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Jean-Louis COLLIOT-THÉLÈNE and Wayne RASKIND

Reprinted from the
JOURNAL OF THE FACULTY OF SCIENCE, THE UNIVERSITY OF TOKYO
Sec. IA, Vol. 33, No. 2, pp. 283-294
October 1986

J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
33 (1986), 283-294.

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(Communicated by Y. Ihara)

Let X be a smooth, proper, geometrically connected variety over a finite field $\boldsymbol{F}_q$, $q=p^f$. Let $Z_0^0(X)$ be the group of zero-cycles of degree zero on X. Frobenius substitutions define a map

$$\Theta: Z_0^0(X) \longrightarrow \pi_1^{\text{geom}}(X)$$

where $\pi_1^{\text{geom}}(X)$ denotes the abelian geometric fundamental group of X which classifies abelian étale covers of X which do not arise from extending the base field. This group is well-known to be finite ([15], Theorem 2), and Lang showed ([16], Statement 5, p. 314) that Θ factors through rational equivalence of cycles to give a surjective map ([21], Théorème 3, p. 146) of torsion groups

$$\theta: A_0(X) \longrightarrow \pi_1^{\text{geom}}(X)$$

where $A_0(X)$ is the group of zero-cycles of degree zero modulo rational equivalence. That $A_0(X)$ is a torsion group is classical, see for example [7], proof of Proposition 4. In particular, θ induces a surjection θ_l on l-primary components for all prime numbers l. One way of stating the reciprocity law for the unramified class field theory of function fields in one variable over a finite field is that if X is a curve then θ is an isomorphism. Kato and Saito showed [14] that *if X is a surface then θ is an isomorphism* and then a standard geometric argument ([14], § 9) allows one to deduce that θ is an isomorphism for X projective of any dimension.

In [7], p. 789, Colliot-Thélène, Sansuc and Soulé gave another proof of the prime-to-p part of the above result of Kato-Saito using the theorem of Merkur'ev-Suslin [17] and the Weil conjectures for étale cohomology with twisted coefficients as proved by Deligne [9]. However, their argument relied on the commutativity of a certain diagram ([7], Proposition 1) which complicated the argument considerably. A proof of the p-part using the theorem of Bloch-Gabber-Kato [3] and the logarithmic cohomology of Bloch-Illusie-Milne was given by M. Gros [10], who used a diagram similar to the one used in [7]. *The purpose of this note is to show how one may* *use a counting argument along with the theorems of Merkur'ev-Suslin, Bloch-Gabber-Kato and Deligne to avoid the complication of the above-mentioned diagrams and hence give a simple (but high-flown) proof of the result of Kato-Saito.* As a matter of fact, a counting argument in the same spirit already appears in a paper of Milne ([19], Corollary 8.4, p. 283). We end the paper with a corollary (Corollary 4) which follows easily from the theorem but which does not seem to be present in the literature. This corollary is essentially equivalent to the theorem and our interest in its stems from Conjecture B in [8], p. 443.

1. Preliminaries

Notations will be as in [6] and [7]. In particular, given an abelian group A, a prime number l and a positive integer n, we set ${}_{l^n}A=\{x\in A, l^n x=0\}$ and $A\{l\}=\bigcup_{n=1}^{\infty} {}_{l^n}A$. Cohomology with coefficients in twisted groups of roots of unity $\mu_{l^n}^{\otimes i}$ ($i\in \mathbf{Z}$) is étale cohomology, but cohomology with coefficients in K-theory sheaves is Zariski cohomology. Let us recall the basic theorems:

THEOREM A (Merkur'ev-Suslin, [17]). *Let k be a field and n a positive integer prime to the characteristic of k. Then the Galois symbol*

$$R_{n,k}: K_2k/nK_2k \longrightarrow H^2(k, \mu_n^{\otimes 2})$$

is an isomorphism.

THEOREM B (Bloch-Kato [3], Gabber). *Let k be a perfect field of characteristic $p>0$ and let X be a smooth variety over k. Then the "differential symbol"*

$$\mathcal{K}_2/p^n\mathcal{K}_2 \longrightarrow \nu_n(2)$$

is an isomorphism of Zariski sheaves for any positive integer n.

