The third unramified cohomology group for varieties over a finite field : On the Tate conjecture for 1-cycles on threefolds

Jean-Louis Colliot-Thélène (CNRS et Université Paris-Saclay)

Motives and invariants : Theory and Applications to Algebraic Groups and their Torsors
Workshop, BIRS, 9th to 13th October, 2023

Let k be a field. Let $g = \mathrm{Gal}(\overline{k}/k)$. Let X be a smooth projective geometrically connected variety over k. Let $k(X)$ be the function field of X. Let $\overline{X} := X \times_k \overline{k}$. Let $\ell \neq \mathrm{char} k$ be a prime. Let $j \in \mathbb{Z}$. Let $\mathbb{Q}_\ell/\mathbb{Z}_\ell(j) = lim_n \, \mu_{\ell^n}^{\otimes j}$.
Let $i \geq 1$. The i-th unramified cohomology group of X/k

$$H^i_{nr}(X, i-1) := H^i_{nr}(k(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(i-1))$$

is the kernel of the sum of residue maps ∂_x on Galois cohomology

$$H^i(k(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(i-1)) \to \oplus_{x \in X^{(1)}} H^{i-1}(k(x), \mathbb{Q}_\ell/\mathbb{Z}_\ell(i-2))$$

for x running through all codimension 1 points of X.
These groups are k-birational invariants of smooth, projective k-varieties.

We have

$$H^1_{nr}(X,0) = H^1_{et}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

$$H^2_{nr}(X,1) = \mathrm{Br}(X)\{\ell\}$$

These groups interact with $\mathrm{CH}^1(X) \simeq \mathrm{Pic}\,(X)$, the group of divisors of codimension 1 modulo rational equivalence. For instance we have an exact sequence :

$$0 \to \mathrm{Pic}\,(X) \to \mathrm{Pic}\,(\overline{X})^g \to \mathrm{Br}\,(k) \to$$
$$\mathrm{Ker}[\mathrm{Br}\,(X) \to \mathrm{Br}\,(\overline{X})] \to H^1(g, \mathrm{Pic}\,(\overline{X}))$$

The next group $H^3_{nr}(X,2)$ interacts with the Chow group $\mathrm{CH}^2(X)$ of codimension 2 cycles modulo rational equivalence. Let me give a first instance of such a relation.

In the following statement, ignore char(k)-torsion.

Theorem

Let X/k be a smooth, projective, geometrically connected variety. Assume $X(k) \neq \emptyset$ and :
(a) $\overline{X}$ unirational, hence $\mathrm{Pic}\,(\overline{X})$ is a lattice ;
(b) $\mathrm{Br}(\overline{X}) = 0$;
(c) $H^3_{nr}(\overline{X}, \mathbb{Q}/\mathbb{Z}(2)) = 0$.
Then there is an exact sequence

$$0 \to \mathrm{Ker}[\mathrm{CH}^2(X) \to \mathrm{CH}^2(\overline{X})^g] \xrightarrow{\alpha} H^1(g, \mathrm{Pic}\,(\overline{X}) \otimes \overline{k}^\times) \to$$
$$\to H^3_{nr}(X, \mathbb{Q}/\mathbb{Z}(2))/H^3(k, \mathbb{Q}/\mathbb{Z}(2)) \to$$
$$\to \mathrm{Coker}[\mathrm{CH}^2(X) \to \mathrm{CH}^2(\overline{X})^g] \xrightarrow{\beta} H^2(g, \mathrm{Pic}\,(\overline{X}) \otimes \overline{k}^\times).$$

The theorem has a long history : S. Bloch 81, CT-Sansuc 81, Merkurjev-Suslin 83, CT-Raskind 85, B. Kahn 93 and 96. Results in algebraic K-theory are crucial to the proof.

Even for $X/\mathbb{C}$ unirational with $H^3_{nr}(X,2)=0$, this theorem may be useful. To disprove the rationality of a unirational variety $X/\mathbb{C}$, even if $H^3_{nr}(X,2)=0$, one may try to disprove the rationality of X by producing a function field $K/\mathbb{C}$ such that $H^3_{nr}(X_K,2)/H^3(K,2)\neq 0$. One here encounters the question about the existence of a "universal codimension 2 cycle" considered by Voisin.

