Hasse principle for torsors over p-adic function fields

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- Review on torsors and Galois cohomology

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- Variants in similar situations

Torsors and Galois cohomology

Let K be a field. Fix a separable closure K_s and denote by Γ_K the Galois group $\mathrm{Gal}(K_s/K)$. Let G be a linear (i.e. smooth and affine) algebraic group over K. A G-**torsor** over K is a K-scheme Y equipped with a faithful and transitive (right) action of $G(K_s)$ on $Y(K_s)$ that is compatible with the natural (left) action of Γ_K :

(1) $\forall\, y_1, y_2 \in Y(K_s)$, $\exists!\, g \in G(K_s)$ such that $y_1.g = y_2$, and

(2) $\forall\, y \in Y(K_s)\,,\, g \in G(K_s)\,,\, \gamma \in \Gamma_K : \gamma(y.g) = \gamma(y).\gamma(g)$.

A G-torsor over K is **trivial** (i.e., isomorphic to G with the action given by translation) if and only if it has a K-rational point. The set of isomorphism classes of G-torsors over K is isomorphic to the first cohomology set

$$H^1(K\,,\,G) := \frac{\{\, z\ :\Gamma_K \to G(K_s) \,|\, z_{\sigma\tau} = z_\sigma . {}^\sigma z_\tau \,\}}{\{\, z \sim z' \iff \exists\, g \in G(K_s)\ :\ z_\sigma = g^{-1} z'_\sigma {}^\sigma g \,\}}\,.$$

This is a pointed set with the distinguished element corresponding to the class of the trivial torsor $Y = G$.

Example (Description of $H^1(K, G)$)

(1) $G = \mathbf{PGL}_n$. The set $H^1(K, G)$ may be identified with the set of isomorphism classes of central simple K-algebras of dimension n^2. Indeed, $\mathbf{PGL}_n$ is the (algebraic) group of automorphisms of the algebra $\mathbf{M}_n$ of $n \times n$ matrices. The set $H^1(K, G)$ classifies the twisted forms of $\mathbf{M}_n$, namely, the central simple algebras of dimension n^2 over K. (Since $G = \mathbf{PGL}_n$ is also the automorphism group of the projective space $\mathbb{P}^{n-1}$, we may also regard $H^1(K, G)$ as the set of isomorphism classes of Severi–Brauer varieties of dimension $n-1$ over K.)

Example (Continued)

(2) $(\mathrm{char}(K) \neq 2)$ $G = \mathbf{O}(q)$, the orthogonal group of a nonsingular quadratic form q over K. Then $H^1(K, G)$ is identified with the set of isomorphism classes of (nonsingular) quadratic forms of dimension $n = \dim q$ over K.

(3) $(\mathrm{char}(K) \neq 2)$ $G = \mathbf{SO}(q)$, the special orthogonal group of a nonsingular quadratic form q over K. Then $H^1(K, G)$ is identified with the set of isomorphism classes of (nonsingular) quadratic forms *that have the same dimension and the same determinant as* q over K.

Example (Continued)

(4) Let A be a central simple algebra (CSA) over K. Let $G = \mathbf{GL}_1(A)$, the general linear group of A. This is the group defined by

$$\mathbf{GL}_1(A)(R) := (A \otimes_K R)^* \quad \text{for every commutative } K\text{-algebra } R\,.$$

Then $H^1(K,\,G) = 1$ (Hilbert's Theorem 90).

(5) Let $\mathrm{Nrd} : A \to K$ be the reduced norm map of a CSA A over K. Let $G = \mathbf{SL}_1(A)$ be the kernel of the induced group homomorphism $\mathrm{Nrd} : \mathbf{GL}_1(A) \longrightarrow \mathbb{G}_m = \mathbf{GL}_1(K)$. The previous example yields an identification

$$K^*/\mathrm{Nrd}(A^*) \xrightarrow{\sim} H^1(K,\,G)\,.$$

Classical problems

It is not difficult to prove (using the description of $H^1(K, \mathbf{PGL}_n)$) that if $H^1(K, G) = 1$ for all connected semisimple groups G over K, then the field K must have cohomological dimension ≤ 1.

Conversely, we have the following theorem of Steinberg, which settles **Serre's Conjecture I**.

