual two-dimensional sphere, $\mathbb{N}_0$ a.e. (or a.s. for any of the conditional measures introduced above).

For $x \in \mathbf{m}_\infty$ and $r > 0$, let $B_r(x)$ denote the closed ball of radius r centered at x in $\mathbf{m}_\infty$. If x and y are distinct points of $\mathbf{m}_\infty$ and $r \in (0, D(x,y))$, the hull $B_r^{\bullet(y)}(x)$ is the closed subset of $\mathbf{m}_\infty$ such that $\mathbf{m}_\infty \backslash B_r^{\bullet(y)}(x)$ is the connected component of the complement of $B_r(x)$ that contains y. We say that $B_r^{\bullet(y)}(x)$ is the hull of radius r centered at x relative to y. Obviously $B_r(x) \subset B_r^{\bullet(y)}(x)$, and every point of the topological boundary $\partial B_r^{\bullet(y)}(x)$ is at distance r from x.

The following fact known as the *cactus bound* [17, Proposition 3.1] is useful to study hulls centered at x_*. Let $x = \Pi(a)$ and $y = \Pi(b)$ be two points of $\mathbf{m}_\infty$, and let $(\gamma(t))_{0 \le t \le 1}$ be a continuous path in $\mathbf{m}_\infty$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Then,

$$\min_{0 \le t \le 1} D(x_*, \gamma(t)) \le \min_{c \in [\![a,b]\!]} \ell_c - W_*, \tag{11}$$

where $[\![a,b]\!]$ denotes the line segment between a and b in $\mathcal{T}_{(\omega)}$.

2.6. *Symmetry properties.* As mentioned above, $\mathbf{m}_\infty$ is viewed as a two-pointed measure metric space, and in this section we will write $(\mathbf{m}_\infty, x_*, x_0)$ to make it explicit that x_* and x_0 are the distinguished points. Our goal here is to observe that x_* and x_0 can be replaced by points chosen uniformly according to the volume measure without changing the distribution of $(\mathbf{m}_\infty, x_*, x_0)$. This will be very important for our applications.

PROPOSITION 3. *Let F be a nonnegative measurable function on the space $\mathbb{M}^{\bullet\bullet}$. Then,*

$$\mathbb{N}_0(F(\mathbf{m}_\infty, x_*, x_0)) = \mathbb{N}_0\Big(\int\!\!\int \frac{\mathrm{Vol}(\mathrm{d}x)}{\sigma}\,\frac{\mathrm{Vol}(\mathrm{d}y)}{\sigma}\, F(\mathbf{m}_\infty, x, y)\Big).$$

The same identity holds if $\mathbb{N}_0$ is replaced by $\mathbb{N}_0^{(s)} = \mathbb{N}_0(\cdot \,|\, \sigma = s)$, for any $s > 0$.

PROOF. It is enough to treat the case of $\mathbb{N}_0^{(s)}$. We fix $s > 0$ and recall that, for every $t \in [0, s]$, the measure $\mathbb{N}_0^{(s)}$ is invariant under the re-rooting operation $\omega \mapsto \omega^{[t]}$. On the other hand, it is easy to verify that the measure metric space $\mathbf{m}_\infty$ is left unchanged if ω is replaced by $\omega^{[t]}$, and that the first distinguished point also remains the same (the minimal label is attained at the "same" point of $\mathcal{T}_{(\omega)}$ and $\mathcal{T}_{(\omega^{[t]})}$). However, the second distinguished point x_0 is replaced by $\Pi(p_{(\omega)}(t))$. It follows from these considerations and the definition of the volume measure that

$$\mathbb{N}_0^{(s)}(F(\mathbf{m}_\infty, x_*, x_0)) = \mathbb{N}_0^{(s)}\Big(\frac{1}{s}\int \mathrm{Vol}(\mathrm{d}y)\, F(\mathbf{m}_\infty, x_*, y)\Big).$$

Then an application of [17, Theorem 8.1] shows that the right-hand side is also equal to

$$\mathbb{N}_0^{(s)}\Big(\iint \frac{\mathrm{Vol}(\mathrm{d}x)}{s}\,\frac{\mathrm{Vol}(\mathrm{d}y)}{s}\,F(\mathbf{m}_\infty,x,y)\Big).$$

To be precise, [17, Theorem 8.1] considers $\mathbf{m}_\infty$ as a metric space, and so we need a slight extension of this result, when $\mathbf{m}_\infty$ is viewed as a *measure* metric space. This extension is obtained by the very same arguments as in [17], using the convergence of rescaled quadrangulations to the Brownian sphere in the Gromov-Hausdorff-Prokhorov sense, as stated in [21, Theorem 7]. This completes the proof. □

Let us mention some immediate consequences of Proposition 3. First, we have

$$\mathbb{N}_0(F(\mathbf{m}_\infty,x_*,x_0)) = \mathbb{N}_0(F(\mathbf{m}_\infty,x_0,x_*)). \tag{12}$$

Since the conditioning defining $\mathbb{N}_0^{[r]} = \mathbb{N}_0(\cdot \mid D(x_0,x_*)>r)$ depends on x_* and x_0 in a symmetric manner, we get for every $r>0$,

$$\mathbb{N}_0^{[r]}(F(\mathbf{m}_\infty,x_*,x_0)) = \mathbb{N}_0^{[r]}(F(\mathbf{m}_\infty,x_0,x_*)).$$

The following consequence of Proposition 3 will also be useful. If F is now defined on the space of three-pointed compact metric spaces (see e.g. [21, Section 2.1]), we have

$$\mathbb{N}_0\Big(\int \frac{\mathrm{Vol}(\mathrm{d}x)}{\sigma}F(\mathbf{m}_\infty,x_*,x_0,x))\Big) = \mathbb{N}_0\Big(\iiint \frac{\mathrm{Vol}(\mathrm{d}x)}{\sigma}\,\frac{\mathrm{Vol}(\mathrm{d}y)}{\sigma}\,\frac{\mathrm{Vol}(\mathrm{d}z)}{\sigma}\,F(\mathbf{m}_\infty,x,y,z)\Big). \tag{13}$$

2.7. *Moments of exit measures and volumes of balls.* We start with a lemma providing bounds on moments of the volume of balls centered at x_*.

LEMMA 4. *Let $p\geq 1$ be an integer. There exists a constant C_p such that, for every $r>0$,*

$$\mathbb{N}_0\Big(\mathrm{Vol}(B_r(x_*))^p\Big) = C_p\,r^{4p-2}.$$

Consequently, for every integer $p\geq 1$, and every $\eta\in(0,1)$, there exists a constant $C_{p,\eta}$ such that, for every $r\in(0,1)$,

$$\mathbb{N}_0^{[1]}\Big(\mathrm{Vol}(B_r(x_*))^p\Big) \leq C_{p,\eta}\,r^{4p-\eta}.$$

PROOF. Using (9) and the scaling property of the measures $\mathbb{N}_x$, we get

$$\mathbb{N}_0\Big(\mathrm{Vol}(B_r(x_*))^p\Big) = \mathbb{N}_0\Big(\Big(\int_0^\sigma \mathrm{d}s\,\mathbf{1}_{\{\widehat{W}_s - W_*\leq r\}}\Big)^p\Big) = r^{4p-2}\,\mathbb{N}_0\Big(\Big(\int_0^\sigma \mathrm{d}s\,\mathbf{1}_{\{\widehat{W}_s - W_*\leq 1\}}\Big)^p\Big).$$

So we only need to verify that

$$\mathbb{N}_0\Big(\Big(\int_0^\sigma \mathrm{d}s\,\mathbf{1}_{\{\widehat{W}_s - W_*\leq 1\}}\Big)^p\Big) < \infty. \tag{14}$$

To this end, we note that, for every $\delta\in(0,1)$, [17, Lemma 6.1] gives

$$\mathbb{N}_0^{(1)}\Big(\Big(\int_0^1 \mathrm{d}s\,\mathbf{1}_{\{\widehat{W}_s - W_*\leq r\}}\Big)^p\Big) \leq c_{p,\delta}\,r^{4p-\delta},$$

with a constant $c_{p,\delta}$ independent of $r>0$. By scaling, we get for $t\geq 1$,

$$\mathbb{N}_0^{(t)}\Big(\Big(\int_0^t \mathrm{d}s\,\mathbf{1}_{\{\widehat{W}_s - W_*\leq 1\}}\Big)^p\Big) = t^p\,\mathbb{N}_0^{(1)}\Big(\Big(\int_0^1 \mathrm{d}s\,\mathbf{1}_{\{\widehat{W}_s - W_*\leq t^{-1/4}\}}\Big)^p\Big) \leq c_{p,\delta}\,t^{\delta/4},$$

and (14) follows by writing

$$\mathbb{N}_0\Big(\Big(\int_0^\sigma \mathrm{d}s\,\mathbf{1}_{\{\widehat{W}_s - W_* \le 1\}}\Big)^p\Big) = \int_0^\infty \frac{\mathrm{d}t}{2\sqrt{2\pi t^3}}\mathbb{N}_0^{(t)}\Big(\Big(\int_0^t \mathrm{d}s\,\mathbf{1}_{\{\widehat{W}_s - W_* \le 1\}}\Big)^p\Big)$$
$$\le \int_0^1 \frac{\mathrm{d}t}{2\sqrt{2\pi t^3}}\,t^p + \int_1^\infty \frac{\mathrm{d}t}{2\sqrt{2\pi t^3}} c_{p,\delta}\,t^{\delta/4}.$$

For the second assertion, let $q \ge 2$ be an integer. Since $\mathbb{N}_0(W_* < -1) = \frac{3}{2}$, we have

$$\mathbb{N}_0^{[1]}\Big(\mathrm{Vol}(B_r(x_*))^p\Big) \le \Big(\mathbb{N}_0^{[1]}\Big(\mathrm{Vol}(B_r(x_*))^{qp}\Big)\Big)^{1/q} \le \Big(\frac{2}{3}\mathbb{N}_0\Big(\mathrm{Vol}(B_r(x_*))^{qp}\Big)\Big)^{1/q}$$
$$\le (C_{qp})^{1/q}\,r^{4p-2/q},$$

which gives the desired result by taking q such that $2/q < \eta$. □

We will need bounds on the moments of $\mathcal{Z}_{W_*+r}$ under $\mathbb{N}_0^{[r]} = \mathbb{N}_0(\cdot \mid W_* < -r)$, and to this end we will use a coupling with the Brownian plane. Let $\mathcal{P}$ stand for the Brownian plane as defined in [8, 9]. According to [9, Proposition 1.1], we can make sense of a quantity Z_r which corresponds to the boundary size of the hull of radius r in $\mathcal{P}$ (see the introduction of [9] for more details).

LEMMA 5. *Let $r > 0$ and $u \ge r$. The random variable $\mathcal{Z}_{W_*+r}$ under $\mathbb{N}_0^{[u]}$ is stochastically dominated by Z_r.*

This lemma is a straightforward consequence of a coupling between the Brownian plane $\mathcal{P}$ and the Brownian sphere $\mathbf{m}_\infty$ under $\mathbb{N}_0(\cdot \mid W_* = -u)$, which relies on the spine decomposition of Section 2.4 and is described in the proof of another technical lemma in Appendix B. For this reason, we also postpone the proof of Lemma 5 to Appendix C.

According to [9, Proposition 1.2], the variable Z_r follows the Gamma distribution with parameter $3/2$ and mean r^2. In particular, for every $p \ge 1$, there exists a constant c_p such that $\mathbf{E}[(Z_r)^p] = c_p\,r^{2p}$, and then Lemma 5 implies that, for every $0 < r \le u$,

$$\mathbb{N}_0^{[u]}\Big((\mathcal{Z}_{W_*+r})^p\Big) \le c_p\,r^{2p}. \tag{15}$$

3. Hulls.

3.1. *The construction of hulls.* Let us start by defining the random compact metric space which will correspond to the conditional distribution of a hull of radius r in the free Brownian sphere, given its boundary size. Throughout this section, $r > 0$ and $z > 0$ are fixed. We let

$$\mathcal{N} = \sum_{i \in I} \delta_{(t_i,\omega_i)}$$

be a Poisson point measure on $[0, z] \times \mathcal{S}_0$ with intensity

$$\mathrm{d}t \otimes \mathbb{N}_0(\mathrm{d}\omega \cap \{W_* > -r\}).$$

Futhermore, let ω_* be a random snake trajectory distributed according to $\mathbb{N}_0(\cdot \mid W_* = -r)$, and let U_* be uniformly distributed over $[0, z]$. We assume that ω_*, U_* and $\mathcal{N}$ are independent. We also set

$$\Sigma := \sigma(\omega_*) + \sum_{i \in I} \sigma(\omega_i).$$

We let $\mathbf{H}$ be derived from the disjoint union

$$[0,z] \cup \Big(\bigcup_{i\in I} \mathcal{T}_{(\omega_i)}\Big) \cup \mathcal{T}_{(\omega_*)}$$

by identifying 0 with z, the root of $\mathcal{T}_{(\omega_*)}$ with the point U_* of $[0,z]$ and, for every $i \in I$, the root of $\mathcal{T}_{(\omega_i)}$ with the point t_i of $[0,z]$. The volume measure on $\mathbf{H}$ is defined by saying that it puts no mass on $[0,z]$ and that its restriction to each of the trees $\mathcal{T}_{(\omega_i)}$, resp. to $\mathcal{T}_{(\omega_*)}$, is the volume measure on this tree.

We then assign labels $(\Lambda_a)_{a\in\mathbf{H}}$ to the points of $\mathbf{H}$. We take $\Lambda_a = 0$ if $a \in [0,z]$, and, for every $a \in \mathcal{T}_{(\omega_i)}$, resp. $a \in \mathcal{T}_{(\omega_*)}$, we take $\Lambda_a = \ell_a(\omega_i)$, resp. $\Lambda_a = \ell_a(\omega_*)$. We let $b_* \in \mathcal{T}_{(\omega_*)}$ be the unique point of $\mathbf{H}$ with label $-r$.

We finally define a cyclic exploration $(\mathcal{E}_s)_{s\in[0,\Sigma]}$ of $\mathbf{H}$. Informally, this cyclic exploration is obtained by concatenating the cyclic explorations of the $\mathcal{T}_{(\omega_i)}$'s, and of $\mathcal{T}_{(\omega_*)}$, in the order prescribed by the reals t_i, and U_*. To get a more precise description, set

$$A_u = \sum_{i\in I, t_i \le u} \sigma(\omega_i) + \mathbf{1}_{\{u\ge U_*\}}\sigma(\omega_*)$$

for every $u \in [0,z]$, and use the notation A_{u-} for the left limit at u. Then:

- If $0 \le s \le A_{U_*-}$ or $A_{U_*} \le s \le \Sigma$, either there is a (unique) $i \in I$ such that $A_{t_i-} \le s \le A_{t_i}$, and we set $\mathcal{E}_s = p_{(\omega_i)}(s - A_{t_i-})$, or there is no such i and we set $\mathcal{E}_s = \sup\{t_i : A_{t_i} \le s\} \in [0,z]$.
- If $A_{U_*-} < s < A_{U_*}$, we set $\mathcal{E}_s = p_{(\omega_*)}(s - A_{U_*-})$.

There is a unique $s_* \in [0,\Sigma]$ such that $\mathcal{E}_{s_*} = b_*$.

If $s, s' \in [0,\Sigma]$ and $s < s'$, we make the convention that $[s',s] = [s',\Sigma] \cup [0,s]$ (and of course $[s,s']$ is the usual interval). Then, for every $a, b \in \mathbf{H}$, we can find $s, s' \in [0,\Sigma]$ such that $\mathcal{E}_s = a$, $\mathcal{E}_{s'} = b$ and $[s,s']$ is as small as possible (note that we may have $s' < s$). We let the interval $[a,b]$ of $\mathbf{H}$ be defined by $[a,b] := \{\mathcal{E}_r : r \in [s,s']\}$. We then define, for every $a, b \in \mathbf{H}$,

$$D^\circ_{\mathbf{H}}(a,b) := \Lambda_a + \Lambda_b - 2\max\Big(\inf_{c\in[a,b]} \Lambda_c, \inf_{c\in[b,a]} \Lambda_c\Big),$$

and

$$D_{\mathbf{H}}(a,b) = \inf_{a_0=a,a_1,\dots,a_{k-1},a_k=b} \sum_{i=1}^{k} D^\circ_{\mathbf{H}}(a_{i-1},a_i). \tag{16}$$

Then, almost surely for every $a, b \in \mathbf{H}$, the property $D_{\mathbf{H}}(a,b) = 0$ holds if and only if $D^\circ_{\mathbf{H}}(a,b) = 0$. This is derived from the analogous property for the Brownian map, which is proved in [16]. The bound $D^\circ_{\mathbf{H}}(a,b) \ge |\Lambda_a - \Lambda_b|$ is immediate from the definition, and it follows that we have also $D_{\mathbf{H}}(a,b) \ge |\Lambda_a - \Lambda_b|$.

The mapping $(a,b) \mapsto D_{\mathbf{H}}(a,b)$ is a pseudo-metric on $\mathbf{H}$. We writte $\mathfrak{H} := \mathbf{H}/\{D_{\mathbf{H}} = 0\}$ for the associated quotient space, which is equipped with the metric (induced by) $D_{\mathbf{H}}$. We write $\Pi_{\mathfrak{H}}$ for the canonical projection from $\mathbf{H}$ onto $\mathfrak{H}$. The restriction of $\Pi_{\mathfrak{H}}$ to $[0,z)$ is continuous and one-to-one, and its range is a simple loop denoted by $\partial\mathfrak{H} = \Pi_{\mathfrak{H}}([0,z))$ (one proves that $\mathfrak{H}$ is homeomorphic to the closed unit disk, and via such a homeomorphism $\partial\mathfrak{H}$ indeed corresponds to the unit circle). The volume measure on $\mathfrak{H}$ is the pushforward of the volume measure on $\mathbf{H}$ under $\Pi_{\mathfrak{H}}$. We view $\mathfrak{H}$ as a random variable with values in the space $\mathbb{M}^{\bullet b}$ of Section 2.1: the distinguished point is $\Pi_{\mathfrak{H}}(b_*)$, and the distinguished "boundary" is $\partial\mathfrak{H}$. Without risk of confusion, we identify b_* and $\Pi_{\mathfrak{H}}(b_*)$ in what follows.

We call the random two-boundary measure metric space $\mathfrak{H}$ the **standard hull of radius r and perimeter** z. This terminology will be justified below by relations with the Brownian sphere. If $x \in \mathfrak{H}$, we can set $\Lambda_x = \Lambda_a$, for a such that $\Pi_{\mathfrak{H}}(a) = x$ (this does not depend on the choice of a) and it easily follows from the definitions that

$$D_{\mathbf{H}}(x, b_*) = \Lambda_x + r \tag{17}$$

for every $x \in \mathfrak{H}$. In particular, all points of $\partial\mathfrak{H}$ are at distance r from b_*.

Let us turn to geodesics in $\mathfrak{H}$. More precisely, we are interested in geodesics between an arbitrary point of $\mathfrak{H}$ and b_*. Let $x \in \mathfrak{H}$, and let $a \in \mathbf{H}$ such that $\Pi_{\mathfrak{H}}(a) = x$, and $s \in [0, \Sigma]$ such that $\mathcal{E}_s = a$. Consider first the case where $s \in [0, s_*]$. We then set, for every $t \in [0, \Lambda_a + r]$,

$$\gamma_s(t) := \Pi_{\mathfrak{H}}\Big(\mathcal{E}_{\inf\{u \geq s : \Lambda_{\mathcal{E}_u} = \Lambda_a - t\}}\Big).$$

If $s \in [s_*, \Sigma]$, we define similarly, for every $t \in [0, \Lambda_a + r]$,

$$\gamma_s(t) := \Pi_{\mathfrak{H}}\Big(\mathcal{E}_{\sup\{u \leq s : \Lambda_{\mathcal{E}_u} = \Lambda_a - t\}}\Big).$$

Then, using (17) and the bound $D_{\mathbf{H}}(a, b) \geq |\Lambda_a - \Lambda_b|$, it is straightforward to verify that $(\gamma_s(t))_{0 \leq t \leq \Lambda_a + r}$ is a geodesic from x to b_* in $\mathfrak{H}$. Such a geodesic is called a *simple geodesic*.

PROPOSITION 6. *All geodesics in $\mathfrak{H}$ that end at b_* are simple geodesics.*

The analog of this result for the Brownian sphere is proved in [17]. The proposition can be derived from this analog by using the relations with the Brownian sphere that are discussed below.

PROPOSITION 7. *Let $\varepsilon \in (0, r)$. Define an integer-valued random variable N_ε by saying that $N_\varepsilon \geq k$ if and only if there exist k geodesics $\phi_1, \phi_2, \ldots, \phi_k$ from $\partial\mathfrak{H}$ to b_* such that the sets $\{\phi_1(t) : 0 \leq t \leq r - \varepsilon\}$, $\{\phi_2(t) : 0 \leq t \leq r - \varepsilon\}, \ldots, \{\phi_k(t) : 0 \leq t \leq r - \varepsilon\}$ are disjoint. Then $N_\varepsilon - 1$ follow a Poisson distribution with parameter*

$$\frac{3z}{2}\Big((r - \varepsilon)^{-2} - r^{-2})\Big).$$

PROOF. Let $u, v \in [0, z]$, and $s, s' \in [0, \Sigma]$ such that $\mathcal{E}_s = u$ and $\mathcal{E}_{s'} = v$. Consider the simple geodesics γ_s and $\gamma_{s'}$ as defined above. Then it follows from this definition (using also the fact that $D_{\mathbf{H}}(a, b) = 0$ if and only if $D^{\circ}_{\mathbf{H}}(a, b) = 0$) that the sets $\{\gamma_s(t) : 0 \leq t \leq r - \varepsilon\}$ and $\{\gamma_{s'}(t) : 0 \leq t \leq r - \varepsilon\}$ are disjoint if and only if we have both

$$\min_{t \in [s, s']} \Lambda_{\mathcal{E}_t} < -r + \varepsilon \ \text{ and } \ \min_{t \in [s', s]} \Lambda_{\mathcal{E}_t} < -r + \varepsilon.$$

Using also Proposition 6, we get that

$$N_\varepsilon - 1 = \#\{i \in I : W_*(\omega_i) < -r + \varepsilon\}. \tag{18}$$

which follows a Poisson distribution with parameter

$$z\,\mathbb{N}_0(-r < W_* < -r + \varepsilon) = \frac{3z}{2}\Big((r - \varepsilon)^{-2} - r^{-2})\Big).$$

This completes the proof. $\square$

3.2. *Hulls in the Brownian sphere.* We now consider the free Brownian sphere $(\mathbf{m}_\infty, D)$, which is defined under the measure $\mathbb{N}_0$. Recall that the two distinguished points of $\mathbf{m}_\infty$ are x_* and x_0.

We will be interested in hulls centered at x_* relative to x_0. We write $B^\bullet_r(x_*) = B^{\bullet(x_0)}_r(x_*)$ to simplify notation: this hull is defined on the event where $D(x_0, x_*) = -W_* > r$. We notice the following useful fact, which is obtained from the cactus bound (11) in a way similar to the proof of formulas (16) and (17) in [9]. A point $x = \Pi(p_{(\omega)}(s))$, for $s \in [0, \sigma]$, belongs to $B^\bullet_r(x_*)$ if and only if $\tau_{W_*+r}(W_s) \leq \zeta_s$, and to the interior of $B^\bullet_r(x_*)$ if and only if $\tau_{W_*+r}(W_s) < \zeta_s$.

The exit measure $\mathcal{Z}_{W_*+r}$, which was introduced in Section 2.4 and is also defined on the event $\{W_* < -r\}$, can be interpreted as the boundary size of $B^\bullet_r(x_*)$. This interpretation is justified by the following approximation, which is a reformulation of (6) using the preceding observations,

$$\mathcal{Z}_{W_*+r} = \lim_{\varepsilon \to 0} \varepsilon^{-2} \mathrm{Vol}((B^\bullet_r(x_*))^c \cap B_{r+\varepsilon}(x_*)) \text{ , a.e. on } \{D(x_*, x_0) > r\}. \tag{19}$$

If O is a connected open subset of $\mathbf{m}_\infty$, the intrinsic distance D^O_{int} on O is defined as follows. For $x, y \in O$, $D^O_{\mathrm{int}}(x, y)$ is the infimum of the lengths of paths connecting x and y and staying inside O.

Recall that $\mathbb{N}^{[r]}_0 = \mathbb{N}_0(\cdot \,|\, D(x_*, x_0) > r)$. We write $\overline{\mathbf{m}_\infty \backslash B^\bullet_r(x_*)}$ for the closure of $\mathbf{m}_\infty \backslash B^\bullet_r(x_*)$.

