September 9th, 2020

A complement to the paper

Bayer-Fluckiger, Eva; Parimala, Raman, On unramified Brauer groups of torsors over tori. Doc. Math. 25 (2020), 1263–1284

Proposition *Let K/k be a finite Galois extension of number fields, with group G. Then there exists a finite field extension L/k linearly disjoint from K/k, such that all decomposition groups of the field extension KL/L are cyclic.*

Fröhlich's proof for this precise statement (Mathematika 9 (1962) 133-134) is given over $k = \mathbf{Q}$, inspection of the proof should give an analogous proof over any number field k.

Theorem *Let G/k be a connected reductive group over a number field k. Let X be a smooth compactification of G. Assume that for any finite field extension E/k we have $Sha^1(E,G) = 0$. Then one has $\mathrm{Br}(X)/\mathrm{Br}(k) = 0$, and G satisfies weak approximation, and similarly over any finite field extension of k.*

Proof. In [RFGA] I defined flasque resolutions of connected linear algebraic groups and proved their existence. Given G/k a connected reductive over a field k one shows that there exist exact sequences

$$1 \to S \to H \to G \to 1$$

with H quasitrivial (see definition there) and S a flasque k-torus – whose character group $\hat{S}$ up to addition of a permuation module can be taken to be $\mathrm{Pic}(X \times_k \overline{k})$.

Extending the ground field from k to a field extension L/k gives a flasque resolution over L.

We have $H^1(k,\hat{S}) \simeq \mathrm{Br}(X)/\mathrm{Br}(k)$.

Using subtle results of Kneser, Harder, Borovoi, one shows ([RFGA] Theorem 9.4) that for k a number field, such a resolution induces an isomorphism of abelian groups

$$A(G) \simeq Coker[H^1(k,S) \to \oplus_v H^1(k_v,S)]$$

and a bijection

$$Sha^1(k,G) \simeq Sha^2(k,S).$$

By class field theory, $Sha^2(k,S)$ is dual to $Sha^1(k,\hat{S})$.

From this one recovers Sansuc's exact sequence [San]

$$0 \to A(G) \to Hom(\mathrm{Br}(X)/\mathrm{Br}(k), \mathbf{Q}/\mathbf{Z}) \to Sha^1(k,G) \to 0.$$

generalizing Voskresenskiĭ's exact sequence for tori.

Suppose now that we know $Sha^1(F,G) = 0$ over any finite field extension F of the number field k.

Let K/k be a finite Galois field extension splitting the k-torus S. Let L/k be a field extension as in the Proposition. We have $H^1(k,\hat{S}) \simeq H^1(L,\hat{S})$.

Because S/k is a flasque torus, we have $H^1(L,\hat{S}) = Sha^1_{cycl}(L,\hat{S})$.

For L as in the Proposition, $Sha^1_{cycl}(L,\hat{S})$ coincides with $Sha^1(L,\hat{S})$. By class field theory, the latter group is dual to $Sha^2(L,S)$ and, by assumption and [RFGA, Thm. 9.4] $Sha^2(L,S)$ vanishes.

Thus $H^1(L,\hat{S}) = 0$, hence $H^1(k,\hat{S}) = 0$, hence $\mathrm{Br}(X)/\mathrm{Br}(k) = 0$, and this also holds over any finite field extension of k, and then weak approximation for G holds over any finite field extension of k.

This proof of course specializes to the proof I had for tori (10 september 2016).

[BFP] Eva Bayer-Fluckiger, Raman Parimala, On unramified Brauer groups of torsors over tori. Doc. Math. 25 (2020), 1263–128

[RFGA] JLCT, Résolutions flasques des groupes algébriques linéaires, Crelle 618 (2008) 77–133.

[San] J.-J. Sansuc, Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres, Crelle 327 (1981) 12–80.