THEOREM C (Gabber, [7], Théorème 3, p. 782). *Let X be a smooth projective variety over a finite field. Then for any non-negative integers i and r such that $i\neq r, r+1$, the group $H^i_{\text{ét}}(X, \nu_\infty(r))$ is finite.*

2. Proof of the reciprocity law

Let X be a smooth geometrically connected variety over a field k, and let $k(X)$ be its function field. Let $l\neq \text{char}. k$ be a prime number, and let n be a positive integer. Bloch ([1], [2], Lecture 5, see also [7], p. 778) showed that combining the Gersten-Quillen resolution of the sheaf $\mathcal{K}_2$ with the results of Bloch-Ogus [4] and with the Merkur'ev-Suslin theorem (Theorem A above) yields an exact sequence (which is functorial in n):

$$(*) \qquad 0 \longrightarrow H^1(X, \mathcal{K}_2)/l^n \longrightarrow NH^3(X, \mu_{l^n}^{\otimes 2}) \longrightarrow {}_{l^n}CH^2(X) \longrightarrow 0$$

where $NH^3(X, \mu_{l^n}^{\otimes 2}) = \mathrm{Ker}[H^3(X, \mu_{l^n}^{\otimes 2}) \to H^3(k(X), \mu_{l^n}^{\otimes 2})]$.

The following proposition is due to Panin ([20], Theorem 2.11, a) and b)).

PROPOSITION 1. *Let X be a smooth, proper, geometrically connected variety over a finite field $\boldsymbol{F}_q$, $q=p^f$. Then for any prime number $l \neq p$, we have*

a) $H^1(X, \mathcal{K}_2) \otimes \boldsymbol{Q}_l/\boldsymbol{Z}_l = 0$.

b) *There is a natural isomorphism of finite groups*

$$NH^3(X, \boldsymbol{Q}_l/\boldsymbol{Z}_l(2)) \xrightarrow{\sim} CH^2(X)\{l\}.$$

PROOF. Passing to the direct limit over n in the exact sequences $(*)$, we get an exact sequence:

$$0 \longrightarrow H^1(X, \mathcal{K}_2) \otimes \boldsymbol{Q}_l/\boldsymbol{Z}_l \longrightarrow NH^3(X, \boldsymbol{Q}_l/\boldsymbol{Z}_l(2)) \longrightarrow CH^2(X)\{l\} \longrightarrow 0\,.$$

By the Weil conjectures as proved by Deligne (cf. [7], Theorem 2, p. 780) the group $H^3(X, \boldsymbol{Q}_l/\boldsymbol{Z}_l(2))$ is finite and hence so is $NH^3(X, \boldsymbol{Q}_l/\boldsymbol{Z}_l(2))$. Since $H^1(X, \mathcal{K}_2) \otimes \boldsymbol{Q}_l/\boldsymbol{Z}_l$ is now l-divisible and finite, it must be zero. This completes the proof of the proposition.

THEOREM 1. *Let X be a smooth, proper, geometrically connected surface over a finite field $\boldsymbol{F}_q$, $q=p^f$ and let $l \neq p$ be a prime number. Then the reciprocity map*

$$\theta_l : A_0(X)\{l\} \longrightarrow \pi_1^{\mathrm{geom}}(X)\{l\}\,,$$

defined as the map θ of the introduction restricted to l-primary torsion, is an isomorphism.

PROOF. Consider the sequence of homomorphisms between finite groups (cf. Introduction and Proposition 1):

$$A_0(X)\{l\} \longrightarrow NH^3(X, \boldsymbol{Q}_l/\boldsymbol{Z}_l(2)) \subset H^3(X, \boldsymbol{Q}_l/\boldsymbol{Z}_l(2)) \longrightarrow H^4(X, \boldsymbol{Z}_l(2))\{l\}$$
$$\longrightarrow \pi_1^{\mathrm{geom}}(X)\{l\}\,.$$