THEOREM 8. *With $\mathbb{N}^{[r]}_0$-probability one, the intrinsic distance on the interior of $B^\bullet_r(x_*)$ has a continuous extension to $B^\bullet_r(x_*)$, which is a metric on $B^\bullet_r(x_*)$, and similarly the intrinsic distance on $\mathbf{m}_\infty \backslash B^\bullet_r(x_*)$ has a continuous extension to $\overline{\mathbf{m}_\infty \backslash B^\bullet_r(x_*)}$, which is a metric on this space. Consider both $B^\bullet_r(x_*)$ and $\overline{\mathbf{m}_\infty \backslash B^\bullet_r(x_*)}$ as metric spaces for these (extended) intrinsic metrics. The metric space $B^\bullet_r(x_*)$ equipped with the restriction of the volume measure on $\mathbf{m}_\infty$, with the distinguished point x_* and with the distinguished boundary $\partial B^\bullet_r(x_*)$ is a random element of $\mathbb{M}^{\bullet b}$ and the same holds for the metric space $\overline{\mathbf{m}_\infty \backslash B^\bullet_r(x_*)}$ equipped with the restriction of the volume measure, with the distinguished point x_0 and with the distinguished boundary $\partial B^\bullet_r(x_*)$. Then, for any nonnegative measurable functions F and G defined on $\mathbb{M}^{\bullet b}$, for every $z > 0$, we have*

$$\mathbb{N}^{[r]}_0\Big(F(B^\bullet_r(x_*))\, G\big(\overline{\mathbf{m}_\infty \backslash B^\bullet_r(x_*)}\big) \,\Big|\, \mathcal{Z}_{W_*+r} = z \Big) = \mathbb{E}[F(\mathfrak{H}_{r,z})]\, \mathbb{E}[G(\mathbb{D}^\bullet_z)],$$

where $\mathfrak{H}_{r,z}$ stands for the standard hull of radius r and perimeter z, and $\mathbb{D}^\bullet_z$ denotes a free pointed Brownian disk of perimeter z.

Remarks. (i) The theorem implies in particular that $B^\bullet_r(x_*)$ and $\overline{\mathbf{m}_\infty \backslash B^\bullet_r(x_*)}$ are independent under $\mathbb{N}^{[r]}_0$ conditionally on $\mathcal{Z}_{W_*+r}$. If now we take $r' > r$, it remains true that $B^\bullet_r(x_*)$ and $\overline{\mathbf{m}_\infty \backslash B^\bullet_r(x_*)}$ are independent under $\mathbb{N}^{[r']}_0$ conditionally on $\mathcal{Z}_{W_*+r}$. The point is that, if one already knows that $D(x_*, x_0) > r$, the event $D(x_*, x_0) > r'$ occurs if and only if the distance from the distinguished point of $\overline{\mathbf{m}_\infty \backslash B^\bullet_r(x_*)}$ to the boundary is greater than $r' - r$, which only depends on $\overline{\mathbf{m}_\infty \backslash B^\bullet_r(x_*)}$.

(ii) The free pointed Brownian disk $\mathbb{D}^\bullet_z$ in the theorem is also viewed as a random variable with values in $\mathbb{M}^{\bullet b}$ (the "boundary" is of course the usual boundary $\partial \mathbb{D}^\bullet_z$, see e.g. [6]).

PROOF. This is very similar to the proof of Theorems 29 and 31 of [22], which give the analogous statements for the Brownian plane (then $\mathbb{D}^\bullet_z$ is replaced by an infinite Brownian disk with perimeter z). For this reason, we only outline certain arguments, especially in the final part of the proof, and we refer to [22] for more details. Throughout the proof, we argue under the measure $\mathbb{N}_0^{[r]}(\mathrm{d}\omega)$. We consider the truncation $\mathrm{tr}_{W_*+r}(\omega)$ and the excursions $(\omega^j)_{j\in J}$ below level W_*+r as defined in Section 2.2. According to Proposition 12 of [21], we know that, conditionally on $\mathcal{Z}_{W_*+r}=z$, $\sum_{j\in J}\delta_{\omega^j}$ is independent of $\mathrm{tr}_{W_*+r}(\omega)$ and distributed as a Poisson point measure with intensity $z\,\mathbb{N}_0$ conditioned to have a minimum equal to $-r$. We need in fact a slightly more precise result involving also the exit local time at level W_*+r, which is denoted by $(L_s^{W_*+r})_{s\geq 0}$. Notice that this exit local time is easily defined by using the spine decomposition in Section 2.4, and that $L_\sigma^{W_*+r}=\mathcal{Z}_{W_*+r}$.

For every $j\in J$, let (α_j,β_j) be the time interval corresponding to the excursion ω^j, and let $l_j=L_{\alpha_j}^{W_*+r}=L_{\beta_j}^{W_*+r}$. Then an application of the special Markov property in the form given in the appendix of [20] (using again the spine decomposition in Section 2.4) shows that, conditionally on $\mathcal{Z}_{W_*+r}=z$, the point measure

$$\sum_{j\in J}\delta_{(l_j,\omega^j)} \tag{20}$$

is independent of $\mathrm{tr}_{W_*+r}(\omega)$ and has the same distribution as $\mathcal{N}+\delta_{(U_*,\omega_*)}$ with the notation of Section 3.1. According to Section 3.1, the hull $\mathfrak{H}_{r,z}$ is constructed as a measurable function of $\mathcal{N}+\delta_{(U_*,\omega_*)}$. We will then verify that, if we perform the construction of Section 3.1 from the point measure (20), we get a pointed compact metric space isometric to $B_r^\bullet(x_*)$ (equipped with its intrinsic metric). We let $(\mathfrak{H}^\star,D^\star)$ stand for the metric space obtained from the point measure in (20) by the construction of Section 3.1 — notice that this construction makes sense even with a random perimeter z. We also use the notation $\mathbf{H}^\star$ for the space constructed from the point measure (20) in a way similar to $\mathbf{H}$ in Section 3.1, and $\Pi^\star$ for the canonical projection from $\mathbf{H}^\star$ onto $\mathfrak{H}^\star$. Note that $D^\star=D_{\mathbf{H}^\star}$ in the notation of Section 3.1. We also define $D^\circ_{\mathbf{H}^\star}$ as in Section 3.1 replacing $\mathbf{H}$ by $\mathbf{H}^\star$.

We first explain that the set $\mathfrak{H}^\star$ can be identified to $B_r^\bullet(x_*)$. To this end, set

$$F_r:=\{p_{(\omega)}(s):0\leq s\leq\sigma,\tau_{W_*+r}(W_s)\leq\zeta_s\},$$
$$\partial F_r:=\{p_{(\omega)}(s):0\leq s\leq\sigma,\tau_{W_*+r}(W_s)=\zeta_s\}.$$

As we already noticed, we have $B_r^\bullet(x_*)=\Pi(F_r)$ and $\partial B_r^\bullet(x_*)=\Pi(\partial F_r)$. We define a mapping $\mathcal{I}$ from F_r onto $\mathbf{H}^\star$ by the following prescriptions. If $p_{(\omega)}(s)\in F_r\backslash\partial F_r$, then s belongs to (α_j,β_j) for some $j\in J$, and we take $\mathcal{I}(p_{(\omega)}(s))=p_{(\omega^j)}(s-\alpha_j)\in\mathcal{T}_{(\omega^j)}\subset\mathbf{H}^\star$. On the other hand, if $p_{(\omega)}(s)\in\partial F_r$, we take $\mathcal{I}(p_{(\omega)}(s))=L_s^{W_*+r}\in[0,\mathcal{Z}_{W_*+r}]=\partial\mathbf{H}^\star$. The reader will easily check that $\mathcal{I}(a)$ is well-defined for every $a\in F_r$ independently of the choice of s such that $p_{(\omega)}(s)=a$. Moreover, we have $D^\circ_{\mathbf{H}^\star}(\mathcal{I}(a),\mathcal{I}(b))=D^\circ(a,b)$ for every $a,b\in F_r$ (we omit the details). The mapping $a\mapsto\mathcal{I}(a)$ is not one-to-one (though its restriction to $F_r\backslash\partial F_r$ is one-to-one) but the latter property shows that $\Pi(a)=\Pi(b)$ implies $\Pi^\star(\mathcal{I}(a))=\Pi^\star(\mathcal{I}(b))$. So one can define a mapping $\mathcal{J}$ from $\Pi(F_r)=B_r^\bullet(x_*)$ onto $\mathfrak{H}^\star$ by declaring that $\mathcal{J}(\Pi(a))=\Pi^\star(\mathcal{I}(a))$ for every $a\in F_r$. This mapping $\mathcal{J}$ is one-to-one since $\Pi^\star(\mathcal{I}(a))=\Pi^\star(\mathcal{I}(b))$ is only possible if $D^\circ_{\mathbf{H}^\star}(\mathcal{I}(a),\mathcal{I}(b))=0$, which implies $D^\circ(a,b)=0$ and $\Pi(a)=\Pi(b)$. The mapping $\mathcal{J}$ provides the desired identification of $B_r^\bullet(x_*)$ with $\mathfrak{H}^\star$, and we also observe that $\partial B_r^\bullet(x_*)$ is identified with $\partial\mathfrak{H}^\star$. From now one we make these identifications, and we notice that we have $D(x,y)\leq D^\star(x,y)$ for $x,y\in B_r^\bullet(x_*)$, by comparing formulas (10) and (16), using the equality $D^\circ_{\mathbf{H}^\star}(\mathcal{I}(a),\mathcal{I}(b))=D^\circ(a,b)$ for $a,b\in F_r$ (the point is that there are more choices for the intermediate points $a_1,\ldots,a_{k-1}$ in (10) than in (16)).

Then, we need to verify that the restriction of $D^\star$ to the interior of $B_r^\bullet(x_*)$ coincides with the intrinsic distance, which we denote by D^{intr}. The bound $D^\star\leq D^{\mathrm{intr}}$ is easy. If

$(\gamma(t))_{0\leq t\leq 1}$ is a path connecting two points x and y of the interior of $B^\bullet_r(x_*)$ that stays in this interior, the length of γ is bounded below by $\sum_{k=1}^n D(\gamma(t_{k-1}),\gamma(t_k))$, where $0=t_0<t_1<\cdots<t_n=1$ is a subdivision of $[0,1]$. For every $k\in\{0,\ldots,n\}$, let $a_k\in\mathcal{T}_{(\omega)}$ be such that $\Pi(a_k)=\gamma(t_k)$. Then,

$$D(\gamma(t_{k-1}),\gamma(t_k))=\inf_{\substack{c_0=a_{k-1},c_1,\ldots,c_p=a_k\\ c_1,\ldots,c_{p-1}\in\mathcal{T}_{(\omega)}}}\sum_{j=1}^p D^\circ(c_{j-1},c_j) \tag{21}$$

but if the mesh of the subdivision is sufficiently small (so that all $D(\gamma(t_{k-1}),\gamma(t_k))$ are small) we can assume that the infimum of the previous display is attained by considering only points c_j such that $\Pi(c_j)\in B^\bullet_r(x_*)$ (otherwise the sum in the right-hand side of (21) is bounded below by the D-distance between the range of γ and $\partial B^\bullet_r(*)$). For such a choice of the c_j's, we have $D^\circ(c_{j-1},c_j)=D^\circ_{\mathbf{H}^\star}(\mathcal{I}(c_{j-1}),\mathcal{I}(c_j))$. It follows that $D(\gamma(t_{k-1}),\gamma(t_k))\geq D^\star(\gamma(t_{k-1}),\gamma(t_k))$, and finally that the length of γ is bounded below by $D^\star(x,y)$ as desired.

The reverse bound $D^{\rm intr}\leq D^\star$ is slightly more delicate, and we only sketch the argument. The difficulty comes from the following observation. In formula (16) giving $D^\star$ in terms of $D^\circ_{\mathbf{H}^\star}$, even if a and b do not belong to the boundary $\partial\mathbf{H}^\star$, we need a priori to consider points $a_1,\ldots,a_{k-1}\in\mathbf{H}^\star$ that may belong to this boundary. However, we leave it as an exercise for the reader to check that the infimum remains the same even if we impose that all points $a_1,\ldots,a_{k-1}$ do not lie on $\partial\mathbf{H}^\star$. In that case, $D^\circ_{\mathbf{H}^\star}(a_{i-1},a_i)$ can be interpreted as the length of a path from $\Pi^\star(a_{i-1})$ to $\Pi^\star(a_i)$ made of the concatenation of two simple geodesics started respectively from $\Pi^\star(a_{i-1})$ and from $\Pi^\star(a_i)$, see e.g. the end of [22, Section 4.1] for a very similar argument. In the identification of $\mathfrak{H}^\star$ with $B^\bullet_r(x_*)$, these simple geodesics remain geodesics for the distance D and stay in the interior of $B^\bullet_r(x_*)$. Summarizing, we obtain that $D^\star(x,y)$ is obtained as an infimum of quantities that are lengths of paths connecting x to y and staying in the interior of $B^\bullet_r(x_*)$. This gives the desired bound $D^{\rm intr}\leq D^\star$.

Once we know that $D^{\rm intr}=D^\star$ in the interior of $B^\bullet_r(x_*)$, the fact that $D^{\rm intr}$ has a continuous extension to the boundary is easy. Suppose that $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are two sequences in the interior of $B^\bullet_r(x_*)$ that converge to x and y respectively (for the metric D). We have to verify that $D^{\rm intr}(x_n,y_n)$ has a limit as $n\to\infty$. We observe that the convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ for the metric D also implies that $D^\star(x_n,x)\longrightarrow 0$ as $n\to\infty$. Indeed, by compactness, we can find a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ and $x'\in B^\bullet_r(x_*)$ such that $D^\star(x_{n_k},x')\longrightarrow 0$. But since $D\leq D^\star$ on $B^\bullet_r(x_*)$, this readily implies that $x'=x$ and we get that $D^\star(x_n,x)\longrightarrow 0$ as $n\to\infty$. We have similarly $D^\star(y_n,y)\longrightarrow 0$ as $n\to\infty$, and we conclude that $D^{\rm intr}(x_n,y_n)=D^\star(x_n,y_n)$ converges to $D^\star(x,y)$.

At this stage we have proved that the intrinsic distance on the interior of $B^\bullet_r(x_*)$ has a continuous extension to $B^\bullet_r(x_*)$, and that, conditionally on $\mathcal{Z}_{W_*+r}=z$, the resulting random (pointed) metric space $B^\bullet_r(x_*)$ has the same distribution as $\mathfrak{H}_{r,z}$ and is independent of $\mathrm{tr}_{W_*+r}(\omega)$. To complete the proof, we need to verify that that the intrinsic metric on $\mathbf{m}_\infty\backslash B^\bullet_r(x_*)$ has a continuous extension to $\overline{\mathbf{m}_\infty\backslash B^\bullet_r(x_*)}$ and that the (two-boundary measure) metric space $\overline{\mathbf{m}_\infty\backslash B^\bullet_r(x_*)}$ is a function of $\mathrm{tr}_{W_*+r}(\omega)$, which, conditionally on $\mathcal{Z}_{W_*+r}=z$, is distributed as $\mathbb{D}_z$. The proof of the first assertion proceeds by minor modifications of the proofs of [21, Theorem 28] or of [22, Theorem 29], and we omit the details. As for the second assertion, we rely on [21, Proposition 12], which gives the conditional distribution of $\mathrm{tr}_{W_*+r}(W)$ given $\mathcal{Z}_{W_*+r}=z$: For every nonnegative measurable function F on $\mathcal{S}_0$,

$$\mathbb{N}_0^{[r]}\Big(F(\mathrm{tr}_{W_*+r}(\omega))\,\Big|\,\mathcal{Z}_{W_*+r}=z\Big)=z^{-2}\,\mathbb{N}_0^{*,z}\Big(\int_0^\sigma \mathrm{d}s\,F(\omega^{[s]})\Big), \tag{22}$$

where we recall that $\omega^{[s]}$ stands for ω re-rooted at s, and $\mathbb{N}_0^{*,z}$ is the law of a positive Brownian snake excursion with boundary size z, as defined in [3]. Write $\widetilde{\mathbb{N}}_0^{*,z}$ for the probability measure on $\mathcal{S}_0$ such that $\widetilde{\mathbb{N}}_0^{*,z}(F)$ is the right-hand side of (22). From the beginning of Section 4.3 in [22], the free pointed Brownian disk $\mathbb{D}_z$ can be constructed as a measurable function of a random snake trajectory distributed according to $\widetilde{\mathbb{N}}_0^{*,z}$. We leave it to the reader to check that the same measurable function applied to $\mathrm{tr}_{W_*+r}(\omega)$ yields the space $\overline{\mathbf{m}_\infty \backslash B_r^\bullet(x_*)}$ equipped with its extended intrinsic metric — here again arguments are very similar to the proof of [22, Theorem 29]. It thus follows from (22) that $\overline{\mathbf{m}_\infty \backslash B_r^\bullet(x_*)}$ under $\mathbb{N}_0^{[r]}(\cdot \mid \mathcal{Z}_{W_*+r} = z)$ has the distribution of $\mathbb{D}_z$. This completes the proof. □

We will need a "two-point version" of Theorem 8, which we now state as a corollary. It is convenient to write

$$\mathcal{Z}_r^{x_*(x_0)} = \mathcal{Z}_{W_*+r}$$

for the boundary size of the hull $B_r^\bullet(x_*)$, which makes sense when $D(x_*, x_0) > r$ or equivalently $W_* < -r$. By interchanging the roles of x_* and x_0 and relying on the symmetry properties of the Brownian sphere (cf. (12)), we can also define on the same event the quantity $\mathcal{Z}_r^{x_0(x_*)}$ now corresponding to the boundary size of the hull $B_r^{\bullet(x_*)}(x_0)$ — one may use the analog of the approximation formula (19).

On the event where $D(x_*, x_0) > 2r$, the hulls $B_r^{\bullet(x_0)}(x_*)$ and $B_r^{\bullet(x_*)}(x_0)$ are disjoint, and we set

$$\mathcal{C}_r^{x_*,x_0} := \mathbf{m}_\infty \backslash \big(B_r^{\bullet(x_0)}(x_*) \cup B_r^{\bullet(x_*)}(x_0)\big).$$

In the next corollary, we view both $B_r^{\bullet(x_0)}(x_*)$ and $B_r^{\bullet(x_*)}(x_0)$ equipped with their (extended) intrinsic metrics as random variables with values in $\mathbb{M}^{\bullet b}$ as stated in Theorem 8 (the fact that this is also legitimate for $B_r^{\bullet(x_*)}(x_0)$ is a consequence of (12)). We use the notation $\Theta_{r,z}$ for the distribution of the standard hull $\mathfrak{H}_{r,z}$ of Section 3.1, so that $\Theta_{r,z}$ is a probability measure on the space $\mathbb{M}^{\bullet b}$.

COROLLARY 9. *A.e. under $\mathbb{N}_0(\cdot \cap \{D(x_*, x_0) > 2r\})$, the intrinsic metric on $\mathcal{C}_r^{x_*,x_0}$ has a continuous extension to its closure $\overline{\mathcal{C}_r^{x_*,x_0}}$, which is a metric on this space. We equip this metric space with the restriction of the volume measure of $\mathbf{m}_\infty$ and with the boundaries $\partial B_r^{\bullet(x_0)}(x_*)$ and $\partial B_r^{\bullet(x_*)}(x_0)$, so that we view $\overline{\mathcal{C}_r^{x_*,x_0}}$ as a random variable with values in $\mathbb{M}^{bb}$. Then, if F_1 and F_2 are two nonnegative measurable functions on $\mathbb{M}^{\bullet b}$, and G is a nonnegative measurable function on $\mathbb{M}^{bb}$, we have*

$$\begin{aligned}\mathbb{N}_0^{[2r]}&\Big(F_1(B_r^{\bullet(x_0)}(x_*))\,F_2(B_r^{\bullet(x_*)}(x_0))\,G\big(\overline{\mathcal{C}_r^{x_*,x_0}}\big)\Big)\\ &= \mathbb{N}_0^{[2r]}\Big(\Theta_{r,\mathcal{Z}_r^{x_*(x_0)}}(F_1)\,\Theta_{r,\mathcal{Z}_r^{x_0(x_*)}}(F_2)\,G\big(\overline{\mathcal{C}_r^{x_*,x_0}}\big)\Big).\end{aligned}$$

PROOF. Write $D^{\mathrm{intr},\mathcal{C}}$ for the intrinsic distance on $\mathcal{C}_r^{x_*,x_0}$. We need to verify that, if $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are two sequences in $\mathcal{C}_r^{x_*,x_0}$ that converge respectively to x and y belonging to $\overline{\mathcal{C}_r^{x_*,x_0}}$, then the sequence $D^{\mathrm{intr},\mathcal{C}}(x_n, y_n)$ converges. To this end, it suffices to prove that $D^{\mathrm{intr},\mathcal{C}}(x_n, x_p)$ converges to 0 as $n, p \to \infty$ (and similarly for $D^{\mathrm{intr},\mathcal{C}}(y_n, y_p)$). If $x \in \mathcal{C}_r^{x_*,x_0}$ this is trivial, so we can suppose that $x \in \partial B_r^{\bullet(x_0)}(x_*)$ — the case $x \in \partial B_r^{\bullet(x_*)}(x_0)$ is treated in a symmetric manner. However, writing D^{intr,x_*} for the intrinsic distance on $\mathbf{m}_\infty \backslash B_r^{\bullet(x_0)}(x_*)$, and recalling that this distance is extended continuously to the closure of $\mathbf{m}_\infty \backslash B_r^{\bullet(x_0)}(x_*)$, we already know from Theorem 8 that $D^{\mathrm{intr},x_*}(x_n,x_p)$ converges to 0 as $n,p\to\infty$. Then the desired result follows from the fact that we have $D^{\mathrm{intr},\mathcal{C}}(x_n,x_p)=D^{\mathrm{intr},x_*}(x_n,x_p)$ as soon as n,p are large enough. Indeed, for any $\varepsilon>0$, if both $D^{\mathrm{intr},x_*}(x_n,x)$ and $D^{\mathrm{intr},x_*}(x_p,x)$ are smaller than ε, we have

$$D(x_n,B^{*(x_*)}(x_0))+D(x_p,B^{*(x_*)}(x_0))>2\min_{u\in B_r^{\bullet(x_0)}(x_*),v\in B_r^{\bullet(x_*)}(x_0)} D(u,v)-2\varepsilon,$$

and therefore, taking ε small enough, we see that, when n and p are sufficiently large, the length of a path from x_n to x_p that hits $B_r^{\bullet(x_*)}(x_0)$ is bounded below by a positive quantity. Since $D^{\mathrm{intr},x_*}(x_n,x_p)\le D^{\mathrm{intr},x_*}(x_n,x)+D^{\mathrm{intr},x_*}(x_p,x)$, which tends to 0 as $n,p\to\infty$, it follows that the infimum that gives $D^{\mathrm{intr},x_*}(x_n,x_p)$ must be attained for paths that do not hit $B_r^{\bullet(x_*)}(x_0)$, and thus $D^{\mathrm{intr},\mathcal{C}}(x_n,x_p)=D^{\mathrm{intr},x_*}(x_n,x_p)$ when n and p are large enough. So $D^{\mathrm{intr},\mathcal{C}}$ can be extended by continuity to $\overline{\mathcal{C}_r^{x_*,x_0}}$, and a similar argument shows that $D^{\mathrm{intr},\mathcal{C}}(x,y)>0$ if x and y are distinct points of $\partial B_r^{\bullet(x_0)}(x_*)$ (resp., of $\partial B_r^{\bullet(x_*)}(x_0)$) since otherwise this would imply $D^{\mathrm{intr},x_*}(x,y)=0$.

Let us turn to the second assertion of the corollary. It follows from Theorem 8, that, for functions F and G as in this statement,

$$\mathbb{N}_0^{[r]}\Big(F(B_r^\bullet(x_*))\,G\big(\overline{\mathbf{m}_\infty\backslash B_r^\bullet(x_*)}\big)\Big)=\mathbb{N}_0^{[r]}\Big(\Theta_{r,\mathcal{Z}_r^{x_*}(x_0)}(F)\,G\big(\overline{\mathbf{m}_\infty\backslash B_r^\bullet(x_*)}\big)\Big)$$

Up to replacing $G\big(\overline{\mathbf{m}_\infty\backslash B_r^\bullet(x_*)}\big)$ by $\mathbf{1}_{\{D(x_0,\partial B_r^\bullet(x_*))>r\}}\,G\big(\overline{\mathbf{m}_\infty\backslash B_r^\bullet(x_*)}\big)$, we see that the preceding display remains valid if $\mathbb{N}_0^{[r]}$ is replaced by $\mathbb{N}_0^{[2r]}$. Next, under the assumptions of the corollary, the quantity

$$F_2(B_r^{\bullet(x_*)}(x_0))\,G\big(\overline{\mathcal{C}_r^{x_*,x_0}}\big)$$

is equal $\mathbb{N}_0^{[2r]}$ a.e. to a measurable function of $\overline{\mathbf{m}_\infty\backslash B_r^\bullet(x_*)}$, and so we get from the preceding considerations that

$$\begin{aligned}&\mathbb{N}_0^{[2r]}\Big(F_1(B_r^{\bullet(x_0)}(x_*))\,F_2(B_r^{\bullet(x_*)}(x_0))\,G\big(\overline{\mathcal{C}_r^{x_*,x_0}}\big)\Big)\\&\quad=\mathbb{N}_0^{[2r]}\Big(\Theta_{r,\mathcal{Z}_r^{x_*}(x_0)}(F_1)\,F_2(B_r^{\bullet(x_*)}(x_0))\,G\big(\overline{\mathcal{C}_r^{x_*,x_0}}\big)\Big).\end{aligned}$$

At this stage, we use (12) to interchange the roles of x_* and x_0. It follows that the preceding quantity is equal to

$$\mathbb{N}_0^{[2r]}\Big(\Theta_{r,\mathcal{Z}_r^{x_0}(x_*)}(F_1)\,F_2(B_r^{\bullet(x_0)}(x_*))\,G\big(\overline{\mathcal{C}_r^{x_0,x_*}}\big)\Big),$$

which, by the same application of Theorem 8, is also equal to

$$\mathbb{N}_0^{[2r]}\Big(\Theta_{r,\mathcal{Z}_r^{x_0}(x_*)}(F_1)\,\Theta_{r,\mathcal{Z}_r^{x_*}(x_0)}(F_2)\,G\big(\overline{\mathcal{C}_r^{x_0,x_*}}\big)\Big).$$

Finally, by interchanging once again the roles of x_0 and x_* in the last display, we obtain the desired result. □

4. The first-moment estimate.

4.1. *Slices.* In this section, we provide a brief description of the random compact metric spaces called slices, which appear as scaling limits of the planar maps with geodesic boundaries considered in [6, 18] (notice that [27] uses a slightly different definition of slices). We fix $h>0$ and argue under $\mathbb{N}_0(\mathrm{d}\omega \,|\, W_* = -h)$. Under this probability measure, the Brownian sphere $\mathbf{m}_\infty(\omega)$ is conditioned on the event that the distance between the two distinguished points is equal to h. According to [17], the unique geodesic from x_0 to x_* is the path $(\gamma(r))_{0\le r\le h}$ defined by

$$\gamma(r) = \Pi\circ p_{(\omega)}(\inf\{s\in[0,\sigma] : \widehat{W}_s = -r\}) = \Pi\circ p_{(\omega)}(\sup\{s\in[0,\sigma] : \widehat{W}_s = -r\}), \tag{23}$$

for every $r\in[0,h]$.