Theorem (Steinberg, 1965)

Let K be a perfect field of cohomological dimension $\mathrm{cd}(K) \leq 1$. Then for every connected linear algebraic group G/K, $H^1(K, G) = 1$ i.e. every G-torsor over K is trivial.

The two-dimensional analog is the famous Serre's Conjecture II.

Conjecture II (Serre, 1962)

Let K be a perfect field of cohomological dimension $\mathrm{cd}(K) \leq 2$. Then $H^1(K, G) = 1$ for every semisimple simply connected group G over K.

This conjecture has been proven in many cases and remains open only for a few exceptional groups (e.g. E_8).

Notice that the conjecture is certainly false if the "simply connectedness" assumption is removed.

By a theorem of Merkurjev and Suslin, the converse of Conjecture II over a perfect field is true.

For fields of cohomological dimension 3, more interesting would be the problem of *Hasse principle*.

Let Ω_K denote the set of equivalence classes of (rank 1) discrete valuations of K and let K_v be the completion of K at v for each $v \in \Omega_K$. A K-variety X is said to satisfy the **Hasse principle** (HP) (with respect to Ω_K) if

$$\prod_{v\in\Omega_K} X(K_v) \neq \emptyset \Longrightarrow X(K) \neq \emptyset\,.$$

For a torsor Y of a linear algebraic group G/K, one has $\prod_{v\in\Omega_K} Y(K_v) \neq \emptyset$ iff the class $[Y] \in H^1(K,\, G)$ is in the kernel of the natural map

$$H^1(K\,,\,G) \longrightarrow \prod_{v\in\Omega_K} H^1(K_v\,,\,G)\,.$$

Why p-adic function fields ?

- Analogous results over $\mathbb{C}((t))(C)$, C : algebraic curve over the field of Laurent series $\mathbb{C}((t))$. (Colliot-Thélène–Gille–Parimala, 2004)

  (1) $H^1(K, G) = 1$ for all semisimple simply connected G over $K = \mathbb{C}((t))(C)$, i.e., Conjecture II is completely solved over $\mathbb{C}((t))(C)$.

  (2) The HP for torsors under semisimple *absolutely almost simple* (not necessarily simply connected) groups over $\mathbb{C}((t))(C)$.

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  (1) $H^1(K, G) = 1$ for all semisimple simply connected G over $K = \mathbb{C}((t))(C)$, i.e., Conjecture II is completely solved over $\mathbb{C}((t))(C)$.
  (2) The HP for torsors under semisimple *absolutely almost simple* (not necessarily simply connected) groups over $\mathbb{C}((t))(C)$.
- Nice properties that can be useful in some simplest cases have been established over p-adic function fields. For example, the arithmetic of quadratic forms over these fields have been well understood. (More details later.)

The Hasse principle conjecture

The main question we'll discuss from now on is the following conjecture :

Conjecture (Colliot-Thélène–Parimala–Suresh, 2008)

Let $K = F(C)$ be the function field of an algebraic curve C over a p-adic field F. Let G be a simply connected semisimple group over K. Then the G-torsors over K satisfy the Hasse principle. In other words, the kernel of the natural map

$$H^1(K, G) \longrightarrow \prod_{v \in \Omega_K} H^1(K_v, G)$$

is trivial.

We may (by Shapiro's lemma) and we will assume G is absolutely simple (i.e. the root system of $G_{\bar{K}}$ is irreducible).

Colliot-Thélène, Parimala and Suresh proved their conjecture for quasi-split groups. (*quasi-split* = having a Borel subgroup defined over the base field. For B_n, C_n, E_7, E_8, F_4 and G_2, quasi-split $\Leftrightarrow$ split.)

- If G is split ($\Leftrightarrow$ has a Borel K-subgroup that admits a composition series with quotients $\mathbb{G}_m$ or $\mathbb{G}_a$ $\Leftrightarrow$ (since G reductive) has a maximal torus which splits over the base field K), they used a patching method developed by Harbater–Hartmann–Krashen. This argument is classification-free (and applies to type E_8).

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- If G is (quasi-split and) not of type E_8, one may use an injectivity property of the **Rost invariant**.

For an absolutely simple simply connected group G over a field K (which can be arbitrary for the moment), the Rost invariant is a "functorial map" of pointed sets

$$R_G \,:\, H^1(K,\, G) \longrightarrow H^3(K,\, \mathbb{Q}/\mathbb{Z}(2))\,.$$

We say that it is a *cohomological invariant of dimension 3* (with value in $\mathbb{Q}/\mathbb{Z}(2)$). The image of R_G lies in the n-torsion part $H^3(K,\, \mathbb{Q}/\mathbb{Z}(2))[n]$ for some integer $n = n_G > 0$ depending on G.