In to-day's talk, I want to concentrate on the case of varieties over a finite field $\mathbb{F}$.

There are cycle maps with value in integral ℓ-adic cohomology

$$\mathrm{CH}^i(X)\hat{\otimes}\mathbb{Z}_\ell \to H^{2i}_{et}(X, \mathbb{Z}_\ell(i))$$

For $i = 1$ and k finite or algebraically closed there is an exact sequence

$$0 \to \mathrm{Pic}\,(X) \otimes \mathbb{Z}_\ell \to H^2_{et}(X, \mathbb{Z}_\ell(1)) \to T_\ell(\mathrm{Br}(X)) \to 0,$$

where $T_\ell(A) = \mathit{limproj}\ A[\ell^n]$ (limit with respect to multiplication by ℓ on torsion) denotes the Tate module of an abelian group.

Conjecture (Tate). *For* $\mathbb{F}$ *finite and,* $X/\mathbb{F}$ *smooth and projective,* $\mathrm{Br}(X)$ *is finite, hence* $T_\ell(\mathrm{Br}(X)) = 0$ *and*

$$\mathrm{Pic}\,(X) \otimes \mathbb{Z}_\ell \simeq H^2_{et}(X, \mathbb{Z}_\ell(1)).$$

For cycles of codimension $i \geq 1$, there is the general :

Conjecture (Tate). *For $\mathbb{F}$ finite and $X/\mathbb{F}$ smooth and projective,* $\mathrm{Coker}[\mathrm{CH}^i(X) \otimes \mathbb{Z}_\ell \to H^{2i}_{et}(X, \mathbb{Z}_\ell(i))]$ *is finite.*

For $i > 1$, there exist examples (Totaro) for which the cokernel is not zero ("Integral Tate conjecture fails"). However :

Open question for 1-cycles : for $X/\mathbb{F}$ smooth and projective of arbitrary dimension d, is the integral cycle class map

$$\mathrm{CH}^{d-1}(X) \otimes \mathbb{Z}_\ell \to H^{2d-2}_{et}(X, \mathbb{Z}_\ell(d-1))$$

onto ?

Theorem (Kahn 2012, CT-Kahn 2013, uses Bloch-Kato) Assume $\ell \neq p$. For $i = 2$, and and $X/\mathbb{F}$ smooth projective, the following finite groups are isomorphic

(i) The torsion subgroup of the finitely generated $\mathbb{Z}_\ell$-module $\mathrm{Coker}[\mathrm{CH}^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2)]$.
(ii) The quotient of $H^3_{nr}(X, 2)$ by its maximal divisible subgroup.

In (i), conjecturally, the cokernel is finite. The integral Tate conjecture for $\mathrm{CH}^2(X)$ would require the group to be zero.
As far as (ii) is concerned, we have :

Question 1. Is $H^3_{nr}(X,2)$ always finite ? This is a higher degree analogue of the question for $\mathrm{Br}\,(X)$. That question is in general open. Even the question whether the ℓ-torsion subgroup $H^3_{nr}(X,\mu_\ell^{\otimes 2})$ of that group is finite is in general open.
Question 2. Is $H^3_{nr}(X,2)=0$?
Problem 3. Give interesting classes of varieties for which $H^3_{nr}(X,2)=0$.

For $dim(X)\leq 2$, we have $H^3_{nr}(X,2)=0$ (higher class field theory, Kato, CT-Sansuc-Soulé 1983).
[For d-dimensional varieties $H^{d+1}_{nr}(X,d)=0$ (Kerz-Saito).]
BUT :
For any $d\geq 5$, there exists X of dimension d with $H^3_{nr}(X,2)\neq 0$ (Pirutka 2011).
There exists X of dimension 4 with $H^3_{nr}(X,2)\neq 0$ (Scavia-Suzuki 2023).

Open question (already raised in 2009, probably earlier) :
For $X/\mathbb{F}$ of dimension 3, do we have $H^3_{nr}(X,2)=0$?

Some known cases :

Quadric bundles over a curve : immediate corollary of Kahn-Rost-Sujatha. The integral Tate conjecture holds for 1-cycles.

Conic bundles X over a surface S (Parimala-Suresh 2016).
Consequence : If the integral Tate conjecture for 1-cycles holds on the surface S (example $S = C \times_{\mathbb{F}} \mathbb{P}^1_{\mathbb{F}}$, C curve), then also for X.

