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LECTURES ON LINEAR ALGEBRAIC GROUPS
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These notes were produced as I was lecturing in Beijing in April 2007. A few misprints have been corrected, but the notes are not in final form.

1. Some general motivation

I described motivation from two sources:

(i) Over an arbitrary field k: the Lüroth theorem for curves and its failure in higher dimension: there exist k-unirational varieties which are not k-rational.

(ii) Over a number field k, for a given k-variety, the question of density of the image of the diagonal map

$$X(k) \to \prod_v X(k_v)$$

where v runs through the set of all places of k, the field k_v is the completion of k at v, each $X(k_v)$ is equipped with the topology induced by that of k_v, and the right hand side is given the product topology. Here is the question is to measure the failure of weak approximation and of the Hasse principle.

These apparently unrelated problems are sometimes connected: some of the invariants used to show that some k-varieties are not k-rational also play a rôle in detecting the failure of density in the number field situation. Among such invariants, we find the Galois action of the absolute Galois group on the Picard group of the variety over an algebraic closure, the Brauer group of the variety, the quotient $X(k)/R$ of the set of k-rational points of X by R-equivalence.

Connected linear algebraic groups and their homogeneous spaces are (very) special examples of geometrically rational varieties, in particular they are special cases of rationally connected varieties (à la Kollár-Miyaoka-Mori).

Connected linear algebraic groups serve both as examples and as building blocks in the study of the arithmetic of these much more general varieties.

2. Notation and background

Let k be a field.

A k-variety X is a separated scheme of finite type over the field k. One writes $k[X] = H^0(X, \mathcal{O}_X)$ for the ring of regular functions on X and $k[X]^* = H^0(X, \mathcal{O}_X^*)$ for the multiplicative group of invertible functions on X. For any field F containing k one lets $X(F) = Hom_{Speck}(SpecF, X)$ be the set of F-rational points of X.

Given two k-varieties X and Y their fibre product over k is denoted $X \times_k Y$. The subscript k is sometimes omitted when the context is clear. It is most often omitted when the field k is algebraically closed.

Let X/k be a k-scheme. For any field extension K/k one writes X_K for the fibre product $X \times_k K = X \times_{Speck} SpecK$.

If $\overline{k}$ is an algebraic closure of k and X is a k-variety one writes $\overline{X} = X \times_k \overline{k}$.

We let $k[X] = H^0(X, \mathcal{O}_X)$ and $\overline{k}[X] = H^0(\overline{X}, \mathcal{O}_{\overline{X}})$.

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A scheme is called integral if it is irreducible and reduced.

A k-variety X is called geometrically something if if $\overline{X}$ is. Something can be any property: connected, reduced, irreducible, integral, normal, regular. A k-variety is smooth if and only if it is geometrically smooth. A smooth k-variety is regular. The converse holds if the ground field is perfect.

A smooth connected k-variety is integral but it need not be geometrically connected. A smooth geometrically connected k-variety is geometrically integral.

Over a nonperfect field k there exist normal, regular k-vareties which are not geometrically normal, hence not geometrically regular, and thus are not smooth over k.

For an integral k-variety we let $k(X)$ denote the function field of X and for a geometrically integral variety, i.e. such that $\overline{X}$ is integral, we let $\overline{k}(X)$ be the function field of $\overline{X}$.

Two integral k-varieties X and Y are said to be k-birationally equivalent if their function fields $k(X)$ and (Y) are isomorphic (over k). This is equivalent to requiring that there exist nonempty open sets $U \subset X$ and $V \subset Y$ which are k-isomorphic.

An integral k-variety is called k-rational if it is k-birational to affine space of the same dimension. This is equivalent to requiring that the function field $k(X)$ of X is purely transcendental over k.

If a geometrically integral k-variety is geometrically rational, i.e. $\overline{X}$ is $\overline{k}$-rational, one often simply says that X is rational. It then need not be k-rational (indeed it might have no k-rational point). For instance a smooth k-conic is a rational variety, whether it has a rational k-point or not. It is k-rational if and only if it has a k-point.

An integral k-variety is called k-unirational if there exists a nonempty open set U of affine space $\mathbb{A}^n_k$ for some integer n and a dominant k-morphism $U \to X$. This is equivalent to requiring that there is a k-embedding of the function field $k(X)$ into a purely transcendental extension of k of some degree n. If k is infinite, the assumption for some n implies the same assumption for $n = dim X$.

I started with a recapitulation of basic definitions on Cartier divisors, Weil divisors on an integral variety, the class group and the Picard group. For regular varieties, Cartier divisors and Weil divisors coincide. A good reference for this is a chapter in Mumford's book *Curves on an algebraic surface.*

For a variety X defined over a field k with separable closure $\overline{k}$, the absolute Galois group $g = Gal(\overline{k}/k)$ acts on to $X \times_k \overline{k}$ hence on various objects attached to $X \times_k \overline{k}$. In this section I collect various general results which will be used later on.

