ON THE RECIPROCITY SEQUENCE IN THE HIGHER CLASS FIELD THEORY OF FUNCTION FIELDS

J.-L. COLLIOT-THÉLÈNE
C.N.R.S., URA D0752
Mathématiques, Bât. 425
Université de Paris-Sud
F–91405 Orsay Cédex
France

ABSTRACT. According to a conjecture of Kato (1986), the classical reciprocity sequence for the Brauer group of a function field in one variable over a finite field F should have analogues for higher dimensional function fields. A more precise form of the conjecture is that on smooth projective varieties of dimension d over F, the homology of a certain Bloch-Ogus complex of length $d+1$ should be trivial except in the last term, where it should be $\mathbf{Q}/\mathbf{Z}$. For surfaces, the conjecture was established some years ago. In the present paper, I prove that for varieties of arbitrary dimension, the complex has the expected homology in its last four terms, thus settling the case of threefolds (attention is restricted to torsion prime to the characteristic).

Introduction.

Let X be a smooth, geometrically irreducible variety over a perfect field k, let n be a positive integer prime to $p = \mathrm{char}(k)$. For any integer $i \in \mathbf{N}$ and $j \in \mathbf{Z}$, there is a natural complex of étale (= Galois) cohomology groups :

$$(\mathcal{C}_n^{i,j}) \quad 0 \to H^i(k(X), \mu_n^{\otimes j}) \to \bigoplus_{x \in X^{(1)}} H^{i-1}(k(x), \mu_n^{\otimes j-1}) \to \bigoplus_{x \in X^{(2)}} H^{i-2}(k(x), \mu_n^{\otimes j-2}) \cdots$$

Here $X^{(t)}$ denotes the set of points of the scheme X with codimension t, the field $k(x)$ is the residue field at such a point, $k(X)$ is the function field of X and $\mu_n^{\otimes j}$ is the étale sheaf defined by the group μ_n of n-th roots of 1, twisted j times. This complex, already considered by Grothendieck ([Gr], p. 165), is discussed at length in the seminal paper of Bloch and Ogus [B/O]; a more explicit construction of it is given by Kato in [K,§ 1].

In [K], Kato put forward the conjecture : if X is a smooth, projective, geometrically integral variety over a finite field F, $d = \dim(X)$, $i = d+1$ and $j = d$, the complex $(\mathcal{C}_n^{d+1,d})$, henceforth denoted $(\mathcal{C}_n)$:

$$(\mathcal{C}_n) \qquad 0 \to H^{d+1}(F(X), \mu_n^{\otimes d}) \to \bigoplus_{x \in X^{(1)}} H^d(F(x), \mu_n^{\otimes d-1}) \to \bigoplus_{x \in X^{(2)}} H^{d-1}(F(x), \mu_n^{\otimes d-2}) \to \cdots \to \bigoplus_{x \in X^{(d)}} H^1(F(x), \mathbf{Z}/n) \to 0$$

is exact except at the last place, where its homology is $\mathbf{Z}/n$ ([K], Conjecture (0.3), p. 144). As a matter of fact, taking norms down to $H^1(F, \mathbf{Z}/n)$ yields a map

$$\bigoplus_{x \in X^{(d)}} H^1(F(x), \mathbf{Z}/n) \to H^1(F, \mathbf{Z}/n).$$

Since X is projective, a suitable reciprocity statement, which is true over any base field, ensures that the composite map

$$\bigoplus_{x \in X^{(d-1)}} H^2(F(x), \mu_n) \to \bigoplus_{x \in X^{(d)}} H^1(F(x), \mathbf{Z}/n) \to H^1(F, \mathbf{Z}/n)$$

is zero. Because F is finite, we may identify $H^1(F, \mathbf{Z}/n) \simeq \mathbf{Z}/n$. We thus have :

Conjecture 1 (Kato). — *For a smooth, projective, geometrically irreducible variety X over a finite field F, the natural complex :*

$$(\mathcal{C}_n) \qquad \underset{\text{(degree 0)}}{H^{d+1}(F(X), \mu_n^{\otimes d})} \to \underset{\text{(degree 1)}}{\bigoplus_{x \in X^{(1)}} H^d(F(x), \mu_n^{\otimes d-1})} \to \cdots \to \underset{\text{(degree } d)}{\bigoplus_{x \in X^{(d)}} H^1(F(x), \mathbf{Z}/n)} \to \underset{\text{(degree } d+1)}{\mathbf{Z}/n}$$

(extended by zero on both sides) is exact.

By letting n vary among the powers l^m of a fixed prime number l and by going over to the limit as m tends to infinity, the previous conjecture leads to :

Conjecture 2 (Kato). — *For a smooth, projective, geometrically irreducible variety X over a finite field F, the natural complex :*

$$(\mathcal{C}) \qquad \underset{\text{(degree 0)}}{H^{d+1}(F(X), \mathbf{Q}_l/\mathbf{Z}_l(d))} \to \underset{\text{(degree 1)}}{\bigoplus_{x \in X^{(1)}} H^d(F(x), \mathbf{Q}_l/\mathbf{Z}_l(d-1))} \to \cdots \to \underset{\text{(degree } d)}{\bigoplus_{x \in X^{(d)}} H^1(F(x), \mathbf{Q}_l/\mathbf{Z}_l)} \to \underset{\text{(degree } d+1)}{\mathbf{Q}_l/\mathbf{Z}_l}$$

(extended by zero on both sides) is exact.

Use of the de Rham-Witt complex allows Kato to formulate analogous conjectures with p-primary torsion coefficients. Kato also states analogous conjectures for function fields of regular proper schemes over $\mathbf{Z}$, but we shall not be concerned with these in this paper.

When $\dim(X) = 1$, i.e. X is a curve, the conjecture boils down to the classical exact sequence for the Brauer group ([Gr],§ 2 ; this sequence itself is the function theoretic analogue of Hasse's exact sequence for the Brauer group of a number field) :

$$0 \to \mathrm{Br}(F(X)) \to \bigoplus_{x \in X^{(1)}} \mathbf{Q}/\mathbf{Z} \to \mathbf{Q}/\mathbf{Z} \to 0.$$

In the case $\dim(X) = 2$, these conjectures were proved some years ago by Sansuc, Soulé and the author ([CT/S/S, p. 790], exactness on the left term, for the prime-to-p part) and by Kato ([K, p. 176/177]).

In this note, I prove part of Kato's conjectures for higher dimensional varieties.(1)

Theorem A. — *Let X be a smooth, projective, geometrically irreducible variety of dimension d over a finite field F. Then the complex $(\mathcal{C})$ of Conjecture 2 is exact in degrees $\geq d-3$. In particular Conjecture 2 holds when $d = \dim(X) \leq 3$.*

Theorem B. — *Let X be a smooth, projective, geometrically irreducible variety of dimension d over a finite field F. Then the complex $(\mathcal{C}_n)$ of Conjecture 1 is exact in degrees $\geq d-2$. If $\dim(X) = 3$, and n is a power of 2, then Conjecture 1 holds.*

(In the case $\dim(X) = 3$, Conjecture 1 would hold for n arbitrary if a certain well-known conjecture on the cohomology of fields held.)

The proof of Theorem A in degree $\geq d-2$ and of Theorem B in degree $\geq d-1$ is a straightforward extension of Kato's proof ([K]) of the 2-dimensional case. To obtain Theorem A in degree $d-3$, which is the main contribution of this paper (§ 4, Theorem 4.2), I first prove (§ 3, Theorem 3.5) that for a smooth projective variety X over a finite field, the group $H^{d-1}(X, \mathcal{K}_d) \otimes \mathbf{Q}/\mathbf{Z}$ vanishes (as a matter of fact, it is generally conjectured that on such a variety the group $H^{d-1}(X, \mathcal{K}_d)$ is torsion, possibly of finite exponent). The proof, which is not entirely straightforward, uses the technique of Lefschetz pencils and the known results on $H^{d-1}(X, \mathcal{K}_d)$ when $d = 2$, as well as results of Kato and Saito. Theorem A then follows from an analysis of the local to global spectral sequence in étale cohomology, together with various vanishing properties deduced from Deligne's result on the Weil conjectures. That one may go back and forth between the above K-theoretical result and some results in étale cohomology must naturally be traced back to the Merkur'ev-Suslin theorem. From a standard conjecture on the Galois cohomology of fields (§ 2) one could deduce Conjecture 1 from Conjecture 2. The known cases of that conjecture enable us to deduce Theorem B from Theorem A (§ 2 and end of § 4)

When X is a threefold, Shuji Saito has an independent and earlier proof of the exactness of the complex $(\mathcal{C})$ in degree 0 ([Sa]). He actually proves an injectivity result for a surface defined over the completion of a global field $F(C)$ at a point P of a curve C (with reasonable bad reduction at P) and he uses recent results of Jannsen ([J1], [J2]) to go from this local

(1) Using logarithmic de Rham-Witt cohomology, Suwa [Sw] has recently extended my proof to cover the p-part of Kato's conjectures (in the same range).

result (at each point P of C) to the global result $H^0(\mathcal{C}) = 0$. My proof avoids this local detour.