We will now define another quotient space of $\mathcal{T}_{(\omega)}$ (later called a slice), which roughly speaking corresponds to cutting the Brownian sphere $\mathbf{m}_\infty(\omega)$ along the geodesic from x_0 to x_* (see the end of Section 3.2 in [18] for more details about this interpretation). We start by setting, for every $s,t\in[0,\sigma]$,

$$\widetilde{D}^\circ(s,t) := \widehat{W}_s + \widehat{W}_t - 2\min_{s\wedge t\le r\le s\vee t} \widehat{W}_r,$$

and then, for every $a,b\in\mathcal{T}_{(\omega)}$,

$$\widetilde{D}^\circ(a,b) := \min\{\widetilde{D}^\circ(s,t) : s,t\in[0,\sigma], p_{(\omega)}(s)=a,\, p_{(\omega)}(t)=b\}. \tag{24}$$

We finally let $\widetilde{D}(a,b)$ be the maximal pseudo-metric on $\mathcal{T}_{(\omega)}$ that is bounded above by $\widetilde{D}^\circ(a,b)$. We define the slice $\mathbf{S}(\omega)$ as the quotient space $\mathcal{T}_{(\omega)}/\{\widetilde{D}=0\}$, which is equipped with the metric induced by $\widetilde{D}$. We write $\widetilde{\Pi}$ for the canonical projection from $\mathcal{T}_{(\omega)}$ onto $\mathbf{S}(\omega)$. We may view $\mathbf{S}(\omega)$ as a 2-pointed measure metric space, with the two distinguished points $\widetilde{x}_* := \widetilde{\Pi}(a_*)$ and $\widetilde{x}_0 := \widetilde{\Pi}(\rho_{(\omega)})$ and the volume measure which is the pushforward of the volume measure on $\mathcal{T}_{(\omega)}$.

It is immediate to verify that $\widetilde{D}(a,b)\ge D(a,b)$, and therefore $\widetilde{D}(a,b)=0$ implies that $D(a,b)=0$. Conversely, suppose that $D(a,b)=0$ and $a\ne b$. We know that $D^\circ(a,b)=0$, and thus (up to interchanging a and b) we can assume that

$$\Lambda_a = \Lambda_b = \min_{c\in[a,b]} \Lambda_c.$$

Pick $s,t\in[0,\sigma]$ such that $p_{(\omega)}(s)=a$, $p_{(\omega)}(t)=b$, and $[s,t]$ is as small as possible, where we use the convention $[s,t]=[s,\sigma]\cup[0,t]$ if $s>t$. Then the equalities of the last display are equivalent to

$$\widehat{W}_s = \widehat{W}_t = \min_{r\in[s,t]} \widehat{W}_r.$$

If $s\le t$, this implies $\widetilde{D}^\circ(s,t)=0$, and thus $\widetilde{D}(a,b)=0$. On the other hand, if $s>t$, we obtain that necessarily

$$\widehat{W}_s = \widehat{W}_t, \qquad \widehat{W}_s = \min_{s\le r\le\sigma} \widehat{W}_r, \qquad \widehat{W}_t = \min_{0\le r\le t} \widehat{W}_r.$$

These equalities imply that $\Pi(a)=\Pi(b)=\gamma(-\Lambda_a)$ belongs to the range of the geodesic γ. On the other hand, if, for every $r\in(0,h)$, we take $a=p_{(\omega)}(\sup\{s\in[0,\sigma]:\widehat{W}_s=-r\})$ and $b=p_{(\omega)}(\inf\{s\in[0,\sigma]:\widehat{W}_s=-r\})$ we have $\Pi(a)=\Pi(b)=\gamma(r)$, but $\widetilde{D}(a,b)>0$ (see Lemma 12 (ii) in [6]) and therefore $\widetilde{\Pi}(a)\ne\widetilde{\Pi}(b)$.

The preceding considerations show that every point of $\mathbf{m}_\infty(\omega)$ that does not belong to the geodesic γ corresponds to a single point of $\mathbf{S}(\omega)$, but every point of the geodesic γ (other than x_0 and x_*) corresponds to two points of $\mathbf{S}(\omega)$. More precisely, if we set for every $r\in[0,h]$,

$$\gamma'(r) := \widetilde{\Pi}\circ p_{(\omega)}(\inf\{s\in[0,\sigma]:\widehat{W}_s=-r\}),\ \gamma''(r) := \widetilde{\Pi}\circ p_{(\omega)}(\sup\{s\in[0,\sigma]:\widehat{W}_s=-r\}),$$

then γ' and γ'' are two geodesics in $\mathbf{S}(\omega)$ from $\widetilde{x}_0$ to $\widetilde{x}_*$ that are disjoint except at their initial and terminal times. We call γ' and γ'' the left and right boundary geodesics of $\mathbf{S}(\omega)$.

We will use the fact that, for every $\kappa \in (0, h/2)$, one has a.s.

$$\inf_{s,t\in[\kappa,h-\kappa]} \widetilde{D}(\gamma'(s), \gamma''(t)) > 0. \tag{25}$$

This is immediate since the function $(s,t) \mapsto \widetilde{D}(\gamma'(s), \gamma''(t))$ is continuous and does not vanish on $[\kappa, 1-\kappa] \times [\kappa, 1-\kappa]$.

Taking $h = 1$, (25) allows us to find a sequence $(\delta_k)_{k\geq 1}$ of positive reals such that the probability (under $\mathbb{N}_0(\cdot \,|\, W_* = -1)$) of the event where

$$\inf_{s,t\in[1-2^{-k},1-2^{-k-4}]} \widetilde{D}(\gamma'(s), \gamma''(t)) > 2\delta_k\,, \quad \text{for every } k \geq 1,$$

is at least $9/10$. Without loss of generality, we may and will assume that $\delta_k < 2^{-k-5}$ for every $k \geq 1$. By scaling, we also obtain that, for every $h \in [3/4, 1]$, the probability under $\mathbb{N}_0(\cdot \,|\, W_* = -h)$ of the event where

$$\inf_{s,t\in[h(1-2^{-k}),h(1-2^{-k-4})]} \widetilde{D}(\gamma'(s), \gamma''(t)) > \delta_k\,, \quad \text{for every } k \geq 1,$$

is at least $9/10$. Finally, fix $\varepsilon \in (0, 1/4)$, and observe that, if $h \in [1-\varepsilon, 1]$ and if the integer $k \geq 1$ is such that $2^{-k-4} > \varepsilon$, we have $[1-2^{-k}, 1-2^{-k-3}] \subset [h(1-2^{-k}), h(1-2^{-k-4})]$. We arrive at the following lemma.

LEMMA 10. *Let $\varepsilon \in (0, 1/4)$. Then, for every $h \in [1-\varepsilon, 1]$, the probability under $\mathbb{N}_0(\cdot \,|\, W_* = -h)$ of the event where*

$$\inf_{s,t\in[1-2^{-k},1-2^{-k-3}]} \widetilde{D}(\gamma'(s), \gamma''(t)) > \delta_k\,, \quad \textit{for every } k \geq 1 \textit{ such that } 2^{-k-4} > \varepsilon, \tag{26}$$

is at least $9/10$.

4.2. *Slices in hulls.* We consider the standard hull $\mathfrak{H}$ of radius $r = 1$ and perimeter z as constructed in Section 3.1 from (U_*, ω_*) and the point measure $\sum_{i\in I} \delta_{(t_i,\omega_i)}$, and we keep the notation of this section. To avoid confusion, we write $[a,b]_{\mathbf{H}}$ for the intervals of $\mathbf{H}$ as defined in Section 3.1. We fix $m \in \{1, 2, 3\}$ and $\varepsilon \in (0, 1/4)$, and we will now condition on the event

$$E^m_\varepsilon := \{\#\{i \in I : W_*(\omega_i) < -1 + \varepsilon\} \geq m\}. \tag{27}$$

Under this conditioning, there are (at least) m indices $i \in I$ such that $W_*(\omega_i) \leq -1+\varepsilon$, and we write $i_1, \ldots, i_m$ for these indices ranked in such a way that $t_{i_1} < \cdots < t_{i_m}$ (if there are more than m indices with the desired property, we keep those corresponding to the smaller values of t_i). We then set

$$\mathbf{R}_j := \Pi_{\mathfrak{H}}(\mathcal{T}_{(\omega_{i_j})})\,, \text{ for } 1 \leq j \leq m\,, \qquad \mathbf{R}_* = \Pi_{\mathfrak{H}}(\mathcal{T}_{(\omega_*)}).$$

Let $r_1 \in [1-\varepsilon, 1]$. We note that, conditionally on $W_*(\omega_{i_1}) = -r_1$, ω_{i_1} is distributed according according to $\mathbb{N}_0(\cdot \,|\, W_* = -r_1)$, and so we can consider the slice $\mathbf{S}(\omega_{i_1})$ constructed in the previous section (with $h = r_1$).

We observe that $\mathbf{R}_1$ and $\mathbf{S}(\omega_{i_1})$ are canonically identified as sets. Indeed both $\mathbf{R}_1$ and $\mathbf{S}(\omega_{i_1})$ are quotient spaces of $\mathcal{T}_{(\omega_{i_1})}$, and the point is to verify that, for $a, b \in \mathcal{T}_{(\omega_{i_1})}$, we have $D^\circ_{\mathbf{H}}(a,b) = 0$ if and only if $\widetilde{D}^\circ(a,b) = 0$ (we abuse notation by still writing $\widetilde{D}^\circ(a,b)$ for the function defined in (24) when $\omega = \omega_{i_1}$). Suppose first that $\widetilde{D}^\circ(a,b) = 0$. Then there exist $s,t \in [0,\sigma(\omega_{i_1})]$ such that $p_{(\omega_{i_1})}(s) = a$, $p_{(\omega_{i_1})}(t) = b$, and

$$\widehat{W}_s(\omega_{i_1}) = \widehat{W}_t(\omega_{i_1}) = \min_{s\wedge t\leq u\leq s\vee t} \widehat{W}_u(\omega_{i_1}).$$

Suppose that $s \leq t$ for definiteness. Then, writing $[u_1, u_1 + \sigma(\omega_{i_1})]$ for the time interval corresponding to $\mathcal{T}_{(\omega_{i_1})}$ in the time scale of the cyclic exploration $\mathcal{E}$ of $\mathbf{H}$, we have $[a,b]_{\mathbf{H}} \subset \{\mathcal{E}_{u_1+u} : s \leq u \leq t\}$, so that we get

$$\Lambda_a = \Lambda_b = \min_{c\in[a,b]_{\mathbf{H}}} \Lambda_c \tag{28}$$

and therefore $D^\circ_{\mathbf{H}}(a,b) = 0$. Conversely, if $D^\circ_{\mathbf{H}}(a,b) = 0$, we can assume without loss of generality that (28) holds, and this is only possible if $[a,b]_{\mathbf{H}} \subset \mathcal{T}_{(\omega_{i_1})}$ (otherwise we would have $b_* \in [a,b]_{\mathbf{H}}$). But then, using the definition of intervals in $\mathbf{H}$, we have

$$\min_{c\in[a,b]_{\mathbf{H}}} \Lambda_c = \max\Big\{ \min_{u\in[s,t]} \widehat{W}_u : s \leq t \leq \sigma(\omega_{i_1}),\, p_{(\omega_{i_1})}(s) = a,\, p_{(\omega_{i_1})}(t) = b\Big\},$$

and (28) implies that $\widetilde{D}^\circ(a,b) = 0$. Similarly, $\mathbf{R}_j$ and $\mathbf{S}(\omega_{i_j})$ are canonically identified as sets, for every $1 \leq j \leq m$ (note however that $\mathbf{S}(\omega_*)$, which also makes sense since ω_* is distributed according to $\mathbb{N}_0(\cdot \,|\, W_* = -1)$, is *not* identified to $\mathbf{R}_*$).

We also observe that we have, for every $a,b \in \mathcal{T}_{(\omega_{i_1})}$,

$$\widetilde{D}(a,b) \geq D_{\mathbf{H}}(a,b). \tag{29}$$

Indeed, it is immediately seen that $\widetilde{D}(a,b)$ is obtained by considering the same infimum as in (16), with the additional restriction that we require $a_1, \dots, a_{k-1} \in \mathcal{T}_{(\omega_{i_1})}$.

Since both metric spaces $(\mathbf{R}_1, D_{\mathbf{H}})$ and $(\mathbf{S}(\omega_{i_1}), \widetilde{D})$ are compact, the bound (29) implies that the identity mapping from $\mathbf{S}(\omega_{i_1})$ (equipped with $\widetilde{D}$) onto $\mathbf{R}_1$ (equipped with $D_{\mathbf{H}}$) is a homeomorphism. See Fig. 1 below for a schematic representation of $\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_*$ in the case where E^2_ε holds.

Consider then the two geodesics γ' and γ'' from $\widetilde{x}_0$ to $\widetilde{x}_*$ in $\mathbf{S}(\omega_{i_1})$, as defined in the previous section. Let $\Gamma' := \{\gamma'(t) : 0 \leq t \leq -W_*(\omega_{i_1})\}$ denote the range of γ', and similarly let Γ'' denote the range of γ''. Via the identification of $\mathbf{R}_1$ and $\mathbf{S}(\omega_{i_1})$, we may and will view Γ' and Γ'' as subsets of $\mathbf{R}_1$. Then, it is easily checked that $\Gamma' \cup \Gamma''$ is the topological boundary of $\mathbf{R}_1$ in $\mathbf{H}$ (note that a point a of $\mathcal{T}_{(\omega_{i_1})}$ different from the root is identified to a point of $\mathbf{H}\backslash\mathcal{T}_{(\omega_{i_1})}$ if and only if $\Pi_{\mathfrak{H}}(a)$ belongs to $\Gamma' \cup \Gamma''$). Furthermore, the restriction of $D_{\mathbf{H}}$ to Γ' (resp. to Γ'') clearly coincides with the restriction of $\widetilde{D}$. We write $\mathrm{Int}(\mathbf{R}_1) = \mathbf{R}_1 \backslash (\Gamma' \cup \Gamma'')$ for the interior of $\mathbf{R}_1$. We call Γ' and Γ'' the left and right boundaries of the slice $\mathbf{R}_1$ and write $\Gamma' = \partial_\ell \mathbf{R}_1$ and $\Gamma'' = \partial_r \mathbf{R}_1$. We use a similar terminology for the slices $\mathbf{R}_j$, $2 \leq j \leq m$.

LEMMA 11. *Let $(\phi(t))_{0\leq t\leq 1}$ be a continuous path in $\mathbf{R}_1$ satisfying the condition:*

(H) *There exist two reals u and v such that $0 \leq u \leq v \leq 1$ such that $\phi(t) \in \Gamma'$ for every $t \in [0,u]$, $\phi(t) \in \Gamma''$ for every $t \in [v,1]$, and $\phi(t) \in \mathrm{Int}(\mathbf{R}_1)$ for every $t \in (u,v)$.*

Then, the length of ϕ with respect to the distance $D_{\mathbf{H}}$ coincides with its length with respect to the distance $\widetilde{D}$.

PROOF. Since the restriction of $D_{\mathbf{H}}$ to Γ' or to Γ'' coincides with the restriction of $\widetilde{D}$, we may assume that condition (H) holds with $u = 0$ and $v = 1$. By a continuity argument, we may even replace (H) by

(H') $\phi(t) \in \mathrm{Int}(\mathbf{R}_1)$ for every $t \in [0,1]$.

Then, we can find $\delta > 0$ such that $D_{\mathbf{H}}(\phi(t), (\mathrm{Int}(\mathbf{R}_1))^c) \geq \delta$ for every $t \in [0,1]$. The conclusion of the lemma follows from the fact that we have $\widetilde{D}(\phi(t), \phi(t')) = D_{\mathbf{H}}(\phi(t), \phi(t'))$ as soon as $t, t' \in [0,1]$ are such that $D_{\mathbf{H}}(\phi(t), \phi(t')) \leq \delta/2$. Indeed, in the definition (16) of $D_{\mathbf{H}}(\phi(t), \phi(t'))$, we may restrict the infimum of the quantities $\sum D^\circ_{\mathbf{H}}(a_{i-1}, a_i)$ to the case where $a_1, \ldots, a_{k-1}$ all belong to $\mathrm{Int}(\mathbf{R}_1)$, because if for instance $a_j \notin \mathrm{Int}(\mathbf{R}_1)$, we have

$$\sum_{i=1}^{j} D^\circ_{\mathbf{H}}(a_{i-1}, a_i) \geq D_{\mathbf{H}}(\phi(t), a_j) \geq \delta.$$

If $a_1, \ldots, a_{k-1}$ all belong to $\mathrm{Int}(\mathbf{R}_1)$, we have $\sum D^\circ_{\mathbf{H}}(a_{i-1}, a_i) = \sum \widetilde{D}^\circ(a_{i-1}, a_i)$, and thus we obtain that $\widetilde{D}(\phi(t), \phi(t')) \leq D_{\mathbf{H}}(\phi(t), \phi(t'))$. The reverse bound follows from (29). $\square$

We can now combine Lemma 11 with the discussion of the end of Section 4.1. For every $0 \leq u \leq v \leq -W_*(\omega_{i_1})$, we use the notation

$$\Gamma'_{[u,v]} = \{\gamma'(t) : u \leq t \leq v\},$$

and we similarly define $\Gamma''_{[u,v]}$. We let the sequence $(\delta_k)_{k \geq 1}$ be as in Lemma 10, and we recall the definition (27) of the event E^m_ε.

LEMMA 12. *Let $\varepsilon \in (0, 1/4)$. The following property holds with probability at least $9/10$ under $\mathbb{P}(\cdot \mid E^m_\varepsilon)$. For every integer $k \geq 1$ such that $2^{-k-4} > \varepsilon$, for every continuous path $(\phi(t))_{0 \leq t \leq 1}$ in $\mathbf{R}_1$ such that $\phi(0) \in \Gamma'_{[1-2^{-k}, 1-2^{-k-3}]}$ and $\phi(1) \in \Gamma''_{[1-2^{-k}, 1-2^{-k-3}]}$, the length (with respect to $D_{\mathbf{H}}$) of ϕ is at least δ_k.*

PROOF. First note that we may restrict our attention to paths ϕ satisfying condition (H) of Lemma 11: Indeed, set $u = \sup\{t \in [0,1] : \phi(t) \in \Gamma'\}$ and $v = \inf\{t \in [u,1] : \phi(t) \in \Gamma''\}$, and observe that replacing ϕ by a portion of the geodesic γ' (resp. of the geodesic γ'') on the interval $[0,u]$ (resp. on $[v,1]$) will only decrease its length. By Lemma 11, if ϕ satisfies (H), its length with respect to $D_{\mathbf{H}}$ is the same as its length with respect to $\widetilde{D}$, provided we view ϕ as a path in $\mathbf{S}(\omega_{i_1})$. In particular, this length is bounded below by $\widetilde{D}(\phi(0), \phi(1))$. The desired result now follows from Lemma 10. $\square$

We also need an analog of Lemma 12 when $\mathbf{R}_1$ is replaced by $\mathbf{R}_*$. The situation in that case is a bit different since $\mathbf{S}(\omega_*)$ is no longer identified bijectively with $\mathbf{R}_*$. Instead, $\mathbf{R}_*$ appears as a quotient space of $\mathbf{S}(\omega_*)$, where, for $u \in (0,1]$, the points $\gamma'(u)$ and $\gamma''(u)$ of the left and right boundary geodesics in $\mathbf{S}(\omega*)$ are identified in $\mathbf{R}_*$ if (and only if) $u \geq \mu_0$, where

$$\mu_0 := \sup\{-W_*(\omega_i) : i \in I\} \geq 1 - \varepsilon. \tag{30}$$

We can nonetheless define the range Γ' (resp. Γ'') of the left boundary geodesic (resp. of the right boundary geodesic) as a closed subset of $\mathbf{R}_*$, and make sense of the sets $\Gamma'_{[u,v]}$ and $\Gamma''_{[u,v]}$ for $0 \leq u \leq v \leq 1$. We have $\Gamma'_{[u,v]} \cap \Gamma''_{[u,v]} = \varnothing$ if $v < \mu_0$. The reader will easily check that an analog of Lemma 11 remains valid when $\mathbf{R}_1$ is replaced by $\mathbf{R}_*$, provided that we add the constraint $D_{\mathbf{H}}(\phi(t), b_*) > \varepsilon$ for every $t \in [0,1]$ in condition (H). Then, the proof of Lemma 12 when $\mathbf{R}_1$ is replaced by $\mathbf{R}_*$ goes through almost without change: recall that we have assumed $\delta_k < 2^{-k-5}$, and therefore a path $(\phi(t))_{0 \leq t \leq 1}$ in $\mathbf{R}_*$ such that $\phi(0) \in \Gamma'_{[1-2^{-k}, 1-2^{-k-3}]}$ will not visit the set $\{x \in \mathbf{R}_* : D(x, x_*) \leq \varepsilon\}$ if its length is bounded by δ_k. The preceding discussion is summarized in the following statement.

LEMMA 13. *The statement of Lemma 12 remains valid if $\mathbf{R}_1$ is replaced by $\mathbf{R}_*$.*

4.3. *The first-moment estimate.* We again fix $m \in \{1,2,3\}$ ($m+1$ will correspond to what we called m in the introduction). For $x,y \in \mathbf{m}_\infty$, we recall our notation $B_r^{\bullet(y)}(x)$ for the hull of radius r centered at x relative to y, which makes sense if $D(x,y) > r$.

For $x,y \in \mathbf{m}_\infty$ and $\varepsilon \in (0,r)$, we let $F^{(m)}_{\varepsilon,r}(x,y)$ be equal to 1 if $D(x,y) > r$ and there exist $m+1$ geodesic paths $(\xi_0(t))_{0\le t\le r}, (\xi_1(t))_{0\le t\le r}, \dots, (\xi_m(t))_{0\le t\le r}$ with $\xi_i(0) \in \partial B_r^{\bullet(y)}(x)$ and $\xi_i(r) = x$, for every $i \in \{0,\dots,m\}$, such that the sets $\{\xi_i(t) : 0 \le t \le r-\varepsilon\}$, for $i \in \{0,\dots,m\}$, are disjoint. If these conditions do not hold, we take $F^{(m)}_{\varepsilon,r}(x,y) = 0$. By convention, if $\varepsilon \ge r$, we take $F^{(m)}_{\varepsilon,r}(x,y) = 1$ if $D(x,y) > r$ and $F^{(m)}_{\varepsilon,r}(x,y) = 0$ otherwise.

In the case $r = 1$, we also define $\widetilde F^{(m)}_{\varepsilon,1}(x,y)$ exactly as $F^{(m)}_{\varepsilon,1}(x,y)$, except that we require the additional property that the geodesics $\xi_0, \xi_1, \dots, \xi_m$ satisfy

$$D(\xi_i(t), \xi_j(t)) \ge \delta_k \tag{31}$$

for every $t \in [1-2^{-k-1}, 1-2^{-k-2}]$, for every $k \ge 1$ such that $2^{-k-4} > \varepsilon$, and for every $0 \le i < j \le m$. Here the sequence $(\delta_k)_{k\ge1}$ is fixed as in Lemma 12.

We let $\widetilde{\mathbb{N}}_0$ denote the measure with density $\frac{1}{\sigma}$ with respect to $\mathbb{N}_0$.