Proposition 2.1. *Let k be field, $\overline{k}$ a separable closure of k and $g = Gal(\overline{k}/k)$. Let X/k be a smooth geometrically integral k-variety, $\overline{X} = X \times_k \overline{k}$.*

Assume that X is smooth.

(i) There is a natural exact sequence

$$1 \to H^1(g, \overline{k}[X]^*) \to \operatorname{Pic} X \to (\operatorname{Pic} \overline{X})^g \to H^2(g, \overline{k}[X]^*) \to H^2(g, \overline{k}(X)^*).$$

(ii) If $\overline{k}^* = \overline{k}[X]^*$ then we have the exact sequence

$$0 \to \operatorname{Pic} X \to (\operatorname{Pic} \overline{X})^g \to H^2(g, \overline{k}^*) \to H^2(g, \overline{k}(X)^*).$$

(iii) If $\overline{k}^* = \overline{k}[X]^*$ and X has a k-point then the map $H^2(g, \overline{k}^*) \to H^2(g, \overline{k}(X)^*)$ is injective and we have

$$\operatorname{Pic} X \simeq (\operatorname{Pic} \overline{X})^g.$$

(iv) Under the same assumptions as in (iii), there is a natural isomorphism

$$H^1(g, \mathrm{Pic}\,\overline{X}) \simeq Ker[H^2(g, \overline{k}(X)^*/\overline{k}^*) \to H^2(g, Div\overline{X})].$$

Proof. On starts from the exact sequence of Galois modules

$$1 \to \overline{k}[X]^* \to \overline{k}(X)^* \to \mathrm{Div}\overline{X} \to \mathrm{Pic}\,\overline{X} \to 0,$$

one breaks it up into two short exact sequences and one applies Galois cohomology.

One uses the following facts. We have $\mathrm{Div}X = (\mathrm{Div}\overline{X})^g$. The Galois module $\mathrm{Div}\overline{X}$ is a direct sum of permutation modules hence satisfies $H^1(g, \mathrm{Div}\overline{X}) = 0$ (Shapiro's lemma). We have $H^1(g, \overline{k}^*) = 0$ and $H^1(g, \overline{k}(X)^*) = 0$ (Hilbert's theorem 90). Statement (ii) is a special case of (i). As for (iii), I refer to [13], Prop. 2.2.2, which proves that for a geometrically integral k-variety with a smooth k-point, the sequence of Galois modules

$$1 \to \overline{k}^* \to \overline{k}(X)^* \to \overline{k}(X)^*/\overline{k}^* \to 1$$

is split. That result is also used in the proof of (iv). □

Remark 2.2. For an arbitrary k-variety X, the spectral sequence

$$E_2^{pq} = H^p_{et}(g, H^q(\overline{X}, \mathbb{G}_m)) \Longrightarrow H^n_{et}(X, \mathbb{G}_m)$$

gives rise to the long exact sequence

$$0 \to H^1(g, \overline{k}[X]^*) \to \mathrm{Pic}\,X \to (\mathrm{Pic}\,\overline{X})^g \to H^2(g, \overline{k}[X]^*) \to Ker[\mathrm{Br}X \to \mathrm{Br}\overline{X}]$$
$$\to H^1(g, \mathrm{Pic}\,\overline{X}) \to H^3(g, \overline{k}[X]^*).$$

Here $\mathrm{BrX} = H^2_{et}(X, \mathbb{G}_m)$ denotes the Brauer group of the scheme X.

There is a more general result. Let S/k be a multiplicative k-group. Then there is an exact sequence

$$0 \to H^1(k, H^0(\overline{X}, \overline{S})) \to H^1(X, S) \to (H^1(\overline{X}, \overline{S}))^g \to H^2(k, H^0(\overline{X}, \overline{S})) \to H^2(X, S).$$

Here cohomology is étale cohomology, and the sequence can be extended to a few more terms, just as the one for $S = \mathbb{G}_m$.

Proposition 2.3. *Let X, Y be two smooth, proper, geometrically integral k-varieties. Let K/k be a Galois extension with group g (finite or not). If X and Y are k-birationally equivalent. Then:*

(i) There exist finitely generated g-permutation modules P_1 and P_2 and an isomorphism of g-modules

$$\mathrm{Pic}\,X_K \oplus P_1 \simeq \mathrm{Pic}\,Y_K \oplus P_2.$$

(ii) For any closed subgroup $h \subset g$, there is an isomorphism

$$H^1(h, \mathrm{Pic}\,X_K) \simeq H^1(h, \mathrm{Pic}\,Y_K).$$

Proof. See [13], p. 461-463 (the elegant proof, due to Moret-Bailly, is independent of the rest of the paper [13]). □

In particular, if X/k is smooth, proper, geometrically integral and geometrically rational, then

(i) For any field extension K/k the group $\mathrm{Pic}\,X_K$ is a torsionfree finitely generated abelian group.