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§ **0. Notation and preliminaries**.

Given an abelian group A and a positive integer n, we denote by A/n the quotient A/nA and by ${}_nA$ the group of elements of A killed by n. By A_{tors} we denote the torsion subgroup of A.

Let X be an algebraic variety over a field k. Given a point x of the scheme X, we shall denote by $k(x)$ the residue field at x.

By $\mu_n^{\otimes j}$ (n prime to char(k)) we denote the étale sheaf on X defined by the group μ_n of n-th roots of 1, twisted j times. Cohomology with values in $\mu_n^{\otimes j}$ will always be étale cohomology (Galois cohomology when X is the spectrum of a field).

For any nonnegative integer i we denote by $CH^i(X)$ the Chow group of codimension i cycles on X modulo rational equivalence.

By $\mathcal{H}^i(\mu_n^{\otimes j})$ we denote the Zariski sheaf on X associated to the Zariski presheaf $U \mapsto H^i_{\text{ét}}(U, \mu_n^{\otimes j})$. Cohomology with values in a sheaf $\mathcal{H}^i(\mu_n^{\otimes j})$ will always be Zariski cohomology.

Let $K_i(R)$ be the i-th Quillen K-group of a commutative ring R. By $\mathcal{K}_i$ we denote the Zariski sheaf on X associated to the Zariski presheaf $U \mapsto K_i(H^0(U, O_X))$. Cohomology with values in a sheaf $\mathcal{K}_i$ will always be Zariski cohomology.

Using dévissage, one obtains the Gersten-Quillen complex :

$$(0.1) \qquad 0 \to \bigoplus_{x \in X^{(0)}} K_i k(x) \to \cdots \to \bigoplus_{x \in X^{(i-j)}} K_j k(x) \to \cdots \to \bigoplus_{x \in X^{(i)}} \mathbf{Z} \to 0 \ .$$

For X smooth over a perfect field k, Quillen proved that this complex is the complex of global sections of a flasque resolution of the sheaf $\mathcal{K}_i$ (Gersten's conjecture).

Similarly, under the same assumption on X, Bloch and Ogus [B/O] proved that the complex $(\mathcal{C}_n^{i,j})$ is the complex of global sections of a flasque resolution of the sheaf $\mathcal{H}^i(\mu_n^{\otimes j})$.

These resolutions give rise to isomorphisms

$$(0.2) \qquad H^i(X, \mathcal{K}_i) \simeq CH^i(X)$$

$$(0.3) \qquad H^i(X, \mathcal{H}^i(\mu_n^{\otimes i})) \simeq CH^i(X)/n \ .$$

A basic idea of Spencer Bloch, combined with the Merkur'ev-Suslin result ([M/S1]), leads from these resolutions to a natural exact sequence :

$$(0.4) \qquad 0 \to H^{i-1}(X, \mathcal{K}_i)/n \to H^{i-1}(X, \mathcal{H}^i(\mu_n^{\otimes i})) \to {}_nCH^i(X) \to 0 \ .$$

For this, we refer to [CT/S/S § 1] and [CT § 3].

Another key ingredient in our proofs will be the local to global spectral sequence

$$E_2^{pq} = H^p_{\text{Zar}}(X, \mathcal{H}^q(\mu_n^{\otimes j})) \Longrightarrow H^n_{\text{ét}}(X, \mu_n^{\otimes j}) \tag{0.5}$$

whose non-zero terms satisfy $0 \le p \le d = \dim(X)$ (trivial) and $p \le q$, the latter inequality being a consequence of the work of Bloch and Ogus.

We thus have maps

$$\alpha_i : H^{i-1}(X, \mathcal{H}^i(\mu_n^{\otimes j})) \to H^{2i-1}_{\text{ét}}(X, \mu_n^{\otimes i})$$

and a diagram :

$$\begin{array}{c} 0 \to H^{i-1}(X, \mathcal{K}_i)/n \to H^{i-1}(X, \mathcal{H}^i(\mu_n^{\otimes i})) \to {}_nCH^i(X) \to 0 \\ \qquad\qquad\qquad\quad \Big\downarrow \alpha_i \\ \qquad\qquad\qquad\quad H^{2i-1}_{\text{ét}}(X, \mu_n^{\otimes i}) \ . \end{array} \tag{0.6}$$

This diagram may be completed in the following manner. For any positive integer m, invertible in k, on the smooth variety X, we have Grothendieck's cycle map

$$\rho : CH^i(X) \to H^{2i}_{\text{ét}}(X, \mu_m^{\otimes i})$$

which Deligne describes in [SGA 4 1/2]. For later use, let us note that the induced map

$$CH^i(X)/m \to H^{2i}_{\text{ét}}(X, \mu_m^{\otimes i}),$$

i.e. :

$$H^i(X, \mathcal{H}^i(\mu_m^{\otimes i})) \to H^{2i}_{\text{ét}}(X, \mu_m^{\otimes i})$$

is precisely the map coming from the local to global spectral sequence ([B/O], (7.2)).

From the exact sequence of étale sheaves on X :

$$1 \to \mu_m^{\otimes i} \to \mu_{nm}^{\otimes i} \to \mu_n^{\otimes i} \to 1$$

we deduce a boundary map :

$$\beta : H^{2i-1}_{\text{ét}}(X, \mu_n^{\otimes i}) \to H^{2i}_{\text{ét}}(X, \mu_m^{\otimes i}).$$

In [CT/S/S, Prop.1, p. 766] we checked that the following diagram commutes (up to sign) :

$$\begin{array}{ccccccc} 0 & \to & H^{i-1}(X, \mathcal{K}_i)/n & \to & H^{i-1}(X, \mathcal{H}^i(\mu_n^{\otimes i})) & \to & {}_nCH^i(X) \to 0 \\ & & & & \Big\downarrow \alpha_i & & \Big\downarrow \rho \\ & & & & H^{2i-1}_{\text{ét}}(X, \mu_n^{\otimes i}) & \xrightarrow{\beta} & H^{2i}_{\text{ét}}(X, \mu_m^{\otimes i}) \ . \end{array} \tag{0.7}$$

Letting n and m run through powers of a prime $l \neq \text{char}(k)$, and going over to the direct limit in n and the inverse limit in m, we get the following diagram, commutative up to sign :

(0.8)

$$\begin{array}{ccccccc} 0 & \longrightarrow H^{i-1}(X,\mathcal{K}_i)\otimes \mathbf{Q}_l/\mathbf{Z}_l & \longrightarrow H^{i-1}(X,\mathcal{H}^i(\mathbf{Q}_l/\mathbf{Z}_l(i))) & \longrightarrow & CH^i(X)_{l-\text{tors}} \to 0 \\ & & \downarrow \alpha_i & & \downarrow \rho \\ & & H^{2i-1}_{\text{ét}}(X,\mathbf{Q}_l/\mathbf{Z}_l(i)) & \xrightarrow{\beta} & H^{2i}_{\text{ét}}(X,\mathbf{Z}_l(i))_{l-\text{tors}} \ . \end{array}$$

§ 1. Homology of the complex $(\mathcal{C})$ in degrees at least $d-2$.

Let X be a smooth, projective, geometrically integral variety over a finite field F of characteristic p. Let $d = \dim(X)$. Let n be a positive integer and assume that p does not divide n.