Here the first map is the inverse of the isomorphism of Proposition 1, b), the second is the natural inclusion, the third is the Bockstein map ([7], p. 774) which is an isomorphism because $H^3(X, \boldsymbol{Q}_l(2))=0$ (Weil conjectures, [7], p. 781) and the fourth is the arithmetic Poincaré duality isomorphism ([7], p. 789, (45)[1]). Now the composite map between finite groups, say

$$\lambda_l:\ A_0(X)\{l\} \longrightarrow \pi_1^{\text{geom}}(X)\{l\}$$

is not a priori the same as the map θ_l. However, θ_l is surjective (Introduction), hence

$$\# A_0(X)\{l\} \geqq \# \pi_1^{\text{geom}}(X)\{l\}\,.$$

But λ_l is injective and hence

$$\# A_0(X)\{l\} \leqq \# \pi_1^{\text{geom}}(X)\{l\}\,.$$

Thus these groups have the same order and so both θ_l and λ_l are isomorphisms. This completes the proof of Theorem 1.

COROLLARY 1 (cf. [7], Remarque 2, p. 790). *For X as above, the natural inclusion*

$$NH^3(X, \boldsymbol{Q}_l/\boldsymbol{Z}_l(2)) \subset H^3(X, \boldsymbol{Q}_l/\boldsymbol{Z}_l(2))$$

is an isomorphism. In other words, the natural map

$$H^3(X, \boldsymbol{Q}_l/\boldsymbol{Z}_l(2)) \longrightarrow H^3(\boldsymbol{F}_q(X), \boldsymbol{Q}_l/\boldsymbol{Z}_l(2))$$

is zero. Similar statements hold for $\bar{X}=X\times_{\boldsymbol{F}_q}\bar{\boldsymbol{F}}_q$.

PROOF. Indeed, λ_l is an isomorphism and all the other maps in the above composite are isomorphisms. The statement for $\bar{X}$ follows by passing to the direct limit over finite extensions of $\boldsymbol{F}_q$ or by noting that $\bar{\boldsymbol{F}}_q(X)$ has cohomological dimension two.

We now turn to the p-part. Let X be a smooth geometrically connected variety over a perfect field k of characteristic $p>0$. In a simpler fashion than above (see the proof of Lemma 1 below) there is an exact sequence:

$$0 \longrightarrow H^1(X, \mathcal{K}_2)/p^n \longrightarrow H^1(X, \mathcal{K}_2/p^n) \longrightarrow {}_{p^n}CH^2(X) \longrightarrow 0$$

which Theorem B allows us to rewrite as:

$$(**)\qquad 0 \longrightarrow H^1(X, \mathcal{K}_2)/p^n \longrightarrow H^1(X_{\text{Zar}}, \nu_n(2)) \longrightarrow {}_{p^n}CH^2(X) \longrightarrow 0\,.$$

[1] In [7], the notation $\pi_1^{ab}(X)\otimes \boldsymbol{Z}_l$ was used as an unfortunate abuse of notation for $\varprojlim_n \pi_1^{ab}(X)/l^n$.

PROPOSITION 2. *Let X be a smooth, projective, geometrically connected variety over a finite field $\boldsymbol{F}_q$, $q=p^f$. Then:*

a) $H^1(X, \mathcal{K}_2)\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p=0$.

b) *There is a natural isomorphism of finite groups:*

$$H^1(X_{\mathrm{Zar}}, \nu_\infty(2)) \xrightarrow{\sim} CH^2(X)\{p\}.$$

PROOF. We have a natural injection:

$$H^1(X_{\mathrm{Zar}}, \nu_\infty(2)) \hookrightarrow H^1(X_{\text{ét}}, \nu_\infty(2)).$$

Now by Theorem C, this last group is finite. Passing to the limit over n in (**), we see that $H^1(X, \mathcal{K}_2)\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p$ is both finite and p-divisible, hence is zero. This completes the proof of the proposition.

THEOREM 2. *Let X be a smooth, projective, geometrically connected surface over a finite field $\boldsymbol{F}_q$, $q=p^f$. Then the map*

$$\theta_p:\ A_0(X)\{p\} \longrightarrow \pi_1^{\mathrm{geom}}(X)\{p\},$$

defined as the map θ of the introduction restricted to p-primary torsion, is an isomorphism.