(ii) If X is k-rational, then for any finite Galois extension K/k with group g, the g-module $\mathrm{Pic}\,X_K$ is stably a permutation module, thus $H^1(h, \mathrm{Pic}\,X_K) = 0$ for any closed subgroup of g. Such statements go back to work of Shafarevich, Manin, Voskresenskiĭ.

3. Units and the Picard group of products

Proposition 3.1. *Let k be an arbitrary field.*

(i) Let X/k be a geometrically integral k-variety. Then the quotient $k[X]^*/k^*$ is a torsionfree abelian group of finite type.

(ii) Let X/k and Y/k be two smooth geometrically integral k-varieties. Then the natural map $k[X]^*/k^* \oplus k[Y]^*/k^* \to k[X \times_k Y]^*/k^*$ is an isomorphism.

(iii) If k is algebraically closed and X/k is a smooth integral variety, then for any field extension K/k the natural map $k[X]^*/k^* \to K[X]^*/K^*$ is an isomorphism.

(iv) Let X/k and Y/k be two smooth geometrically integral k-varieties. Assume that Y is k-birational to affine space. Then the natural map

$$\operatorname{Pic} X \oplus \operatorname{Pic} Y \to \operatorname{Pic} X \times_k Y$$

is an isomorphism.

(v) Let X/k be a smooth geometrically integral variety. Assume that X is k-rational and that k is algebraically closed. Then for any field extension K/k the natural map $\operatorname{Pic} X \to \operatorname{Pic} X_K$ is an isomorphism.

Proof. (i) To prove the result, one may replace X by a normal affine open set. One then takes an integral projective model $X \subset Z$ and then one replaces Z by its normalisation. This is a proper, normal, geometrically integral variety. It is actually a projective variety. There are only finitely many points of codimension 1 in $Z \setminus X$, let the associated valuation be $v_1, \ldots, v_r$. One then has the exact sequence

$$1 \to k[Z]^* \to k[X]^* \to \oplus_{i=1}^r \mathbb{Z}$$

and $k^* = k[Z]^*$. This proves (i). (Note that the result is false for nonreduced schemes such as $\operatorname{Spec} k[t, \varepsilon]$ with t a variable and $\varepsilon^2 = 0$.)

(ii) Let us first assume k separably closed. That the map $k[X]^*/k^* \oplus k[Y]^*/k^* \to k[X \times_k Y]^*/k^*$ is injective is essentially obvious. Let $f(x,y) \in k[X \times Y]^*$. Since X/k is smooth the k-points are Zariski dense. Let us pick $(a,b) \in X(k) \times Y(k)$. Let $F(x,y) = f(a,b)^{-1}.f(x,b).f(a,y)$. The quotient $f(x,y)/F(x,y)$ takes the value 1 on $a \times Y$ and on $X \times b$. We want to show that the quotient is equal to 1.

Let X_c resp. Y_c be normal projective compactifications of X resp. Y (see above). The divisor Δ of the rational function F/f on $Z_c = X_c \times Y_c$ is supported in the union of $(X_c \setminus X) \times Y_c$ and $X_c \times (Y_c \setminus Y)$. Assume for instance that f/F has a zero along an irreducible divisor of the shape $D \times Y_c$ for D a (Weil) divisor of X. Let U be the complement in $X_c \times Y_c$ of the union of all the other components of Δ. This open set contains $X \times Y$ and it meets $D \times b$. The function f/F is a regular function on U which vanishes on the trace of $D \times Y_c$ on U. In particular the restriction of f/F on $U \cap (D \times b)$ vanishes. But this restriction is equal to 1. Thus f/F has no zero of this shape. The same proof, applies to F/f, shows that f/F has no pole along an irreducible divisor of the shape $D \times Y_c$. And the same argument applies to divisors of the shape $X_c \times D$ with D supported in $Y_c \setminus Y$. Evaluation at (a,b) shows that $f(x,y) = F(x,y)$, which proves the claim.

It remains to handle the case of an arbitrary ground field. Let k_s be a separable closure of k and $g = Gal(k_s/k)$. Hilbert's theorem 90 applied to the exact sequence of g-modules

$$1 \to k_s^* \to k_s[X]^* \to k_s[X]^*/k_s^* \to 1$$

gives $k[X]^*/k^* \simeq [k_s[X]^*/k_s^*]^g$. Applying this to X, Y and $X \times_k Y$ yields the result.

(iii) Since any field K/k is a union of fields of finite type over k it is enough to prove the statement for K the function field of a smooth k-variety Y. The group $K[X]^*$ is the union of the groups $k[U \times X]^*$ for U running through the nonempty open sets of Y. For $V \subset U$ open sets of Y the restriction map $k[U\times X]^*/k[U]^* \to k[V\times X]^*/k[V]^*$ is an injection. The quotient $K[X]^*/K^*$ is the direct limit of these quotients. The statement now follows from (ii).