PROPOSITION 14. *There exists a constant $c > 0$ such that, for every $\varepsilon \in (0, 1/4)$, we have*

$$\widetilde{\mathbb{N}}_0\Big(\int \mathrm{Vol}(\mathrm{d}x)\, \mathbf{1}_{\{D(x,x_*)<2\}}\, \widetilde F^{(m)}_{\varepsilon,1}(x,x_*)\Big) \ge c\,\varepsilon^m.$$

PROOF. By the symmetry properties of the Brownian sphere (Section 2.6), we have

$$\begin{aligned}\widetilde{\mathbb{N}}_0\Big(\int \mathrm{Vol}(\mathrm{d}x)\; \mathbf{1}_{\{D(x,x_*)<2\}}\, \widetilde F^{(m)}_{\varepsilon,1}(x,x_*)\Big) &= \mathbb{N}_0\Big(\mathbf{1}_{\{D(x_*,x_0)<2\}}\, \widetilde F^{(m)}_{\varepsilon,1}(x_*,x_0)\Big)\\ &= \frac{3}{2}\,\mathbb{N}_0^{[1]}\Big(\mathbf{1}_{\{D(x_*,x_0)<2\}}\, \widetilde F^{(m)}_{\varepsilon,1}(x_*,x_0)\Big).\end{aligned}$$

We then use Theorem 8, noting that the indicator function $\mathbf{1}_{\{D(x_*,x_0)<2\}}$ is a function of $\overline{\mathbf{m}_\infty \backslash B_1^{\bullet(x_0)}(x_*)}$ and that $\widetilde F^{(m)}_{\varepsilon,1}(x_*,x_0)$ can be written as a function of the hull $B_1^{\bullet(x_0)}(x_*)$: for this last fact, observe that geodesics from the boundary of $B_1^{\bullet(x_0)}(x_*)$ to the center x_* are the same for D and for the intrinsic distance of the hull, and that in condition (31) we can also replace D by the intrinsic distance, since if $D(\xi_i(t),\xi_j(t))$ is smaller than the intrinsic distance between $\xi_i(t)$ and $\xi_j(t)$, this means that a geodesic (for D) from $\xi_i(t)$ to $\xi_j(t)$ has to intersect the boundary of the hull, and thus $D(\xi_i(t),\xi_j(t))$ is at least $1/2$. It follows from Theorem 8 and these observations that

$$\mathbb{N}_0^{[1]}\Big(\mathbf{1}_{\{D(x_*,x_0)<2\}}\, \widetilde F^{(m)}_{\varepsilon,1}(x_*,x_0)\Big) = \mathbb{N}_0^{[1]}\Big(\theta(\mathcal{Z}_{W_*+1})\, \mathbb{N}_0^{[1]}(\widetilde F^{(m)}_{\varepsilon,1}(x_*,x_0)\,|\,\mathcal{Z}_{W_*+1})\Big),$$

where $\theta(z)$ is the probability for a free pointed Brownian disk of perimeter z that the distance from the distinguished point to the boundary is smaller than 1. Since we also know that, under $\mathbb{N}_0^{[1]}(\cdot\,|\,\mathcal{Z}_{W_*+1} = z)$, $B_1^{\bullet(x_0)}(x_*)$ is distributed as a standard hull of radius 1 and perimeter z (as defined in Section 3.1), the proof of Proposition 14 reduces to establishing the following claim.

Claim. Let $\mathfrak{H}$ be a standard hull of radius 1 and perimeter z as in Section 3.1, and, for every $\varepsilon \in (0,1/4)$, let $\mathcal{A}^m_\varepsilon$ be the event where there exist $m+1$ geodesics $(\eta_0(t))_{0\le t\le1}$, $(\eta_1(t))_{0\le t\le1}, \dots, (\eta_m(t))_{0\le t\le1}$ from the boundary $\partial\mathfrak{H}$ to the center of the hull, such that the sets $\{\eta_j(t) : 0 \le t \le 1-\varepsilon\}$ are disjoint, and moreover

$$D_{\mathbf{H}}(\eta_i(t), \eta_j(t)) \ge \delta_k \tag{32}$$

for every $t \in [1-2^{-k-1}, 1-2^{-k-2}]$, for every $k \geq 1$ such that $2^{-k-4} > \varepsilon$, and for every $0 \leq i < j \leq m$. Then there exists a constant $c > 0$ (depending on z) such that $\mathbb{P}(\mathcal{A}_\varepsilon^m) \geq c\,\varepsilon^m$.

In the remaining part of the proof, we establish the preceding claim. We consider the event E_ε^m defined in (27). Then $\mathbb{P}(E_\varepsilon^m)$ is just the probability that a Poisson variable with parameter $\frac{3z}{2}((1-\varepsilon)^{-2}-1)$ is greater than or equal to m, and thus there exists a constant c' such that $\mathbb{P}(E_\varepsilon^m) \geq c'\varepsilon^m$. To get the desired lower bound, it remains to verify that we have also $\mathbb{P}(\mathcal{A}_\varepsilon^m \,|\, E_\varepsilon^m) \geq c''$ with another constant $c'' > 0$. To this end, we rely on Lemma 12 and Lemma 13. For every $1 \leq j \leq m$ (resp. for $j = 0$), we write $\mathcal{B}_\varepsilon^{m,j}$ for the intersection of E_ε^m with the event where the property of Lemma 12 holds when $\mathbf{R}_1$ is replaced by $\mathbf{R}_j$ (resp. by $\mathbf{R}_*$). We then know that

$$\mathbb{P}((\mathcal{B}_\varepsilon^{m,j})^c \,|\, E_\varepsilon^m) \leq \frac{1}{10}, \quad \text{for every } 0 \leq j \leq m,$$

and, if

$$\mathcal{B}_\varepsilon^m = \bigcap_{j=0}^m \mathcal{B}_\varepsilon^{m,j},$$

it follows that $\mathbb{P}(\mathcal{B}_\varepsilon^m \,|\, E_\varepsilon^m) \geq 1/2$. So to get our claim we only need to verify that $\mathcal{B}_\varepsilon^m \subset \mathcal{A}_\varepsilon^m$.

From now on, we argue on the event E_ε^m, and we let $i_1, \ldots, i_m$ be the indices such that $W_*(\omega_i) < -1+\varepsilon$, as defined at the beginning of Section 4.2. Then $t_{i_1}, \ldots, t_{i_m}$ are elements of $[0,z] \subset \mathbf{H}$, and, recalling the definition of the exploration process $(\mathcal{E}_s)_{s\in[0,\Sigma]}$, we set

$$s'_j := \inf\{s \in [0,\Sigma] : \mathcal{E}_s = t_{i_j}\}\,, \quad s''_j := \sup\{s \in [0,\Sigma] : \mathcal{E}_s = t_{i_j}\}, \quad \text{for } 1 \leq j \leq m,$$
$$s'_0 := \inf\{s \in [0,\Sigma] : \mathcal{E}_s = U_*\}\,, \quad s''_0 := \sup\{s \in [0,\Sigma] : \mathcal{E}_s = U_*\}.$$

We can then consider the simple geodesics $(\gamma_{s'_j}(t))_{0\leq t\leq 1}$ and $(\gamma_{s''_j}(t))_{0\leq t\leq 1}$, for every $0 \leq j \leq m$, as defined in Section 3.1. We note that $\gamma_{s'_j}(0) = \gamma_{s''_j}(0)$ and $\gamma_{s'_j}(r) \neq \gamma_{s''_j}(r)$ if $0 < r < \mu_j$, where μ_0 was defined in (30) and $\mu_j := -W_*(\omega_j) \in (1-\varepsilon, 1)$ if $1 \leq j \leq m$. On the other hand, $\gamma_{s'_j}(r) = \gamma_{s''_j}(r)$ if $\mu_j \leq r \leq 1$. The set

$$\{\gamma_{s'_j}(r) : 0 \leq r \leq \mu_j\} \cup \{\gamma_{s''_j}(r) : 0 \leq r \leq \mu_j\}$$

is the range of a simple cycle, and (the closure of) the connected component of the complement of this simple cycle that does not contain the point $\Pi_{\mathfrak{H}}(0)$ of $\partial\mathfrak{H}$ coincides with the slice $\mathbf{R}_j = \Pi_{\mathfrak{H}}(\mathcal{T}_{(\omega_{i_j})})$ (resp. with $\mathbf{R}_* = \Pi_{\mathfrak{H}}(\mathcal{T}_{(\omega_*)})$ when $j = 0$). More precisely, for $1 \leq j \leq m$, the set $\{\gamma_{s'_j}(r) : 0 \leq r \leq \mu_j\}$ is the left boundary of $\mathbf{R}_j$, and the set $\{\gamma_{s''_j}(r) : 0 \leq r \leq \mu_j\}$ is the right boundary of $\mathbf{R}_j$. Similarly, we can interpret $\{\gamma_{s'_0}(r) : 0 \leq r \leq \mu_0\}$ as the left boundary of $\mathbf{R}_*$, and $\{\gamma_{s''_0}(r) : 0 \leq r \leq \mu_0\}$ as the right boundary of $\mathbf{R}_*$. Since $\gamma_{s'_j}$ and $\gamma'_{s'_j}$ are geodesics to b_*, we have

$$D_{\mathbf{H}}(b_*, \gamma_{s'_j}(r)) = 1 - r = D_{\mathbf{H}}(b_*, \gamma_{s''_j}(r))$$

for every $r \in [0,1]$.

To complete the proof, we verify that, if $\mathcal{B}_\varepsilon^m$ holds, then $\mathcal{A}_\varepsilon^m$ also holds, and we can take $\eta_j = \gamma_{s'_j}$ for $0 \leq j \leq m$. We first observe that, thanks to the fact that $W_*(\omega_{i_j}) < -1+\varepsilon$ for $1 \leq j \leq m$, the sets $\{\gamma_{s'_j}(t) : 0 \leq t \leq 1-\varepsilon\}$, $0 \leq j \leq 1-\varepsilon$, are disjoint. Then, we use the defining property of $\mathcal{B}_\varepsilon^m$ to get that, for every integer $k \geq 1$ such that $2^{-k-4} > \varepsilon$, the length of any continuous path starting from $\{\gamma_{s'_j}(t) : 1-2^{-k} \leq t \leq 1-2^{-k-3}\}$, ending in $\{\gamma_{s''_j}(t) : 1-2^{-k} \leq t \leq 1-2^{-k-3}\}$ and staying in the slice $\mathbf{R}_j$ (in $\mathbf{R}_*$ if $j = 0$) is bounded below by δ_k. Let us explain why this implies that

$$D_{\mathbf{H}}(\gamma_{s'_i}(t), \gamma_{s'_j}(t)) \geq \delta_k \tag{33}$$

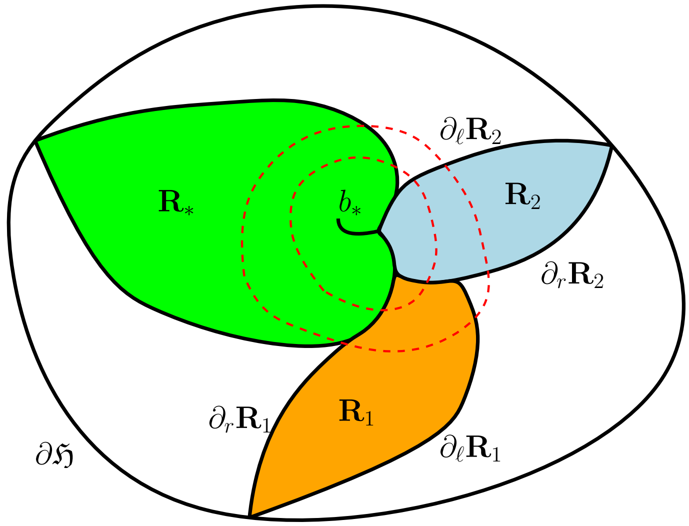

FIG 1. *A schematic representation of the standard hull $\mathfrak{H}$ and the slices $\mathbf{R}_*$, $\mathbf{R}_1$, $\mathbf{R}_2$ in the case when E_ε^m holds with $m = 2$. The two dashed cycles are the boundaries of the hulls of respective radii $1 - 2^{-k}$ and $1 - 2^{-k-3}$ centered at b_*. The end of the proof of Proposition 14 relies on the fact that a continuous path starting on the left boundary $\partial_\ell \mathbf{R}_1$ and ending on $\partial_\ell \mathbf{R}_2$ and staying in the annulus delimited by the two dashed cycles will have to cross one of the two slices $\mathbf{R}_1$ and $\mathbf{R}_2$.*

for every $t \in [1 - 2^{-k-1}, 1 - 2^{-k-2}]$, for every $k \geq 1$ such that $2^{-k-4} > \varepsilon$, and for every $0 \leq i < j \leq m$.

For simplicity we take $i = 1$ and $j = 2$ but the same argument works as well for any choice of i and j. Fix $k \geq 1$ such that $2^{-k-4} > \varepsilon$ and $t_0 \in [1 - 2^{-k-1}, 1 - 2^{-k-2}]$. Consider a continuous path $(\phi(r))_{0 \leq r \leq 1}$ such that $\phi(0) = \gamma_{s'_1}(t_0) \in \partial_\ell \mathbf{R}_1$ and $\phi(1) = \gamma_{s'_2}(t_0) \in \partial_\ell \mathbf{R}_2$. We need to show that the length of ϕ is bounded below by δ_k. We may and will assume that $D_{\mathbf{H}}(b_*, \phi(r)) \in [2^{-k-3}, 2^{-k}]$ for every $r \in [0, 1]$, because otherwise the length of ϕ will be bounded below by $2^{-k-3} \geq \delta_k$.

We then set

$$r_0 := \sup\{r \in [0,1] : \phi(r) \in \partial_\ell \mathbf{R}_1\}, \quad r_1 := \inf\{r \in [r_0, 1] : \phi(r) \in \partial_r \mathbf{R}_1 \cup \partial_r \mathbf{R}_2\}.$$

We note that the set $\{r \in [0,1] : \phi(r) \in \partial_r \mathbf{R}_1 \cup \partial_r \mathbf{R}_2\}$ is not empty, so that the definition of r_1 makes sense. Indeed, from the construction of $\mathfrak{H}$ (or via a planarity argument), it is easy to verify that a path $(\psi(r))_{r \in [0,1]}$ starting on $\partial_\ell \mathbf{R}_1$ and ending on $\partial_\ell \mathbf{R}_2$, such that $D_{\mathbf{H}}(b_*, \psi(r)) \geq \varepsilon$ for every $r \in [0, 1]$, must hit $\partial_r \mathbf{R}_1 \cup \partial_r \mathbf{R}_2$ before (or at the same time) it hits $\partial_\ell \mathbf{R}_2$ — see Fig. 1 for an illustration.

Then we consider two cases:

- Either $\phi(r_1) \in \partial_r \mathbf{R}_1$, and then the restriction of ϕ to $[r_0, r_1]$ stays in $\mathbf{R}_1$, and is a path to which we can apply the definition of the event $\mathcal{B}_\varepsilon^{m,1}$. It follows that the length of this restricted path is at least δ_k.
- Or $\phi(r_1) \in \partial_r \mathbf{R}_2$, then we set

$$r_2 := \sup\{r \in [r_1, 1] : \phi(r) \in \partial_r \mathbf{R}_2\}, \quad r_3 := \inf\{r \in [r_2, 1] : \phi(r) \in \partial_\ell \mathbf{R}_2\},$$

  and the restriction of ϕ to $[r_2, r_3]$ is a path that stays in $\mathbf{R}_2$ and to which we can apply the definition of the event $\mathcal{B}_\varepsilon^{m,2}$. Again it follows that the length of this restricted path is bounded below by δ_k.

In both cases, we conclude that the length of ϕ is at least δ_k. Hence the lower bound (33) holds (for $i=1$ and $j=2$). We have thus obtained that $\mathcal{B}^m_\varepsilon \subset \mathcal{A}^m_\varepsilon$, which completes the proof of the proposition. $\square$

5. The key estimate. As previously in Corollary 9, it will be convenient to use the notation $\mathcal{Z}_r^{x_*(x_0)} = \mathcal{Z}_{W_*+r}$, for $r>0$ such that $W_* < -r$. It will also be useful to consider the boundary size $\mathcal{Z}_r^{z(x_*)}$ of the hull $B_r^{\bullet(x_*)}(z)$, for every $z \in \mathbf{m}_\infty$ such that $D(x_*,z) > r$. To this end, we can set

$$\mathcal{Z}_r^{z(x_*)} = \liminf_{\varepsilon\to 0} \varepsilon^{-2}\mathrm{Vol}((B_r^{\bullet(x_*)}(z))^c \cap B_{r+\varepsilon}(z))\,, \tag{34}$$

and we know from (19) and the symmetry properties of the Brownian sphere that the liminf in the last display is indeed a limit for a.e. z (such that $D(x_*,z)>r$), with respect to the volume measure on $\mathbf{m}_\infty$. We can similarly consider the boundary size $\mathcal{Z}_r^{x_*(z)}$ of the hull $B_r^{\bullet(z)}(x_*)$.

As in the previous section, the integer $m \in \{1,2,3\}$ is fixed and we recall the definition of $F^{(m)}_{\varepsilon,r}(x,y)$ and $\widetilde{\mathbb{N}}_0$ from the beginning of Section 4.3. Also recall that $F^{(m)}_{\varepsilon,r}(x,y) = 0$ if $D(x,y) \le r$.

LEMMA 15. *Let $\delta \in (0,1)$. There exists a constant $C_{(\delta)}$ such that, for every $\varepsilon \in (0,1/8)$ and every integer $k \ge 0$ such that $2^{-k} > 2\varepsilon$,*

$$\widetilde{\mathbb{N}}_0\left(\iint \mathrm{Vol}(\mathrm{d}x)\mathrm{Vol}(\mathrm{d}y)\,\mathbf{1}_{\{D(x,y)\in[2^{-k+2},2^{-k+3}]\}} F^{(m)}_{\varepsilon,1}(x,x_*)\,F^{(m)}_{\varepsilon,1}(y,x_*)\right) \le C_{(\delta)}\,2^{-(4-m)k+\delta k}\,\varepsilon^{2m}. \tag{35}$$

PROOF. To simply notation, we write $F_{\varepsilon,r}(x,y)$ instead of $F^{(m)}_{\varepsilon,r}(x,y)$ in the proof. Let $A_{\varepsilon,k}$ denote the left-hand side of (35). We can write $A_{\varepsilon,k}$ in the form

$$A_{\varepsilon,k} = \mathbb{N}_0\Big(\sigma \iint \frac{\mathrm{Vol}(\mathrm{d}x)}{\sigma}\,\frac{\mathrm{Vol}(\mathrm{d}y)}{\sigma}\,\Gamma_{\varepsilon,k}(x_*,x,y)\Big),$$

with an appropriate function $\Gamma_{\varepsilon,k}$. Thanks to (13), we have also

$$A_{\varepsilon,k} = \mathbb{N}_0\Big(\sigma \int \frac{\mathrm{Vol}(\mathrm{d}z)}{\sigma}\,\Gamma_{\varepsilon,k}(z,x_*,x_0)\Big),$$

which leads to

$$A_{\varepsilon,k} = \mathbb{N}_0\left(\mathbf{1}_{\{D(x_*,x_0)\in[2^{-k+2},2^{-k+3}]\}}\int \mathrm{Vol}(\mathrm{d}z)\,F_{\varepsilon,1}(x_*,z)\,F_{\varepsilon,1}(x_0,z)\right).$$

We write $A_{\varepsilon,k} = A'_{\varepsilon,k} + A''_{\varepsilon,k}$, where $A'_{\varepsilon,k}$ is obtained from the right-hand side of the last display by restricting the integral with respect to $\mathrm{Vol}(\mathrm{d}z)$ to the set $(B^{\bullet(x_0)}_{2^{-k}}(x_*) \cup B^{\bullet(x_*)}_{2^{-k}}(x_0))^c$.

First step. We start by estimating $A'_{\varepsilon,k}$. Note that the property $z \notin B^{\bullet(x_0)}_{2^{-k}}(x_*) \cup B^{\bullet(x_*)}_{2^{-k}}(x_0)$ means that z and x_0 are in the same connected component of $(B_{2^{-k}}(x_*))^c$, and similarly z and x_* are in the same connected component of $(B_{2^{-k}}(x_0))^c$. Consequently, under the condition $z \notin B^{\bullet(x_0)}_{2^{-k}}(x_*) \cup B^{\bullet(x_*)}_{2^{-k}}(x_0)$, we have

$$B^{\bullet(z)}_{2^{-k}}(x_*) = B^{\bullet(x_0)}_{2^{-k}}(x_*)\,, \qquad B^{\bullet(z)}_{2^{-k}}(x_0) = B^{\bullet(x_*)}_{2^{-k}}(x_0). \tag{36}$$

We next observe that

$$F_{\varepsilon,1}(x_*,z) \le F_{\varepsilon,2^{-k}}(x_*,z) \times F_{2^{-k+4},1}(x_*,z),$$

where we recall our convention for $F_{2^{-k+4},1}(x_*,z)$ when $2^{-k+4} \ge 1$. We have also $F_{\varepsilon,1}(x_0,z) \le F_{\varepsilon,2^{-k}}(x_0,z)$, so that $A'_{\varepsilon,k}$ is bounded above by

$$\mathbb{N}_0\Bigg(\mathbf{1}_{\{D(x_*,x_0)\in[2^{-k+2},2^{-k+3}]\}} \times \int_{(B^{\bullet(x_0)}_{2^{-k}}(x_*)\cup B^{\bullet(x_*)}_{2^{-k}}(x_0))^c} \mathrm{Vol}(\mathrm{d}z)\, F_{\varepsilon,2^{-k}}(x_*,z)\, F_{2^{-k+4},1}(x_*,z)\, F_{\varepsilon,2^{-k}}(x_0,z)\Bigg).$$

Under the condition $z \notin B^{\bullet(x_0)}_{2^{-k}}(x_*) \cup B^{\bullet(x_*)}_{2^{-k}}(x_0)$, (36) holds, which implies

$$\mathbf{1}_{\{D(x_0,x_*)>2^{-k}\}}\, F_{\varepsilon,2^{-k}}(x_*,z) \le F_{\varepsilon,2^{-k}}(x_*,x_0),$$
$$\mathbf{1}_{\{D(x_0,x_*)>2^{-k}\}}\, F_{\varepsilon,2^{-k}}(x_0,z) \le F_{\varepsilon,2^{-k}}(x_0,x_*).$$

It follows that

$$A'_{\varepsilon,k} \le \mathbb{N}_0\Big(\mathbf{1}_{\{D(x_*,x_0)\in[2^{-k+2},2^{-k+3}]\}}\, F_{\varepsilon,2^{-k}}(x_*,x_0)\, F_{\varepsilon,2^{-k}}(x_0,x_*) \times \int_{(B^{\bullet(x_0)}_{2^{-k}}(x_*)\cup B^{\bullet(x_*)}_{2^{-k}}(x_0))^c} \mathrm{Vol}(\mathrm{d}z)\, F_{2^{-k+4},1}(x_*,z)\Big). \tag{37}$$

We now want to apply Corollary 9 to the right-hand side. We observe that $F_{\varepsilon,2^{-k}}(x_*,x_0)$ is a function of the hull $B^{\bullet(x_0)}_{2^{-k}}(x_*)$, and $F_{\varepsilon,2^{-k}}(x_0,x_*)$ is the same function applied to the hull $B^{\bullet(x_*)}_{2^{-k}}(x_0)$. On the other hand, the quantity

$$\mathbf{1}_{\{D(x_*,x_0)\in[2^{-k+2},2^{-k+3}]\}} \int_{(B^{\bullet(x_0)}_{2^{-k}}(x_*)\cup B^{\bullet(x_*)}_{2^{-k}}(x_0))^c} \mathrm{Vol}(\mathrm{d}z)\, F_{2^{-k+4},1}(x_*,z) \tag{38}$$

is a function of $\overline{\mathcal{C}^{x_*,x_0}_{2^{-k}}}$, with the notation of Corollary 9. Let us explain this in the case where $2^{-k+4} < 1$ (the case $2^{-k+4} \ge 1$ is easier and left to the reader). The indicator function $\mathbf{1}_{\{D(x_*,x_0)\in[2^{-k+2},2^{-k+3}]\}}$ is the indicator function of the event where the (intrinsic) distance between the two boundaries of $\overline{\mathcal{C}^{x_*,x_0}_{2^{-k}}}$ lies between $2^{-k+2} - 2^{-k+1}$ and $2^{-k+3} - 2^{-k+1}$. Furthermore, if $D(x_*,x_0) \in [2^{-k+2}, 2^{-k+3}]$ and $D(x_*,z) > 1$, we have $(B^{\bullet(x_0)}_{2^{-k}}(x_*) \cup B^{\bullet(x_*)}_{2^{-k}}(x_0)) \subset B^{\bullet(z)}_{2^{-k+4}}(x_*)$. Let Δ_1 stand for the first boundary of $\overline{\mathcal{C}^{x_*,x_0}_{2^{-k}}}$ (that is, $\Delta_1 = \partial B^{\bullet(x_0)}_{2^{-k}}(x_*)$) and, using our notation $D^{\mathrm{intr},\mathcal{C}}$ for the (extended) intrinsic distance on $\overline{\mathcal{C}^{x_*,x_0}_{2^{-k}}}$, define the hull of radius r centered at Δ_1 relative to z as the complement of the connected component of $\{x \in \overline{\mathcal{C}^{x_*,x_0}_{2^{-k}}} : D^{\mathrm{intr},\mathcal{C}}(x,\Delta_1) > r\}$ that contains z (this makes sense if $D^{\mathrm{intr},\mathcal{C}}(\Delta_1,z) > r$). It follows from the preceding considerations that the integral with respect to $\mathrm{Vol}(\mathrm{d}z)$ in (38) can be rewritten as

$$\int_{\overline{\mathcal{C}^{x_*,x_0}_{2^{-k}}}} \mathrm{Vol}(\mathrm{d}z)\, G_k(z),$$

where $G_k(z) \in \{0,1\}$ and $G_k(z) = 1$ if and only if the (intrinsic) distance between z and Δ_1 is greater than $1 - 2^{-k}$, and if there are $m+1$ disjoint paths of (intrinsic) length $1 - 2^{-k+4}$ between the boundary of the hull of radius $1 - 2^{-k}$ centered at Δ_1 relative to z and the boundary of the same hull of radius $2^{-k+4} - 2^{-k}$.