Lemma 1.1. — *The local to global spectral sequence*

$$E_2^{pq} = H^p(X,\mathcal{H}^q(\mu_n^{\otimes j})) \Longrightarrow H^n_{\text{ét}}(X,\mu_n^{\otimes j})$$

is concentrated in the range $0 \leq p \leq d$, $p \leq q$, *and* $q \leq d+1$.

Proof : As already mentioned, the vanishing of E_2^{pq} for $p > d$ is trivial for dimension reasons, and is due to Bloch and Ogus for $p \leq q$. Let us prove the vanishing for $q \geq d+2$. As a matter of fact, the sheaves $\mathcal{H}^q(\mu_n^{\otimes j})$ themselves are zero for $q \geq d+2$, since for U affine over F, we already have $H^q(U,\mu_n^{\otimes j}) = 0$. Indeed, since the cohomological dimension $cd(F)$ equals 1, the Hochschild-Serre spectral sequence ($G = \text{Gal}(\overline{F}/F)$)

$$E_2^{pq} = H^p(G, H^q_{\text{ét}}(U \times_F \overline{F},\mu_n^{\otimes j})) \Longrightarrow H^n_{\text{ét}}(U,\mu_n^{\otimes j}),$$

yields short exact sequences

$$0 \to H^1(G, H^{q-1}_{\text{ét}}(U \times_F \overline{F},\mu_n^{\otimes j})) \to H^q(U,\mu_n^{\otimes j}) \to (H^q_{\text{ét}}(U \times_F \overline{F},\mu_n^{\otimes j}))^G \to 0 \ .$$

But over an algebraically closed field, the étale cohomological dimension of an affine variety is at most its (Zariski) dimension, hence all groups in this sequence are zero for $q \geq d+2$. □

Proposition 1.2. — *There are natural isomorphisms*

$$H^d(X,\mathcal{H}^{d+1}(\mu_n^{\otimes d})) \simeq \mathbf{Z}/n$$

$$H^d(X,\mathcal{K}_{d+1})/n \simeq H^d(X,\mathcal{H}^{d+1}(\mu_n^{\otimes d+1})) \simeq F^*/F^{*n}.$$

Proof : From the previous lemma and the local to global spectral sequence we deduce

$$H^d(X, \mathcal{H}^{d+1}(\mu_n^{\otimes j})) \simeq H^{2d+1}_{\text{ét}}(X, \mu_n^{\otimes j}).$$

The Hochschild-Serre spectral sequence, the fact that $cd(F) = 1$ and Poincaré duality for $X \times_F \overline{F}$ (i.e. $H^{2d}(X \times_F \overline{F}, \mu_n^{\otimes d})) \simeq \mathbf{Z}/n$) yield

$$H^{2d+1}_{\text{ét}}(X, \mu_n^{\otimes j}) \simeq H^1(G, H^{2d}_{\text{ét}}(X \times_F \overline{F}, \mu_n^{\otimes j})) \simeq H^1(G, \mu_n^{\otimes j-d})$$

hence $H^d(X, \mathcal{H}^{d+1}(\mu_n^{\otimes d})) \simeq H^1(G, \mathbf{Z}/n)$ and $H^d(X, \mathcal{H}^{d+1}(\mu_n^{\otimes d+1})) \simeq H^1(G, \mu_n) \simeq F^*/F^{*n}$. As for the isomorphism

$$H^d(X, \mathcal{K}_{d+1})/n \simeq H^d(X, \mathcal{H}^{d+1}(\mu_n^{\otimes d+1})),$$

it follows trivially from exact sequence (0.4) applied to $i = d+1$. ∎

This proposition establishes the degree d and $d+1$ part of Theorem B and, by going over to direct limits, the degree d and $d+1$ part of Theorem A.

Let us now study the line $p + q = 2d$ in the local to global spectral sequence, which will require less trivial arguments. Lemma 1.1 and that spectral sequence give rise to an exact sequence

$$H^{2d+1}_{\text{ét}}(X, \mu_n^{\otimes d}) \to H^{d-2}(X, \mathcal{H}^{d+1}(\mu_n^{\otimes d})) \to H^d(X, \mathcal{H}^d(\mu_n^{\otimes d})) \to$$
$$H^{2d}_{\text{ét}}(X, \mu_n^{\otimes d}) \to H^{d-1}(X, \mathcal{H}^{d+1}(\mu_n^{\otimes d})) \to 0 \ .$$

The arrow $H^d(X, \mathcal{H}^d(\mu_n^{\otimes d})) \to H^{2d}_{\text{ét}}(X, \mu_n^{\otimes d})$ is the cycle map

$$CH^d(X)/n \to H^{2d}_{\text{ét}}(X, \mu_n^{\otimes d}).$$

Combining Poincaré duality over $\overline{F}$ and arithmetic duality for Galois cohomology of F (cf. [CT/S/S] Lemme 5 p. 790) yields an isomorphism

$$H^{2d}_{\text{ét}}(X, \mu_n^{\otimes d}) \simeq \text{Hom}(H^1_{\text{ét}}(X, \mathbf{Z}/n), \mathbf{Z}/n) \simeq \pi_1^{ab}(X)/n \ ,$$

and the main theorem of unramified class field theory for smooth projective varieties over a finite field precisely says that the composite map

$$CH^d(X)/n \to \pi_1^{ab}(X)/n$$

is an isomorphism ([K/S1] Theorem 1, [CT/S/S] Théorème 5 p. 792, [CT/R2]). We thus conclude :

$$H^{d-1}(X, \mathcal{H}^{d+1}(\mu_n^{\otimes d})) = 0$$

and by going over to the direct limit of powers of a fixed prime l :

$$H^{d-1}(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d))) = 0$$

which proves the degree $d-1$ part of Theorems A and B.

We also deduce that the differential

$$H^{d-2}(X, \mathcal{H}^{d+1}(\mu_n^{\otimes d})) \to H^d(X, \mathcal{H}^d(\mu_n^{\otimes d}))$$

is zero, and that the map

$$H^{2d-1}_{\text{ét}}(X, \mu_n^{\otimes d}) \to H^{d-2}(X, \mathcal{H}^{d+1}(\mu_n^{\otimes d}))$$

coming from the spectral sequence is surjective. Going over to the direct limit of powers of a fixed prime l, we find that the map

$$H^{2d-1}_{\text{ét}}(X, \mathbf{Q}_l/\mathbf{Z}_l(d)) \to H^{d-2}(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d)))$$

is also surjective. Deligne's theorem on the Weil conjectures implies that the group $H^{2d-1}_{\text{ét}}(X, \mathbf{Q}_l/\mathbf{Z}_l(d))$ is finite ([CT/S/S], Théorème 2, p. 780). We thus conclude that the group $H^{d-2}(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d)))$ is finite.

Now the vanishing of $H^{d-1}(X, \mathcal{H}^{d+1}(\mu_n^{\otimes d}))$ (above) and Lemma 2.2 a) below (in the trivial case $i = d-2$) imply that $H^{d-2}(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d)))$ is divisible. Being divisible and finite, this group is zero, thus completing the proof of Theorem A in degrees $\geq d-2$. The proof of Theorem B in degree $(d-2)$ is postponed to the end of § 4.

§ 2. Going back and forth between finite and infinite coefficients.

Let L be a field, and let n and m be positive integers. Assume that $\text{char}(L)$ does not divide n. A well-known conjecture, attributed to Milnor and Kato, claims that the Galois symbol from Milnor K-theory to Galois cohomology

$$K_m^M(L)/n \to H^m(L, \mu_n^{\otimes m})$$

is an isomorphism. This would imply the

Cohomological Conjecture. — *For any positive integers r and s prime to* $\text{char}(L)$, *the map*

$$H^{m+1}(L, \mu_r^{\otimes m}) \to H^{m+1}(L, \mu_{rs}^{\otimes m})$$

induced by the inclusion $\mu_r^{\otimes m} \to \mu_{rs}^{\otimes m}$ is an injection. Equivalently, for l prime different from $\text{char}(L)$, *the natural map :*

$$H^{m+1}(L, \mu_{l^n}^{\otimes m}) \to H^{m+1}(L, \mathbf{Q}_l/\mathbf{Z}_l(m))$$

is an injection.

This conjecture is known for $m = 0$ (obvious), $m = 1$ (Kummer theory) and $m = 2$ (Merkur'ev-Suslin [M/S1]). It is also known when $m = 3$ and n is a power of 2 (Merkur'ev-Suslin [M/S2], Rost).