Statement (iii) is wrong if one does not assume k algebraically closed, the quotient $k[X]^*/k^*$ may increase under finite field extensions K/k.

(iv) One proves that the statement is invariant under restriction of X or Y to a nonempty Zariski open set. One is then reduced to the statement $\mathrm{Pic}\, A \simeq \mathrm{Pic}\, A[t]$ for A a regular ring. This statement is reduced to the statement that $\mathrm{Pic}\, K[T] = 0$ for K a field (the fraction field of A).

(v) One argues just as for (iii). The field K is a union of smooth k-algebras of finite type over k. For any such algebra A statement (iv) gives an isomorphism

$$\mathrm{Pic}\, X \oplus \mathrm{Pic}\, A \simeq \mathrm{Pic}\,(X \times_k A).$$

If one passes over to the limit over all such A's, one gets

$$\mathrm{Pic}\, X \simeq \mathrm{Pic}\, X_K.$$

□

Exercise. Investigate to which extent the previous proposition extends to geometrically normal k-varieties.

We shall be interested in more general situations than products.

Proposition 3.2. *Let k be a field, $f : X \to Y$ be a faithfully flat k-morphism of smooth integral k-varieties. Assume all fibres of f are geometrically integral. Let $K = k(X)$ be the function field of Y. Let $Z = X \times_Y K$ be the generic fibre. There is a natural exact sequence*

$$0 \to k[Y]^*/k^* \to k[X]^*/k^* \to K[Z]^*/K^* \to \mathrm{Pic}\, Y \to \mathrm{Pic}\, X \to \mathrm{Pic}\, Z \to 0.$$

Proof. Exercise. □

Proposition 3.3. *Assume k is algebraically closed. With notation as in the previous proposition, assume there exists a smooth integral, k-rational variety W/k such that $Z = W \times_k K$ (which is to say that there exists a nonempty open set $U \subset Y$ such that $X \times_Y U \simeq W \times_k U$ as varieties fibred over $U \subset Y$). Then there is a natural exact sequence*

$$0 \to k[Y]^*/k^* \to k[X]^*/k^* \to k[W]^*/k^* \to \mathrm{Pic}\, Y \to \mathrm{Pic}\, X \to \mathrm{Pic}\, W \to 0.$$

Proof. Given the previous proposition, it is enough to assume that $X = Y \times_k W$ and to show that the natural maps $k[W]/k^* \simeq K[W]^*/K^*$ and $\mathrm{Pic}\, W \simeq \mathrm{Pic}\, W_K$ are isomorphisms. These statements have been proven above. □

References for these results: Rosenlicht, Fossum-Iversen [19], Iversen [23], the papers in [30], [10].

There are further results on the Picard group of fibrations which are not covered by the above results, because they handle fibrations which in general are not locally trivial for the Zariski topology.

Proposition 3.4. *(Sansuc) Let k be a field. Let H/k be a connected linear algebraic group, assumed reductive if $char(k) > 0$. Given a torsor X over a smooth integral k-variety Y with group H there is a natural exact sequence of abelian groups*

$$1 \to k[Y]^*/k^* \to k[X]^*/k^* \to \hat{H}(k) \to \mathrm{Pic}\, Y \to \mathrm{Pic}\, X \to \mathrm{Pic}\, H \to \mathrm{Br}\, Y \to \mathrm{Br}\, X.$$

Here $\hat{H}(k)$ is the group of characters of H defined over k.

Proof. See [33], Prop. 6.10. □

The proof is rather abstract. It would be nice to have a concrete description of the map $\mathrm{Pic}\, H \to \mathrm{Br}\, Y$.

Proposition 3.5. *(Sansuc) Let k be a field. Let $1 \to G_1 \to G_2 \to G_3 \to 1$ be an exact sequence of connected linear algebraic groups, assumed reductive if $char(k) > 0$. The natural maps give rise to a natural exact sequence of Galois modules*

$$0 \to \hat{G}_3(k) \to \hat{G}_2(k) \to \hat{G}_1(k) \to \mathrm{Pic}\, G_3 \to \mathrm{Pic}\, G_2 \to \mathrm{Pic}\, G_1.$$

If $k = \overline{k}$ then the last map is onto.

Proof. Sansuc [33], Corollaire 6.11 et Remarque 6.11.3. □

Proposition 3.6. *(Knop–Kraft–Vust) Let k be a field of char. zero, G/k a connected linear algebraic group and $H \subset G$ a not necessarily connected closed subgroup. The natural maps give rise to a natural exact sequence*

$$\hat{G}(k) \to \hat{H}(k) \to \mathrm{Pic}\, G/H \to \mathrm{Pic}\, G.$$

Proof. See [29], Prop. 3.2. The ideas in the proof bear some analogy with the arguments developed in section 4.4 hereafter. □

Remark 3.7. When H is connected, the above result is covered by Sansuc's general result on torsors. The case when H is a finite central group of G will be considered later in these notes. But the case where H is not connected and not central is not covered by these other methods.

