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Real rational surfaces without a real point

By

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A smooth projective geometrically connected variety X defined over a field k is called rational if it becomes birational to a projective space after some extension of the ground field k. In this note we are interested in rational varieties over the field $\mathbb{R}$ of real numbers which do not possess a real point.

In dimension 1, up to $\mathbb{R}$-isomorphism, there is a unique such variety, namely the real plane conic C without a real point in projective plane $\mathbb{P}^2_{\mathbb{R}}$, defined in homogeneous coordinates by the equation

$$\sum_{i=0}^{2} X_i^2 = 0.$$

In dimension 2, three obvious such surfaces are: the product $C \times_{\mathbb{R}} \mathbb{P}^1_{\mathbb{R}}$, the product $C \times_{\mathbb{R}} C$, and the quadric $Q \subset \mathbb{P}^3_{\mathbb{R}}$ defined in homogeneous coordinates by the equation

$$\sum_{i=0}^{3} X_i^2 = 0.$$

An easy exercise shows that these three surfaces are $\mathbb{R}$-birational to one another. We provide a "modern" proof for a fact which was already known to Annibale Comessatti [4]:

Theorem. *Let X be a smooth projective geometrically connected surface over $\mathbb{R}$. Assume that $X_{\mathbb{C}} = X \times_{\mathbb{R}} \mathbb{C}$ is rational, and that X has no real point. Then X is $\mathbb{R}$-birational to the quadric $Q \subset \mathbb{P}^3_{\mathbb{R}}$ without a real point.*

The theorem implies the following corollary, due to Parimala and Sujatha [13]:

Corollary. *In the function field $\mathbb{R}(X)$ of a real rational surface without a real point, (-1) is a sum of two squares.*

Parimala and Sujatha give a unified proof for the corollary, based on some K-theoretical facts. The proof of our Theorem is based on the birational classification of rational surfaces over an arbitrary field, due to Enriques, Manin, Iskovskih [9] (see also Mori [12]). It also relies on classical facts regarding nonsingular plane quartics. We shall freely use methods and results from the birational theory of surfaces, as may be found in [5] or [11].

the genus of Y is 3, hence odd, implies that all Tate cohomology groups $\hat{H}^i(G, J(\mathbb{C}))$, where $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$, are equal to $\mathbb{Z}/2$.

In terms of the period lattice M of the abelian variety $J(\mathbb{C})$, lattice which fits into an exact sequence of G-modules:

$$0 \to M \to \mathbb{C}^3 \to J(\mathbb{C}) \to 0,$$

the G-lattice M is G-isomorphic to a sum $\mathbb{Z} \oplus \mathbb{Z}[G]/(1+\sigma) \oplus \mathbb{Z}[G]^2$. From this we deduce that the group ${}_2J(\mathbb{R})$ of real 2-torsion points is isomorphic to $(M/2M)^G \cong (\mathbb{Z}/2)^4$.

Let L_0 be a real bitangent, which cuts out on Y twice the real divisor $(P+Q)$. If L_1 is another real bitangent, cutting out on Y twice the real divisor $(R+S)$, then the class of the real divisor $(R+S-P-Q)$ defines a point in ${}_2J(\mathbb{R}) \subset \mathrm{Pic}(Y_{\mathbb{C}})^G$. If $L_1 \neq L_0$, this point is non trivial; indeed, it is a well-known fact that a smooth plane quartic curve is not hyperelliptic. For the same reason, distinct bitangents L_1 and L_2 give rise to distinct points in ${}_2J(\mathbb{R})$. Thus the total number of real bitangents certainly cannot exceed 16.

As a matter of fact, this number is exactly 4 (Atiyah ([1], p. 62); this reference was pointed out to me by Parimala.

We may therefore find two distinct bitangents L_1 and L_2 of $Y_{\mathbb{C}}$ which are each defined over $\mathbb{C}$ and are conjugate. If we let $E_1 \subset X_{\mathbb{C}}$ be one the components of $f^{-1}(L_1)$, its conjugate $E_2 \subset X_{\mathbb{C}}$ lies above L_2. Thus E_1 and E_2 do not meet and they may therefore be simultaneously blown down over $\mathbb{R}$, contradicting the assumption that X is $\mathbb{R}$-minimal.