If we grant this conjecture for a minute, for any function field L of transcendence degree t over a finite field F, and any power l^n of a prime $l \neq \text{char}(F)$, the exact sequence of Galois modules

$$0 \to \mu_{l^n}^{\otimes t} \to \mathbf{Q}_l/\mathbf{Z}_l(t) \xrightarrow{l^n} \mathbf{Q}_l/\mathbf{Z}_l(t) \to 0$$

would give rise to exact sequences of Galois cohomology :

$$0 \to H^{t+1}(L, \mu_{l^n}^{\otimes t}) \to H^{t+1}(L, \mathbf{Q}_l/\mathbf{Z}_l(t)) \xrightarrow{l^n} H^{t+1}(L, \mathbf{Q}_l/\mathbf{Z}_l(t)) \to 0$$

(note that $H^{t+2}(F, \mu_{l^n}^{\otimes d}) = 0$ since $cd(L) \leq t+1$).

If now X is a smooth d-dimensional variety over a finite field F and we apply this argument to the function fields of all irreducible subvarieties of X, we find that the complexes appearing in Conjectures 1 and 2 would fit into an exact sequence :

$$0 \to \mathcal{C}_{l^n} \to \mathcal{C} \xrightarrow{l^n} \mathcal{C} \to 0 \ .$$

These complexes $\mathcal{C}_{l^n}$ and $\mathcal{C}$ are none other than the global sections of the Bloch-Ogus flasque resolutions of the sheaves $\mathcal{H}^{d+1}(\mu_{l^n}^{\otimes d})$ and $\mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d))$. Thus we would get a long exact sequence

$$H^i(X, \mathcal{H}^{d+1}(\mu_{l^n}^{\otimes d})) \to H^i(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d))) \xrightarrow{l^n} H^i(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d))) \to$$
$$H^{i+1}(X, \mathcal{H}^{d+1}(\mu_{l^n}^{\otimes d})) \to H^{i+1}(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d))) \to \cdots$$

hence :

Lemma 2.1. — *Let X be a smooth d-dimensional variety over a finite field F and let i be an integer, $i \geq -1$. If the cohomological conjecture is true, then :*

a) *The vanishing of $H^{i+1}(X, \mathcal{H}^{d+1}(\mu_{l^n}^{\otimes d}))$ for some positive integer n implies that $H^i(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d)))$ is a divisible group.*

b) *If $H^{i+1}(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d))) = 0$ and $H^i(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d)))$ is a divisible group, then $H^{i+1}(X, \mathcal{H}^{d+1}(\mu_{l^n}^{\otimes d})) = 0$.* □

Actually writing the complexes and chasing through them reveals :

Lemma 2.2. — *Let X be a smooth d-dimensional variety over a finite field F and let i be an integer, $i \geq -1$.*

a) *If the cohomological conjecture holds for $H^{d-i-1}(F(Y), \mu_{l^n}^{\otimes d-i-2})$ and Y an integral variety over F of dimension $d-i-1$, and if $H^{i+1}(X, \mathcal{H}^{d+1}(\mu_{l^n}^{\otimes d}))$ vanishes for some n, then $H^i(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d)))$ is divisible.*

b) *If the cohomological conjecture holds for $H^{d-i}(F(Y), \mu_{l^n}^{\otimes d-i-1})$ and Y an integral variety over F of dimension $d-i$, if $H^{i+1}(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d))) = 0$ and $H^i(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d)))$ is divisible, then $H^{i+1}(X, \mathcal{H}^{d+1}(\mu_{l^n}^{\otimes d})) = 0$ for any positive integer n.* □

Note that by the results of Merkur'ev-Suslin and Rost, the assumption on the cohomological conjecture in Lemma 2.2 a) is satisfied if $i \geq d-4$, and it is also satisfied if $l = 2$ and $i \geq d-5$ (the case $i = d-2$, used in the preceding proof, is trivial and $i = d-3$ only uses Kummer theory). The assumption on the cohomological conjecture in Lemma 2.2 b) is satisfied if $i \geq d-3$ (the case $i = d-3$ uses the Merkur'ev-Suslin result) and also if $i = d-4$ and $l = 2$.

Remark : Of course, we could have tried to use the cohomology of the sequence

$$0 \to \mathcal{H}^{d+1}(\mu_{l^n}^{\otimes d}) \to \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d)) \xrightarrow{l^n} \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d)) \to 0 \ ,$$

but a direct proof of the exactness of this sequence would require knowledge of the cohomological conjecture for $m = d$. Replacing the sheaves in the above sequence by their flasque resolutions enable us to build upon the low degree cases of the conjecture.

§ 3. On the structure of $H^{d-1}(X, \mathcal{K}_d)$.

In the next section, I shall prove that the homology of the complex $(\mathcal{C})$ vanishes in degree $d-3$. The key will be the vanishing of the group $H^{d-1}(X, \mathcal{K}_d) \otimes \mathbf{Q}/\mathbf{Z}$, which will be proved in the present section.

We shall start by studying the general situation :

(3.1) X is a smooth, projective, geometrically integral variety of dimension $d \geq 2$ over a finite field F. We are given a smooth, projective, geometrically integral curve C over F, and a proper, dominant F-morphism $f : X \to C$ whose generic fibre X_η is smooth and geometrically integral over the generic point $\eta = \mathrm{Spec}(F(C))$ of C.

Comparison of the Quillen resolution of the sheaf $\mathcal{K}_d$ on X and on the generic fibre X_η yields a long exact sequence (cf. [Sh], Theorem 2.1)

$$(3.2) \quad \bigoplus_{P \in C^{(1)}} H^{d-1}_{X_P}(X, \mathcal{K}_d) \to H^{d-1}(X, \mathcal{K}_d) \to H^{d-1}(X_\eta, \mathcal{K}_d) \to \bigoplus_{P \in C^{(1)}} H^{d}_{X_P}(X, \mathcal{K}_d) \to H^{d}(X, \mathcal{K}_d) \to 0 \ .$$

Here $H^{i}_{X_P}(X, \mathcal{K}_d)$ is just a notation for the homology in dimension i of the subcomplex of the Gersten complex of $\mathcal{K}_d$ supported on X_P. If X_P is smooth over the residue field $k(P)$, then the Gersten conjecture, as proved by Quillen, yields :

$$H^{i}_{X_P}(X, \mathcal{K}_d) = H^{i-1}(X_P, \mathcal{K}_{d-1}).$$

For any point P of $C^{(1)}$, the group $H^{d}_{X_P}(X, \mathcal{K}_d)$ coincides with the Chow group $CH_0(X_P)$. The projection $X_P \to \mathrm{Spec}(F(P))$ induces a degree map $CH_0(X_P) \to \mathbf{Z}$. Similarly, we have the degree map $H^d(X, \mathcal{K}_d) \simeq CH_0(X) \to \mathbf{Z}$.

Let $A_0(X_P) = \mathrm{Ker}[CH_0(X_P) \to \mathbf{Z}]$. Each of the groups $A_0(X)$ and $A_0(X_P)$ is a finite group. Indeed, it is a theorem of Kato and Saito (also in the singular case) that for any connected proper, possibly singular, variety Z over a finite field, the group $A_0(Z)$ is finite ([K/S2], Theorem 6.1). (In proving Theorem 3.5 below, we shall only need the case where X_P is projective and has at worst an isolated quadratic singularity. In that case, the result immediately follows from the smooth case ([K/S1], [CT/S/S]).)

We have the reciprocity map $H^{d-1}(X_\eta, \mathcal{K}_d) \to F(C)^*$, induced by the norm maps from the residue field at a closed point down to the ground field. One checks that the diagram of complexes

$$
\begin{array}{ccccc}
H^{d-1}(X_\eta, \mathcal{K}_d) & \to & \bigoplus_{P\in C^{(1)}} H^d_{X_P}(X, \mathcal{K}_d) & \to & H^d(X, \mathcal{K}_d) \\
\downarrow & & \downarrow & & \downarrow \\
F(C)^* & \xrightarrow{\text{div}} & \bigoplus_{P\in C^{(1)}} \mathbf{Z} & \xrightarrow{\Sigma} & \mathbf{Z}
\end{array} \tag{3.3}
$$

commutes (the map Σ sends the element $(n_P)_{P\in C^{(1)}}$ to the sum $\sum_P n_P[F(P) : F]$.)