If G is quasi-split and not of type E_8 and if K is of cohomological dimension ≤ 3, then R_G has a trivial kernel, by the work of Chernousov, Garibaldi and Gille.

We have the commutative diagram

$$\begin{array}{ccc} H^1(K\,,\,G) & \longrightarrow & \prod_v H^1(K_v\,,\,G) \\ \downarrow & & \downarrow \\ H^3(K,\,\mathbb{Q}/\mathbb{Z}(2)) & \longrightarrow & \prod_v H^3(K_v,\,\mathbb{Q}/\mathbb{Z}(2)) \end{array}$$

Theorem (Kato, 1986)

For $K/\mathbb{Q}_p(t)$ finite, the map

$$H^3(K,\,\mathbb{Q}/\mathbb{Z}(2)) \longrightarrow \prod_v H^3(K_v,\,\mathbb{Q}/\mathbb{Z}(2))$$

is injective.

This proves the HP conjecture for quasi-split groups (not of type E_8).

Remark

For quasi-split/split groups of type E_8 over p-adic function fields, one proves the injectivity (i.e. triviality of the kernel) of the Rost invariant by an argument in the reverse direction : The HP for torsors + extra (nontrivial) work $\Longrightarrow \mathrm{Ker}(R_G) = 1$. (CT-P-S)

Main theorem

For not necessarily quasi-split groups, we have the following theorem :

Main Theorem (H., 2012)

Over a finite extension K of $\mathbb{Q}_p(t)$, the Hasse principle for G-torsors holds if G is a semisimple simply connected group which has no factors other than the following

$$^1A_n^*,\ ^2A_n^*\,,\ B_n\,,\ C_n^*\,,\ D_n^*\,,\ F_4^{red}\,,\ G_2\,.$$

Explicit description of the groups in the theorem :

(1) ${}^1A_n^*$: the special linear group $\mathbf{SL}_1(A)$ of a CSA A over K whose index $\mathrm{ind}(A)$ is **square-free** ;

(2) ${}^2A_n^*$: the special unitary group $\mathbf{SU}(h)$ of a hermitian form h over a pair (D, τ), where D is a central division algebra of **square-free** index over a quadratic extension L/K and τ is an L/K-*involution*on D ;

(3) B_n : the spin group $\mathbf{Spin}(q)$ of a quadratic form q of rank $2n+1$ over K ;

(4) C_n^* : the unitary group $\mathbf{U}(h)$ of a hermitian form h over a pair (D, τ) where D is a **quaternion algebra** over K and τ is the canonical (symplectic) involution on D ;

(5) D_n^* : the spin group **Spin**(h) of a hermitian form h over a pair (D, τ), where D is a **quaternion algebra** over K and τ is an orthogonal involution on D (If D is split, we get the spin group **Spin**(q) of an even dimensional quadratic form q/K) ;

(6) F_4^{red} : the group of automorphisms **Aut**(J) of a **reduced** exceptional Jordan algebra J of dimension 27 over K ;

(7) G_2 : the group of automorphisms **Aut**(C) of a Cayley algebra C over K.

Remark

By a theorem of Saltman, a central division algebra of exponent 2 over a p-adic function field K is either a quaternion algebra or a biquaternion algebra. So for a group of type C_n, say $G = \mathbf{U}(h)$ with h a hermitian form over a symplectic pair (D, τ), the case not covered by our theorem is the case where D is a biquaternion algebra. Similarly, for a group of classical type D_n (trialitarian D_4 excluded), say $D = \mathbf{Spin}(h)$ with h a hermitian form over an orthogonal pair (D, τ), the remaining case is the one with D a biquaternion algebra.

Strategies for the proof

Easy cases : ${}^1A_n^*$, C_n^*, F_4^{red} and G_2.

- ${}^1A_n^*$ and F_4^{red} : use cohomological invariants.

For $\mathbf{SL}_1(A)$, the Rost invariant map

$$H^1(K, \mathbf{SL}_1(A)) = K^*/\mathrm{Nrd}(A^*) \longrightarrow H^3(K, \mathbb{Q}/\mathbb{Z}(2))\,; \quad \lambda \longmapsto (\lambda) \cup (A)$$

is injective when $\mathrm{ind}(A)$ is square-free (Suslin).