If $d = 4$, the linear system associated to the anticanonical line bundle makes X into a smooth intersection of two quadrics in $\mathbb{P}^4_{\mathbb{R}}$. There are 16 exceptional curves of the first kind on $X_{\mathbb{C}}$, which are none other than the 16 lines lying over $X_{\mathbb{C}}$. Let $E_1 \subset X_{\mathbb{C}}$ be such a line. It may not be defined over $\mathbb{R}$, since $X(\mathbb{R}) = \emptyset$. Let E_2 be its conjugate. If the two lines E_1 and E_2 meet, they meet in one point which is clearly a real point. Since $X(\mathbb{R}) = \emptyset$, this is impossible. Thus E_1 and E_2 do not meet, but then they may be simultaneously blown down over $\mathbb{R}$, contradicting the assumption that X is $\mathbb{R}$-minimal.

If $d = 6$, there are 6 exceptional curves of the first kind ("lines") on $X_{\mathbb{C}}$, whose configuration is well-known: If lines be represented by dots, and two dots be connected by a segment if the lines meet, one gets a hexagon. Defining the distance in an obvious way, one sees that there are exactly 3 sets of pairs of lines (L_1, L_2) such that the distance of L_1 to L_2 is equal to 3. Thus one of these sets must be defined over $\mathbb{R}$, and X is not $\mathbb{R}$-minimal.

We are thus left with the case $d = 8$. Two possibilities may occur. Either $X_{\mathbb{C}}$ is the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ in one point, or $X_{\mathbb{C}}$ is $\mathbb{C}$-isomorphic to $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$. In the first case, there exists a unique exceptional curve E of the first kind on $X_{\mathbb{C}}$. This curve is then defined over $\mathbb{R}$ and isomorphic to $\mathbb{P}^1_{\mathbb{R}}$, hence $X(\mathbb{R})$ is not empty, which we excluded. Thus we may assume that $X_{\mathbb{C}}$ is $\mathbb{C}$-isomorphic to $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$. The structure of the Picard group of such a surface is well-known:

$$\mathrm{Pic}(X_{\mathbb{C}}) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2,$$

with $(e_1.e_2) = 1$, $(e_1.e_1) = 0$, $(e_2.e_2) = 0$, and $\omega = -2e_1 - 2e_2$, the classes e_1 and e_2 being given by the line bundles $O(1)$ on each of the factors $\mathbb{P}^1_{\mathbb{C}}$. From these formulas, one sees

Proof of the theorem. We may assume that X is $\mathbb{R}$-minimal. According to the classification of rational surfaces over a perfect field, X is then $\mathbb{R}$-isomorphic to a surface of one of the following types:

I) A standard conic bundle X/Y: there is a fibration $p: X \to Y$ of X over a smooth projective curve Y of genus zero, any geometric fibre of which is isomorphic to a plane conic, which is either smooth or is a union of two exceptional curves of the first kind, meeting transversally in one point (such fibres are called degenerate fibres).

II) A Del Pezzo surface X, i.e. a rational surface whose anticanonical bundle ω^{-1} is ample. We then let $d = (\omega.\omega)$ be the "degree" of the surface X. Here $1 \leqq d \leqq 9$.

Assume first that X is a standard conic bundle $p: X \to C$.

If $C(\mathbb{R}) = \emptyset$, the function field $\mathbb{R}(C)$ is of cohomological dimension 1, and the generic fibre of p, which is a conic over $\mathbb{R}(C)$, has a rational point, hence is isomorphic to the projective line over $\mathbb{R}(C)$, so that X is $\mathbb{R}$-birational to $\mathbb{P}^1_{\mathbb{R}} \times C$, hence to Q.