PROOF. Consider the sequence of homomorphisms between finite groups (cf. Introduction and Proposition 2):

$$A_0(X)\{p\} \longrightarrow H^1(X_{\mathrm{Zar}}, \nu_\infty(2)) \hookrightarrow H^1(X_{\text{ét}}, \nu_\infty(2)) \longrightarrow H^2(X, \mathfrak{p}(2))\{p\} \longrightarrow \pi_1^{\mathrm{geom}}(X)\{p\}.$$

Here the first map is the inverse of the isomorphism of Proposition 2, the second is the natural injection, the third is the "Bockstein map" which is an isomorphism because $H^1(X, \mathfrak{p}(2))$ is torsion (cf. [7], proof of Théorème 3 and [10], Lemme 2.1.16) and the last map is given by the "Milne duality isomorphism" ([18], Theorem 1.9 combined with [7], Lemme 3 and induction). Now the composite map between finite groups

$$\lambda_p:\ A_0(X)\{p\} \longrightarrow \pi_1^{\mathrm{geom}}(X)\{p\}$$

is again not a priori the same as the map θ_p. However, θ_p is surjective (Introduction), hence

$$\# A_0(X)\{p\} \geqq \# \pi_1^{\mathrm{geom}}(X)\{p\}.$$

But λ_p is injective and hence

$$\sharp A_0(X)\{p\} \leq \sharp \pi_1^{\mathrm{geom}}(X)\{p\}.$$

Thus these groups have the same order and so both θ_p and λ_p are isomorphisms. This completes the proof of Theorem 2.

COROLLARY 2. *For X as in Theorem 2, the natural injection*

$$H^1(X_{\mathrm{Zar}}, \nu_\infty(2)) \hookrightarrow H^1(X_{\acute{e}t}, \nu_\infty(2))$$

is an isomorphism. A similar statement is true for $\bar{X}=X\times_{\boldsymbol{F}_q}\bar{\boldsymbol{F}}_q$.

PROOF. Indeed, λ_p is an isomorphism and all of the other maps in the above composite are isomorphisms. The statement for $\bar{X}$ now follows by passage to the direct limit over finite extensions of $\boldsymbol{F}_q$.

COROLLARY 3. *For X as above, we have:*

a) $$\mathrm{Ker}'[A_0(X) \longrightarrow A_0(\bar{X})] \xrightarrow{\sim} \bigoplus_{l\neq p} H^1(\boldsymbol{F}_q, H^3(\bar{X}, \boldsymbol{Z}_l(2))\{l\})$$

where Ker′ *denotes the prime-to-p part of the torsion group*

$$\mathrm{Ker}[A_0(X) \longrightarrow A_0(\bar{X})].$$

b) $$\mathrm{Ker}[A_0(X) \longrightarrow A_0(\bar{X})]\{p\} \xrightarrow{\sim} H^1(\boldsymbol{F}_q, H^1(\bar{X}_{\acute{e}t}, \nu(2))\{p\}).$$

c) *The natural map*

$$A_0(X) \longrightarrow A_0(\bar{X})^G$$

is surjective, where $G=\mathrm{Gal}(\bar{\boldsymbol{F}}_q/\boldsymbol{F}_q)$.

PROOF. Consider the following diagram:

$$\begin{array}{ccccccc}
0\longrightarrow H^1(\boldsymbol{F}_q, H^2(\bar{X}, \boldsymbol{Q}_l/\boldsymbol{Z}_l(2))) & \longrightarrow & H^3(X, \boldsymbol{Q}_l/\boldsymbol{Z}_l(2)) & \longrightarrow & H^3(\bar{X}, \boldsymbol{Q}_l/\boldsymbol{Z}_l(2))^G & \longrightarrow 0 \\
\downarrow & & \downarrow\wr & & \downarrow\wr & \\
0\longrightarrow \mathrm{Ker}\,[A_0(X)\longrightarrow A_0(\bar{X})]\{l\} & \longrightarrow & A_0(X)\{l\} & \longrightarrow & A_0(\bar{X})\{l\}^G. &
\end{array}$$