Smooth cubic threefolds (corollary of Parimala-Suresh).

Let $X/\mathbb{F}$ be smooth and projective, of arbitrary dimension. Exact sequence (CT-Kahn 2013)

$$0 \to \mathrm{Ker}[\mathrm{CH}^2(X)\{\ell\} \to \mathrm{CH}^2(\overline{X})\{\ell\}] \xrightarrow{\phi_\ell} H^1(g, H^3(\overline{X}, \mathbb{Z}_\ell(2))_{tors})$$

$$\to \mathrm{Ker}[H^3_{nr}(X,2) \to H^3_{nr}(\overline{X},2)]$$

$$\to \mathrm{Coker}[\mathrm{CH}^2(X)\{\ell\} \to \mathrm{CH}^2(\overline{X})^g\{\ell\}] \to 0$$

For cycles algebraically equivalent to zero, there is a refined version by Scavia-Suzuki 2023.

The would-be surjectivity of the map ϕ_ℓ may be interpreted in various ways (CT-Scavia 2021). We proved the surjectivity for the product of one surface by an arbitrary product of curves.

Scavia-Suzuki 2023 have now given examples, also for $dim(X) = 4$, where the map ϕ_ℓ is not onto, hence in particular $H^3_{nr}(X,2) \neq 0$.

Let $X/\mathbb{F}$ be smooth and projective, of arbitrary dimension. To study the surjectivity of the cycle map

$$\mathrm{CH}^2(X) \otimes \mathbb{Z}_\ell \to H^4_{et}(X, \mathbb{Z}_\ell(2))$$

one may use the exact sequence (Hochschild-Serre)

$$0 \to H^1(\mathbb{F}, H^3_{et}(\overline{X}, \mathbb{Z}_\ell(2))) \to H^4_{et}(X, \mathbb{Z}_\ell(2)) \to H^4_{et}(\overline{X}, \mathbb{Z}_\ell(2))^g \to 0$$

where $H^1(\mathbb{F}, H^3_{et}(\overline{X}, \mathbb{Z}_\ell(2)))$ is finite (Deligne), and try to show that the composite map $\mathrm{CH}^2(X) \otimes \mathbb{Z}_\ell \to H^4_{et}(\overline{X}, \mathbb{Z}_\ell(2))^g$ is onto, which gives a weak but already powerful version of the integral Tate conjecture, then try to cover $H^1(\mathbb{F}, H^3_{et}(\overline{X}, \mathbb{Z}_\ell(2)))$ by arithmetic cycle classes with trivial image in $H^4_{et}(\overline{X}, \mathbb{Z}_\ell(2))$.

For the rest of the talk, I very roughly sketch recent and also ongoing work, by Scavia-Suzuki, Kollár-Tian and Tian.
Recent work in complex algebraic geometry (Benoist-Ottem) has attracted attention on subtleties regarding the filtration of integral cohomology by codimension of support. Let X be smooth and projective over an algebraically closed field k. Already in codimension 1, given a class in $H^i(X, \mathbb{Z}_\ell)$ one may ask whether it vanishes on some nonempty Zariski open set, a property which defines the classical $N^1H^i(X, \mathbb{Z}_\ell)$, or whether it belongs to the subgroup spanned by the images of the $H^i_Y(X, \mathbb{Z}_\ell) \to H^i(X, \mathbb{Z}_\ell)$ for Y varying among the *smooth* closed subvarieties of X. One denotes this subgroup

$$\tilde{N}^1H^i(X, \mathbb{Z}_\ell) \subset N^1H^i(X, \mathbb{Z}_\ell).$$

Theorem SS23 (Scavia-Suzuki 2023) *Let $X/\mathbb{F}$ be smooth and projective. Assume*

$$\tilde{N}^1 H^3(\overline{X}, \mathbb{Z}_\ell(2)) = N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2)).$$

Then the ℓ-adic Abel-Jacobi map on homologically trivial cycles in $\mathrm{CH}^2(X) \otimes \mathbb{Z}_\ell$, with values in $H^1(\mathbb{F}, H^3(\overline{X}, \mathbb{Z}_\ell(2)))$, induces an isomorphism

$$\mathrm{CH}^2(X)_{alg} \otimes \mathbb{Z}_\ell(2) \to H^1(\mathbb{F}, N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2)))$$

and the map ϕ_ℓ is an isomorphism.