Now the kernel of the map $F(C)^* \xrightarrow{\text{div}} \bigoplus_{P\in C^{(1)}} \mathbf{Z}$ is the finite group F^*, and the middle homology of the lower complex is equal to the group of F-rational points of the jacobian of C, hence is finite.

Let us define :

$$V(X_\eta) = \operatorname{Ker}[H^{d-1}(X_\eta, \mathcal{K}_d) \to F(C)^*] \ .$$

There is an induced map :

$$V(X_\eta) \to \bigoplus_{P\in C^{(1)}} A_0(X_P)$$

with finite cokernel.

Putting all the quoted finiteness and compatibility results together, one finds that sequence (3.2) and diagram (3.3) induce exact sequences

$$0 \to G \to V(X_\eta) \to \bigoplus_{P\in C^{(1)}} A_0(X_P) \to K \to 0 \ , \tag{3.4}$$

where K is a finite group, and

$$\bigoplus_{P\in C^{(1)}} H^{d-1}_{X_P}(X, \mathcal{K}_d) \to H^{d-1}(X, \mathcal{K}_d) \to G_1 \to 0 \tag{3.5}$$

where

$$G \text{ is a subgroup of finite index in } G_1 \ . \tag{3.6}$$

Although we are ultimately going to prove $H^{d-1}(X, \mathcal{K}_d) \otimes \mathbf{Q}/\mathbf{Z} = 0$, for the purpose of the following proof, it will be convenient to consider a more refined property of abelian groups, and to give it a name.

Lemma–Definition 3.1. — 1) *The following properties of an abelian group A are equivalent :*

(a) *The quotient of A by its maximal divisible subgroup is a torsion group ;*

(b) *A is the direct sum of its maximal divisible subgroup and a torsion group ;*

(c) *A is the direct sum of a divisible group and a torsion group.*

Such a group will be called **torsion-by-divisible**.

2) *Any quotient of a torsion-by-divisible group is torsion-by-divisible.*

3) *Any subgroup of finite index in a torsion-by-divisible group is torsion-by-divisible.*

4) *Any extension of a torsion group by a torsion-by-divisible group is torsion-by-divisible.*

5) *Any torsion-by-divisible group A satisfies $A \otimes \mathbf{Q}/\mathbf{Z} = 0$.* □

PROPOSITION 3.2. — *Let X be a smooth, projective, geometrically integral surface X/F equipped with a morphism $f : X \to C$ as in (3.1). Then, with notation as above :*

(i) *The group $V(X_\eta)$ is torsion-by-divisible. In particular, $V(X_\eta) \otimes \mathbf{Q}/\mathbf{Z} = 0$.*

(ii) *The map $V(X_\eta)_{\mathrm{tors}} \to \bigoplus_{P \in C^{(1)}} A_0(X_P)$ has finite cokernel.*

Proof : Gros and Suwa have proved that $H^1(X, \mathcal{K}_2)$ is the direct sum of a uniquely divisible group (conjecturally zero) and an explicit finite group ([G/S], Thm. 4.19). The proof of their result relies on Lemme 1.15 of [G/S], whose proof is incorrect as it stands, but whose statement is correct when the ground field is finite – one only needs to adapt the proofs of Thm. 1.8 and Thm. 2.2 in [CT/R1].

In particular $H^1(X, \mathcal{K}_2)$ is torsion-by-divisible. This result is a refined version of earlier work (Panin [P]; [CT/R 1986]) which had already proved $H^1(X, \mathcal{K}_2) \otimes \mathbf{Q}/\mathbf{Z} = 0$. These various works in turn build upon basic results of Merkur'ev-Suslin [M/S1] and Suslin [S].

With notation as above, from (3.5) we deduce that G_1 is torsion-by-divisible; then from (3.6) we deduce that G is torsion-by-divisible; finally, from (3.4) and from the fact that $\bigoplus_{P \in C^{(1)}} A_0(X_P)$ is a torsion group, we conclude that $V(X_\eta)$ is torsion-by-divisible.

Since each $A_0(X_P)$ is a finite group, any divisible subgroup of $V(X_\eta)$ is mapped to zero in each $A_0(X_P)$, hence also in $\bigoplus_{P \in C^{(1)}} A_0(X_P)$, and since $V(X_\eta)$ is the direct sum of a divisible group and a torsion group, and the group K in (3.4) is finite, statement (ii) follows. □

PROPOSITION 3.3. — *Let F be a finite field, C a geometrically integral curve over F. Let $Z/F(C)$ be a smooth, projective, geometrically integral variety over the function field $F(C)$. Then the group $V(Z)$ is torsion-by-divisible. In particular $V(Z) \otimes \mathbf{Q}/\mathbf{Z} = 0$.*

Proof : Let us first assume that Z is a curve. Then one may extend the map $Z \to \mathrm{Spec}(F(C))$ to a morphism $f : X \to C$ as above ([Ab]), and the result has just been proved. One may reduce the higher dimensional case to the case of curves – as was done once in a local context by Salberger (unpublished). Let $k = F(C)$. Let $\alpha \in V(Z)$ be represented by a finite sum $\sum_M f_M$, where f_M belongs to $k(M)^*$ (multiplicative group of the residue field $k(M)$ at M), and

$$\prod_M N_{k(M)/k}(f_M) = 1 \in k^* \ .$$

Let L/k be a finite normal (not necessarily separable) field extension of k over which all the closed points M with $f_M \neq 1$ become rational.

Restriction from k to L defines a map $i : V(Z) \to V(Z_L)$, and taking norms from L to k defines a map $N : V(Z_L) \to V(Z)$. The composite map $N \circ i$ is multiplication by the degree $m = [L : k]$. Under the map i, the class of the sum $\sum_M f_M$ goes to the class of a sum $\sum_N f_N$ where all closed points $N \in Z_L$ with $f_N \neq 1$ are L-rational. Now by a suitable variant of the Bertini theorem ([A/K]; actually, one would have been content with L/k purely inseparable), one may find a smooth, projective, geometrically integral curve Y over L, lying inside Z_L and going through the finitely many rational points N with non-trivial f_N. Thus α_L comes from a class in $V(Y)$. Considering the composite map :

$$V(Y) \to V(Z_L) \xrightarrow{N} V(Z),$$

we find that $m\alpha$ lies in the image of $V(Y)$. Since $V(Y)$ is torsion-by-divisible, there is a positive integer $n > 0$ such that $nm\alpha$ lies in the image of the maximal divisible subgroup of $V(Y)$, hence in the maximal divisible subgroup of $V(Z)$. The quotient of $V(Z)$ by its maximal divisible subgroup is thus a torsion group, as was to be proved. ∎

Remark 3.3.1. — Related results have been proved by W. Raskind ([R]). For X/C as in proposition 3.2 above, his paper and the above discussion establish a link between the conjecture that $H^{d-1}(X, \mathcal{K}_d)$ is a torsion group and Bloch's conjecture that $V(X_\eta)$ is a torsion group. At least when X is a surface, I wonder whether the maximal divisible subgroup of $V(X_\eta)$ can contain torsion elements.

Proposition 3.4. — *Let X be a smooth, projective, geometrically integral variety and assume given a fibration $f : X \to C$ as in (3.1). Then the cokernel of the map*

$$V(X_\eta)_{\mathrm{tors}} \to \bigoplus_{P \in C^{(1)}} A_0(X_P)$$

is finite, and the group

$$G = \mathrm{Ker}[V(X_\eta) \to \bigoplus_{P \in C^{(1)}} A_0(X_P)]$$

is torsion-by-divisible.

Proof: The map $V(X_\eta) \to \bigoplus_{P \in C^{(1)}} A_0(X_P)$ has finite cokernel, and we have just seen that $V(X_\eta)$ is the direct sum of its maximal divisible subgroup and a torsion group. The divisible group goes to 0 in each of the finite groups $A_0(X_P)$, hence the first statement. Also, the maximal divisible subgroup G_{div} of G coincides with that of $V(X_\eta)$, and G/G_{div} is a torsion group, hence the second statement. ∎

We may finally prove :

Theorem 3.5. — *Let X/F be a smooth, projective, geometrically integral variety of dimension d over a finite field F. Then $H^{d-1}(X, \mathcal{K}_d) \otimes \mathbf{Q}/\mathbf{Z} = 0$.*

Proof : We shall prove the result by induction on the dimension of X. We already know it for $\dim(X) \leq 2$ (see the proof of Theorem 3.2). Assume $\dim(X) = d \geq 3$, and assume that the result has been proved for varieties of dimension at most $d-1$.