Here the top row is deduced from the Hochschild-Serre spectral sequence and the two right vertical maps are the isomorphisms of Proposition 1 combined with Corollary 1 (that the right vertical map is an isomorphism follows from Prop. 1 and Cor. 1 by passage to the direct limit over X_L for $L/\boldsymbol{F}_q$ a finite extension). This proves the prime-to-p part of c). The right square is clearly commutative and this induces the left vertical arrow which is then seen to be an isomorphism. An argument with the Weil conjectures (cf. [7], Théorème 2 and its proof) yields the isomorphism of finite groups

$$H^1(\boldsymbol{F}_q, H^2(\bar{X}, \boldsymbol{Q}_l/\boldsymbol{Z}_l(2))) \longrightarrow H^1(\boldsymbol{F}_q, H^3(\bar{X}, \boldsymbol{Z}_l(2))\{l\})$$

which proves a).

To prove b) and the p-part of c), consider the diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow H^1(\boldsymbol{F}_q, H^0(\bar{X}, \nu_\infty(2))) & \longrightarrow & H^1(X_{\text{ét}}, \nu_\infty(2)) & \longrightarrow & H^1(\bar{X}_{\text{ét}}, \nu_\infty(2))^G \longrightarrow 0 \\ \downarrow & & \downarrow \wr & & \downarrow \wr \\ 0 \longrightarrow \operatorname{Ker}[A_0(X) \longrightarrow A_0(\bar{X})]\{p\} & \longrightarrow & A_0(X)\{p\} & \longrightarrow & A_0(\bar{X})\{p\}^G \end{array}$$

which is constructed in the same way as the diagram above. The two right vertical maps are isomorphisms by Proposition 2 combined with Corollary 2. This proves the p-part of c). The diagram induces the left vertical arrow which is also seen to be an isomorphism. Now another argument with the Weil conjectures for crystalline cohomology (cf. [7], (35) and statement (iii) p. 784) yields the isomorphism of finite groups:

$$H^1(\boldsymbol{F}_q, H^0(\bar{X}_{\text{ét}}, \nu_\infty(2))) \xrightarrow{\sim} H^1(\boldsymbol{F}_q, H^1(\bar{X}_{\text{ét}}, \mathfrak{D}(2))\{p\})$$

which proves b). This completes the proof of the corollary.

Before stating the next corollary we record two lemmas which will be needed for its proof.

LEMMA 1. *Let X be a smooth, proper, geometrically connected variety over a finite field $\boldsymbol{F}_q$, $q=p^f$. Then*

(i) $H^i(\boldsymbol{F}_q, H^0(\bar{X}, \mathcal{K}_2))=0$ *for* $i \geqq 2$

(ii) $H^1(\boldsymbol{F}_q, K_2\bar{\boldsymbol{F}}_q(X)/H^0(\bar{X}, \mathcal{K}_2))=0$

(iii) $H^2(\boldsymbol{F}_q, K_2\bar{\boldsymbol{F}}_q(X)) \xrightarrow{\sim} H^2(\boldsymbol{F}_q, K_2\bar{\boldsymbol{F}}_q(X)/H^0(\bar{X}, \mathcal{K}_2))$

(iv) $H^0(\bar{X}, \nu_\infty(2)) \cong H^1(\bar{X}, \mathcal{K}_2)\{p\}$.

PROOF. By ([5], Theorem B and Remark 5.2), we have

$$H^1(\boldsymbol{F}_q, K_2\bar{\boldsymbol{F}}_q(X))=0$$

and then taking cohomology of the obvious exact sequence of $\mathrm{Gal}(\bar{\boldsymbol{F}}_q/\boldsymbol{F}_q)$-modules:

$$0 \longrightarrow H^0(\bar{X}, \mathcal{K}_2) \longrightarrow K_2\bar{\boldsymbol{F}}_q(X) \longrightarrow K_2\bar{\boldsymbol{F}}_q(X)/H^0(\bar{X}, \mathcal{K}_2) \longrightarrow 0$$

shows that (i) implies (ii) and (iii). Let us prove (i), which is clear for $i \geqq 3$.