Thanks to the previous observations, we can apply Corollary 9 to the right-hand side of (37), and we get that

$$A'_{\varepsilon,k} \leq \mathbb{N}_0\bigg(\mathbf{1}_{\{D(x_*,x_0)\in[2^{-k+2},2^{-k+3}]\}}\,\Theta_{2^{-k},\mathcal{Z}^{x_*(x_0)}_{2^{-k}}}(H_\varepsilon)\,\Theta_{2^{-k},\mathcal{Z}^{x_0(x_*)}_{2^{-k}}}(H_\varepsilon)$$

$$\times\int_{(B^{\bullet(x_0)}_{2^{-k}}(x_*)\cup B^{\bullet(x_*)}_{2^{-k}}(x_0))^c}\mathrm{Vol}(\mathrm{d}z)\,F_{2^{-k+4},1}(x_*,z)\bigg), \tag{39}$$

where H_ε (applied to a space belonging to $\mathbb{M}^{\bullet b}$) denotes the event where there are $m+1$ geodesic paths from the boundary to the distinguished point, that are disjoint up to the time when there are at distance ε from the distinguished point. We note that, thanks to Proposition 7, we have

$$\Theta_{2^{-k},z}(H_\varepsilon) \leq C\,(\varepsilon z 2^{3k})^m \wedge 1, \tag{40}$$

with a universal constant C. To simplify notation, we write $\varphi_{\varepsilon,k}(z) = \Theta_{2^{-k},z}(H_\varepsilon)$.

Thanks to the symmetry properties of the Brownian sphere (13), we can interchange the roles of z and x_0 in (39), and we arrive at

$$A'_{\varepsilon,k} \leq \mathbb{N}_0\Big(F_{2^{-k+4},1}(x_*,x_0)\int\mathrm{Vol}(\mathrm{d}z)\,\mathbf{1}_{\{D(x_*,z)\in[2^{-k+2},2^{-k+3}]\}}$$

$$\times\,\varphi_{\varepsilon,k}(\mathcal{Z}^{x_*(z)}_{2^{-k}})\,\varphi_{\varepsilon,k}(\mathcal{Z}^{z(x_*)}_{2^{-k}})\mathbf{1}_{\{x_0\notin B^{\bullet(z)}_{2^{-k}}(x_*)\cup B^{\bullet(x_*)}_{2^{-k}}(z)\}}\Big). \tag{41}$$

Under the condition $x_0\notin B^{\bullet(z)}_{2^{-k}}(x_*)\cup B^{\bullet(x_*)}_{2^{-k}}(z)$, we have $B^{\bullet(x_*)}_{2^{-k}}(z) = B^{\bullet(x_0)}_{2^{-k}}(z)$ and $B^{\bullet(z)}_{2^{-k}}(x_*) = B^{\bullet(x_0)}_{2^{-k}}(x_*)$. It follows in particular that $\mathcal{Z}^{x_*(z)}_{2^{-k}} = \mathcal{Z}^{x_*(x_0)}_{2^{-k}}$. Hence, the right-hand side of (41) is equal to

$$\mathbb{N}_0\Big(F_{2^{-k+4},1}(x_*,x_0)\,\varphi_{\varepsilon,k}(\mathcal{Z}^{x_*(x_0)}_{2^{-k}})\int\mathrm{Vol}(\mathrm{d}z)\,\mathbf{1}_{\{D(x_*,z)\in[2^{-k+2},2^{-k+3}]\}}$$

$$\times\,\varphi_{\varepsilon,k}(\mathcal{Z}^{z(x_*)}_{2^{-k}})\mathbf{1}_{\{x_0\notin B^{\bullet(z)}_{2^{-k}}(x_*)\cup B^{\bullet(x_*)}_{2^{-k}}(z)\}}\Big). \tag{42}$$

Let us assume that $2^{-k+4}<1$ and argue on the event where $D(x_*,x_0)>1$. We observe that the quantity

$$\varphi_{\varepsilon,k}(\mathcal{Z}^{x_*(x_0)}_{2^{-k}})\int\mathrm{Vol}(\mathrm{d}z)\,\mathbf{1}_{\{D(x_*,z)\in[2^{-k+2},2^{-k+3}]\}}\,\varphi_{\varepsilon,k}(\mathcal{Z}^{z(x_*)}_{2^{-k}})\mathbf{1}_{\{x_0\notin B^{\bullet(z)}_{2^{-k}}(x_*)\cup B^{\bullet(x_*)}_{2^{-k}}(z)\}}$$

is a function of $B^{\bullet(x_0)}_{2^{-k+4}}(x_*)$. To this end, note that the condition $D(x_*,z)\in[2^{-k+2},2^{-k+3}]$ implies that $z\in B_{2^{-k+4}}(x_*)$, and also that $B^{\bullet(x_0)}_{2^{-k}}(z)\subset B^{\bullet(x_0)}_{2^{-k+4}}(x_*)$ (if $y\in B^{\bullet(x_0)}_{2^{-k}}(z)$, any path from y to x_0 has to intersect $\partial B^{\bullet(x_0)}_{2^{-k}}(z)$, which is contained in $B_{2^{-k+4}}(x_*)$, and this exactly means that $y\in B^{\bullet(x_0)}_{2^{-k+4}}(x_*)$). Using (19), it follows that, under the conditions $x_0\notin B^{\bullet(z)}_{2^{-k}}(x_*)\cup B^{\bullet(x_*)}_{2^{-k}}(z)$ and $D(x_*,z)\in[2^{-k+2},2^{-k+3}]$, the quantity $\mathcal{Z}^{z(x_*)}_{2^{-k}} = \mathcal{Z}^{z(x_0)}_{2^{-k}}$ is determined by the hull $B^{\bullet(x_0)}_{2^{-k+4}}(x_*)$, and clearly the same is true for $\mathcal{Z}^{x_*(x_0)}_{2^{-k}}$. In addition, the condition $x_0\notin B^{\bullet(z)}_{2^{-k}}(x_*)\cup B^{\bullet(x_*)}_{2^{-k}}(z)$ can also be expressed in terms of the hull $B^{\bullet(x_0)}_{2^{-k+4}}(x_*)$ since it is equivalent to saying that z is in the same connected component of $B_{2^{-k}}(x_*)^c$ as the boundary of this hull, and similarly if x_* and z are interchanged.

On the other hand, $F_{2^{-k+4},1}(x_*,x_0)$ is a function of $\mathbf{m}_\infty\backslash\overline{B^{\bullet(x_0)}_{2^{-k+4}}(x_*)}$ with the notation of Theorem 8. This theorem, or more precisely the remark following the statement of the theorem, shows that $B^{\bullet(x_0)}_{2^{-k+4}}(x_*)$ and $\overline{\mathbf{m}_\infty\backslash B^{\bullet(x_0)}_{2^{-k+4}}(x_*)}$ are conditionally independent given $\mathcal{Z}^{x_*(x_0)}_{2^{-k+4}}$ under $\mathbb{N}^{[1]}_0$. Recalling that $F_{2^{-k+4},1}(x_*,x_0)$ can be nonzero only if $D(x_*,x_0)>1$, it follows that the quantity (42) is equal to

$$\frac{3}{2}\,\mathbb{N}^{[1]}_0\bigg(\mathbb{N}^{[1]}_0\Big(F_{2^{-k+4},1}(x_*,x_0)\,\Big|\,\mathcal{Z}^{x_*(x_0)}_{2^{-k+4}}\Big)\varphi_{\varepsilon,k}(\mathcal{Z}^{x_*(x_0)}_{2^{-k}}) \times\int \mathrm{Vol}(\mathrm{d}z)\,\mathbf{1}_{\{D(x_*,z)\in[2^{-k+2},2^{-k+3}]\}}\,\varphi_{\varepsilon,k}(\mathcal{Z}^{z(x_*)}_{2^{-k}})\mathbf{1}_{\{x_0\notin B^{\bullet(z)}_{2^{-k}}(x_*)\cup B^{\bullet(x_*)}_{2^{-k}}(z)\}}\bigg). \tag{43}$$

We now need a lemma.

LEMMA 16. *There exists a constant C_m such that, for every $\varepsilon\in(0,1/2]$ and $z>0$*

$$\mathbb{N}^{[1]}_0\Big(F_{\varepsilon,1}(x_*,x_0)\,\Big|\,\mathcal{Z}^{x_*(x_0)}_{\varepsilon}=z\Big)\le C_m\,z^{m/2}.$$

We postpone the proof of this lemma to Appendix A. Thanks to the lemma, the quantity (43) can be bounded above by

$$C'_m\mathbb{N}^{[1]}_0\Big((\mathcal{Z}^{x_*(x_0)}_{2^{-k+4}})^{m/2}\,\varphi_{\varepsilon,k}(\mathcal{Z}^{x_*(x_0)}_{2^{-k}})\int \mathrm{Vol}(\mathrm{d}z)\,\mathbf{1}_{\{D(x_*,z)\in[2^{-k+2},2^{-k+3}]\}}\,\varphi_{\varepsilon,k}(\mathcal{Z}^{z(x_*)}_{2^{-k}})\Big) \tag{44}$$

where C'_m is a constant.

We have assumed that $2^{-k+4}<1$, but if $2^{-k+4}\ge 1$, replacing $F_{2^{-k+4},1}(x_*,x_0)$ by $\mathbf{1}_{\{D(x_*,x_0)>1\}}$ immediately shows that (42) is also bounded by a quantity similar to (44) without the term $(\mathcal{Z}^{x_*(x_0)}_{2^{-k+4}})^{m/2}$. For simplicity, we assume until the end of the first step that $2^{-k+4}<1$, but clearly the bounds that follow are also valid when $2^{-k+4}\ge 1$.

We use the Cauchy-Schwarz inequality to bound the quantity (44) by

$$C'_m\,\mathbb{N}^{[1]}_0\Big((\mathcal{Z}^{x_*(x_0)}_{2^{-k+4}})^{m}\,\varphi_{\varepsilon,k}(\mathcal{Z}^{x_*(x_0)}_{2^{-k}})^2\Big)^{1/2} \times\mathbb{N}^{[1]}_0\Big(\Big(\int \mathrm{Vol}(\mathrm{d}z)\,\mathbf{1}_{\{D(x_*,z)\in[2^{-k+2},2^{-k+3}]\}}\,\varphi_{\varepsilon,k}(\mathcal{Z}^{z(x_*)}_{2^{-k}})\Big)^2\Big)^{1/2}. \tag{45}$$

Consider the first term of the product, recalling that, by definition, $\mathcal{Z}^{x_*(x_0)}_{2^{-k}}=\mathcal{Z}_{W_*+2^{-k}}$. Using the bound (40) and again the Cauchy-Schwarz inequality, we get that

$$\mathbb{N}^{[1]}_0\Big((\mathcal{Z}^{x_*(x_0)}_{2^{-k+4}})^{m}\,\varphi_{\varepsilon,k}(\mathcal{Z}^{x_*(x_0)}_{2^{-k}})^2\Big)^{1/2}\le C\,(\varepsilon 2^{3k})^m\,\mathbb{N}^{[1]}_0((\mathcal{Z}^{x_*(x_0)}_{2^{-k}})^{4m})^{1/4}\mathbb{N}^{[1]}_0((\mathcal{Z}^{x_*(x_0)}_{2^{-k+4}})^{2m})^{1/4}.$$

Thanks to the bound (15), we arrive at

$$\mathbb{N}^{[1]}_0\Big((\mathcal{Z}^{x_*(x_0)}_{2^{-k+4}})^{m}\,\varphi_{\varepsilon,k}(\mathcal{Z}^{x_*(x_0)}_{2^{-k}})^2\Big)^{1/2}\le C(\varepsilon 2^{3k})^m c_{4m}^{1/4}\,2^{-2mk}\,c_{2m}^{1/4}\,2^{-m(k-4)}=C''_m\,\varepsilon^m. \tag{46}$$

We then estimate the second term in the product of (45). By the bound (40), we have

$$\begin{aligned}&\mathbb{N}^{[1]}_0\Big(\Big(\int \mathrm{Vol}(\mathrm{d}z)\,\mathbf{1}_{\{D(x_*,z)\in[2^{-k+2},2^{-k+3}]\}}\,\varphi_{\varepsilon,k}(\mathcal{Z}^{z(x_*)}_{2^{-k}})\Big)^2\Big)^{1/2}\\
&\quad\le C\,(\varepsilon 2^{3k})^m\,\mathbb{N}^{[1]}_0\Big(\Big(\int \mathrm{Vol}(\mathrm{d}z)\,\mathbf{1}_{\{D(x_*,z)\in[2^{-k+2},2^{-k+3}]\}}\,(\mathcal{Z}^{z(x_*)}_{2^{-k}})^m\Big)^2\Big)^{1/2}\\
&\quad\le C\,(\varepsilon 2^{3k})^m\,\mathbb{N}^{[1]}_0\Big(\mathrm{Vol}(B_{2^{-k+3}}(x_*))\int \mathrm{Vol}(\mathrm{d}z)\,\mathbf{1}_{\{D(x_*,z)\in[2^{-k+2},2^{-k+3}]\}}\,(\mathcal{Z}^{z(x_*)}_{2^{-k}})^{2m}\Big)^{1/2}\end{aligned}$$

$$= \frac{2}{3} C\,(\varepsilon 2^{3k})^m\,\mathbb{N}_0\Big(\mathbf{1}_{\{D(x_*,x_0)>1\}}\mathrm{Vol}(B_{2^{-k+3}}(x_*))\\ \times \int \mathrm{Vol}(\mathrm{d}z)\,\mathbf{1}_{\{D(x_*,z)\in[2^{-k+2},2^{-k+3}]\}}\,(\mathcal{Z}^{z(x_*)}_{2^{-k}})^{2m}\Big)^{1/2},$$

using the Cauchy-Schwarz inequality and then the definition of $\mathbb{N}_0^{[1]}$. In the last integral under $\mathbb{N}_0$, we can use (13) to interchange the roles of x_0 and z, and then get that the right-hand side of the last display is equal to

$$\frac{2}{3} C\,(\varepsilon 2^{3k})^m \mathbb{N}_0\bigg(\Big(\int \mathrm{Vol}(\mathrm{d}z)\,\mathbf{1}_{\{D(x_*,z)>1\}}\Big)\mathrm{Vol}(B_{2^{-k+3}}(x_*))\\ \times \mathbf{1}_{\{D(x_*,x_0)\in[2^{-k+2},2^{-k+3}]\}}\,(\mathcal{Z}^{x_0(x_*)}_{2^{-k}})^{2m}\bigg)^{1/2}. \tag{47}$$

Fix an integer $p \geq 4$ and set $q = p/(p-1)$. We assume that p is chosen sufficiently large so that $\frac{2}{q} - \frac{1}{pq} - \frac{1}{p} > 2 - \delta$. By the Hölder inequality, we have

(48)

$$\mathbb{N}_0\bigg(\Big(\int \mathrm{Vol}(\mathrm{d}z)\,\mathbf{1}_{\{D(x_*,z)>1\}}\Big)\mathrm{Vol}(B_{2^{-k+3}}(x_*))\,\mathbf{1}_{\{D(x_*,x_0)\in[2^{-k+2},2^{-k+3}]\}}\,(\mathcal{Z}^{x_0(x_*)}_{2^{-k}})^{2m}\bigg)\\ \leq \mathbb{N}_0\Big(\mathbf{1}_{\{D(x_*,x_0)\in[2^{-k+2},2^{-k+3}]\}}\,(\mathrm{Vol}(B_1(x_*)^c))^q\Big)^{1/q}\\ \times \mathbb{N}_0\Big(\mathbf{1}_{\{D(x_*,x_0)\in[2^{-k+2},2^{-k+3}]\}}\,(\mathrm{Vol}(B_{2^{-k+3}}(x_*))^p(\mathcal{Z}^{x_0(x_*)}_{2^{-k}})^{2pm}\Big)^{1/p}\\ = A^{1/q} \times B^{1/p}.$$

Let us estimate A and B separately. Using the fact that (conditionally on σ) x_0 is uniformly distributed over $\mathbf{m}_\infty$, we get

$$\begin{aligned} A &\leq \mathbb{N}_0\Big(\frac{1}{\sigma}\mathrm{Vol}(B_{2^{-k+3}}(x_*)\,(\mathrm{Vol}(B_1(x_*)^c))^q\Big) \\ &\leq \mathbb{N}_0\Big(\sigma^{-q}(\mathrm{Vol}(B_1(x_*)^c))^{q^2}\Big)^{1/q} \times \mathbb{N}_0\Big(\mathrm{Vol}(B_{2^{-k+3}}(x_*))^p\Big)^{1/p} \\ &\leq K_q \times \Big(c_{(p)}(2^{-k+3})^{4p-2}\Big)^{1/p} \\ &= K'_q\, 2^{-4k+2k/p}, \end{aligned} \tag{49}$$

using the bounds on moments of the volume of balls (Lemma 4), together with the fact that

$$\mathbb{N}_0\Big(\sigma^{-q}(\mathrm{Vol}(B_1(x_*)^c))^{q^2}\Big) < \infty.$$

The latter estimate is easily obtained by writing, for every $s > 0$,

$$\mathbb{N}_0\Big(\sigma^{-q}(\mathrm{Vol}(B_1(x_*)^c))^{q^2}\,\Big|\,\sigma = s\Big) \leq s^{q^2-q}\,\mathbb{N}_0(B_1(x_*)^c \neq \varnothing \,|\, \sigma = s) \leq C\, s^{q^2-q}\exp(-c\, s^{-1/3})$$

by an estimate of [29, Proposition 14]. The right-hand side of the last display is integrable with respect to $s^{-3/2}\mathrm{d}s$ as soon as $q^2 - q < 1/2$, which holds here because $p \geq 4$.

Let us consider now the quantity B. By the Cauchy-Schwarz inequality, $B \leq \sqrt{B'B''}$, where

$$B' = \mathbb{N}_0\Big(\mathbf{1}_{\{D(x_*,x_0)\in[2^{-k+2},2^{-k+3}]\}}\,(\mathrm{Vol}(B_{2^{-k+3}}(x_*))^{2p}\Big),$$

$$B'' = \mathbb{N}_0\Big(\mathbf{1}_{\{D(x_*,x_0)\in[2^{-k+2},2^{-k+3}]\}}\,(\mathcal{Z}^{x_0(x_*)}_{2^{-k}})^{4pm}\Big).$$

We have first

$$B' \leq \mathbb{N}_0\Big((\mathrm{Vol}(B_{2^{-k+3}}(x_*))^{2p}\Big) \leq c_{(2p)}(2^{-k+3})^{8p-2} \leq \tilde{c}_p\,2^{-8pk+2k},$$

using again Lemma 4. Then,

$$B'' \leq \mathbb{N}_0\Big((\mathcal{Z}^{x_0(x_*)}_{2^{-k}})^{4pm}\Big) = \mathbb{N}_0\Big((\mathcal{Z}^{x_*(x_0)}_{2^{-k}})^{4pm}\Big) = \mathbb{N}_0((\mathcal{Z}_{W_*+2^{-k}})^{4pm}),$$

where we made the convention that all quantities $\mathcal{Z}^{x_0(x_*)}_{2^{-k}}$, $\mathcal{Z}^{x_*(x_0)}_{2^{-k}}$, $\mathcal{Z}_{W_*+2^{-k}}$ are equal to 0 if $D(x_*,x_0) = -W_* \leq 2^{-k}$. From (15), we now get

$$B'' \leq \mathbb{N}_0(W_* < -2^{-k}) \times \mathbb{N}_0^{[2^{-k}]}\Big((\mathcal{Z}_{W_*+2^{-k}})^{4pm}\Big) \leq \frac{3}{2}\,2^{2k} \times c_{4pm}\,2^{-8pmk} = c''_p\,2^{-8pmk+2k}.$$

By combining our estimates on B' and B'', we arrive at

$$B \leq \sqrt{B'B''} \leq \bar{c}_p\,2^{-4p(m+1)k+2k}.$$

Using also (48) and (49), we get that the quantity (47) is bounded above by

$$\begin{aligned}\frac{2}{3}C\,(\varepsilon 2^{3k})^m &\times \Big((K'_q\,2^{-4k+2k/p})^{1/q}\,(\bar{c}_p\,2^{-4p(m+1)k+2k})^{1/p}\Big)^{1/2}\\ &= \overline{C}_p\,\varepsilon^m\,2^{-k(-3m+\frac{2}{q}-\frac{1}{pq}+2(m+1)-\frac{1}{p})}\\ &\leq \overline{C}'_p\,\varepsilon^m\,2^{-(4-m)k+\delta k},\end{aligned}$$

by our choice of p. By combining this estimate with (46), we obtain that the quantity (45) is bounded above by

$$C'_m \times C''_m\varepsilon^m \times \overline{C}'\,\varepsilon^m\,2^{-(4-m)k+\delta k} = \widetilde{C}_m\varepsilon^{2m}\,2^{-(4-m)k+\delta k}.$$

Since $A'_{\varepsilon,k}$ was bounded above by the quantity (45), we have obtained the desired bound for $A'_{\varepsilon,k}$.

Second step. We now need to get a similar bound for

$$A''_{\varepsilon,k} = \mathbb{N}_0\left(\mathbf{1}_{\{D(x_*,x_0)\in[2^{-k+2},2^{-k+3}]\}}\int_{B^{\bullet(x_0)}_{2^{-k}}(x_*)\cup B^{\bullet(x_*)}_{2^{-k}}(x_0)}\mathrm{Vol}(\mathrm{d}z)\,F_{\varepsilon,1}(x_*,z)\,F_{\varepsilon,1}(x_0,z)\right).$$

For obvious symmetry reasons (we can interchange x_* and x_0), it is enough to bound

$$\begin{aligned}(50)\quad A'''_{\varepsilon,k} &:= \mathbb{N}_0\left(\mathbf{1}_{\{D(x_*,x_0)\in[2^{-k+2},2^{-k+3}]\}}\int_{B^{\bullet(x_0)}_{2^{-k}}(x_*)}\mathrm{Vol}(\mathrm{d}z)\,F_{\varepsilon,1}(x_*,z)\,F_{\varepsilon,1}(x_0,z)\right)\\ &\leq \mathbb{N}_0\left(\mathbf{1}_{\{D(x_*,x_0)\in[2^{-k+2},2^{-k+3}]\}}\int_{B^{\bullet(x_0)}_{2^{-k}}(x_*)}\mathrm{Vol}(\mathrm{d}z)\,F_{\varepsilon,1}(x_*,z)\,F_{\varepsilon,2^{-k}}(x_0,z)\right).\end{aligned}$$

Let us argue on the event where $D(x_*,x_0) \in [2^{-k+2},2^{-k+3}]$, and note that this property implies $B^{\bullet(x_0)}_{2^{-k}}(x_*) \cap B^{\bullet(x_*)}_{2^{-k}}(x_0) = \varnothing$. In the integral with respect to $\mathrm{Vol}(\mathrm{d}z)$, we consider

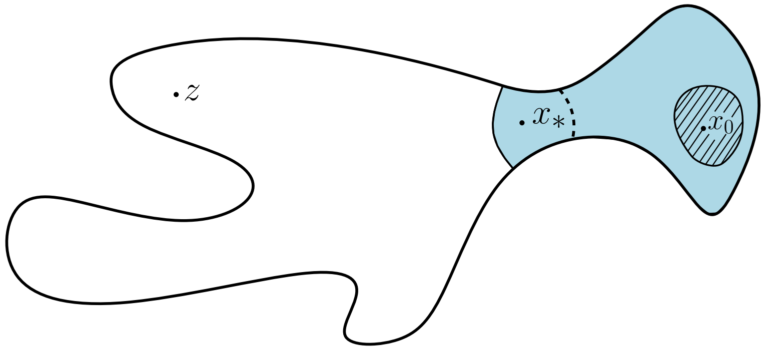

FIG 2. *Illustration of a "bottleneck" case where $z \in B^{\bullet(x_0)}_{2^{-k}}(x_*)$ and $B^{\bullet(x_*)}_{2^{-k}}(x_0) \subset B^{\bullet(z)}_{2^{-k}}(x_*)$. The shaded part represents $B^{\bullet(z)}_{2^{-k}}(x_*)$, and the hatched part represents $B^{\bullet(x_*)}_{2^{-k}}(x_0)$. The dashed curve is meant to represent the boundary of $B^{\bullet(x_0)}_{2^{-k}}(x_*)$.*

points z such that $D(x_*, z) > 1$ (otherwise $F_{\varepsilon,1}(x_*, z) = 0$) and the fact that $z \in B^{\bullet(x_0)}_{2^{-k}}(x_*)$ is equivalent to saying that x_0 and z are different components of $B_{2^{-k}}(x_*)^c$. Furthermore, we have then

$$B^{\bullet(x_*)}_{2^{-k}}(x_0) \subset B^{\bullet(z)}_{2^{-k}}(x_*) \tag{51}$$

because the condition $D(x_*, x_0) \in [2^{-k+2}, 2^{-k+3}]$ ensures that every point of $\partial B^{\bullet(x_*)}_{2^{-k}}(x_0)$ belongs to the same connected component of $B_{2^{-k}}(x_*)^c$ as x_0, and by the preceding observations this boundary is entirely contained in $B^{\bullet(z)}_{2^{-k}}(x_*)$. See Fig. 2 for an illustration. It now follows that $B^{\bullet(z)}_{2^{-k}}(x_0) = B^{\bullet(x_*)}_{2^{-k}}(x_0)$ because clearly z and x_* are in the same connected component of $B_{2^{-k}}(x_0)^c$ (a geodesic path from x_* to $\partial B^{\bullet(z)}_{2^{-k}}(x_*)$ does not intersect $B_{2^{-k}}(x_0)$ since it stays within distance 2^{-k} from x_*, and then (51) shows that x_* is connected to z by a path avoiding $B_{2^{-k}}(x_0)$).