If $C(\mathbb{R}) \neq \emptyset$, then C is isomorphic to $\mathbb{P}^1_{\mathbb{R}}$. No fibre of p above an $\mathbb{R}$-point of C may be degenerate, since $X(\mathbb{R})$ would then contain the singular point of such a fibre. On the other hand, if a fibre of p above a nonreal closed point of C were degenerate, it would give rise to a pair of disjoint conjugate exceptional curves of the first kind, which could then be contracted, contradicting the assumption that X is $\mathbb{R}$-minimal. Thus no fibre of p is degenerate, and standard facts on conic bundles over the projective line (one may also use Milnor's exact sequence for the Witt group ([10], IX, § 3)) then imply that the generic fibre of p is a (smooth) conic over $\mathbb{R}(\mathbb{P}^1_{\mathbb{R}})$ which comes from a smooth conic Z over $\mathbb{R}$. Thus X is $\mathbb{R}$-birational to $Z \times_{\mathbb{R}} \mathbb{P}^1_{\mathbb{R}}$. From $X(\mathbb{R}) = \emptyset$, we conclude $Z(\mathbb{R}) = \emptyset$, i.e. $Z \cong C$, hence X is $\mathbb{R}$-birational to $C \times_{\mathbb{R}} \mathbb{P}^1_{\mathbb{R}}$, itself $\mathbb{R}$-birational to Q.

We now assume that X is a Del Pezzo surface.

The degree $d = (\omega.\omega)$ must be even. Indeed, d is the image of the pair $(\omega, \omega) \in \operatorname{Pic}(X) \times \operatorname{Pic}(X)$ under the composite map

$$\operatorname{Pic}(X) \times \operatorname{Pic}(X) \xrightarrow{\text{int}} CH_0(X) \xrightarrow{\text{deg}} \mathbb{Z},$$

where $\operatorname{Pic}(X)$ denotes the Picard group of X and $CH_0(X)$ denotes the Chow group of zero-cycles on X modulo rational equivalence, and "int" denotes the intersection of cycle classes (see [6]). If d is odd, one may produce a 0-cycle of odd degree on X, hence also an $\mathbb{R}$-point, since finite field extensions of $\mathbb{R}$ are of degree either 1 or 2.

Thus the degree d may be either 2, 4, 6 or 8. We shall use well-known facts about Del Pezzo surfaces, which may be read off from [11] or [5].

If $d = 2$, then the linear system associated to the anticanonical line bundle ω^{-1} gives rise to a finite morphism $f: X \to \mathbb{P}^2_{\mathbb{R}}$ which makes X into a double cover of $\mathbb{P}^2_{\mathbb{R}}$ ramified along a smooth plane quartic Y. The quartic $Y_{\mathbb{C}}$ possesses 28 bitangents, and for each bitangent L, the inverse image $f^{-1}(L)$ consists of two exceptional curves of the first kind on $X_{\mathbb{C}}$, which meet transversally in 2 points. One thus gets all the 56 exceptional curves of the first kind on $X_{\mathbb{C}}$. The hypothesis $X(\mathbb{R}) = \emptyset$ implies $Y(\mathbb{R}) = \emptyset$.

Now a smooth plane quartic Y without real points cannot have all its 28 bitangents real. This is most easily deduced from classical facts in the following manner. Let J be the jacobian of Y. Thus $J(\mathbb{C})$ is the subgroup of the Picard group $\operatorname{Pic}(Y_{\mathbb{C}})$ consisting of classes of divisors of degree zero. As was shown by Klein and Weichold, a more algebraic approach being due to Witt and Geyer [7], the hypothesis $Y(\mathbb{R}) = \emptyset$ and the fact that

that complex conjugation acting on $\mathrm{Pic}(X_{\mathbb{C}})$ either permutes e_1 and e_2, or it acts trivially on $\mathrm{Pic}(X_{\mathbb{C}})$.

Let $L_1 \subset X_{\mathbb{C}}$ be a curve corresponding to a section of the line bundle e_1. This is none other than a line $\mathbb{P}^1_{\mathbb{C}} \times N$, for some point $N \in \mathbb{P}^1_{\mathbb{C}}$. If e_1 be transformed into e_2 by complex conjugation, then L_1 is transformed by this same conjugation into a curve L_2 which is a line $M \times \mathbb{P}^1_{\mathbb{C}}$, for some point $M \in \mathbb{P}^1_{\mathbb{C}}$, and the intersection point of L_1 and L_2 is a real point of X, and there are no such points. Thus complex conjugation acts trivially on $\mathrm{Pic}(X_{\mathbb{C}})$.