To prove the prime-to-p part of (i) we use the following sublemma:

SUBLEMMA. *Let l be a prime number and let G be a group of* *cohomological dimension one for torsion modules. Let M be a G-module which is an extension of a torsion G-module by an l-divisible G-module. Then the l-primary component of $H^2(G, M)$ is zero.*

The proof of the sublemma is easy and will be omitted. It follows from Theorem 1.8 of [6] that for X as in the lemma and $l \neq p$, the group $H^0(\bar{X}, \mathcal{K}_2)$ satisfies the hypotheses of the sublemma. This proves the prime-to-p part of (i).

To prove the p-part of (i) we proceed as follows: By [22], Theorem 1.10, $K_2\boldsymbol{F}_q(X)$ has no p-torsion and hence $H^0(X, \mathcal{K}_2)$ has no p-torsion. Also, $K_2\boldsymbol{F}_q(X)$ having no p-torsion implies that the sheaf ${}_{p^n}\mathcal{K}_2$ is zero. Thus there is an exact sequence

$$0 \longrightarrow \mathcal{K}_2 \xrightarrow{p^n} \mathcal{K}_2 \longrightarrow \mathcal{K}_2/p^n \longrightarrow 0\,.$$

Taking cohomology of this sequence, using Theorem B and passing to the direct limit over n, we get an exact sequence:

$$0 \longrightarrow H^0(X, \mathcal{K}_2)\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p \longrightarrow H^0(X_{\mathrm{Zar}}, \nu_\infty(2)) \longrightarrow H^1(X, \mathcal{K}_2)\{p\} \longrightarrow 0\,.$$

Now $H^0(X_{\mathrm{Zar}}, \nu_\infty(2)) \cong H^0(X_{\text{ét}}, \nu_\infty(2))$ and by Theorem C, this last group is finite. Hence $H^0(X, \mathcal{K}_2)\otimes \boldsymbol{Q}_p/\boldsymbol{Z}_p = 0$. This and the above remarks imply that $H^0(X, \mathcal{K}_2)$ is uniquely p-divisible. A limit argument yields (iv) which is the p-primary analogue of [20], Corollary 2.3, or [6], Theorem 2.1. Another limit argument shows that $H^0(\bar{X}, \mathcal{K}_2)$ is uniquely p-divisible (compare with [6], Theorem 1.8) and hence the p-primary component of $H^2(\boldsymbol{F}_q, H^0(\bar{X}, \mathcal{K}_2))$ is zero. This proves (i) and completes the proof of the lemma.

LEMMA 2. *For X a surface as above, the group $H^1(\boldsymbol{F}_q, H^1(\bar{X}, \mathcal{K}_2))$ is finite and isomorphic to the group*

$$\bigoplus_{l \neq p} H^1(\boldsymbol{F}_q, H^3(\bar{X}, \boldsymbol{Z}_l(2))\{l\}) \oplus H^1(\boldsymbol{F}_q, H^1(\bar{X}, \mathfrak{v}(2))\{p\})\,.$$

PROOF. As usual we break up the proof into the prime-to-p part and the p-part. Let $l \neq p$ be a prime number. Then it follows from Theorems 2.1 and 2.2 of [6], or [20], Corollary 2.3 and Theorem 2.11, that we have an exact sequence of $\mathrm{Gal}(\bar{\boldsymbol{F}}_q/\boldsymbol{F}_q)$-modules:

$$0 \longrightarrow H^2(\bar{X}, \boldsymbol{Q}_l/\boldsymbol{Z}_l(2)) \longrightarrow H^1(\bar{X}, \mathcal{K}_2) \longrightarrow H^1(\bar{X}, \mathcal{K}_2)\otimes \boldsymbol{Z}[1/l] \longrightarrow 0\,.$$