From the equality $B^{\bullet(z)}_{2^{-k}}(x_0) = B^{\bullet(x_*)}_{2^{-k}}(x_0)$ which holds for the relevant values of z, we conclude that, in the right-hand side of (50), we can replace $F_{\varepsilon,2^{-k}}(x_0, z)$ by $F_{\varepsilon,2^{-k}}(x_0, x_*)$, and we have thus

$$A'''_k \leq \mathbb{N}_0\left(\mathbf{1}_{\{D(x_*,x_0)\in[2^{-k+2},2^{-k+3}]\}}\, F_{\varepsilon,2^{-k}}(x_0, x_*) \int_{B^{\bullet(x_0)}_{2^{-k}}(x_*)} \mathrm{Vol}(\mathrm{d}z)\, F_{\varepsilon,1}(x_*, z)\right) \tag{52}$$

At this point, we observe that $F_{\varepsilon,2^{-k}}(x_0, x_*)$ is a function of the hull $B^{\bullet(x_*)}_{2^{-k}}(x_0)$, whereas one can verify that

$$\mathbf{1}_{\{D(x_*,x_0)\in[2^{-k+2},2^{-k+3}]\}} \int_{B^{\bullet(x_0)}_{2^{-k}}(x_*)} \mathrm{Vol}(\mathrm{d}z)\, F_{\varepsilon,1}(x_*, z)$$

is a function of $\overline{\mathbf{m}_\infty \backslash B^{\bullet(x_*)}_{2^{-k}}(x_0)}$ (in both cases, spaces are equipped with their intrinsic distances). Let us explain this. We know that (under the condition $D(x_*, x_0) \in [2^{-k+2}, 2^{-k+3}]$) we have $B^{\bullet(x_0)}_{2^{-k}}(x_*) \subset \mathbf{m}_\infty \backslash B^{\bullet(x_*)}_{2^{-k}}(x_0)$, and moreover the property $z \in B^{\bullet(x_0)}_{2^{-k}}(x_*)$ holds if and only if z is not in the same component of $B_{2^{-k}}(x_*)^c$ as the boundary of $\mathbf{m}_\infty \backslash B^{\bullet(x_*)}_{2^{-k}}(x_0)$. Then, assuming that $z \in B^{\bullet(x_0)}_{2^{-k}}(x_*)$ and $D(x_*, z) > 1$, we observe that any geodesic from $\partial B^{\bullet(z)}_1(x_*)$ to x_* stays in the same connected component of $B_{2^{-k}}(x_*)^c$ as z until it comes within distance 2^{-k} from x_*, and this component is contained in $\mathbf{m}_\infty \backslash B^{\bullet(x_*)}_{2^{-k}}(x_0)$ by (51).

We then apply Theorem 8 (together with (12)) to get that, under $\mathbb{N}_0(\cdot \mid D(x_0,x_*) > 2^{-k})$ and conditionally on $\mathcal{Z}^{x_0(x_*)}_{2^{-k}}$, $B^{\bullet(x_*)}_{2^{-k}}(x_0)$ and $\overline{\mathbf{m}_\infty \backslash B^{\bullet(x_*)}_{2^{-k}}(x_0)}$ are independent, and the conditional distribution of $B^{\bullet(x_*)}_{2^{-k}}(x_0)$ is the law of a standard hull of radius 2^{-k} and perimeter $\mathcal{Z}^{x_0(x_*)}_{2^{-k}}$. It follows that the right-hand side of (52) is equal to

$$\mathbb{N}_0\left(\mathbf{1}_{\{D(x_*,x_0)\in[2^{-k+2},2^{-k+3}]\}}\,\varphi_{\varepsilon,k}\big(\mathcal{Z}^{x_0(x_*)}_{2^{-k}}\big)\int_{B^{\bullet(x_0)}_{2^{-k}}(x_*)}\mathrm{Vol}(\mathrm{d}z)\,F_{\varepsilon,1}(x_*,z)\right). \tag{53}$$

Thanks to (13), we can now interchange the roles of x_0 and z and get that the quantity (53) is also equal to

$$\mathbb{N}_0\left(F_{\varepsilon,1}(x_*,x_0)\int_{B^{\bullet(x_0)}_{2^{-k}}(x_*)}\mathrm{Vol}(\mathrm{d}z)\,\mathbf{1}_{\{D(x_*,z)\in[2^{-k+2},2^{-k+3}]\}}\,\varphi_{\varepsilon,k}\big(\mathcal{Z}^{z(x_*)}_{2^{-k}}\big)\right) \tag{54}$$

using once again the fact that $z \in B^{\bullet(x_0)}_{2^{-k}}(x_*)$ is equivalent to $x_0 \in B^{\bullet(z)}_{2^{-k}}(x_*)$. Now we observe that

$$F_{\varepsilon,1}(x_*,x_0) \le F_{2^{-k},1}(x_*,x_0)\,F_{\varepsilon,2^{-k}}(x_*,x_0).$$

On one hand, $F_{2^{-k},1}(x_*,x_0)$ is a function of $\overline{\mathbf{m}_\infty \backslash B^{\bullet(x_0)}_{2^{-k}}(x_*)}$, and on the other hand,

$$F_{\varepsilon,2^{-k}}(x_*,x_0)\int_{B^{\bullet(x_0)}_{2^{-k}}(x_*)}\mathrm{Vol}(\mathrm{d}z)\,\mathbf{1}_{\{D(x_*,z)\in[2^{-k+2},2^{-k+3}]\}}\,\varphi_{\varepsilon,k}\big(\mathcal{Z}^{z(x_*)}_{2^{-k}}\big)$$

is a function of the hull $B^{\bullet(x_0)}_{2^{-k}}(x_*)$. For this last point, we observe that, for points z that are relevant in the integral with respect to $\mathrm{Vol}(\mathrm{d}z)$, we have $B^{\bullet(x_*)}_{2^{-k}}(z) \subset B^{\bullet(x_0)}_{2^{-k}}(x_*)$ (by (51), recalling that we interchanged the roles of x_0 and z), and we also note that, if z belongs to the hull $B^{\bullet(x_0)}_{2^{-k}}(x_*)$, a geodesic (with respect to the distance D) from z to x_* may hit the boundary of the hull but cannot exit the hull, so that the intrinsic distance between z and x_* relative to the hull indeed coincides with $D(x_*,z)$. Using Theorem 8, and replacing $\mathbb{N}_0$ by $\mathbb{N}_0^{[1]}$ (recall that $F_{\varepsilon,1}(x_*,x_0)=1$ implies $D(x_*,x_0)>1$), we get that (54) is bounded above by

$$\begin{aligned}
&\frac{3}{2}\mathbb{N}_0^{[1]}\Big(\mathbb{N}_0^{[1]}\big(F_{2^{-k},1}(x_*,x_0)\,\big|\,\mathcal{Z}^{x_*(x_0)}_{2^{-k}}\big)\,F_{\varepsilon,2^{-k}}(x_*,x_0)\\
&\qquad\qquad\times\int_{B^{\bullet(x_0)}_{2^{-k}}(x_*)}\mathrm{Vol}(\mathrm{d}z)\,\mathbf{1}_{\{D(x_*,z)\in[2^{-k+2},2^{-k+3}]\}}\,\varphi_{\varepsilon,k}\big(\mathcal{Z}^{z(x_*)}_{2^{-k}}\big)\Big)\\
&\le \frac{3C_m}{2}\mathbb{N}_0^{[1]}\Big(\big(\mathcal{Z}^{x_*(x_0)}_{2^{-k}}\big)^{m/2}\,F_{\varepsilon,2^{-k}}(x_*,x_0)\\
&\qquad\qquad\qquad\times\int_{B^{\bullet(x_0)}_{2^{-k}}(x_*)}\mathrm{Vol}(\mathrm{d}z)\,\mathbf{1}_{\{D(x_*,z)\in[2^{-k+2},2^{-k+3}]\}}\,\varphi_{\varepsilon,k}\big(\mathcal{Z}^{z(x_*)}_{2^{-k}}\big)\Big)
\end{aligned}$$

by Lemma 16. Comparing with (44), we see that the same arguments as in the first step would allow us to complete the proof if we could replace $F_{\varepsilon,2^{-k}}(x_*,x_0)$ by $\varphi_{\varepsilon,k}\big(\mathcal{Z}^{x_*(x_0)}_{2^{-k}}\big)$ in the right-hand side of the last display. Unfortunately, it is not so easy to justify this replacement.

Let $a>0$. To simplify notation, we write $\mathbb{N}_0^{[1],a} := \mathbb{N}_0^{[1]}(\cdot \mid \mathcal{Z}^{x_*(x_0)}_{2^{-k}} = a)$, so that, under $\mathbb{N}_0^{[1],a}$, $B^{\bullet(x_0)}_{2^{-k}}(x_*)$ is distributed as a standard hull of radius 2^{-k} and perimeter a, and we have to evaluate

$$M_{\varepsilon,k}(a) := \mathbb{N}_0^{[1],a}\left(F_{\varepsilon,2^{-k}}(x_*,x_0)\int_{B^{\bullet(x_0)}_{2^{-k}}(x_*)}\mathrm{Vol}(\mathrm{d}z)\,\mathbf{1}_{\{D(x_*,z)\in[2^{-k+2},2^{-k+3}]\}}\,\varphi_{\varepsilon,k}\big(\mathcal{Z}^{z(x_*)}_{2^{-k}}\big)\right).$$

As we already explained in the proof of Theorem 8, $B^{\bullet(x_0)}_{2^{-k}}(x_*)$ corresponds in the Brownian snake representation to the excursions of W below level W_*+2^{-k}. Let us write $(\omega^i)_{i\in I}$ and ω_* for these excursions, in a way similar to Section 3.1. Under $\mathbb{N}^{[1],a}_0$, ω_* is distributed according to $\mathbb{N}_0(\cdot\,|\,W_*=-2^{-k})$, and $\sum_{i\in I}\delta_{\omega^i}$ is an independent Poisson point measure with intensity $a\,\mathbb{N}_0(\cdot\,|\,W_*>-2^{-k})$. As in Section 4.1, we can associate a slice $\mathbf{S}(\omega^i)$ with each excursion ω^i, resp. a slice $\mathbf{S}(\omega_*)$ with ω_*, and this slice corresponds to a subset of $B^{\bullet(x_0)}_{2^{-k}}(x_*)$ (this correspondence preserves the volume and is bijective except in the case of $\mathbf{S}(\omega_*)$, as we explained in Section 4.2), in such a way that the union of these subsets is the whole hull $B^{\bullet(x_0)}_{2^{-k}}(x_*)$, up to a set of zero volume. Then, we have,

$$\int_{B^{\bullet(x_0)}_{2^{-k}}(x_*)}\mathrm{Vol}(\mathrm{d}z)\,\mathbf{1}_{\{D(x_*,z)\in[2^{-k+2},2^{-k+3}]\}}\,\varphi_{\varepsilon,k}(\mathcal{Z}^{z(x_*)}_{2^{-k}})=\Psi_{\varepsilon,k}(\omega_*)+\sum_{i\in I}\Psi_{\varepsilon,k}(\omega^i),\tag{55}$$

where, for $\omega\in\mathcal{S}_0$ such that $W_*(\omega)\geq -2^{-k}$,

$$\Psi_{\varepsilon,k}(\omega):=\int_{\mathbf{S}(\omega)}\mathrm{Vol}(\mathrm{d}\widetilde{z})\,\mathbf{1}_{\{\tilde{D}(\tilde{x}_*,\tilde{z})\in[2^{-k+2}-(W_*+2^{-k}),2^{-k+3}-(W_*+2^{-k})]\}}\,\varphi_{\varepsilon,k}(\mathcal{Z}^{\tilde{z}(\tilde{x}_*)}_{2^{-k}}).$$

We have used the notation of Section 4.1 and the fact that, for every point $\widetilde{z}$ of the slice $\mathbf{S}(\omega)$ at distance greater than 2^{-k} from the distinguished point $\widetilde{x}_*$, we can define the hull of radius 2^{-k} centered at $\widetilde{z}$ (relative to $\widetilde{x}_*$) and its boundary size $\mathcal{Z}^{\tilde{z}(\tilde{x}_*)}_{2^{-k}}$ via the analog of formula (34). The point in (55) is to observe that any point z of $B^{\bullet(x_0)}_{2^{-k}}(x_*)$ such that $D(x_*,z)\in[2^{-k+2},2^{-k+3}]$ corresponds to a point $\widetilde{z}$ of the slice $\mathbf{S}(\omega^i)$ for some $i\in I$ (or of $\mathbf{S}(\omega_*)$), such that $\widetilde{D}(\widetilde{x}_*,\widetilde{z})=D(x_*,z)-(2^k+W_*(\omega_i))$ (or $\widetilde{D}(\widetilde{x}_*,\widetilde{z})=D(x_*,z)$), and moreover the hull of radius 2^{-k} centered at z in $\mathbf{m}_\infty$ is contained in $B^{\bullet(x_0)}_{2^{-k}}(x_*)$ and identified with the same hull centered at $\widetilde{z}$ in the slice, so that these two hulls have the same boundary size. To check the last property, we also use the easy fact that the identification of the slice $\mathbf{S}(\omega^i)$ with a subset of $B^{\bullet(x_0)}_{2^{-k}}(x_*)$ is isometric on the set $\{\widetilde{z}\in\mathbf{S}(\omega^i):\widetilde{D}(\widetilde{x}_*,\widetilde{z})\geq 2^{-k+1}\}$ (and similarly for $\mathbf{S}(\omega_*)$).

As in the proof of Proposition 7 (cf. formula (18)), we have $F_{\varepsilon,2^{-k}}(x_*,x_0)=\mathbf{1}_{\{N\geq m\}}$, where $N:=\#\{i\in I:W_*(\omega^i)<-2^{-k}+\varepsilon\}$. Therefore, we can write

$$M_{\varepsilon,k}(a)=\mathbb{N}^{[1],a}_0\Bigg(\mathbf{1}_{\{N\geq m\}}\Big(\Psi_{\varepsilon,k}(\omega_*)+\sum_{i\in I}\Psi_{\varepsilon,k}(\omega^i)\Big)\Bigg).\tag{56}$$

Under $\mathbb{N}^{[1],a}_0$, the variable N is Poisson with parameter $\frac{3a}{2}((2^{-k}-\varepsilon)^{-2}-(2^{-k})^{-2})$ and thus, as in (40),

$$\mathbb{N}^{[1],a}_0(N\geq m)\leq C(\varepsilon a2^{3k})^m.$$

Since ω_* is independent of $\sum_{i\in I}\delta_{\omega^i}$, we have plainly

$$\mathbb{N}^{[1],a}_0\Big(\mathbf{1}_{\{N\geq m\}}\Psi_{\varepsilon,k}(\omega_*)\Big)=\mathbb{N}^{[1],a}_0(N\geq m)\,\mathbb{N}^{[1],a}_0(\Psi_{\varepsilon,k}(\omega_*))\leq C(\varepsilon a2^{3k})^m\mathbb{N}^{[1],a}_0(\Psi_{\varepsilon,k}(\omega_*)).\tag{57}$$

Set $I'=\{i\in I:W_*(\omega^i)<-2^{-k}+\varepsilon\}$ (so that $N=\#I'$). Since N is also independent of the point measure $\sum_{i\in I\setminus I'}\delta_{\omega^i}$, we get similarly

$$\mathbb{N}^{[1],a}_0\Big(\mathbf{1}_{\{N\geq m\}}\sum_{i\in I\setminus I'}\Psi_{\varepsilon,k}(\omega^i)\Big)\leq C(\varepsilon a2^{3k})^m\mathbb{N}^{[1],a}_0\Big(\sum_{i\in I\setminus I'}\Psi_{\varepsilon,k}(\omega^i)\Big).\tag{58}$$

The delicate part is to estimate

(59)
$$\mathbb{N}_0^{[1],a}\Big(\mathbf{1}_{\{N\geq m\}}\sum_{i\in I'}\Psi_{\varepsilon,k}(\omega^i)\Big)=\sum_{p=m}^{\infty}p\,\mathbb{N}_0^{[1],a}(N=p)\,\mathbb{N}_0(\Psi_{\varepsilon,k}(\omega)\,|\,-2^{-k}<W_*<-2^{-k}+\varepsilon)$$
$$\leq C'(\varepsilon a 2^{3k})^m\,\mathbb{N}_0(\Psi_{\varepsilon,k}(\omega)\,|\,-2^{-k}<W_*<-2^{-k}+\varepsilon).$$

We use the spine decomposition of Section 2.4 to verify that, for every $r\in(2^{-k}-\varepsilon,2^{-k})$, one can couple a snake trajectory $\omega_{(r)}$ distributed according to $\mathbb{N}_0(\cdot\,|\,W_*=-r)$ with a snake trajectory $\omega_{(2^{-k})}$ distributed according to $\mathbb{N}_0(\cdot\,|\,W_*=-2^{-k})$ in such a way that the following holds. The slice $\mathbf{S}(\omega_{(r)})$ is identified isometrically (and in a manner preserving both the volume measure and the first distinguished point $\widetilde{x}_*$) to a closed subset of $\mathbf{S}(\omega_{(2^{-k})})$, and moreover, if z is a point of $\mathbf{S}(\omega_{(r)})$ whose distance from $\widetilde{x}_*$ is greater than 2^{-k+1}, the hull of radius 2^{-k} centered at z in $\mathbf{S}(\omega_{(r)})$ is identified with the same hull in $\mathbf{S}(\omega_{(1)})$, and these two hulls have the same boundary size — here we omit a few details that are left to the reader. It follows that $\Psi_{\varepsilon,k}(\omega_{(r)})\leq\widetilde{\Psi}_{\varepsilon,k}(\omega_{(2^{-k})})$, where $\widetilde{\Psi}_{\varepsilon,k}\geq\Psi_{\varepsilon,k}$ is defined as $\Psi_{\varepsilon,k}$, except that the interval $[2^{-k+2},2^{-k+3}]$ is replaced by $[2^{-k+2}-2^{-k},2^{-k+3}]$. Hence we have

(60)
$$\mathbb{N}_0(\Psi_{\varepsilon,k}(\omega)\,|\,-2^{-k}<W_*<-2^{-k}+\varepsilon)\leq\mathbb{N}_0(\widetilde{\Psi}_{\varepsilon,k}(\omega)\,|\,W_*=2^{-k})=\mathbb{N}_0^{[1],a}(\widetilde{\Psi}_{\varepsilon,k}(\omega_*)).$$

Finally, using (56), (57), (58), (59), (60) and the analog of (55) where $[2^{-k+2},2^{-k+3}]$ is replaced by $[2^{-k+2}-2^{-k},2^{-k+3}]$ and $\Psi_{\varepsilon,k}$ is replaced by $\widetilde{\Psi}_{\varepsilon,k}$, we get the existence of a constant C'' such that

$$M_{\varepsilon,k}(a)\leq C''(\varepsilon a2^{3k})^m\,\mathbb{N}_0^{[1],a}\left(\int_{B^{\bullet(x_0)}_{2^{-k}}(x_*)}\mathrm{Vol}(\mathrm{d}z)\,\mathbf{1}_{\{D(x_*,z)\in[2^{-k+2}-2^{-k},2^{-k+3}]\}}\,\varphi_{\varepsilon,k}\big(\mathcal{Z}^{z(x_*)}_{2^{-k}}\big)\right).$$

It follows that $A'''_{\varepsilon,k}$ is bounded by a constant times

$$\mathbb{N}_0^{[1]}\Bigg((\mathcal{Z}^{x_*(x_0)}_{2^{-k}})^{m/2}\times(C''(\varepsilon\mathcal{Z}^{x_*(x_0)}_{2^{-k}}2^{3k})^m$$
$$\times\int_{B^{\bullet(x_0)}_{2^{-k}}(x_*)}\mathrm{Vol}(\mathrm{d}z)\,\mathbf{1}_{\{D(x_*,z)\in[2^{-k+2}-2^{-k},2^{-k+3}]\}}\,\varphi_{\varepsilon,k}\big(\mathcal{Z}^{z(x_*)}_{2^{-k}}\big)\Bigg),$$

and we get an upper bound by replacing the integral over $B^{\bullet(x_0)}_{2^{-k}}(x_*)$ by the same integral over $\mathbf{m}_\infty$. The very same arguments that we used in the first step to bound the quantity (44), now show that $A'''_{\varepsilon,k}$ is bounded above by a constant times $\varepsilon^{2m}2^{-(4-m)k+\delta k}$. This completes the proof of Lemma 15. □

LEMMA 17. *Let $\alpha\in(0,4-m)$. There exists a constant C_α such that, for every $\varepsilon\in(0,1/2)$,*

$$\widetilde{\mathbb{N}}_0\left(\iint\mathrm{Vol}(\mathrm{d}x)\mathrm{Vol}(\mathrm{d}y)\,\mathbf{1}_{\{D(x,y)<\varepsilon\}}\,D(x,y)^{-\alpha}\,F^{(m)}_{\varepsilon,1}(x,x_*)\right)\leq C_\alpha\,\varepsilon^{2m}. \tag{61}$$

PROOF. As previously, we write $F_{\varepsilon,r}(x,y)$ instead of $F^{(m)}_{\varepsilon,r}(x,y)$ in the proof. Let $\widehat{A}_{\varepsilon,k}$ denote the left-hand side of (61). In a way similar to the beginning of the proof of Lemma

15, we can use the symmetry properties of the Brownian sphere to write $\widehat{A}_{\varepsilon,k}$ in a different form. We write

$$\widehat{A}_{\varepsilon,k} = \mathbb{N}_0\Big(\sigma \iint \frac{\mathrm{Vol}(\mathrm{d}x)}{\sigma}\,\frac{\mathrm{Vol}(\mathrm{d}y)}{\sigma}\,\widehat{\Gamma}_{\varepsilon,k}(x_*,x,y)\Big),$$

with an appropriate function $\widehat{\Gamma}_{\varepsilon,k}$, and observe that we have also

$$\begin{aligned}\widehat{A}_{\varepsilon,k} &= \mathbb{N}_0\Big(\sigma \int \frac{\mathrm{Vol}(\mathrm{d}z)}{\sigma}\,\widehat{\Gamma}_{\varepsilon,k}(x_0,x_*,z)\Big)\\ &= \mathbb{N}_0\Big(F_{\varepsilon,1}(x_*,x_0)\int \mathrm{Vol}(\mathrm{d}z)\,\mathbf{1}_{\{D(x_*,z)<\varepsilon\}}D(x_*,z)^{-\alpha}\Big)\\ &= \frac{3}{2}\,\mathbb{N}_0^{[1]}\Big(F_{\varepsilon,1}(x_*,x_0)\int \mathrm{Vol}(\mathrm{d}z)\,\mathbf{1}_{\{D(x_*,z)<\varepsilon\}}D(x_*,z)^{-\alpha}\Big).\end{aligned}$$

We then note that the quantity $F_{\varepsilon,1}(x_*,x_0)$ is a function of $\overline{\mathbf{m}_\infty\backslash B_\varepsilon^{\bullet(x_0)}(x_*)}$, whereas $\int \mathrm{Vol}(\mathrm{d}z)\,\mathbf{1}_{\{D(x_*,z)<\varepsilon\}}D(x_*,z)^{-\alpha}$ is a function of $B_\varepsilon^{\bullet(x_0)}(x_*)$. We can thus apply Theorem 8 to obtain that the right-hand side of the last display is also equal to

$$\frac{3}{2}\,\mathbb{N}_0^{[1]}\Big(\mathbb{N}_0^{[1]}(F_{\varepsilon,1}(x_*,x_0)\,|\,\mathcal{Z}_\varepsilon^{x_*(x_0)})\int \mathrm{Vol}(\mathrm{d}z)\,\mathbf{1}_{\{D(x_*,z)<\varepsilon\}}D(x_*,z)^{-\alpha}\Big).$$

From Lemma 16, we get the bound

$$\widehat{A}_{\varepsilon,k} \leq \frac{3}{2}\,C_m\,\mathbb{N}_0^{[1]}\Big((\mathcal{Z}_\varepsilon^{x_*(x_0)})^{m/2}\int \mathrm{Vol}(\mathrm{d}z)\,\mathbf{1}_{\{D(x_*,z)<\varepsilon\}}D(x_*,z)^{-\alpha}\Big).$$

The remaining part of the argument is now easy. Write $k(\varepsilon)\geq 1$ for the smallest integer such that $2^{-k(\varepsilon)}<\varepsilon$. Fix $\kappa\in(0,1)$ such that $\alpha-(4-m)+\kappa<0$. For every $k\geq k(\varepsilon)$, we have

$$\begin{aligned}&\mathbb{N}_0^{[1]}\Big((\mathcal{Z}_\varepsilon^{x_*(x_0)})^{m/2}\int \mathrm{Vol}(\mathrm{d}z)\,\mathbf{1}_{\{2^{-k}<D(x_*,z)\leq 2^{-k+1}\}}D(x_*,z)^{-\alpha}\Big)\\ &\quad\leq 2^{k\alpha}\,\mathbb{N}_0^{[1]}\Big((\mathcal{Z}_\varepsilon^{x_*(x_0)})^{m/2}\,\mathrm{Vol}(B_{2^{-k+1}}(x_*))\Big)\\ &\quad\leq 2^{k\alpha}\,\mathbb{N}_0^{[1]}\Big((\mathcal{Z}_\varepsilon^{x_*(x_0)})^{m}\Big)^{1/2}\,\mathbb{N}_0^{[1]}\Big((\mathrm{Vol}(B_{2^{-k+1}}(x_*)))^2\Big)^{1/2}\\ &\quad\leq C_{m,\kappa}\,2^{k\alpha}\times\varepsilon^m\times 2^{-(4-\kappa)k},\end{aligned}$$

using Lemma 4 and the bound (15). By summing over $k\geq k(\varepsilon)$, we arrive at

$$\begin{aligned}\mathbb{N}_0^{[1]}\Big((\mathcal{Z}_\varepsilon^{x_*(x_0)})^{m/2}\int \mathrm{Vol}(\mathrm{d}z)\,\mathbf{1}_{\{D(x_*,z)<\varepsilon\}}D(x_*,z)^{-\alpha}\Big) &\leq C_{m,\kappa}\,\varepsilon^m\sum_{k=k(\varepsilon)}^{\infty}2^{-k(4-\alpha-\kappa)}\\ &\leq C'_{m,\kappa}\,\varepsilon^m\,2^{-k(\varepsilon)(4-\alpha-\kappa)}\\ &\leq C'_{m,\kappa}\,\varepsilon^{2m}.\end{aligned}$$