To prove the result, we may allow finite field extensions. Indeed, if K/F is a finite field extension, the composite map

$$H^{d-1}(X, \mathcal{K}_d) \to H^{d-1}(X_K, \mathcal{K}_d) \xrightarrow{N_{K/F}} H^{d-1}(X, \mathcal{K}_d)$$

of the restriction from F to K and the norm is multiplication by the degree $[K : F]$. Thus if $H^{d-1}(X_K, \mathcal{K}_d) \otimes \mathbf{Q}/\mathbf{Z} = 0$, then the divisible group $H^{d-1}(X, \mathcal{K}_d) \otimes \mathbf{Q}/\mathbf{Z}$ is killed by $[K : F]$, hence is zero.

Recall ([SGA7 II], exp. XVII) that a map

$$f : X \to C$$

as in (3.1) is called a Lefschetz pencil if each of the finitely many singular fibres $X_P/F(P)$ is a $(d-1)$-dimensional, projective, geometrically integral variety with an isolated rational singularity $M_P \in X_P(F(P)) \subset X(F(P))$. One may resolve the singularity of X_P by a single blow-up $Z_P \to X_P$, and the inverse image of M_P is a smooth projective quadric over $F(P)$ of dimension $d-2$.

Allowing for a finite extension of the ground field, the theory of Lefschetz pencils ([SGA7 II], loc. cit.) shows that one may blow up a smooth closed subvariety Z of codimension 2 in the smooth projective variety X to produce a variety X' which is equipped with a Lefschetz pencil $f : X' \to C$ (where C is the projective line over F).

Let $Z \subset X$ be a smooth subvariety of pure codimension 2 in X and let X' be the blow-up of X along Z. The proper map $r : X' \to X$ induces a map r_* on Gersten complexes. Analyzing this map in the last two terms of the complexes, one shows that there is an exact sequence :

$$H^{d-2}(Z, \mathcal{K}_{d-1}) \longrightarrow H^{d-1}(X', \mathcal{K}_d) \longrightarrow H^{d-1}(X, \mathcal{K}_d) \longrightarrow 0.$$

The proof, which is rather simple except for the writing down of a huge diagram, involves the following two facts. Firstly, above any point M of Z, the fibre $r^{-1}(M)$ is the projective line over the residue field $k(M)$. Secondly, for the projective line $\mathbf{P}^1_K$ over a field K, the homology of the Gersten complex

$$K_2K(\mathbf{P}^1) \longrightarrow \bigoplus_{M \in \mathbf{P}_K^{1(1)}} K(M)^*$$

on the right hand side is none other than K^*, the maps $K(M)^* \to K^*$ being the obvious norm maps (identity when M is a K-rational point).

(The formula for the $\mathcal{K}$-cohomology of a blow-up does not seem to be in the literature; the above special result is enough for our purposes.)

The group $H^{d-2}(Z, \mathcal{K}_{d-1})$ is a quotient of the direct sum $\oplus F(M)^*$, where M runs over all closed points of Z. Since $F(M)^* \otimes \mathbf{Q}/\mathbf{Z} = 0$, the field $F(M)$ being finite, we certainly have $H^{d-2}(Z, \mathcal{K}_{d-1}) \otimes \mathbf{Q}/\mathbf{Z} = 0$, hence $H^{d-1}(X', \mathcal{K}_d) \otimes \mathbf{Q}/\mathbf{Z} \simeq H^{d-1}(X, \mathcal{K}_d) \otimes \mathbf{Q}/\mathbf{Z}$.

We are thus reduced to the case where X is equipped with an F-morphism $f : X \to C$ to a smooth, projective, geometrically integral curve C/F which is a Lefschetz pencil.

Let us show that each group $H^{d-1}_{X_P}(X, \mathcal{K}_d)$ in sequence (3.5) satisfies

$$H^{d-1}_{X_P}(X, \mathcal{K}_d) \otimes \mathbf{Q}/\mathbf{Z} = 0 \ . \qquad (3.7)$$

If X_P is smooth, then, as has been mentioned above, $H^{d-1}_{X_P}(X, \mathcal{K}_d) = H^{d-2}(X_P, \mathcal{K}_{d-1})$ and (3.7) holds by the induction assumption. If X_P is not smooth, let $Z_P \to X_P$ be the blow-up of X_P at its singular point, and let $E/F(P)$ be the exceptional divisor.

The projection $Z_P \to X_P$ induces a diagram of complexes :

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
\bigoplus_{x\in X_{P(2)}} K_2F(x) & \to & \bigoplus_{x\in X_{P(1)}} F(x)^* & \to & \bigoplus_{x\in X_{P(0)}} \mathbf{Z} & & & & \\
\uparrow & & \uparrow & & \uparrow & & & & \\
\bigoplus_{x\in Z_{P(2)}} K_2F(x) & \to & \bigoplus_{x\in Z_{P(1)}} F(x)^* & \to & \bigoplus_{x\in Z_{P(0)}} \mathbf{Z} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \bigoplus_{x\in E_{(1)}} F(x)^* & \to & \bigoplus^0_{x\in E_{(0)}} \mathbf{Z} & \to & A_0(E) & \to & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

In this diagram, indices between parentheses denote the dimension of the points under consideration. The group $\bigoplus^0_{x\in E_{(0)}} \mathbf{Z}$ is the group of zero-cycles of degree zero on E, and $A_0(E)$ is the quotient of this group by rational equivalence.

The vertical columns of the diagram are exact, and the bottom line is also exact. The middle homology of the top horizontal complex is $H^{d-1}_{X_P}(X, \mathcal{K}_d)$ and the middle homology of the middle horizontal complex is $H^{d-2}(Z_P, \mathcal{K}_{d-1})$. The variety E is a smooth quadric of dimension $d-2 \geq 1$, hence satisfies $A_0(E) = 0$ (this is a general result on quadrics, valid over any ground field; however in the case under consideration, since any quadric over a finite field has a rational point, the smooth quadric E is birational over its ground field to projective space and the result follows from the birational invariance of the group A_0 on smooth projective varieties, together with the well-known vanishing of that group on projective space). A simple diagram chase now reveals that the map

$$H^{d-2}(Z_P, \mathcal{K}_{d-1}) \to H^{d-1}_{X_P}(X, \mathcal{K}_d)$$

is surjective. This result and the induction assumption applied to Z_P imply (3.7) for X_P.

In the short exact sequence

$$(3.5) \qquad \bigoplus_{P\in C^{(1)}} H^{d-1}_{X_P}(X, \mathcal{K}_d) \to H^{d-1}(X, \mathcal{K}_d) \to G_1 \to 0 \ ,$$

the group G_1 is an extension of a finite group by the torsion-by-divisible group G ((3.6) and Prop. 3.4). In particular G_1, just as G, is the direct sum of its maximal divisible subgroup and a torsion group, and $G_1 \otimes \mathbf{Q}/\mathbf{Z} = 0$. Tensoring (3.5) by $\mathbf{Q}/\mathbf{Z}$ and using (3.7), we find :

$$H^{d-1}(X, \mathcal{K}_d) \otimes \mathbf{Q}/\mathbf{Z} = 0. \quad \square$$

Remark 3.5.1. — The proof actually shows that $H^{d-1}(X, \mathcal{K}_d)$ is obtained by successive extensions of torsion-by-divisible groups. But it does not say whether $H^{d-1}(X, \mathcal{K}_d)$ itself is torsion-by-divisible.

Remark 3.5.2. — A consequence of the theory of characteristic classes for higher K-theory and of the Riemann-Roch theorem (Gillet, Shekhtman) is that for any smooth variety X over a perfect field F, the Brown-Gersten-Quillen spectral sequence degenerates up to torsion on the K_1 (and K_0) line. More precisely, letting $d = \dim(X)$:

$$H^i(X, \mathcal{K}_{i+1}) \otimes \mathbf{Z}[1/d!] \simeq \mathrm{Gr}^i K_1(X) \otimes \mathbf{Z}[1/d!] \ .$$

Here $\mathrm{Gr}^i K_1(X) = F^i(X)/F^{i+1}(X)$, where $F^i(X)$ denotes the filtration on $K_1(X)$ coming from the spectral sequence.