Since $H^1(\bar{X}, \mathcal{K}_2)\otimes \boldsymbol{Z}[1/l]$ is uniquely l-divisible, so is $[H^1(\bar{X}, \mathcal{K}_2)\otimes \boldsymbol{Z}[1/l]]^G$ $(G=\mathrm{Gal}(\bar{\boldsymbol{F}}_q/\boldsymbol{F}_q))$ and $H^1(\boldsymbol{F}_q, H^1(\bar{X}, \mathcal{K}_2)\otimes \boldsymbol{Z}[1/l])\{l\}=0$. By the last part of the proof of Corollary 3, a), we have $H^1(\boldsymbol{F}_q, H^2(\bar{X}, \boldsymbol{Q}_l/\boldsymbol{Z}_l(2))) \cong H^1(\boldsymbol{F}_q, H^3(\bar{X}, \boldsymbol{Z}_l(2))\{l\})$. Hence the boundary map

$$[H^1(\bar{X}, \mathcal{K}_2)\otimes \boldsymbol{Z}[1/l]]^G \longrightarrow H^1(\boldsymbol{F}_q, H^2(\bar{X}, \boldsymbol{Q}_l/\boldsymbol{Z}_l(2)))$$

maps a uniquely l-divisible group to a finite l-primary group and so is zero. This gives the desired isomorphism

$$H^1(\boldsymbol{F}_q, H^1(\bar{X}, \mathcal{K}_2))\{l\} \xrightarrow{\sim} H^1(\boldsymbol{F}_q, H^3(\bar{X}, \boldsymbol{Z}_l(2))\{l\}).$$

The prime-to-p part of the lemma now follows from the fact that $H^3(\bar{X}, \boldsymbol{Z}_l(2))\{l\}$ is zero for almost all l. This last fact is most easily seen by noting that $H^3(\bar{X}, \boldsymbol{Z}_l(2))\{l\}$ may be identified (non-canonically) with the l-primary component of the torsion subgroup of the Néron-Severi group of $\bar{X}$ (cf. [11], 8.11, p. 147).

To prove the p-part, we observe that it follows from Proposition 2, a), (going over to $\bar{\boldsymbol{F}}_q$) and Lemma 1, (iv), that there is an exact sequence of G-modules:

$$0 \longrightarrow H^0(\bar{X}_{\text{ét}}, \nu_\infty(2)) \longrightarrow H^1(\bar{X}, \mathcal{K}_2) \longrightarrow H^1(\bar{X}, \mathcal{K}_2)\otimes \boldsymbol{Z}[1/p] \longrightarrow 0.$$

By the proof of Corollary 3, b), the group $H^1(\boldsymbol{F}_q, H^0(\bar{X}_{\text{ét}}, \nu_\infty(2)))$ is finite and isomorphic to $H^1(\boldsymbol{F}_q, H^1(\bar{X}_{\text{ét}}, \mathfrak{D}(2))\{p\})$. Now the exact same argument we used in the proof of the prime-to-p part shows that

$$H^1(\boldsymbol{F}_q, H^1(\bar{X}, \mathcal{K}_2))\{p\} \cong H^1(\boldsymbol{F}_q, H^1(\bar{X}_{\text{ét}}, \mathfrak{D}(2))\{p\}).$$

This completes the proof of Lemma 2.

REMARK 1. Although it is not in general true that the group $H^i(\bar{X}, \mathfrak{D}(j))\{p\}$ is finite, this is true for $i=0$ or 1 (cf. [13], p. 194, comments after Corollaire IV 3.5). We thank M. Gros for this remark.

We may now state the promised corollary to Theorems 1 and 2.

COROLLARY 4. *For X a surface as above, the map*

$$H^2(\boldsymbol{F}_q, K_2\bar{\boldsymbol{F}}_q(X)) \longrightarrow H^2\Big(\boldsymbol{F}_q, \bigoplus_{x\in\bar{X}^1} \bar{\boldsymbol{F}}_q(x)^*\Big),$$

induced by the tame symbol, is injective.