This completes the proof. □

6. Proof of Theorem 1. As previously, $m\in\{1,2,3\}$ is fixed. Recall the definition of $\widetilde{F}_{\varepsilon,1}^{(m)}$ in Section 4.3. For every $\varepsilon\in(0,1/32)$, we introduce the measure ν_ε on $\mathbf{m}_\infty$ defined by

$$\nu_\varepsilon(\mathrm{d}x) := \varepsilon^{-m}\,\widetilde{F}_{\varepsilon,1}^{(m)}(x,x_*)\mathbf{1}_{\{D(x,x_*)<2\}}\,\mathrm{Vol}(\mathrm{d}x).$$

We use the notation

$$R^{\max} = \max\{D(x,x_*) : x \in \mathbf{m}_\infty\}$$

and note that ν_ε is the zero measure if $R^{\max} < 1$ (recall that $\widetilde{F}^{(m)}_{\varepsilon,1}(x,x_*) = 0$ if $D(x,x_*) < 1$). We will then argue under the finite measure

$$\widetilde{\mathbb{N}}_0^\star := \widetilde{\mathbb{N}}_0(\cdot \cap \{R^{\max} \geq 1\}).$$

As an immediate consequence of Proposition 14, we have

$$\widetilde{\mathbb{N}}_0^\star(\langle \nu_\varepsilon, 1\rangle) \geq c, \tag{62}$$

with a positive constant c independent of ε. On the other hand, if $\delta \in (0,1)$ is fixed, we can use Lemma 15 and Lemma 17 to bound the integral under $\widetilde{\mathbb{N}}_0^\star$ of the quantity

$$\begin{aligned}&\iint \nu_\varepsilon(\mathrm{d}x)\nu_\varepsilon(\mathrm{d}y)\, D(x,y)^{-(4-m-\delta)}\\ &\quad\leq \varepsilon^{-2m} \iint \mathrm{Vol}(\mathrm{d}x)\mathrm{Vol}(\mathrm{d}y)\, F^{(m)}_{\varepsilon,1}(x,x_*)\, F^{(m)}_{\varepsilon,1}(y,x_*)\, \mathbf{1}_{\{D(x,y)<4\}}\, D(x,y)^{-(4-m-\delta)},\end{aligned}$$

where we used the trivial bound $\widetilde{F}^{(m)}_{\varepsilon,1}(x,x_*) \leq F^{(m)}_{\varepsilon,1}(x,x_*)$, and the fact that $D(x,x_*) < 2$ and $D(y,x_*) < 2$ imply $D(x,y) < 4$. Let $k(\varepsilon)$ be the greatest integer such that $2^{-k} > 2\varepsilon$. Using Lemma 15, we have

$$\begin{aligned}&\widetilde{\mathbb{N}}_0^\star\Big(\varepsilon^{-2m}\iint \mathrm{Vol}(\mathrm{d}x)\mathrm{Vol}(\mathrm{d}y)\, F^{(m)}_{\varepsilon,1}(x,x_*)\, F^{(m)}_{\varepsilon,1}(y,x_*)\, \mathbf{1}_{\{2^{-k(\varepsilon)+2}\leq D(x,y)<4\}}\, D(x,y)^{-(4-m-\delta)}\Big)\\ &= \sum_{k=1}^{k(\varepsilon)} \widetilde{\mathbb{N}}_0^\star\Big(\varepsilon^{-2m} \iint \mathrm{Vol}(\mathrm{d}x)\mathrm{Vol}(\mathrm{d}y)\, F^{(m)}_{\varepsilon,1}(x,x_*)\, F^{(m)}_{\varepsilon,1}(y,x_*)\\ &\qquad\qquad \times \mathbf{1}_{\{2^{-k+2}\leq D(x,y)<2^{-k+3}\}}\, D(x,y)^{-(4-m-\delta)}\Big)\\ &\leq \sum_{k=1}^{k(\varepsilon)} C_{(\delta/2)}\, 2^{-(4-m)k+(\delta/2)k} \times 2^{-(-k+2)(4-m-\delta)}\\ &\leq C_{(\delta/2)} \sum_{k=1}^{\infty} 2^{-k\delta/2}\\ &= C'_{(\delta)}\end{aligned}$$

with some constant $C'_{(\delta)}$. On the other hand, using the trivial bound $F^{(m)}_{\varepsilon,1}(y,x_*) \leq 1$, the fact that $2^{-k(\varepsilon)} \leq 4\varepsilon$, and Lemma 17, we have

$$\begin{aligned}&\widetilde{\mathbb{N}}_0^\star\Big(\varepsilon^{-2m} \iint \mathrm{Vol}(\mathrm{d}x)\mathrm{Vol}(\mathrm{d}y)\, F^{(m)}_{\varepsilon,1}(x,x_*)\, F^{(m)}_{\varepsilon,1}(y,x_*)\, \mathbf{1}_{\{D(x,y)<2^{-k(\varepsilon)+2}\}}\, D(x,y)^{-(4-m-\delta)}\Big)\\ &\leq \widetilde{\mathbb{N}}_0^\star\Big(\varepsilon^{-2m} \iint \mathrm{Vol}(\mathrm{d}x)\mathrm{Vol}(\mathrm{d}y)\, F^{(m)}_{16\varepsilon,1}(x,x_*)\, \mathbf{1}_{\{D(x,y)\leq 16\varepsilon\}}\, D(x,y)^{-(4-m-\delta)}\Big)\\ &\leq C''_{(\delta)},\end{aligned}$$

with some constant $C''_{(\delta)}$. Summarizing, we have

$$\widetilde{\mathbb{N}}_0^\star\Big(\iint \nu_\varepsilon(\mathrm{d}x)\nu_\varepsilon(\mathrm{d}y)\, D(x,y)^{-(4-m-\delta)}\Big) \leq K_{(\delta)} \tag{63}$$

for a certain constant $K_{(\delta)}$ depending only on δ. Since the measure $\nu_\varepsilon(\mathrm{d}x)\nu_\varepsilon(\mathrm{d}y)$ is supported on pairs (x,y) such that $D(x,y)<4$, the bound (63) also implies that

$$\widetilde{\mathbb{N}}^\star_0(\langle \nu_\varepsilon,1\rangle^2)\le 64\,K_{(\delta)}. \tag{64}$$

From (62) and (64), a standard application of the Cauchy-Schwarz inequality shows that we can find two positive constants a and c_0 such that

$$\widetilde{\mathbb{N}}^\star_0(\langle \nu_\varepsilon,1\rangle\ge a)\ge c_0.$$

Finally, using (63) and (64), we can find $A>0$ large enough such that

$$\widetilde{\mathbb{N}}^\star_0\Bigg(\Big\{\iint \nu_\varepsilon(\mathrm{d}x)\nu_\varepsilon(\mathrm{d}y)\,D(x,y)^{-(4-m-\delta)}\le A\Big\}\cap\Big\{a\le\langle\nu_\varepsilon,1\rangle\le A\Big\}\Bigg)\ge c_0/2.$$

Then, let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence in $(0,1/32)$ that converges to 0. The event

$$\Theta:=\limsup_{n\to\infty}\Bigg(\Big\{\iint \nu_{\varepsilon_n}(\mathrm{d}x)\nu_{\varepsilon_n}(\mathrm{d}y)\,D(x,y)^{-(4-m-\delta)}\le A\Big\}\cap\Big\{a\le\langle\nu_{\varepsilon_n},1\rangle\le A\Big\}\Bigg)$$

has $\widetilde{\mathbb{N}}^\star_0$-measure at least $c_0/2$. Let us argue on the event Θ. On this event, we can find a (random) subsequence $(\nu_{\varepsilon_{n_p}})_{p\in\mathbb{N}}$ that converges weakly to a limiting nonzero finite measure ν_0 such that

$$\iint \nu_0(\mathrm{d}x)\nu_0(\mathrm{d}y)\,D(x,y)^{-(4-m-\delta)}\le A<\infty.$$

We claim that ν_0 is supported on the set $\mathfrak{S}^{(m+1)}$ of $(m+1)$-geodesic stars. If our claim holds, an application of the classical Frostman lemma shows that $\dim(\mathfrak{S}^{(m+1)})\ge 4-m-\delta$ on the event Θ.

Let us justify our claim. Let x belong to the topological support of ν_0. If V is an open neighborhood of x in $\mathbf{m}_\infty$, then, for p large enough, we must have $\nu_{\varepsilon_{n_p}}(V)>0$ and consequently there exists a point y of V such that $\widetilde{F}^{(m)}_{\varepsilon_{n_p},1}(y,x_*)=1$. It follows that we can find a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbf{m}_\infty$ that converges to x, and a sequence $(\varepsilon'_n)_{n\in\mathbb{N}}$ of positive reals converging to 0, such that, for every $n\in\mathbb{N}$, $\widetilde{F}^{(m)}_{\varepsilon'_n,1}(x_n,x_*)=1$. This means that there exist geodesics $(\xi^{(n)}_0(t))_{t\in[0,1]},(\xi^{(n)}_1(t))_{t\in[0,1]},\ldots,(\xi^{(n)}_m(t))_{t\in[0,1]}$ that terminate at x_n and are such that, for every $0\le i<j\le m$, we have

$$D(\xi^{(n)}_i(t),\xi^{(n)}_j(t))\ge\delta_k$$

for every $t\in[1-2^{-k-1},1-2^{-k-2}]$ and $k\ge1$ such that $2^{-k-4}\ge\varepsilon'_n$. By a compactness argument, up to extracting subsequences, we may assume that, for every $i\in\{0,\ldots,m\}$,

$$\xi^{(n)}_i(t)\underset{n\to\infty}{\longrightarrow}\xi^{(\infty)}_i(t)\,,\qquad\text{uniformly in } t\in[0,1],$$

where the limit $(\xi^{(\infty)}_i(t))_{t\in[0,1]}$ must be a geodesic path that terminates at x. Furthermore, we have for every $0\le i<j\le m$,

$$D(\xi^{(\infty)}_i(t),\xi^{(\infty)}_j(t))\ge\delta_k,$$

for every $t\in[1-2^{-k-1},1-2^{-k-2}]$ and every integer $k\ge1$, and this ensures that the sets $\{\xi^{(\infty)}_i(t):t\in[3/4,1)\}$ are disjoint, so that x is an $(m+1)$-geodesic star, proving our claim.

At this point, we have proved that the dimension of the set of all $(m+1)$-geodesic stars is at least $4-m-\delta$ on an event of positive $\widetilde{\mathbb{N}}^\star_0$-measure. Clearly, we can replace $\widetilde{\mathbb{N}}^\star_0$ by $\widetilde{\mathbb{N}}_0$ or $\mathbb{N}_0$.

To simplify notation, set $\mathbb{N}_0^{\{a\}} := \mathbb{N}_0(\cdot \,|\, W_* = -a)$ for every $a > 0$. Via a scaling argument, we also get that the dimension of the set of $(m+1)$-geodesic star is at least $4 - m - \delta$ on an event of positive $\mathbb{N}_0^{\{a\}}$-probability. We now want to argue that the latter property even holds on an event of full $\mathbb{N}_0^{\{a\}}$-probability, and we need to a kind of zero-one law argument, for which it is more convenient to use the Brownian plane.

LEMMA 18. *On the same probability space, we can construct both the Brownian plane $\mathcal{P}$ and a two-pointed random metric space $\mathbf{m}_\infty^{\{1\}}$ distributed according to the law of $\mathbf{m}_\infty$ under $\mathbb{N}_0^{\{1\}}$, in such a way that, for every $\varepsilon \in (0,1)$, there is an event E_ε of positive probability and independent of $\mathbf{m}_\infty^{\{1\}}$, such that the following holds. Write $x_*^{\{1\}}$ and $x_0^{\{1\}}$ for the two distinguished points of $\mathbf{m}_\infty^{\{1\}}$, and let $B^\bullet_{1-\varepsilon}(\mathbf{m}_\infty^{\{1\}})$ stand for the hull defined as the complement of the connected component containing $x_0^{\{1\}}$ of the complement of the closed ball of radius $1-\varepsilon$ centered at $x_*^{\{1\}}$ in $\mathbf{m}_\infty^{\{1\}}$. Similarly, write $B^\bullet_{1-\varepsilon}(\mathcal{P})$ for the complement of the unbounded component of the complement of the closed ball of radius $1-\varepsilon$ centered at the distinguished point of $\mathcal{P}$. On the event E_ε, the interior $\mathrm{Int}(B^\bullet_{1-\varepsilon}(\mathbf{m}_\infty^{\{1\}}))$ equipped with its intrinsic metric is isometric to $\mathrm{Int}(B^\bullet_{1-\varepsilon}(\mathcal{P}))$ equipped with its intrinsic metric.*

This lemma is obtained by comparing the construction of the Brownian plane in [9] with the spine decomposition of $\mathbb{N}_0^{\{1\}}$ in Section 2.4. We refer to Appendix B below for a detailed argument.

With the notation of the lemma, we have a.s.,

$$\mathbf{m}_\infty^{\{1\}} \setminus \bigcup_{\varepsilon>0} B^\bullet_{1-\varepsilon}(\mathbf{m}_\infty^{\{1\}}) = \{x_0^{\{1\}}\}. \tag{65}$$

To justify this, argue under $\mathbb{N}_0^{\{1\}}(\mathrm{d}\omega)$ and observe that if $x = \Pi(a) \in \mathbf{m}_\infty \setminus \{x_0\}$, labels along the line segment from a to $\rho_{(\omega)}$ in $\mathcal{T}_{(\omega)}$ must take negative values, which ensures by the bound (11) that x belongs to $B^\bullet_{1-\varepsilon}(x_*)$ for $\varepsilon > 0$ small enough.

It now follows from (65) and the considerations preceding the lemma that, for $\varepsilon > 0$ small enough, the set of all $(m+1)$-geodesic stars of $\mathbf{m}_\infty^{\{1\}}$ that lie in $\mathrm{Int}(B^\bullet_{1-\varepsilon}(\mathbf{m}_\infty^{\{1\}}))$ has dimension at least $4 - m - \delta$ with positive probability. Hence (here we use the fact that E_ε is independent of $\mathbf{m}_\infty^{\{1\}}$), the set of all $(m+1)$-geodesic stars of $\mathcal{P}$ that lie in $\mathrm{Int}(B^\bullet_{1-\varepsilon}(\mathcal{P}))$ has also dimension at least $4 - m - \delta$ with positive probability, for $\varepsilon > 0$ small enough. The scaling invariance of $\mathcal{P}$ now shows that, for every $a > 0$, the event where the set of all $(m+1)$-geodesic stars of $\mathcal{P}$ that lie in $\mathrm{Int}(B^\bullet_a(\mathcal{P}))$ has dimension at least $4 - m - \delta$ has the same (positive) probability. Writing $\mathfrak{S}^{(m+1)}(\mathcal{P})$ for the set of all $(m+1)$-geodesic stars of $\mathcal{P}$, we get that the event

$$\bigcap_{a>0} \Big\{ \dim(\mathfrak{S}^{(m+1)}(\mathcal{P}) \cap B^\bullet_a(\mathcal{P})) \geq 4 - m - \delta \Big\}$$

has also positive probability. However, using the construction of $\mathcal{P}$ given in [9] (see below the proof of Lemma 18), it is not hard to verify that the latter event belongs to an asymptotic σ-field which contains only events of probability 0 or 1. We thus get that the property

$$\dim(\mathfrak{S}^{(m+1)}(\mathcal{P}) \cap B^\bullet_a(\mathcal{P})) \geq 4 - m - \delta$$

holds for every $a > 0$, a.s. Since $\delta \in (0,1)$ was arbitrary, we conclude that

$$\dim(\mathfrak{S}^{(m+1)}(\mathcal{P}) \cap B^\bullet_a(\mathcal{P})) \geq 4 - m,$$

for every $a > 0$, a.s. Finally, using the coupling between the Brownian sphere and the Brownian plane found in [8, Theorem 1], one gets that the same property holds for the Brownian sphere.

APPENDIX A: PROOF OF LEMMA 16

On the event $\{W_* < -1\}$, we define M_ε as the number of excursions below $W_* + 1$ that hit $W_* + \varepsilon$. As in the proof of Proposition 7, we have $\mathbb{N}_0^{[1]}$ a.e.,

$$F^{(m)}_{\varepsilon,1}(x_*, x_0) = \mathbf{1}_{\{M_\varepsilon \geq m+1\}}.$$

So we have to bound $\mathbb{N}_0^{[1]}(M_\varepsilon \geq m+1 \,|\, \mathcal{Z}_{W_*+\varepsilon} = z)$ for $m \in \{1,2,3\}$ (recall that $\mathcal{Z}_\varepsilon^{x_*(x_0)} = \mathcal{Z}_{W_*+\varepsilon}$). To this end, we will rely on explicit calculations. For every $a > 0$, we write $h_a(z)$ for the density of the law of $\mathcal{Z}_{-a}$ under $\mathbb{N}_0(\cdot \cap \{\mathcal{Z}_{-a} \neq 0\})$, as given in [22, Proposition 3]:

$$h_a(z) := \Big(\frac{3}{2a^2}\Big)^2\, \psi(\frac{3z}{2a^2}) \tag{66}$$

where

$$\psi(x) = \frac{2}{\sqrt{\pi}}(x^{1/2} + x^{-1/2}) - 2(x + \frac{3}{2})\, e^x \operatorname{erfc}(\sqrt{x}), \quad x > 0. \tag{67}$$

We also recall from [21, Corollary 13] that, for every $a > 0$, the density of $\mathcal{Z}_{W_*+a}$ under $\mathbb{N}_0^{[a]}$ is the function

$$z \mapsto \frac{1}{a}\sqrt{\frac{3}{2\pi z}} \exp\Big(-\frac{3z}{2a^2}\Big). \tag{68}$$

LEMMA 19. *For every $\varepsilon \in (0,1)$, we have $\mathbb{N}_0^{[1]}(M_\varepsilon = 1) = 1 - \varepsilon$ and*

$$\mathbb{N}_0^{[1]}(M_\varepsilon = 2) = \frac{1}{2}(1-\varepsilon)\Big(1 - (1-\varepsilon)^2\Big), \quad \mathbb{N}_0^{[1]}(M_\varepsilon = 3) = \frac{3}{8}(1-\varepsilon)\Big(1 - (1-\varepsilon)^2\Big)^2.$$

PROOF. By [21, Proposition 12] (see also the remark after this proposition), the conditional distribution of $M_\varepsilon - 1$ under $\mathbb{N}_0^{[1]}$ knowing that $\mathcal{Z}_{W_*+1} = z$ is Poisson with parameter

$$z\, \mathbb{N}_0(-1 < W_* < -1 + \varepsilon) = z\Big(\frac{3}{2(1-\varepsilon)^2} - \frac{3}{2}\Big).$$

Since the distribution of $\mathcal{Z}_{W_*+1}$ is given by (68), the formulas of the lemma follow by straightforward calculations. □

LEMMA 20. *Let $\varepsilon \in (0,1)$. The law of $\mathcal{Z}_{-1+\varepsilon}$ under $\mathbb{N}_0(\cdot \,|\, W_* = -1)$ has density*

$$f_\varepsilon(z) := \varepsilon^{-3}\, z \exp\Big(-\frac{3z}{2\varepsilon^2}\Big) h_{1-\varepsilon}(z).$$

The law of $\mathcal{Z}_{-1+\varepsilon}$ under $\mathbb{N}_0(\cdot \,|\, -1 < W_* < -1+\varepsilon)$ has density

$$\tilde{f}_\varepsilon(z) := \Big(\frac{3}{2(1-\varepsilon)^2} - \frac{3}{2}\Big)^{-1} \exp\Big(-\frac{3z}{2\varepsilon^2}\Big) h_{1-\varepsilon}(z).$$

PROOF. Let $v > 0$ and $a > v$. An application of the special Markov property gives, for any nonnegative measurable function φ on $[0, \infty)$ such that $\varphi(0) = 0$,

$$\begin{aligned}\mathbb{N}_0\Big(\mathbf{1}_{\{W_* > -a\}}\varphi(\mathcal{Z}_{-v})\Big) &= \mathbb{N}_0\Big(\varphi(\mathcal{Z}_{-v}) \exp\Big(-\frac{3\mathcal{Z}_{-v}}{2(a-v)^2}\Big)\Big)\\ &= \int_0^\infty \mathrm{d}z\, h_v(z)\varphi(z) \exp\Big(-\frac{3z}{2(a-v)^2}\Big).\end{aligned}$$

Hence the joint density of the pair $(-W_*, \mathcal{Z}_{-v})$ under $\mathbb{N}_0(\cdot \cap \{W_* < -v\})$ is the function

$$(a,z) \mapsto \mathbf{1}_{\{a>v\}} \frac{3z}{(a-v)^3} \exp\Big(-\frac{3z}{2(a-v)^2}\Big) h_v(z).$$

On the other hand, the density of $-W_*$ under $\mathbb{N}_0(\cdot \cap \{W_* < -v\})$ is the function $a \mapsto \mathbf{1}_{\{a>v\}} 3a^{-3}$. It follows that, for $a > v$, the density of $\mathcal{Z}_{-v}$ under $\mathbb{N}_0(\cdot \mid -W_* = a)$ is

$$z \mapsto \frac{a^3}{(a-v)^3} z \exp\Big(-\frac{3z}{2(a-v)^2}\Big) h_v(z).$$

The case $a = 1$, $v = 1-\varepsilon$ gives the first assertion of the lemma.

The proof of the second assertion is straightforward. For a function φ as above,

$$\begin{aligned}\mathbb{N}_0\Big(\varphi(\mathcal{Z}_{-1+\varepsilon})\mathbf{1}_{\{-1<W_*<-1+\varepsilon\}}\Big) &= \mathbb{N}_0\Big(\varphi(\mathcal{Z}_{-1+\varepsilon}) \exp(-\mathcal{Z}_{-1+\varepsilon}\mathbb{N}_{-1+\varepsilon}(W_* \le -1))\Big)\\ &= \mathbb{N}_0\Big(\varphi(\mathcal{Z}_{-1+\varepsilon}) \exp(-\frac{3}{2\varepsilon^2}\mathcal{Z}_{-1+\varepsilon})\Big)\\ &= \int_0^\infty \mathrm{d}z\, h_{1-\varepsilon}(z)\,\varphi(z) \exp(-\frac{3z}{2\varepsilon^2})\end{aligned}$$

and the desired result follows since $\mathbb{N}_0(-1 < W_* < -1+\varepsilon) = \frac{3}{2(1-\varepsilon)^2} - \frac{3}{2}$. □

LEMMA 21. *Let $\varepsilon \in (0,1)$. The law of $\mathcal{Z}_{W_*+\varepsilon}$ under $\mathbb{N}_0^{[1]}$ has density*

$$g_\varepsilon(z) := 2\,\varepsilon^{-3}\, z \exp\Big(-\frac{3z}{2\varepsilon^2}\Big) \int_{1-\varepsilon}^\infty \mathrm{d}a\, h_a(z).$$

PROOF. We rely on a formula found in [21, Proposition 12], which gives for any nonnegative measurable function φ on $[0,\infty)$,

$$\begin{aligned}\mathbb{N}_0(\mathbf{1}_{\{W_*<-1\}}\, \varphi(\mathcal{Z}_{W_*+\varepsilon})) &= 3\varepsilon^{-3} \int_{-\infty}^{-1+\varepsilon} \mathrm{d}b\, \mathbb{N}_0\Big(\mathcal{Z}_b \exp\Big(-\frac{3\mathcal{Z}_b}{2\varepsilon^2}\Big)\varphi(\mathcal{Z}_b)\Big)\\ &= 3\varepsilon^{-3} \int_{1-\varepsilon}^\infty \mathrm{d}a \int_0^\infty \mathrm{d}z\, z\varphi(z) \exp\Big(-\frac{3z}{2\varepsilon^2}\Big) h_a(z)\\ &= 3\varepsilon^{-3} \int_0^\infty \mathrm{d}z\, z\varphi(z) \Big(\int_{1-\varepsilon}^\infty \mathrm{d}a\, h_a(z)\Big) \exp\Big(-\frac{3z}{2\varepsilon^2}\Big).\end{aligned}$$

The desired result follows since $\mathbb{N}_0(W_* < -1) = 3/2$. □

Let us use the preceding lemmas to evaluate $\mathbb{N}_0^{[1]}(M_\varepsilon = 1 \mid \mathcal{Z}_{W_*+\varepsilon} = z)$. For any nonnegative measurable function φ on $[0,\infty)$, we have

$$\begin{aligned}\mathbb{N}_0^{[1]}(\mathbf{1}_{\{M_\varepsilon=1\}}\, \varphi(\mathcal{Z}_{W_*+\varepsilon})) &= \mathbb{N}_0^{[1]}(M_\varepsilon = 1) \times \mathbb{N}_0^{[1]}(\varphi(\mathcal{Z}_{W_*+\varepsilon}) \mid M_\varepsilon = 1)\\ &= (1-\varepsilon) \int_0^\infty \mathrm{d}z\, f_\varepsilon(z)\varphi(z),\end{aligned}$$

by Lemma 19 and Lemma 20, using also the fact that the law of $\mathcal{Z}_{W_*+\varepsilon}$ under $\mathbb{N}_0^{[1]}(\cdot \mid M_\varepsilon = 1)$ coincides with the law of $\mathcal{Z}_{-1+\varepsilon}$ under $\mathbb{N}_0(\cdot \mid W_* = -1)$ (see [21, Proposition 12]). On the other hand,

$$\begin{aligned}\mathbb{N}_0^{[1]}(\mathbf{1}_{\{M_\varepsilon=1\}}\, \varphi(\mathcal{Z}_{W_*+\varepsilon})) &= \mathbb{N}_0^{[1]}\Big(\varphi(\mathcal{Z}_{W_*+\varepsilon})\, \mathbb{N}_0^{[1]}(M_\varepsilon = 1 \mid \mathcal{Z}_{W_*+\varepsilon})\Big)\\ &= \int_0^\infty \mathrm{d}z\, g_\varepsilon(z)\,\varphi(z)\, \mathbb{N}_0^{[1]}(M_\varepsilon = 1 \mid \mathcal{Z}_{W_*+\varepsilon} = z),\end{aligned}$$

by Lemma 21. By comparing the last two displays, we get

$$\mathbb{N}_0^{[1]}(M_\varepsilon=1\,|\,\mathcal{Z}_{W_*+\varepsilon}=z)=(1-\varepsilon)\,\frac{f_\varepsilon(z)}{g_\varepsilon(z)}=\frac{(1-\varepsilon)\,h_{1-\varepsilon}(z)}{2\int_{1-\varepsilon}^\infty \mathrm{d}a\,h_a(z)}=\frac{h_1((1-\varepsilon)^{-2}z)}{2\int_1^\infty \mathrm{d}a\,h_a((1-\varepsilon)^{-2}z)},$$

where the last equality is a consequence of (66).