An old conjecture due to Parshin (see [J3] 12.2) claims that for any smooth projective variety X over a finite field F, the higher K-groups $K_i(X)$, $i \geq 1$, are torsion groups (and even finite groups). That result for $K_1(X)$ would therefore imply that all groups $H^i(X, \mathcal{K}_{i+1})$ are torsion (and of finite exponent if finiteness for $K_1(X)$ is assumed). For some interesting cases where this is known, see Soulé [So] (see also [G/S], 4.29).

Since $H^0(X, \mathcal{K}_1) \otimes \mathbf{Q}/\mathbf{Z} = 0$ (trivial), $H^1(X, \mathcal{K}_2) \otimes \mathbf{Q}/\mathbf{Z} = 0$ ([P], [CT/R2], ([G/S]) and $H^d(X, \mathcal{K}_{d+1}) \otimes \mathbf{Q}/\mathbf{Z} = 0$ (Proposition 1.2), Theorem 3.5 implies at any rate that $K_1(X) \otimes \mathbf{Q}/\mathbf{Z} = 0$ for $\dim(X) \leq 3$.

§ 4. Homology of the complex $(\mathcal{C})$ in degree $(d-3)$.

Let us now look at the lines $p+q = 2d-1$ and $p+q = 2d-2$ in the local to global spectral sequence. These lines give rise to an exact sequence

$$H^{2d-2}_{\text{ét}}(X, \mu_{l^n}^{\otimes d}) \to H^{d-3}(X, \mathcal{H}^{d+1}(\mu_{l^n}^{\otimes d})) \to H^{d-1}(X, \mathcal{H}^d(\mu_{l^n}^{\otimes d})) \to H^{2d-1}_{\text{ét}}(X, \mu_{l^n}^{\otimes d})$$

hence to a similar exact sequence at the level of $\mathbf{Q}_l/\mathbf{Z}_l(d)$:

$$H^{2d-2}_{\text{ét}}(X, \mathbf{Q}_l/\mathbf{Z}_l(d)) \to H^{d-3}(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d))) \to$$
$$H^{d+1}(X, \mathcal{H}^d(\mathbf{Q}_l/\mathbf{Z}_l(d))) \to H^{2d-1}_{\text{ét}}(X, \mathbf{Q}_l/\mathbf{Z}_l(d)) \ .$$

It is a consequence of Deligne's proof of the Weil conjectures (see [CT/S/S], Thm. 2) that the groups $H^{2d-2}_{\text{ét}}(X, \mathbf{Q}_l/\mathbf{Z}_l(d))$ and $H^{2d-1}_{\text{ét}}(X, \mathbf{Q}_l/\mathbf{Z}_l(d))$ are finite. Thus the group $H^{d-3}(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d)))$ is finite if and only if the group $H^{d-1}(X, \mathcal{H}^d(\mathbf{Q}_l/\mathbf{Z}_l(d)))$ is finite.

On the other hand, we have the diagram (0.8), which commutes up to sign :

$$\begin{array}{ccccccc} 0 \to H^{d-1}(X, \mathcal{K}_d) \otimes \mathbf{Q}_l/\mathbf{Z}_l & \to & H^{d-1}(X, \mathcal{H}^d(\mathbf{Q}_l/\mathbf{Z}_l(d))) & \to & CH^d(X)_{l-\text{tors}} \to 0 \\ & & \downarrow & & \downarrow \\ & & H^{2d-1}_{\text{ét}}(X, \mathbf{Q}_l/\mathbf{Z}_l(d)) & \xrightarrow{\beta} & H^{2d}_{\text{ét}}(X, \mathbf{Z}_l(d))_{l-\text{tors}} \ . \end{array}$$

In this diagram, the lower horizontal arrow is an isomorphism of finite groups, and the right vertical map is an isomorphism by unramified class field theory (see [CT/S/S], § 2).

THEOREM 4.1. — *Let X be a smooth, projective, geometrically integral variety over a finite field F, $\dim(X) = d$, and let $l \neq \mathrm{char}(F)$. Then the following properties hold :*

a) $H^{d-1}(X, \mathcal{K}_d) \otimes \mathbf{Q}_l/\mathbf{Z}_l$ *is zero.*

b) $H^{d-1}(X, \mathcal{H}^d(\mathbf{Q}_l/\mathbf{Z}_l(d)))$ *is finite.*

c) *The map* $H^{d-1}(X, \mathcal{H}^d(\mathbf{Q}_l/\mathbf{Z}_l(d))) \to CH^d(X)_{l-\mathrm{tors}}$ *is an isomorphism.*

d) *The map* $H^{d-1}(X, \mathcal{H}^d(\mathbf{Q}_l/\mathbf{Z}_l(d))) \to H^{2d-1}_{\text{ét}}(X, \mathbf{Q}_l/\mathbf{Z}_l(d))$ *is an isomorphism.*

e) *The differential* $H^{d-3}(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d)))) \to H^{d-1}(X, \mathcal{H}^d(\mathbf{Q}_l/\mathbf{Z}_l(d)))$ *vanishes.*

f) *The map* $H^{2d-2}_{\text{ét}}(X, \mathbf{Q}_l/\mathbf{Z}_l(d)) \to H^{d-3}(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d)))$ *is surjective.*

g) *The group* $H^{d-3}(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d)))$ *is finite.*

Proof : Note that the divisible group $H^{d-1}(X, \mathcal{K}_d) \otimes \mathbf{Q}_l/\mathbf{Z}_l$ is zero if and only if it is finite. The following implications are easily deduced from the shape of the Bloch-Ogus spectral sequence and from the results recalled above :

$$\text{a)} \Longleftrightarrow \text{b)} \text{ and } \text{a)} \Longleftrightarrow \text{c)} \Longleftrightarrow \text{d)} \Longrightarrow \text{e)} \Longrightarrow \text{f)} \Longleftrightarrow \text{g)} \Longrightarrow \text{b)}.$$

Now $H^{d-1}(X, \mathcal{K}_d) \otimes \mathbf{Q}_l/\mathbf{Z}_l = 0$ by Theorem 3.5 above, hence all properties actually hold. □

REMARK 4.1.1. — As already mentioned, when $d = \dim(X) = 3$, Saito [Sa] has an independent proof that $H^0(X, \mathcal{H}^4(\mathbf{Q}_l/\mathbf{Z}_l(3))) = 0$. In that case one may thus reverse the argument and starting from Saito's result deduce the other properties stated in Theorem 4.1, in particular $H^2(X, \mathcal{K}_3) \otimes \mathbf{Q}_l/\mathbf{Z}_l = 0$ for X a threefold and $l \neq \mathrm{char}(F)$.

THEOREM 4.2. — *The homology of the complex* $(\mathcal{C})$ *in degree* $d-3$ *is zero. In other words,* $H^{d-3}(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d))) = 0$.

Proof : We shall prove this by induction on $d = \dim(X)$, starting with $d = 2$ where the statement is clear. So let us assume $d = \dim(X) \geq 3$ and assume that the theorem has been proved (over an arbitrary finite field) for all smooth projective varieties of dimension strictly less that d. Assume that we can find a smooth, projective, geometrically integral hyperplane section $Y \subset X$ ($\dim(Y) = d-1$) defined over F, and let U be the complement of X. Note that U is an affine variety.

LEMMA 4.2.1. — *Let U be as above. Let $i \in \mathbf{N}$ and $j \in \mathbf{Z}$ be integers. Then the groups* $H^i_{\text{ét}}(U, \mathbf{Q}_l/\mathbf{Z}_l(j))$ *vanish for* $i \geq d+2$, *and they vanish for* $i = d+1$ *and* $d \neq 2j, 2j-1$. *In particular the group* $H^{2d-2}_{\text{ét}}(U, \mathbf{Q}_l/\mathbf{Z}_l(d))$ *vanishes for* $d \geq 3$.(2)

Proof: Let n be a positive integer and let j be an arbitrary integer. Since $\overline{U} = U \times_F \overline{F}$ is affine, $H^q_{\text{ét}}(\overline{U}, \mu_{l^n}^{\otimes j}) = 0$ for $q \geq d+1$. Thus the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(\mathrm{Gal}(\overline{F}/F), H^q_{\text{ét}}(\overline{U}, \mu_{l^n}^{\otimes j})) \Longrightarrow H^n(U, \mu_{l^n}^{\otimes j})$$

(2) I had initially formulated a weaker version of Lemma 4.2.1. The present version was suggested by N. Suwa. Once it is established, it enables one to give a uniform proof of Theorem 4.2 in the cases $d = 3$ and $d > 3$.

is concentrated in the range $0 \leq p \leq 1$ (since cd(F)=1) and $q \leq d$.