PROOF. Consider the Gersten-Quillen complex which computes the $\mathcal{K}_2$-cohomology of $\bar{X}$:

$$K_2\bar{\boldsymbol{F}}_q(X) \longrightarrow \bigoplus_{x\in\bar{X}^1} \bar{\boldsymbol{F}}_q(x)^* \xrightarrow{\text{div}} \bigoplus_{x\in\bar{X}^2} \boldsymbol{Z}.$$

Let $\mathcal{Z}=\mathrm{Ker}(\mathrm{div})$, $\mathcal{I}=\mathrm{Im}(\mathrm{div})$, so that there are exact sequences:

a) $$0 \longrightarrow K_2\bar{F}_q(X)/H^0(\bar{X}, \mathcal{K}_2) \longrightarrow \mathcal{Z} \longrightarrow H^1(\bar{X}, \mathcal{K}_2) \longrightarrow 0$$

b) $$0 \longrightarrow \mathcal{Z} \longrightarrow \bigoplus_{x\in \bar{X}^1} \bar{F}_q(x)^* \longrightarrow \mathcal{I} \longrightarrow 0$$

c) $$0 \longrightarrow \mathcal{I} \longrightarrow \bigoplus_{x\in \bar{X}^2} \boldsymbol{Z} \longrightarrow CH^2(\bar{X}) \longrightarrow 0 .$$

Taking cohomology of a) and using Proposition 3.6 of [6], we get a long exact sequence:

$$(*) \qquad H^1(\boldsymbol{F}_q, K_2\bar{\boldsymbol{F}}_q(X)/H^0(\bar{X}, \mathcal{K}_2)) \longrightarrow \mathrm{Ker}[CH^2(X)\to CH^2(\bar{X})]$$
$$\longrightarrow H^1(\boldsymbol{F}_q, H^1(\bar{X}, \mathcal{K}_2)) \longrightarrow H^2(\boldsymbol{F}_q, K_2\bar{\boldsymbol{F}}_q(X)/H^0(\bar{X}, \mathcal{K}_2)) \longrightarrow H^2(\boldsymbol{F}_q, \mathcal{Z}) .$$

By Lemma 1, (ii), the first group vanishes. We claim that the map

$$\mathrm{Ker}[CH^2(X)\to CH^2(\bar{X})] \longrightarrow H^1(\boldsymbol{F}_q, H^1(\bar{X}, \mathcal{K}_2))$$

in the above cohomology sequence is an isomorphism. Indeed, by a combination of Corollary 3, a), b) and Lemma 2, there is *some* isomorphism between these finite groups. But the map in question is injective so it must be an isomorphism as well! This and Lemma 1, (iii), yield the injection:

(i) $$H^2(\boldsymbol{F}_q, K_2\bar{\boldsymbol{F}}_q(X)) \hookrightarrow H^2(\boldsymbol{F}_q, \mathcal{Z}) .$$

Now taking Galois cohomology of sequence c) and using Corollary 3, c), Shapiro's lemma $H^1(\boldsymbol{F}_q, \bigoplus_{x\in \bar{X}^2} \boldsymbol{Z})=0$ and the well-known existence of a 0-cycle of degree 1 on X, we get that $H^1(\boldsymbol{F}_q, \mathcal{I})=0$, and then taking Galois cohomology of sequence b), we get the injection

(ii) $$H^2(\boldsymbol{F}_q, \mathcal{Z}) \hookrightarrow H^2\Big(\boldsymbol{F}_q, \bigoplus_{x\in \bar{X}^1} \bar{\boldsymbol{F}}_q(x)^*\Big).$$

Putting (i) and (ii) together yields the corollary.

REMARK 2. It is easy to see that Corollary 4 implies the isomorphism

$$\mathrm{Ker}[CH^2(X)\to CH^2(\bar{X})] \xrightarrow{\sim} H^1(\boldsymbol{F}_q, H^1(\bar{X}, \mathcal{K}_2)),$$

hence Corollary 3. However, we have been unable to deduce Corollary 4 except by this somewhat tortuous route.

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(Received September 18, 1985)

Jean-Louis Colliot-Thélène
Mathématiques
C.N.R.S. et Université de Paris-Sud
Bâtiment 425
F-91405
France

Wayne Raskind
Department of Mathematics
Harvard University
Cambridge, MA 02138
U.S.A.