4. Units, Picard groups and isogenies for linear algebraic groups

4.1. **Units of a connected linear algebraic group.**

Proposition 4.1. *Let G/k be a smooth connected linear algebraic group.*

(i) The character group $\hat{G} = Hom_{k-gp}(G, \mathbb{G}_{m,k})$ is a torsionfree abelian group of finite type.

(ii) We have $k[G]^* = k^*.\hat{G}$: any invertible function which takes the value 1 at the neutral element is a character.

Proof. It is enough to prove that any $f \in k[G]^*$ which takes the value 1 at the neutral element $e \in G(k)$ is a character. By (ii) of proposition 3.1, the function $f(g_1.g_2)$ on $G \times_k G$ may be written as $h_1(g_1).h_2(g_2)$ with $h_1 \in k[G]^*$ and $h_2 \in k[G]^*$. Evaluating at $g_2 = e$ yields $h_1 = f$, evaluating at $g_1 = e$ yields $h_1 = f$. □

4.2. **A consequence of Bruhat decomposition.**

Proposition 4.2. *Let G be a connected linear algebraic group over a separably closed field k. If $char(k) > 0$, assume G is reductive (hence smooth).*

(i) Then there exists a nonempty open set V of G which is isomorphic to a product of copies of $\mathbb{G}_{a,k}$'s and $\mathbb{G}_{m,k}$'s.

(ii) The function field $k(G)$ of the k-variety G is purely transcendental.

(iii) The Picard group $\mathrm{Pic}\, G$ is finitely generated.

Proof. (i) Let U be the unipotent radical of G and let G^{red} denote the quotient of G by U. The group U can be presented as a sucession of extensions by $\mathbb{G}_a$. This implies that the k-variety G is k-isomorphic to $U \times_k G^{red}$ and that the k-variety U is k-isomorphic to a product of copies of $G_{a,k}$. One is thus reduced to the case where G is reductive. If one fixes a maximal torus T of G, there is an associated system of roots. If one fixes an ordering, this determines a system of positive roots and a system of negative roots, then an associated Borel subgroup B_+ of G, its unipotent radical U_+, the opposite Borel subgroup B_-, its unipotent radical U_-. There is an open set V of G, called the big cell, which is isomorphic to the product $U_+ \times T \times U_-$, the map $U_+ \times T \times U_- \to G$ being the obvious product. The group U_+ is isomorphic to a product of unipotent groups G_a associated to the positive roots, similarly for the group U_- with the negative roots. For more details, see [1], Cor. 14.14, see also [38], chapter 8, Cor. 8.3.11. This is connected with the topics discussed in Gille's lecture 3 (Bruhat decomposition). The big cell in G is Bw_0B where w_0 is an element of the Weyl group (the largest element). For instance, for SL_2 it is $SL_2 \setminus B$, where B denotes the group of upper triangular matrices.

Assertions (ii) and (iii) immediately follow. □

Remark 4.3. Let G be semisimple. The fibration $G \to G/B$ is locally trivial for the Zariski topology. By a proposition seen earlier this leads to an exact sequence

$$0 \to \hat{B} \to \operatorname{Pic} G/B \to \operatorname{Pic} G \to 0,$$

which may be rewitten as

$$0 \to \hat{T} \to \operatorname{Pic} G/B \to \operatorname{Pic} G \to 0.$$

The map of lattices $\hat{T} \to \operatorname{Pic} G/B$ is the characteristic map. The dimension of T is called the rank of G. If one goes more closely into the combinatorial description of the cells, one shows that the rank of $\operatorname{Pic} G/B$ is also r, from which follows that $\operatorname{Pic} G$ is a finite group. We shall see another proof of this finiteness statement below. Suppose G almost simple. Let R be the root system associated to B. The lattice $\operatorname{Pic} G/B$ is actually the lattice $P(R)$ of weights of the group G with respect to B. The weight lattice contains the root lattice $Q(R)$, one has inclusions $Q(R) \subset \hat{T} \subset P(R)$. One shows that $\hat{T} = P(R)$ if and only if G is simply connected, and that $Q(R) = \hat{T}$ if and only if G is adjoint.

(In further developments of these notes, various statements in these notes should receive complements in the light of the present remark.)

Remark 4.4. Over a field k which is not algebraically closed, a reductive k-group is k-unirational (as may be proved by using the generic torus over the k-rational variety of maximal tori), but it need not be k-rational (Chevalley). We shall come back to such problems.