Hence $H^i(U, \mu_{l^n}^{\otimes j}) = 0$ for $i \geq d+2$ (see the proof of Lemma 1.1), hence also $H^i_{\text{ét}}(U, \mathbf{Q}_l/\mathbf{Z}_l(j)) = 0$ for $i \geq d+2$. This gives the first part of the lemma.

The case $i = d+1$ is more subtle. From the shape of the spectral sequence we have isomorphisms

$$H^{d+1}_{\text{ét}}(U, \mu_{l^n}^{\otimes j}) \simeq H^1(F, H^d_{\text{ét}}(\overline{U}, \mu_{l^n}^{\otimes j}))$$

hence also

$$H^{d+1}_{\text{ét}}(U, \mathbf{Q}_l/\mathbf{Z}_l(j)) \simeq H^1(F, H^d_{\text{ét}}(\overline{U}, \mathbf{Q}_l/\mathbf{Z}_l(j))).$$

From the exact sequence of sheaves on U

$$0 \to \mu_l^{\otimes d} \to \mathbf{Q}_l/\mathbf{Z}_l(d) \xrightarrow{\times l} \mathbf{Q}_l/\mathbf{Z}_l(d) \to 0$$

and the vanishing of $H^{d+1}_{\text{ét}}(\overline{U}, \mu_l^{\otimes j})$ we deduce that $H^d_{\text{ét}}(\overline{U}, \mathbf{Q}_l/\mathbf{Z}_l(j))$ is a divisible group. Standard arguments ([CT/S/S] p. 774) then show that this group is a quotient of $H^d_{\text{ét}}(\overline{U}, \mathbf{Q}_l(j))$. On the other hand, the localisation sequence for étale cohomology, combined with the purity theorem for the smooth closed subvariety $\overline{Y}$ of $\overline{X}$, yields an exact sequence

$$H^d_{\text{ét}}(\overline{X}, \mathbf{Q}_l(j)) \to H^d_{\text{ét}}(\overline{U}, \mathbf{Q}_l(j)) \to H^{d-1}_{\text{ét}}(\overline{Y}, \mathbf{Q}_l(j-1)).$$

Let φ be the Frobenius endomorphism. Since X and Y are smooth and projective, Deligne's theorem implies that $\varphi - 1$ is invertible on $H^d_{\text{ét}}(\overline{X}, \mathbf{Q}_l(j))$ if $d \neq 2j$, and it is invertible on $H^{d-1}_{\text{ét}}(\overline{Y}, \mathbf{Q}_l(j))$ if $d-1 \neq 2(j-1)$. From the above exact sequence we find that if $d \neq 2j-1, 2j$ then $\varphi - 1$ is invertible on $H^d_{\text{ét}}(\overline{U}, \mathbf{Q}_l(j))$. Thus for $d \neq 2j-1, 2j$ the map $\varphi - 1$ induces a surjection on the quotient $H^d_{\text{ét}}(\overline{U}, \mathbf{Q}_l/\mathbf{Z}_l(j))$, and the group $H^1(F, H^d_{\text{ét}}(\overline{U}, \mathbf{Q}_l/\mathbf{Z}_l(j)))$ vanishes, since it is none other than the group of coinvariants of the module $H^d_{\text{ét}}(\overline{U}, \mathbf{Q}_l/\mathbf{Z}_l(j))$ under the action of Gal($\overline{F}/F$). ∎

Let us now complete the proof of Theorem 4.2. Taking the localisation sequence of the sheaf $\mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d))$ for the inclusion $Y \subset X$ yields an exact sequence

$$H^{d-3}_Y(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d))) \to H^{d-3}(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d))) \to H^{d-3}(U, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d))).$$

Replacing $\mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d))$ by its Bloch-Ogus resolution by flasque sheaves, and using

$$H^i_Y(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d))) \simeq H^{i-1}(Y, \mathcal{H}^d(\mathbf{Q}_l/\mathbf{Z}_l(d-1)))$$

which immediately follows from the structure of that resolution together with the Bloch-Ogus result applied to the smooth variety Y, we obtain the exact sequence

$$H^{d-4}(Y, \mathcal{H}^d(\mathbf{Q}_l/\mathbf{Z}_l(d-1))) \to H^{d-3}(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d))) \to H^{d-3}(U, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d))).$$

This exact sequence fits into the commutative diagram

$$\begin{array}{ccc} & & H^{d-4}(Y, \mathcal{H}^d(\mathbf{Q}_l/\mathbf{Z}_l(d-1))) \\ & & \downarrow \\ H^{2d-2}_{\text{ét}}(X, \mathbf{Q}_l/\mathbf{Z}_l(d)) & \to & H^{d-3}(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d))) \\ \downarrow & & \downarrow \\ H^{2d-2}_{\text{ét}}(U, \mathbf{Q}_l/\mathbf{Z}_l(d)) & \to & H^{d-3}(U, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d)))), \end{array}$$

where the middle commutative square is obtained by functoriality of the local to global spectral sequence.

By the induction assumption applied to Y, we have $H^{d-4}(Y, \mathcal{H}^d(\mathbf{Q}_l/\mathbf{Z}_l(d-1))) = 0$. According to Lemma 4.2.1, $H^{2d-2}_{\text{ét}}(U, \mathbf{Q}_l/\mathbf{Z}_l(d)) = 0$. Thus the above commutative diagram reduces to

$$\begin{array}{ccc} H^{2d-2}_{\text{ét}}(X, \mathbf{Q}_l/\mathbf{Z}_l(d)) & \rightarrow & H^{d-3}(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d))) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & H^{d-3}(U, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d))). \end{array}$$

In this diagram the right vertical arrow is one-to-one, as we have just seen, and the top map is surjective by Theorem 4.1. We thus conclude $H^{d-3}(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d))) = 0$.

We have postulated the existence of a good section Y of X. A priori, it is only over an infinite perfect field that such a section can be found (e.g. [A/K]). So we resort to the old trick (cf. [CT/S/S] p. 788) of finding two finite extensions F_1/F and F_2/F of coprime degrees over which we may find such good sections $Y_1 \subset X \times_F F_1$ and $Y_2 \subset X \times_F F_2$. The above argument shows that each of the groups $H^{d-3}(X \times_F F_i, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d)))$ $(i = 1, 2)$ is zero, and a transfer argument then shows that $H^{d-3}(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d)))$ is killed by each degree $[F_i : F]$, hence is zero. $\square$

We have now completed the proof of Theorem A. Let us complete that of Theorem B. We have $H^{d-3}(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d))) = 0$ (above), $H^{d-2}(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d))) = 0$ (§ 2). We may therefore apply Lemma 2.2 b) for $i = d-3$ (here we use the cohomological conjecture for $H^3(L, \mu_{l^n}^{\otimes 2})$, which we may according to Merkur'ev-Suslin), and thus obtain

$$H^{d-2}(X, \mathcal{H}^{d+1}(\mu_{l^n}^{\otimes d})) = 0 \ ,$$

i.e. Theorem B in degree $d-2$.

Finally, if $d = \dim(X) = 3$, we have $H^{d-3}(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d))) = 0$ (above) and $H^{d-4}(X, \mathcal{H}^{d+1}(\mathbf{Q}_l/\mathbf{Z}_l(d))) = 0$ for the trivial reason $d-4 < 0$. If $l = 2$, the cohomological conjecture for $H^4(L, \mu_{l^n}^{\otimes 3})$ is known (Merkur'ev-Suslin [M/S2], Rost). By a direct argument (a simple case of Lemma 2.2 b)), we conclude $H^0(X, \mathcal{H}^4(\mu_{2^n}^{\otimes 3})) = 0$. $\square$

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