Quite generally, there is an exact sequence (e.g. [3])

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(X_{\mathbb{C}})^{\sigma} \to \mathrm{Br}(\mathbb{R})$$

where σ denotes complex conjugation. Since σ acts trivially on $\mathrm{Pic}(X_{\mathbb{C}})$ and $\mathrm{Br}(\mathbb{R})$ is killed by 2, each of the classes $2e_i$ $(i = 1, 2)$ actually belongs to $\mathrm{Pic}(X)$. The linear system associated to $2e_i$ $(i = 1, 2)$ defines an $\mathbb{R}$-morphism $X \to \mathbb{P}^2_{\mathbb{R}}$, whose image is a smooth conic $C_i \subset \mathbb{P}^2_{\mathbb{R}}$, and the product of these two morphisms defines an $\mathbb{R}$-isomorphism $X \cong C_1 \times_{\mathbb{R}} C_2$. All this is easily checked by going over to $\mathbb{C}$. Now since $X(\mathbb{R})$ is empty, at least one of the $C_i(\mathbb{R})$ is empty, and X is $\mathbb{R}$-isomorphic either to $C \times_{\mathbb{R}} \mathbb{P}^1_{\mathbb{R}}$ or to $C \times_{\mathbb{R}} C$, both of which are $\mathbb{R}$-birational to Q. □

Remark 1. Let $G = \{1, \sigma\}$ where σ denotes complex conjugation. As any G-lattice, the G-lattice $\mathrm{Pic}(X_{\mathbb{C}})$ admits a decomposition

$$\mathrm{Pic}(X_{\mathbb{C}}) \cong \mathbb{Z}^a \oplus (\mathbb{Z}[G]/(1+\sigma))^b \oplus (\mathbb{Z}[G])^c$$

where the natural integers a, b, c are well-defined, in view of:

$$H^1(G, \mathrm{Pic}(X_{\mathbb{C}})) \cong (\mathbb{Z}/2)^b$$
$$H^2(G, \mathrm{Pic}(X_{\mathbb{C}})) \cong (\mathbb{Z}/2)^a.$$

In the course of their proof, Partimala and Sujatha show that $X(\mathbb{R}) = \emptyset$ implies $a = 2$ and $b = 0$ (for a different approach, see Wall [15]). This result also follows from our approach. Indeed, if Q is the smooth quadric without a real point, the computation made in our study of Del Pezzo surfaces of degree 8 shows that G acts trivially on $\mathrm{Pic}(Q_{\mathbb{C}})$. Hence $H^2(G, \mathrm{Pic}(Q_{\mathbb{C}})) \cong (\mathbb{Z}/2)^2$ and $H^1(G, \mathrm{Pic}(Q_{\mathbb{C}})) \cong 0$. Any smooth projective surface over $\mathbb{R}$ without a real point which is birational to Q can be deduced from Q by a sequence of blowing-ups and blowing-downs, each elementary blowing-up being by necessity in a pair of complex conjugate points, since there are no real points. Now for such an elementary blowing-up say $U \to V$, we have $\mathrm{Pic}(U_{\mathbb{C}}) \cong \mathrm{Pic}(V_{\mathbb{C}}) \oplus Z[G]$, hence $H^i(G, \mathrm{Pic}(U_{\mathbb{C}})) \cong H^i(G, \mathrm{Pic}(V_{\mathbb{C}}))$ for $i = 1, 2$.

Remark 2. Hilbert ([8], see also Choi/Lam [2]) showed that any positive definite polynomial $P(x, y) \in \mathbb{R}[x, y]$ (x, y two independent variables) of total degree at most 4 is the sum of *three* squares of functions (actually polynomials) in the rational function field $\mathbb{R}(x, y)$.