For F_4^{red}, use some cohomological invariants of dimension 3 and 5 (f_3, f_5).

• C_n^* and G_2 : use classification of relevant algebraic objects via quadratic forms (trace forms of hermitian forms over a quaternion algebra, norm forms of Calyer algebras) and the injectivity of

$$I^3(K) \longrightarrow \prod_v I^3(K_v)$$

(easy consequence of Kato's theorem).

(Here $I^3(K)$=the third power of the fundamental ideal $I(K)$ of the Witt ring $W(K)$ of quadratic forms = the subgroup generated by classes of 3-fold Pfister forms =the subgroup consisting of classes of quadratic forms with even rank, trivial discriminant and trivial Clifford invariant.)

Harder cases : B_n, D_n^* and ${}^2A_n^*$ of odd index

Sketch of the proof for $\mathbf{Spin}(q)$ ($\in B_n \cup D_n^*$, q a quadratic form.)

The exact sequence

$$1 \longrightarrow \mu_2 \longrightarrow \mathbf{Spin}(q) \xrightarrow{\mathrm{Sn}} \mathbf{SO}(q) \longrightarrow 1$$

where Sn is the "*spinor norm*" map, induces an exact sequence

$$1 \longrightarrow \frac{K^*/K^{*2}}{\mathrm{Sn}(q_K)} \longrightarrow H^1(K, \mathbf{Spin}(q)) \xrightarrow{\eta} H^1(K, \mathbf{SO}(q)) \,.$$

It suffices to prove 2 HP : one for

$\mathrm{Im}(H^1(K, \mathbf{Spin}(q)) \xrightarrow{\eta} H^1(K, \mathbf{SO}(q)))$, the other for $\frac{K^*/K^{*2}}{\mathrm{Sn}(q_K)}$.

The HP for $\mathrm{Im}(\eta)$ can be proved using the HP for $I^3(K)$: If $\xi \in H^1(K, \mathbf{Spin}(q))$ has image $\eta(\xi)$ corresponding to a form q', then $q' - q \in I^3(K)$. (So this HP relies on Kato's theorem.)

For the HP for the spinor norms $(\mathrm{Ker}(\eta) \cong \frac{K^*/K^{*2}}{\mathrm{Sn}(q_K)})$,

- If $\dim q < 5$, the problem can be reduced to the case of ${}^1A_n^*$: the spinor norms can be described in terms of reduced norms of an associated quaternion algebra D (or of D_L, for a quadratic extension L/K).

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- If $\dim q < 5$, the problem can be reduced to the case of ${}^1A_n^*$: the spinor norms can be described in terms of reduced norms of an associated quaternion algebra D (or of D_L, for a quadratic extension L/K).
- If $\dim q \geq 5$, the group $\frac{K^*/K^{*2}}{\mathrm{Sn}(q_K)}$ is in fact trivial for our base field. This is not too difficult to show since we know the u-invariant of K. (Consider the isotropy of $q - \alpha.q$ for $\alpha \in K^*$.)

Theorem (Parimala–Suresh, 2007)

The u-invariant of a finite extension K of $\mathbb{Q}_p(t)$ for p an odd prime is $u(K) = 8$ ($\Rightarrow$ every quadratic form of rank ≥ 9 over K has a nontrivial zero).

Two other proofs : Harbater–Hartmann–Krashen, 2008 (for p odd) and Leep, 2009 (for p arbitrary).

For groups of the form

Spin(h) ($\in D_n^*$, h over a quaternion division algebra with an orthogonal involution) ; and

SU(h) (type ${}^2A_n^*$ of odd index, h over a unitary pair (D, τ) with $\mathrm{ind}(D)$ odd and square-free) :

(1) Relative versions of the Clifford invariant and the Rost invariant for hermitian forms (with respect to involutions of appropriate types).

(2) An exact sequence of Witt groups of Parimala–Sridharan–Suresh, or at least its simplest form due to Scharlau.

These have been used (and developed) by Bayer-Fluckiger–Parimala in a series of papers on Serre's conjecture II and some similar Hasse principle conjectures.