We can similarly compute $\mathbb{N}_0^{[1]}(M_\varepsilon=2\,|\,\mathcal{Z}_{W_*+\varepsilon}=z)$. Observing that the law of $\mathcal{Z}_{W_*+\varepsilon}$ under $\mathbb{N}_0^{[1]}(\cdot\,|\,M_\varepsilon=2)$ has density $f_\varepsilon*\tilde f_\varepsilon$ (use again [21, Proposition 12]), and recalling the formula for $\mathbb{N}_0^{[1]}(M_\varepsilon=2)$ in Lemma 20, the same argument shows that

$$\begin{aligned}\mathbb{N}_0^{[1]}(M_\varepsilon=2\,|\,\mathcal{Z}_{W_*+\varepsilon}=z)&=\mathbb{N}_0^{[1]}(M_\varepsilon=2)\,\frac{f_\varepsilon*\tilde f_\varepsilon(z)}{g_\varepsilon(z)}\\&=\frac{(1-\varepsilon)^3\int_0^z \mathrm{d}y\,y h_{1-\varepsilon}(y)h_{1-\varepsilon}(z-y)}{6z\int_{1-\varepsilon}^\infty \mathrm{d}a\,h_a(z)}\\&=\frac{\int_0^{(1-\varepsilon)^{-2}z}\mathrm{d}y\,y h_1(y)h_1((1-\varepsilon)^{-2}z-y)}{6(1-\varepsilon)^{-2}z\int_1^\infty \mathrm{d}a\,h_a((1-\varepsilon)^{-2}z)}.\end{aligned}$$

Similarly, since the law of $\mathcal{Z}_{W_*+\varepsilon}$ under $\mathbb{N}_0^{[1]}(\cdot\,|\,M_\varepsilon=3)$ has density $f_\varepsilon*\tilde f_\varepsilon*\tilde f_\varepsilon$, we get

$$\begin{aligned}\mathbb{N}_0^{[1]}(M_\varepsilon=3\,|\,\mathcal{Z}_{W_*+\varepsilon}=z)&=\mathbb{N}_0^{[1]}(M_\varepsilon=3)\,\frac{f_\varepsilon*\tilde f_\varepsilon*\tilde f_\varepsilon(z)}{g_\varepsilon(z)}\\&=\frac{\int_0^{(1-\varepsilon)^{-2}z}\mathrm{d}y\,y\,h_1(y)\,h_1*h_1((1-\varepsilon)^{-2}z-y)}{12(1-\varepsilon)^{-2}z\int_1^\infty \mathrm{d}a\,h_a((1-\varepsilon)^{-2}z)}.\end{aligned}$$

We note that $\mathbb{N}_0^{[1]}(M_\varepsilon=m\,|\,\mathcal{Z}_{W_*+\varepsilon}=z)$ only depends on the quantity $(1-\varepsilon)^{-2}z$, which we could have seen from a scaling argument.

At this stage, we use the explicit formula for the functions h_a in (66) and (67) to get asymptotic expansions as $z\to 0$. We have first

$$h_1(z)=\frac{3^{3/2}2^{-1/2}}{\sqrt{\pi}}\,\frac{1}{\sqrt{z}}-\frac{27}{4}+\frac{3^{5/2}2^{1/2}}{\sqrt{\pi}}\,\sqrt{z}+O(z)$$

and

$$2\int_1^\infty \mathrm{d}a\,h_a(z)=\frac{3^{3/2}2^{-1/2}}{\sqrt{\pi}}\,\frac{1}{\sqrt{z}}-\frac{9}{2}+\frac{3^{5/2}2^{-1/2}}{\sqrt{\pi}}\,\sqrt{z}+O(z). \tag{69}$$

From the preceding formula for $\mathbb{N}_0^{[1]}(M_\varepsilon=1\,|\,\mathcal{Z}_{W_*+\varepsilon}=z)$, it follows that

$$\mathbb{N}_0^{[1]}(M_\varepsilon=1\,|\,\mathcal{Z}_{W_*+\varepsilon}=z)=1+O(\sqrt{z}),\quad \text{as } z\to 0,$$

where the remainder $O(\sqrt{z})$ is uniform in $\varepsilon\in(0,1/2]$. Hence $\mathbb{N}_0^{[1]}(M_\varepsilon\geq 2\,|\,\mathcal{Z}_{W_*+\varepsilon}=z)=O(\sqrt{z})$, which gives the case $m=1$ of Lemma 16.

Similarly, tedious but straightforward calculations show that

$$\int_0^z \mathrm{d}y\,y\,h_1(y)\,h_1(z-y)=\frac{27}{4}z-\frac{3^{9/2}2^{-3/2}}{\sqrt{\pi}}\,z^{3/2}+O(z^2)$$

$$\int_0^z \mathrm{d}y\,y\,h_1(y)\,h_1*h_1(z-y)=\frac{3^{7/2}2^{-1/2}}{\sqrt{\pi}}\,z^{3/2}+O(z^2).$$

To simplify notation, write $z' = (1-\varepsilon)^{-2}z$. Then we get

$$\Big(2\int_1^\infty \mathrm{d}a\, h_a(z')\Big)\,\mathbb{N}_0^{[1]}(M_\varepsilon \le 2 \,|\, \mathcal{Z}_{W_*+\varepsilon} = z) = h_1(z') + \frac{1}{3z'}\int_0^{z'} \mathrm{d}y\, y h_1(y) h_1(z'-y)$$
$$= \frac{3^{3/2}2^{-1/2}}{\sqrt{\pi}}\,\frac{1}{\sqrt{z'}} - \frac{9}{2} + \frac{3^{5/2}2^{-3/2}}{\sqrt{\pi}}\,z'^{1/2} + O(z').$$

Comparing with (69), we obtain that $\mathbb{N}_0^{[1]}(M_\varepsilon \le 2 \,|\, \mathcal{Z}_{W_*+\varepsilon} = z) = 1 + O(z)$ as $z \to 0$, and thus $\mathbb{N}_0^{[1]}(M_\varepsilon \ge 3 \,|\, \mathcal{Z}_{W_*+\varepsilon} = z) = O(z)$, giving the case $m = 2$ of Lemma 16. Finally, we have also

$$\Big(2\int_1^\infty \mathrm{d}a\, h_a(z')\Big)\,\mathbb{N}_0^{[1]}(M_\varepsilon \le 3 \,|\, \mathcal{Z}_{W_*+\varepsilon} = z)$$
$$= h_1(z') + \frac{1}{3z'}\int_0^{z'} \mathrm{d}y\, y h_1(y) h_1(z'-y) + \frac{1}{6z'}\int_0^{z'} \mathrm{d}y\, y h_1(y)\, h_1 * h_1(z'-y)$$
$$= \frac{3^{3/2}2^{-1/2}}{\sqrt{\pi}}\,\frac{1}{\sqrt{z'}} - \frac{9}{2} + \frac{3^{5/2}2^{-1/2}}{\sqrt{\pi}}\,z'^{1/2} + O(z').$$

Comparing again with (69), we get that $\mathbb{N}_0^{[1]}(M_\varepsilon \le 3 \,|\, \mathcal{Z}_{W_*+\varepsilon} = z) = 1 + O(z^{3/2})$ as $z \to 0$, and thus $\mathbb{N}_0^{[1]}(M_\varepsilon \ge 4 \,|\, \mathcal{Z}_{W_*+\varepsilon} = z) = O(z^{3/2})$, which gives the case $m = 3$ of Lemma 16 and completes the proof of this lemma.

APPENDIX B: PROOF OF LEMMA 18

We start by recalling the construction of the Brownian plane as described in [9]. We consider a nine-dimensional Bessel process $R = (R_t)_{t\ge 0}$ started at 0, and, conditionally on R, two independent Poisson point measures $\mathcal{N}$ and $\mathcal{N}'$ on $[0,\infty) \times \mathcal{S}$ with the same intensity

$$\mathrm{d}t\, \mathbb{N}_{R_t}(\mathrm{d}\omega \cap \{W_*(\omega) > 0\}).$$

It is convenient to write

$$\mathcal{N} = \sum_{i\in I} \delta_{(t_i,\omega_i)}\,,\ \mathcal{N}' = \sum_{i\in J} \delta_{(t_i,\omega_i)},$$

where the indexing sets I and J are disjoint. We then consider the (non-compact) $\mathbb{R}$-tree $\mathcal{T}_\infty$ defined by

$$\mathcal{T}_\infty := [0,\infty) \cup \Big(\bigcup_{i\in I\cup J} \mathcal{T}_{(\omega_i)}\Big),$$

where for every $i \in I \cup J$, the root of $\mathcal{T}_{(\omega_i)}$ is identified with the point t_i of $[0,\infty)$: we view $\mathcal{T}_{(\omega_i)}$ as grafted on the "spine" $[0,\infty)$ at height t_i. In fact, we consider the trees $\mathcal{T}_{(\omega_i)}$ for $i \in I$ as grafted to the left side of the spine, and the trees $\mathcal{T}_{(\omega_i)}$ for $i \in J$ as grafted to the right side of the spine. This is reflected in the exploration process $(\mathcal{E}^\infty_s)_{s\in\mathbb{R}}$ of $\mathcal{T}_\infty$, which is such that $\{\mathcal{E}^\infty_s : s \le 0\}$ is exactly the union of the spine and of the trees $\mathcal{T}_{(\omega_i)}$ for $i \in I$, whereas $\{\mathcal{E}^\infty_s : s \ge 0\}$ is the union of the spine and of the trees $\mathcal{T}_{(\omega_i)}$ for $i \in J$ (we refer to [22, Section 2.4] for a more precise definition of $(\mathcal{E}^\infty_s)_{s\in\mathbb{R}}$). The exploration process allows us to define intervals on the tree $\mathcal{T}_\infty$. We make the convention that, if $s, s' \in \mathbb{R}$ and $s > s'$, the "interval" $[s,s']$ is equal to $[s,\infty) \cup (-\infty, s']$. Then if $a, b \in \mathcal{T}_\infty$, there is a smallest "interval" $[s,s']$ such that $\mathcal{E}^\infty_s = a$ and $\mathcal{E}^\infty_{s'} = b$, and we take $[a,b]_\infty = \{\mathcal{E}^\infty_r : r \in [s,s']\}$.

We also assign labels $(\Lambda_a)_{a\in\mathcal{T}_\infty}$ to the points of $\mathcal{T}_\infty$. If $a=t$ belongs to the spine $[0,\infty)$, we take $\Lambda_a=R_t$. If $a\in\mathcal{T}_{(\omega_i)}$ for some $i\in I\cup J$, we let Λ_a be the label of a in $\mathcal{T}_{(\omega_i)}$. We then define, for every $a,b\in\mathcal{T}_\infty$,

$$D^{\infty,\circ}(a,b):=\Lambda_a+\Lambda_b-2\max\Big(\min_{c\in[a,b]_\infty}\Lambda_c,\min_{c\in[b,a]_\infty}\Lambda_c\Big),$$

and we let $D^\infty(a,b)$ be the maximal symmetric function of the pair (a,b) that is bounded above by $D^{\infty,\circ}(a,b)$ and satisfies the triangle inequality. It turns out that the property $D^\infty(a,b)=0$ holds if and only if $D^{\infty,\circ}(a,b)=0$. The Brownian plane $\mathcal{P}$ can then be defined as the quotient space $\mathcal{T}_\infty/\{D^\infty=0\}$, which is equipped with the distance induced by D^∞ and with a distinguished point which is the point 0 of the spine. We write Π_∞ for the canonical projection from $\mathcal{T}_\infty$ onto $\mathcal{P}$.

Thanks to Section 2.4, the Bessel process R and the point measures $\mathcal{N}$ and $\mathcal{N}'$ can also be used to construct a random snake trajectory distributed according to $\mathbb{N}_0(\cdot\,|\,W_*=-1)$, whose genealogical tree is identified to

$$\mathcal{T}_1:=[0,L_1]\cup\Big(\bigcup_{i\in I\cup J,t_i\le L_1}\mathcal{T}_{(\omega_i)}\Big),$$

where we make the same identifications as for $\mathcal{T}_\infty$ and, for every $r>0$, we have set

$$L_r:=\sup\{t\ge 0:R_t=r\}.$$

We may and will view $\mathcal{T}_1$ as the subset of $\mathcal{T}_\infty$ obtained by removing the part of the spine above height L_1 (and of course the trees $\mathcal{T}_{(\omega_i)}$ grafted to this part). Write $[a,b]_1$ for the intervals on the tree $\mathcal{T}_1$. Following the construction of the Brownian sphere in Section 2.5, we define $D^{1,\circ}(a,b)$ for $a,b\in\mathcal{T}_1$ by the very same formula as $D^{\infty,\circ}(a,b)$ above, but replacing the intervals $[a,b]_\infty$ and $[b,a]_\infty$ by $[a,b]_1$ and $[b,a]_1$ respectively. We note that $D^{1,\circ}(a,b)\le D^{\infty,\circ}(a,b)$ for $a,b\in\mathcal{T}_1$ since we have clearly $[a,b]_1\subset[a,b]_\infty$. Finally, we let $D^1(a,b)$ be the maximal symmetric function of $a,b\in\mathcal{T}_1$ that is bounded above by $D^{1,\circ}(a,b)$ and satisfies the triangle inequality, and we define $\mathbf{m}_\infty^{\{1\}}$ as the quotient space $\mathcal{T}_1/\{D^1=0\}$, which is equipped with the metric induced by D^1 and the two distinguished points which are the points 0 and L_1 (bottom and top of the spine). Then $\mathbf{m}_\infty^{\{1\}}$ has the distribution of $\mathbf{m}_\infty$ under $\mathbb{N}_0(\cdot\,|\,W_*=-1)$. We write Π_1 for the canonical projection from $\mathcal{T}_1$ onto $\mathbf{m}_\infty^{\{1\}}$.

We then claim that the conclusion of Lemma 18 holds if we take

$$E_\varepsilon:=\{W_*(\omega_i)>1-\frac{\varepsilon}{2}\text{ for every }i\in I\cup J\text{ such that }t_i>1\}.$$

We note that $\mathbb{P}(E_\varepsilon)>0$ (as a simple consequence of the formula for $\mathbb{N}_0(W_*<-r)$) and that E_ε is independent of $\mathbf{m}_\infty^{\{1\}}$. We then verify that, if E_ε holds, $B^\bullet_{1-\varepsilon}(\mathbf{m}_\infty^{\{1\}})$ and $B^\bullet_{1-\varepsilon}(\mathcal{P})$ can be identified as sets. For every $r>0$, let F_r^∞ be the set of all $a\in\mathcal{T}_\infty$ such that the minimal label along the geodesic from a to ∞ in $\mathcal{T}_\infty$ is smaller than or equal to r. Then $B^\bullet_r(\mathcal{P})=\Pi_\infty(F_r^\infty)$ (see formula (16) in [9]). Similarly, for every $r\in(0,1)$, let F_r^1 be the set of all $a\in\mathcal{T}_1$ such that the minimal label along the geodesic from a to L_1 in $\mathcal{T}_1$ is smaller than or equal to r. Then, we have $B^\bullet_r(\mathbf{m}_\infty^{\{1\}})=\Pi_1(F_r^1)$ as a consequence of the bound (11). Next, on the event E_ε, one immediately gets that $F^\infty_{1-\varepsilon}=F^1_{1-\varepsilon}$. Furthermore, still on the event E_ε, for $a,b\in F^\infty_{1-\varepsilon}$, we have $\Pi_\infty(a)=\Pi_\infty(b)$ if and only $\Pi_1(a)=\Pi_1(b)$. The fact that $\Pi_\infty(a)=\Pi_\infty(b)$ implies $\Pi_1(a)=\Pi_1(b)$ is trivial since $D^1\le D^\infty$. Conversely, if $\Pi_1(a)=\Pi_1(b)$, the only case where we do not immediately get $\Pi_\infty(a)=\Pi_\infty(b)$ is when $[a,b]_1\ne[a,b]_\infty$ and $\Lambda_a=\Lambda_b=\min\{\Lambda_c:c\in[a,b]_1\}$. In that case however, the interval $[a,b]_1$ must contain the point L_1 (top of the spine), and also the geodesic from a to L_1 in $\mathcal{T}_1$, so that $\Lambda_a=\Lambda_b\le 1-\varepsilon$ (by the definition of $F^1_{1-\varepsilon}$), and then $\min\{\Lambda_c:c\in[a,b]_1\}=\min\{\Lambda_c:c\in[a,b]_\infty\}$ (because labels on $\mathcal{T}_\infty\backslash\mathcal{T}_1$ are greater than $1-\varepsilon/2$). Finally we have also $\Pi_\infty(a)=\Pi_\infty(b)$ in that case.

The preceding considerations show that, on the event E_ε, the sets $B^\bullet_{1-\varepsilon}(\mathbf{m}^{\{1\}}_\infty)$ and $B^\bullet_{1-\varepsilon}(\mathcal{P})$ are identified. Very similar arguments (using formula (17) in [9], and its analog for $\mathbf{m}^{\{1\}}_\infty$) show that the boundary of $B^\bullet_{1-\varepsilon}(\mathbf{m}^{\{1\}}_\infty)$ in $\mathbf{m}^{\{1\}}_\infty$ is also identified to the boundary of $B^\bullet_{1-\varepsilon}(\mathcal{P})$ in $\mathcal{P}$. Note that the topology induced by D^∞ on $B^\bullet_{1-\varepsilon}(\mathbf{m}^{\{1\}}_\infty)=B^\bullet_{1-\varepsilon}(\mathcal{P})$ must be the same as the one induced by D^1 since both are compact and $D^1\leq D^\infty$. Finally, using the definitions of the distances D^∞ and D^1, one checks that, for every compact subset K of $\mathrm{Int}(B^\bullet_{1-\varepsilon}(\mathbf{m}^{\{1\}}_\infty))=\mathrm{Int}(B^\bullet_{1-\varepsilon}(\mathcal{P}))$, for every $x,y\in K$ such that $D^\infty(x,y)$ is small enough, resp. such that $D^1(x,y)$ is small enough, one has $D^\infty(x,y)=D^1(x,y)$ (we omit a few details here). It follows that the intrinsic distance on $\mathrm{Int}(B^\bullet_{1-\varepsilon}(\mathbf{m}^{\{1\}}_\infty))$ coincides with the intrinsic distance on $\mathrm{Int}(B^\bullet_{1-\varepsilon}(\mathcal{P}))$, and this completes the proof.

APPENDIX C: PROOF OF LEMMA 5

We assume that the Brownian plane $\mathcal{P}$ is constructed as explained in Appendix B. Let $r>0$. According to formula (18) in [9], the random variable Z_r can be obtained as

$$Z_r=\sum_{i\in I\cup J,t_i>L_r}\mathcal{Z}_r(\omega_i).$$

If $0<r<u$, the spine decomposition of $\mathbb{N}_0(\cdot\,|\,W_*=-u)$ in Section 2.4 shows that

$$\sum_{i\in I\cup J,L_r<t_i<L_u}\mathcal{Z}_r(\omega_i) \tag{70}$$

has the distribution of $\mathcal{Z}_{W_*+r}$ under $\mathbb{N}_0(\cdot\,|\,W_*=-u)$ (compare (70) with the right-hand side of (5)). From the last two displays, we immediately obtain that the random variable $\mathcal{Z}_{W_*+r}$ under $\mathbb{N}_0(\cdot\,|\,W_*=-u)$ is stochastically dominated by Z_r. The lemma follows.

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REFERENCES

[1] C. Abraham, Rescaled bipartite planar maps converge to the Brownian map. *Ann. Inst. H. Poincaré Probab. Stat.* 52, 575-595 (2016)
[2] R. Abraham, J.-F. Delmas, P. Hoscheit, A note on the Gromov-Hausdorff-Prokhorov distance between (locally) compact metric measure spaces. *Electron. J. Probab.* 18, no. 14, 21 pp. (2013)
[3] C. Abraham, J.-F. Le Gall, Excursion theory for Brownian motion indexed by the Brownian tree. *J. Eur. Math. Soc. (JEMS)* 20, 2951–3016 (2018)
[4] L. Addario-Berry, M. Albenque, The scaling limit of random simple triangulations and random simple quadrangulations. *Ann. Probab.* 45, 2767–2825 (2017)
[5] O. Angel, B. Kolesnik, G. Miermont, Stability of geodesics in the Brownian map. *Ann. Probab.* 45, 3451–3479 (2017)
[6] J. Bettinelli, G. Miermont, Compact Brownian surfaces I. Brownian disks. *Probab. Theory Related Fields* 167, 555-614 (2017)
[7] J. Bettinelli, E. Jacob, G. Miermont, The scaling limit of uniform random plane maps, via the Ambjørn-Budd bijection. *Electron. J. Probab.* 19, no. 74, 1-16 (2014)
[8] N. Curien, J.-F. Le Gall, The Brownian plane. *J. Theoret. Probab.* 27, 1240–1291 (2014)
[9] N. Curien, J.-F. Le Gall, The hull process of the Brownian plane. *Probab. Theory Related Fields* 166, 187–231 (2016)
[10] E. Gwynne, Geodesic networks in Liouville quantum gravity surfaces. Preprint, arXiv:2010.11260

[11] E. Gwynne, J. Miller, Scaling limit of the uniform infinite half-plane quadrangulation in the Gromov-Hausdorff-Prokhorov-uniform topology. *Electron. J. Probab.* 22, Paper No 84 (2017)

[12] E. Gwynne, J. Miller, Confluence of geodesics in Liouville quantum gravity for $\gamma \in (0,2)$. Preprint, arXiv:1905.00381

[13] E. Gwynne, J. Miller, Existence and uniqueness of the Liouville quantum gravity metric for $\gamma \in (0,2)$. *Invent. Math.* 223, 213–333 (2021)

[14] E. Gwynne, J. Pfeffer, S. Sheffield, Geodesics and metric ball boundaries in Liouville quantum gravity. Preprint, arXiv:2010.07889

[15] J.-F. Le Gall, *Spatial Branching Processes, Random Snakes and Partial Differential Equations*. Lectures in Mathematics ETH Zürich. Birkhäuser, Boston, 1999.

[16] J.-F. Le Gall, The topological structure of scaling limits of large planar maps. *Invent. Math.* 169, 621–670 (2007)

[17] J.-F. Le Gall, Geodesics in large planar maps and in the Brownian map. *Acta Math.* 205, 287–360 (2010)

[18] J.-F. Le Gall, Uniqueness and universality of the Brownian map. *Ann. Probab.* 41, 2880–2960 (2013)

[19] J.-F. Le Gall, Bessel processes, the Brownian snake and super-Brownian motion. In: Séminaire de Probabilités XLVII - In Memoriam Marc Yor, Lecture Notes Math. 2137, 89–105. Springer 2015.

[20] J.-F. Le Gall, Subordination of trees and the Brownian map. *Probab. Theory Related Fields* 171, 819–864 (2018)

[21] J.-F. Le Gall, Brownian disks and the Brownian snake. *Ann. Inst. H. Poincaré Probab. Stat.* 55, 237–313 (2019)

[22] J.-F. Le Gall, A. Riera, Spine representations for non-compact models of random geometry. Preprint, arXiv:2006.12066

[23] J.-F. Le Gall, M. Weill, Conditioned Brownian trees. *Ann. Inst. H. Poincaré Probab. Stat.* 42, 455–489 (2006)

[24] C. Marzouk, Scaling limits of random bipartite planar maps with a prescribed degree sequence. *Random Struct. Alg.* 53, 448–503 (2018)

[25] G. Miermont, Tessellations of random maps of arbitrary genus. *Ann. Sci. École Norm. Sup.* 42, 735–781 (2009)

[26] G. Miermont, The Brownian map is the scaling limit of uniform random plane quadrangulations. *Acta Math.*, 210, 319-401 (2013)

[27] J. Miller, W. Qian, Geodesics in the Brownian map: Strong confluence and geometric structure. Preprint, arXiv:2008.02242

[28] J. Miller, W. Qian, Personal communication.

[29] L. Serlet, A large deviation principle for the Brownian snake. *Stochastic Process. Appl.* 67, 101–115 (1997)