As we shall now see, this result also follows from the result of Parimala and Sujatha (Corollary 1 above) – which itself relies on deep facts from algebraic K-theory. Let us assume that the affine quartic defined by $P(x, y) = 0$ in the affine plane extends to a non-singular quartic in $\mathbb{P}^2_{\mathbb{R}}$ (the other cases are left to the reader). The double cover of the real affine plane defined by $z^2 + P(x, y) = 0$ extends to a double cover $X \to \mathbb{P}^2_{\mathbb{R}}$, where X is a (smooth) Del Pezzo surface of degree 2 (see [5]), and $X(\mathbb{R}) = \emptyset$. Corollary 1 above ensures that (-1) is a sum of two squares in the field $\mathbb{R}(X)$. It only remains to use the well-known lemma (see [10] XI.2.6):

Lemma. *Let k be a field, $\mathrm{char}(k) \neq 2$. Let a be in k, $a \neq 0$. Let $K = k(\sqrt{-a})$. If (-1) is a sum of two squares in K, then a is a sum of three squares in k.* □

Remark 3. The converse of the above lemma also holds. One may thus use Hilbert's result to prove that for any Del Pezzo surface X of degree 2 without a real point, (-1) is a sum of two squares in $\mathbb{R}(X)$. Given any Del Pezzo surface X with $X(\mathbb{R})$ empty, one may blow up pairs of conjugate complex points (in general position) until one gets a Del Pezzo surface of degree 2. Combining this with the classification argument, one gets another proof of Corollary 1, which avoids the discussion of real bitangents to a smooth plane quartic without a real point.

Remark 4. Let X be a smooth projective real rational surface such that $X(\mathbb{R})$ is non empty and consists of exactly one connected component. Some time ago, I conjectured that X is then $\mathbb{R}$-birational to $\mathbb{P}^2_{\mathbb{R}}$ (I have now realized that this also was known to Comessatti [4].) A modern proof was given by Silhol ([14], VI.6.5). His proof is also based on the birational classification of rational surfaces.

References

[1] M. F. ATIYAH, Riemann surfaces and spin structures. Ann. Sci. École Norm. Sup. (4) **4**, 47–62 (1971).
[2] M.-D. CHOI and T.-Y. LAM, Extremal positive semidefinite forms. Math. Ann. **231**, 1–18 (1977).
[3] J.-L. COLLIOT-THÉLÈNE, Quelques propriétés arithmétiques des surfaces rationnelles (d'après Manin). Sém. de théorie des nombres, Bordeaux (1971–1972), exposé 13.
[4] A. COMESSATTI, Fondamenti per la geometria sopra le superficie razionali dal punto di vista reale. Math. Ann. **73**, 1–72 (1912).
[5] M. DEMAZURE, Surfaces de Del Pezzo. In: Séminaire sur les singularités des surfaces, LNM **777**, 21–69, Berlin-Heidelberg-New York 1980.
[6] W. FULTON, Intersection Theory. Ergeb. Math. Grenzgeb. (3) **2**, Berlin-Heidelberg-New York 1984.
[7] W.-D. GEYER, Ein algebraischer Beweis des Satzes von Weichold über reelle algebraische Funktionenkörper. In: Algebraische Zahlentheorie, 83–98, Oberwolfach 1964, Bibliographisches Institut, Mannheim.
[8] D. HILBERT, Über die Darstellung definiter Formen als Summe von Formenquadraten. Math. Ann. **32**, 342–350 (1888).
[9] V. A. ISKOVSKIH, Minimal models of rational surfaces over arbitrary fields. Math. USSR-Izv. **14**, 17–39 (1980).
[10] T.-Y. LAM, The algebraic theory of quadratic forms. Benjamin/Cummings 1973, 1980.
[11] YU. I. MANIN, Cubic Forms, Algebra, Geometry, Arithmetic. Nauka, Moscow 1972; Amsterdam 1974, 1986.
[12] S. MORI, Threefolds whose canonical bundles are not numerically effective, Ann. of Math. **116**, 133–176 (1982).
[13] R. PARIMALA and R. SUJATHA, Levels of non-real function fields of real rational surfaces. Amer. J. Math. **113**, 757–761 (1991).
[14] R. SILHOL, Real algebraic surfaces, LNM **1392**, Berlin-Heidelberg-New York 1989.
[15] C. T. C. WALL, Real forms of smooth Del Pezzo surfaces. J. Reine Angew. Math. **375/376**, 47–66 (1987).

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