(3) Merkurjev's norm principle (for spinor norms of an orthogonal pair) ;

and

(4) A theorem of Bayer-Fluckiger–Lenstra (needed for the case of ${}^2A_n^*$ of odd index) : the natural map on the Witt groups of a unitary pair induced by the base change to an odd degree extension is injective.

The most complicated case turns out to be the case ${}^2A_n^*$ of even index.

Essentially we have to solve the problem for the group $G = \mathbf{SU}(h)$, where h is a hermitian form over a unitary pair (D, τ) of the following form :

$D = D_0 \otimes_K K(\sqrt{d})$ for a quaternion division algebra $D_0 = (a, b)_K$ over K ; $\tau = \tau_0 \otimes \iota$, where τ_0 is the canonical (symplectic) involution of D_0 and ι is the nontrivial element of $\mathrm{Gal}(K(\sqrt{d})/K)$.

A basic tool : Suresh's exact sequence (which was used in a paper of Parimala and Preeti on a similar HP conjecture over functions fields of curves over number fields)

$$W(L) \xrightarrow{\pi_1} W(D_0\,,\,\tau_0) \xrightarrow{\rho} W(D\,,\,\tau) \xrightarrow{p_2} W^{-1}(D_0,\,\tau_0)\,.$$

(Here W^{-1}=Witt group of skew-hermitian forms.)

Construction of the maps in Suresh's sequence

Since $D = D_0 \oplus D_0\sqrt{d}$, for any (V, h) over (D, τ), we may write

$$h(x, y) = h_1(x, y) + h_2(x, y)\sqrt{d} \quad \text{with} \quad h_i(x, y) \in D_0\,, \quad \text{for} \quad i = 1, 2\,.$$

The projection $h \mapsto h_2$ defines a group homomorphism

$$p_2 \;:\; W(D, \tau) \longrightarrow W^{-1}(D_0\,, \tau_0)\,.$$

Similarly, we have projections

$$\widetilde{\pi}_i \;:\; W(L) = W(K(\sqrt{d})) \longrightarrow W(K)\,;\;\; q \longmapsto q_i\,, \quad i = 1, 2\,.$$

We denote by $\pi_1 \;:\; W(L) \to W(D_0, \tau_0)$ the composite map

$$W(L) \xrightarrow{\widetilde{\pi}_1} W(K) \longrightarrow W(D_0, \tau_0)$$

where the map $W(K) \to W(D_0, \tau_0)$ is induced by base change.

Finally, for a hermitian form (V_0, f) over (D_0, τ_0), set

$$V = V_0 \otimes_{D_0} D = V_0 \otimes_K L = V_0 \oplus V_0\sqrt{d}$$

and let $\rho(f) : V \times V \to D$ be the map extending $f : V_0 \times V_0 \to D_0$ by τ-sesquilinearity. One checks that this defines a group homomorphism

$$\rho : \quad W(D_0, \tau_0) \longrightarrow W(D, \tau); \quad (V_0, f) \longmapsto (V_0 \oplus V_0\sqrt{d}, \rho(f)).$$

We thus obtain the sequence

$$W(L) \xrightarrow{\pi_1} W(D_0, \tau_0) \xrightarrow{\rho} W(D, \tau) \xrightarrow{p_2} W^{-1}(D_0, \tau_0),$$

which is exact as proved by Suresh.

Proof of the HP for the group $G = \mathbf{SU}(h)$ (of type ${}^2A_n^*$, with h over (D, τ) defined as before).

Let $T = R^1_{L/K}\mathbb{G}_m$ be the norm 1 torus associated to the quadratic extension $L/K := K(\sqrt{d})/K$, i.e.,

$$T := \mathrm{Ker}\left(R_{L/K}\mathbb{G}_{m,\,L} \xrightarrow{N_{L/K}} \mathbb{G}_{m,\,K}\right)$$

where $R_{L/K}$ denotes the Weil restriction functor.