Over a nonperfect field k a k-group may have only finitely many k-points, in which case it is certainly not k-unirational. Example:

$$x^p - x - ty^p = 0$$

over $\mathbb{F}_p(t)$ where $\mathbb{F}_p$ is the finite field with p elements.

4.3. **Commutativity for π_1.**

Proposition 4.5. *Assume $char(k) = 0$. Let G/k be a connected algebraic group and let $N \subset G$ be a finite normal k-group. Then N is central in G, and in particular it is a finite diagonalisable group.*

Proof. We may over to an algebraic closure of k. The conjugation action defines a k-algebraic homomorphism $G \to Aut_k(N)$. Since the characteristic of k is zero, the last group is an abstract finite group. Since the group G is connected, this map is trivial. □

Remark 4.6. In this proposition the group G need not be linear: it could be an extension of an abelian variety by a connected linear algebraic group.

A similar result is

Proposition 4.7. *Let G/k be a connected algebraic group and let $M \subset G$ be a normal k-group of multiplicative type. Then M is central in G.*

Proof. The proof is similar, the group of automorphisms of such an M is a discrete group. Indeed in view of the well-known duality between groups of multiplicative type and finitely generated Galois modules of finite type, this reduces to the statement that the group of automorphism of an abelian group of finite type is a discrete group. For M smooth over k, for instance if $char(k) = 0$, one could also argue as follows. One may assume that k is algebraically closed. For any positive integer n, the group of n-torsion points in S is a finite group, hence central in G by the previous result. But the union of all n-torsion points is Zariski dense in S, thus S is central. □

There is actually a result more general than Prop. 4.5.

Theorem 4.8. *(Miyanishi) Let G be a connected linear algebraic group over an algebraically closed field k. If $char(k) > 0$ assume G reductive. Let $X \to G$ be a connected, étale Galois cover, with group g of order prime to the characteristic exponent of k. Then the group g is abelian.*

Remark 4.9. Over a field k of characteristic zero one may reduce the statement to the case $k = \mathbb{C}$. Since Grothendieck's fundamental group of $G/\mathbb{C}$ is the profinite completion of the topological fundamental group of $G(\mathbb{C})$, and since $G(\mathbb{C})$ is connected as soon as the $\mathbb{C}$-variety G is connected, the result follows from the classical statement: the (topological) fundamental group π_1 of a connected Lie group is commutative. Note that this proof holds for any connected algebraic group, linear or not.

Remark 4.10. Let $\mathbb{F}_q$ be a finite field with q elements. On any $\mathbb{F}_q$-scheme X, raising coordinates to the power q yields an $\mathbb{F}_q$-morphism $\varphi_q : X \to X$, the Frobenius morphism. Let G be a connected algebraic group over a finite field $\mathbb{F}_q$. The Frobenius map $\varphi_q : G \to G$ is an $\mathbb{F}_q$-homomorphism. The morphism τ given by $x \mapsto \varphi_q(x).x^{-1}$ was considered by Lang, It is some generalisation of the Artin-Schreier map. It is not in general a homomorphism (check on SL_2). It makes G into a torsor over G under the finite split group $G(\mathbb{F}_q)$. If we go over to an algebraic closure k of $\mathbb{F}_q$, this provides a finite, connected étale, Galois cover of $G \times_{\mathbb{F}_q} k$ with group $G(\mathbb{F}_q)$. Together with Miyanishi's theorem this leads to a hilarious proof of the following statement:

Ldet $G/\mathbb{F}_q$ be a connected linear algebraic group. Assume that $G(\mathbb{F}_q)$ is not commutative, then either it is a p-group or its p-Sylow subgroups are not normal, in other words there is more than one such subgroup. In particular p divides the order of $G(\mathbb{F}_q)$.

Remark 4.11. Over an abelian variety, the analogue of Miyanishi's theorem is a theorem of Lang and Serre. I do not know if an algebraic proof for the case of arbitrary connected algebraic groups is available in the literature.

From now on by an isogeny of connected linear algebraic groups over a field k of characteristic zero we shall mean a faithfully flat k-homomorphism $G_1 \to G_2$ with (automatically) central finite kernel.

4.4. Torsors over a connected linear algebraic group with structural group a group of multiplicative type.

Theorem 4.12. *Let k be a field. Let G/k be a connected linear algebraic group, assumed reductive if $char(k) > 0$. Let S be a smooth k-group of multiplicative type. Let $p : H \to G$ be a torsor over G under S. Assume there exists a k-point $e_H \in H(k)$ above the unit element $e_G \in G(k)$. There then exists a structure of linear algebraic group on H with neutral element e_H, such that the map $p : H \to G$ is a homomorphism of linear algebraic groups with central kernel S.*

Proof. We shall give the proof of [8], Thm. 5.6, which is adapted from an argument of Serre in [34] (over an algebraically closed field, see also [28], Lemma 4.3, and Miyanishi [31]).