Kollár and Tian 2023 using very elaborate deformation arguments prove :

Theorem KT23. *Let $X \to C$ be a dominant morphism of smooth projective varieties over $\mathbb{F}$, with C a curve and with generic fibre a smooth geometrically separably rationally connected variety over $\mathbb{F}(C)$ (e.g. a geometrically rational variety). The map*

$$\mathrm{CH}_1(X)/alg \to (\mathrm{CH}_1(\overline{X})/alg)^g$$

is an isomorphism.
Here algebraic equivalence is over $\mathbb{F}$, resp. over $\overline{\mathbb{F}}$.

By a delicate combination of geometric theorems of Kollár-Tian 2023 (on arXiv) and of motivic tools (Bloch's higher Chow groups, Merkurjev-Suslin, Suslin-Voevodsky) Tian 2023 shows :

Theorem T23. *Let $C/\mathbb{F}$ be a curve. For a threefold X equipped with a morphism $X \to C$ whose generic fibre is a geometrically rational surface, the Scavia-Suzuki condition is fulfilled :*

$$\tilde{N}^1 H^3(\overline{X}, \mathbb{Z}_\ell(2)) = N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2)).$$

For a 3-fold X/C as above, earlier work gives

$$\mathrm{CH}^2(\overline{X})/alg \otimes \mathbb{Z}_\ell \simeq H^4(\overline{X}, \mathbb{Z}_\ell(2))$$

$$N^1 H^3(\overline{X}, \mathbb{Z}_\ell(2)) = H^3(\overline{X}, \mathbb{Z}_\ell(2)),$$

and $H^3_{nr}(\overline{X}, 2) = 0$. Putting everything theorem gives the final theorem :

Theorem (Z. Tian, T23, on arXiv) *Let* $X \to C$ *be a dominant morphism of smooth projective varieties over a finite field* $\mathbb{F}$, *with* $dim(X) = 3$ *and* C *a curve, and with generic fibre a smooth geometrically rationally surface over* $\mathbb{F}(C)$. *Then*

$$\mathrm{CH}^2(X) \otimes \mathbb{Z}_\ell \to H^4_{et}(X, \mathbb{Z}_\ell(2))$$

is surjective, and $H^3_{nr}(X, 2) = 0$.

Earlier work of Shuji Saito 1989 and CT 1999 then gives

Corollary. *Let* $k = \mathbb{F}(C)$, *let* X/C *as in the theorem and* $Y/\mathbb{F}(C)$ *the generic fibre. Then*
(i) The Brauer-Manin obstruction to the existence of a zero-cycle of degree prime to ℓ *on* $Y/\mathbb{F}(C)$ *is the only obstruction.*
(ii) The more precise conjecture of CT-Sansuc for zero-cycles on a geometrically rational surface over a global field holds over the global field $k = \mathbb{F}(C)$.

Note : for statement (i), it is enough to use Kollár-Tian KT23.

This applies to smooth cubic surfaces over $k = \mathbb{F}(C)$, for which the analogue over a number field is not known.
For conic bundles over $\mathbb{P}^1_k$ and k a number field, the corollary goes back to Salberger 1988.
For conic bundles over $\mathbb{P}^1_k$ and $k = \mathbb{F}(C)$, the theorem and the corollary follow from Parimala-Suresh 2016.

https ://www.imo.universite-paris-saclay.fr/ jean-louis.colliot-thelene/liste-cours-exposes.html

The beamer presentations 47, 48

Luminy (Merkurjev conference, september 2015), slides available on my homepage and a similar one in English at Schoß Elmau in April 2016 contain information on

– computation of the group H^3_{nr} for (smooth compactification of) homogeneous spaces of connected linear algebraic groups G and for classifying varieties BG

– relation with Chow groups of codimension 2 on complex varieties

– use of H^3_{nr} as obstruction to the local-global principle for homogeneous spaces of tori over a function field in one variable over a p-adic field. There is ongoing progress in this direction for more general linear algebraic groups.

For the Brauer-Manin aspect, see the beamer presentation 25 (VU Amsterdam Feb. 25, 2010).