The exact sequence

$$1 \longrightarrow \mathbf{SU}(h) \longrightarrow \mathbf{U}(h) \xrightarrow{\mathrm{Nrd}} T \longrightarrow 1$$

yields an exact sequence

$$1 \longrightarrow \frac{T(K)}{\mathrm{Nrd}(\mathbf{U}(h)(K))} \longrightarrow H^1(K, \mathbf{SU}(h)) \xrightarrow{\eta} H^1(K, \mathbf{U}(h)) \longrightarrow 1\,.$$

First step. The HP for $\mathrm{Ker}(\eta) \cong \frac{T(K)}{\mathrm{Nrd}(\mathbf{U}(h)(K))}$:

Fact (Merkurjev) : letting $h_0 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, one has

$$\mathrm{Nrd}(\mathbf{U}(h)(K)) = \mathrm{Nrd}\left(\mathbf{U}(h_0)(K)\right) .$$

$\Longrightarrow$

$$\frac{T(K)}{\mathrm{Nrd}(\mathbf{U}(h)(K))} \hookrightarrow H^1(K, \mathbf{SU}(h_0)) = H^1(K, \mathbf{Spin}(q)) ,$$

where

$$q := \langle 1, -d \rangle \otimes n_{D_0} = \langle 1, -d \rangle \otimes \langle 1 , -a , -b , ab \rangle .$$

The result follows from the HP for the group $\mathbf{Spin}(q)$.

Now it suffices to prove :

Let $\xi \in H^1(K, \mathbf{SU}(h))$, $[h'] := \eta(\xi) \in H^1(K, \mathbf{U}(h))$ and $h_2 := h' \perp (-h)$.

Assume $\xi_v = 1$, $\forall\, v$.

Then $h' \cong h$ over K, or equivalently, $[h_2] = 0$ in the Witt group $W(D\,,\,\tau)$

(of hermitian forms over the unitary pair $(D,\ \tau)$).

We use Suresh's exact sequence

$$W(L) \xrightarrow{\pi_1} W(D_0\,,\,\tau_0) \xrightarrow{\rho} W(D\,,\,\tau) \xrightarrow{p_2} W^{-1}(D_0,\,\tau_0)\,,$$

Second step. Prove $p_2([h_2]) = 0 \in W^{-1}(D_0, \tau_0)$.

Idea : Use the commutative diagram

$$\begin{array}{ccc} W(D, \tau) & \xrightarrow{p_2} & W^{-1}(D_0, \tau_0) \\ \downarrow & & \downarrow \\ W(D \otimes_K K(C), \tau) & \xrightarrow{p_2} & W^{-1}(D_0 \otimes_K K(C), \tau_0) \end{array}$$

where $C \subseteq \mathbb{P}^2_K$ is the plane conic associated to the quaternion algebra D_0/K. The vertical map on the right is injective (Parimala–Sridharan–Suresh). It suffices : $[h_2] = 0$ over $K(C)$.

Since D_0 splits over $K(C)$, the question can be transformed into a question about quadratic forms over $K(C)$. Recall that h_2 comes from a locally trivial class $\xi \in H^1(K, \mathbf{SU}(h))$. It turns out that the injectivity of the map

$$I^4(K(C)) \longrightarrow \prod_v I^4(K_v(C))$$

yields the desired result.

This HP for I^4 can be proved using :

A Hochschild-Serre spectral sequence+ Merkurjev–Suslin, Voevodsky + Kato's theorem.

(The situation here is simplified a lot as C is simply a plane conic !)

Recall (once again) Suresh's exact sequence

$$W(L) \xrightarrow{\pi_1} W(D_0\,,\,\tau_0) \xrightarrow{\rho} W(D\,,\,\tau) \xrightarrow{p_2} W^{-1}(D_0,\,\tau_0)\,,$$

Step 2 $\Longrightarrow$ $[h_2] = \rho([h_0])$ for some $[h_0] \in W(D_0,\,\tau_0)$. $[h_2]$ is locally trivial, so $[h_0]$ lies locally in the image of π_1.

Step 3.

Key Lemma

For a hermitian form $h_0/(D_0,\,\tau_0)$ of even rank $2n$, one has $[h_0] \in \mathrm{Im}(\pi_1)$ if and only if the Pfaffian norm $\mathrm{Pf}(h_0) \in K^*/\mathrm{Nrd}(D_0^*)$ lies in the subgroup generated by $N_{L/K}(L^*)$.

Proof of the key lemma :

The "only if" part follows from direct computation.

For the "if" part, use induction on the rank $\mathrm{rank}(h_0) = 2n$. If $n = 1$, direct computation. When $n \geq 2$, one can show that $\rho(h_0)$ must be isotropic using $u(K) < 12$. (Restricting $\rho(h_0)$ to the 3-dimensional subspace $K.i\sqrt{d} + K.j\sqrt{d} + K.ij\sqrt{d} \subseteq \mathrm{Sym}(D\,,\,\tau)$ yields a quadratic form of rank $3.\mathrm{rank}(h_0)$.) Then split out a hyperbolic factor from h_0 and use induction hypothesis to conclude.