Recall that torsors over a k-variety X under S are classified by the étale cohomology group $H^1_{et}(X, S)$.

Given a torsor Y over X under the commutative group S, any endomorphism of Y as a torsor over X is an automorphism given by an element of the group $S(X) = Mor_k(X, S)$ (proof by faithfully flat descent). Note that the structural map $X \to Speck$ gives rise to an embedding of groups $S(k) \subset S(X)$.

We shall be interested in k-varieties equipped with a marked k-rational point x_0. We then consider the subgroup $\tilde{H}^1_{et}(X, S) \subset H^1_{et}(X, S)$ consisting of classes trivial at the point x_0: this classifies torsors over X under S whose fibre at the point x_0 is in the trivial class, i.e. has a k-point (but the k-point is not fixed, the set of such points is then a nonempty principal homogeneous space under $S(k)$).

For $S = \mathbb{G}_m$, one has $\operatorname{Pic} X = H^1_{et}(X, \mathbb{G}_m)$ (this is one form of Grothendieck's Hilbert's theorem 90). Let k_s be a separable closure of k.

Combining Prop. 3.1 (ii) and (iii) over k_s, using the Kummer sequence and using some Galois cohomology, one checks that Prop. 3.1. (iv) extends to the following statement: *if X and Y are two smooth varieties with a k-point and one of the varieties becomes rational over a separable closure of k then*

$$\tilde{H}^1_{et}(X, S) \oplus \tilde{H}^1_{et}(Y, S) \simeq \tilde{H}^1_{et}(X \times Y, S).$$

That is to say: any S-torsor over $X \times_k Y$ trivial at $x_0 \times y_0$ is isomorphic (as S-torsor) to the sum (as S-torsors) of its restriction to $X = X \times y_0$ and its restriction to $Y = x_0 \times Y$.

Given our assumptions on G, we know that G becomes rational over a separable closure of k. Let us consider multiplication on the rational variety G.

From the above isomorphism applied to $X = G$ and $Y = G$, one concludes that the inverse image of the torsor $H \to G$ by the multiplication map $m : G \times_k G \to G$ is an S-torsor over $G \times_k G$ which is *isomorphic as an S-torsor* to the "sum" of the S torsors $H \to G$ under S, gotten by restriction to $e_G \times G$ and to $G \times e_G$, each of them naturally equal to the original $H \to G$. This is sometimes expressed by saying that any class in $\tilde{H}^1_{et}(G, S)$ is "primitive".

There thus exists a commutative diagram of morphisms of varieties

$$\begin{array}{ccccccc} \varphi & : & H & \times & H & \to & H \\ & & \downarrow p & & \downarrow p & & \downarrow p \\ m & : & G & \times & G & \to & G, \end{array} \tag{4.1}$$

where m denotes multiplication on G and φ is compatible with multiplication $S \times S \to S$, that is

$$\varphi(t_1 h_1, t_2 h_2) = t_1 t_2 \varphi(h_1, h_2).$$

Any vertical morphism $H \to G$ in this diagram may be modified by composing with a morphism $H \to H$ given by $h \to f(h).h$, where f is the composition of projection $H \to G$ with a morphism of K-varieties $G \to S$. Indeed such an operation does not change the class of the S-torsor $H \to G$, and it respects the map p. The only thing it does is that it changes the morphism φ.

We need not have $\varphi(e_H, e_H) = e_H \in H(k)$. But we may force this (for a new φ) by using multiplication by $\varphi(e_H, e_H)^{-1} \in S(k) \subset S(G)$ on the right hand side H.

The morphism $\varphi(e_H, h) : H \to H$ is now a morphism of pointed S-torsors over G. Thus $\varphi(e_H, h) = \chi_2(p(h)).h$, where $p : H \to G$ denotes the natural projection and $\chi_2 : G \to S$ is a morphism which sends e_G to $e_S \in S(k)$, hence is a character (Rosenlicht). In a similar fashion, we have $\varphi(h, e_H) = \chi_1(p(h)).h$, with $\chi_1 : G \to S$ a character.

If we replace $\varphi(h_1, h_2)$ by

$$[(\chi_1(p(h_1)))^{-1}.(\chi_2(p(h_2)))^{-1}].\varphi(h_1, h_2),$$

this gives a new morphism $\varphi = H \times H \to H$ which satisfies $\varphi(e_H, h) = h = \varphi(h, e_H)$. In other words, e_H is now a neutral element for φ. This morphism still gives rise to a commutative diagram as above, that is projection $H \to G$ is compatible with $\varphi : H \times H \to H$, with the action of $S \times S \to S$ and it is compatible (under projection) with multiplication $G \times G \to G$.

Let us prove that the map φ is associative.