Last step (Step 4) : Known $[h_0] \in \mathrm{Im}(\pi_1)$ locally. To Prove :

$h_0 \in \mathrm{Im}(\pi_1) = \mathrm{Ker}(\rho)$ ($\Longrightarrow [h_2] = \rho([h_0]) = 0$!)

Proof : $\lambda := \mathrm{Pf}(h_0) \in \mathrm{Nrd}(D_0^*).N_{L/K}(L^*)$ if and only if the 6-dimensional form

$$\lambda.\langle 1, -d\rangle - n_{D_0}$$

is isotropic.

Immediate from the following

Theorem (CT-P-S, 2008)

The HP for the isotropy of quadratic forms of rank ≥ 3 over K holds.

Final remarks

- In the theorem we have only used local assumptions at divisorial valuations : v divisorial $\Leftrightarrow \exists\ \mathcal{X}$ regular proper model over $\mathrm{Spec}(\mathcal{O}_F)$, such that $v = v_x$ for some $x \in \mathcal{X}^{(1)}$. (Here F is the field of constants of K, i.e., $K = F(C)$ for a geometrically integral curve C/F.)

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- It seems unlikely that the HP conjecture of CT-P-S holds for an arbitrary reductive (or semisimple) group G, for instance a torus. But no counterexample is known.

- Using their patching method, Harbater, Hartmann and Krashen proved a local-global principle with respect to some other families of overfields than the family $\{ K_v \}$ we consider here. Their results apply to function fields over a general completely valued field and assume the group G has some rationality property.

  For example, if $K = \mathbb{Q}_p(t)$ and G is a connected K-rational group, then a G-torsor over K is trivial if and only if it is trivial over the following two fields :

  K_1=the fraction field of the p-adic completion of the ring $\mathbb{Z}_p[t^{-1}]$;

  $K_2 = \mathrm{Frac}(\mathbb{Z}_p[\![t]\!])$. (Here the family $\{ K_1, K_2 \}$ of overfields of $K = \mathbb{Q}_p(t)$ is the simplest one, but there are many other choices for the family, even over a fixed model.)

Open question : does the HP with respect to discrete valuations hold for torsors under connected rational groups over $K = \mathbb{Q}_p(t)$?

- Our proof of the main theorem makes use of the following special properties of p-adic function fields :

  (1) Kato's theorem about the HP for $H^3(\mathbb{Q}/\mathbb{Z}(2))$;

  (2) $u(K) = 8$;

  (3) the HP for smooth quadrics of dimension ≥ 1 (i.e. for quadratic forms of rank ≥ 3). In fact, this last property is only used to treat the case ${}^2A_n^*$ of **even** index and only the case of 6-dimensional forms is needed.

It is known that properties (1) and (2) also hold in a more local situation :

Let $K = \mathrm{Frac}(R)$, R=2-dimensinoal, henselian, excellent local domain with **finite** residue field of characteristic p.

Example : $R = \mathbb{Z}_p[\![t]\!]$, $R = \mathbb{F}_p[\![x, y]\!]$, R=henselization at a closed point of an algebraic surface over $\mathbb{F}_p$, or R=henselization at a closed point of a relative curve over $\mathbb{Z}_p$.

In this **local henselian** case, (suppressing mention of characteristic restrictions)

property (1) : S. Saito ($+\ \varepsilon$) ; and property (2) [H., 2011].

If $R = A[\![t]\!]$ (with A=complete DVR with finite residue field), property (3) is also known ([H., 2010]).

So, in the local henselian case, (under mild restriction on the residue characteristic) the HP for G-torsors is true, if G is of one of the following types :

$$^1A_n^*\,,\ ^2A_n^* \text{ of odd index}\,,\ B_n\,,\ C_n^*\,,\ D_n^*\,,\ F_4^{red}\,,\ G_2\,.$$

If moreover the HP for 6-dimensional quadratic forms holds over K (e.g. $K=\mathrm{Frac}(\mathbb{Z}_p[\![t]\!])$ or $K=\mathbb{F}_p(\!(x\,,\,y)\!)$), then the case $^2A_n^*$ of even index is also OK.

The End !

Thank you !