Let $h_1, h_2, h_3 \in H$. If we use the above diagram, the associativity of multiplication for G and we use Rosenlicht's lemma, we see that there exists a morphism of groups $\sigma : H \times_k H \times_k H \to S$ such that

$$\varphi(h_1, \varphi(h_2, h_3)) = \sigma(h_1, h_2, h_3).\varphi(\varphi(h_1, h_2), h_3) \in H.$$

Now φ is compatible with the action of S. Thus $\sigma(t_1 h_1, t_2 h_2, t_3 h_3) = \sigma(h_1, h_2, h_3)$. Hence σ mods out by the action $S \times_k S \times_k S$, and it is induced by a morphism of groups $\tau : G \times_k G \times_k G \to S$. Rosenlicht's lemma then insures that $\tau(g_1, g_2, g_3) = \chi_1(g_1).\chi_2(g_2).\chi_3(g_3)$, where the $\chi_i : G \to S$ are characters.

From $\varphi(e_H, h) = h = \varphi(h, e_H)$, we get $\tau(e_G, g_2, g_3) = 1 \in S$. This proves $\chi_2 = 1$ and $\chi_3 = 1$. By symmetry, we get $\chi_1 = 1$. Thus φ is associative.

It remains to show that this law has an inverse. We adapt an argument of Serre ([Se59], Chap. VII, §3, no. 15, Théorème 5, p. 183). Let $i_G : G \to G$ be the morphism defined by $g \mapsto g^{-1}$. We first show that the map $i_G^* : H^1(G, S) \to H^1(G, S)$ is the map $x \to -x$. This is not formal, for G an elliptic curve this does not hold in general. Let $\alpha \in H^1(G, S)$ be the class of the torsor $p : H \to G$. If we compose the map $G \to G \times G$ given by $u \mapsto (u, u^{-1})$ with multiplication $m : G \times G \to G$ we get the constant map $G \to e_G$.

Since any class $\alpha \in \tilde{H}^1(G \times_k G, S)$ is equal to $p_1^*(\alpha_1) + p_2^*(\alpha_2)$, where α_1 is the restriction to to $G = G \times e_G$ and α_2 is the restriction to $G = e_G \times G$, and p_1, resp. p_2 is the projection of $G \times G$ onto the first, resp. the second factor, we get $i_G^*(\alpha) = -\alpha$.

The isomorphism of commutative groups $i_S : S \to S$ given by $s \mapsto s^{-1}$ induces a map $(i_S)_* : H^1(G, S) \to H^1(G, S)$ which is clearly $x \mapsto -x$. Thus $i_G^*(\alpha) = i_{S*}(\alpha) \in H^1(G, S)$. This gives a morphism $\theta : H \to H$ above the inverse map $G \to G$, and for this map we have $\theta(s.h) = s^{-1}.\theta(h)$. As at the beginning of the argument, we may modifiy θ so that it sends e_H to e_H. The morphism $\lambda : x \mapsto \varphi(\theta(x), x)$ sends H to the inverse image of e_G under $H \to G$, i.e. into S. It satisfies $\lambda(s.x) = \lambda(x)$. It is thus a pointed morphism from G to S, i.e. a character $\chi : G \to S$. Any character $\chi : G \to S$ defines an isomorphism of S-torsors $H \to H$ under $x \to \chi(p(x)).x$. Let us replace the pointed isomorphism (of varieties) θ by the map $\xi : H \to H$ defined by the pointed isomorphism (of varieties) $\xi(x) = \chi(p(x))^{-1}.\theta(x)$. We then have $\varphi(\xi(x), x) = e_H$. This defines a left inverse for the composition law φ on H. As is well known, this is then also a right inverse for φ, which is already associative and has a neutral element.

That the morphism $p : H \to G$ is a homomorphism follows from the above diagram.

The group H is now an extension of the connected linear algebraic group G by the k-group S. It is thus a linear algebraic group.

The conjugation action of H on S mods out to define an action of the connected group G on S. Any such action is trivial. Thus S is central in H. □

Remark 4.13. There is an analogue of the above result for abelian varieties. But here one has to restrict attention to certain classes. Typically for an elliptic curve E and $S = \mathbb{G}_m$ one would have to restrict attention to elements of $H^1(E, \mathbb{G}_m) = \operatorname{Pic} E$ of degree 0. For an abelian variety A, one would restrict attention to $\operatorname{Pic}^0 A \subset \operatorname{Pic} A$.

4.5. **The Picard group and central extensions of a connected linear algebraic group.**

4.5.1. *Algebraically closed ground field.*

Proposition 4.14. *Let k be an algebraically closed field of characteristic zeo. Let G/k be a connected linear algebraic group. The group* $\operatorname{Pic} G$ *is finite.*

Proof. Let L be a line bundle on G, which we view as a $\mathbb{G}_m$-torsor over $H \to G$ under $\mathbb{G}_{m,k}$. By Hilbert's theorem 90 the fibre above the point e has a rational point. By the previous theorem we may equip the k-variety H with the structure of a (connected) linear algebraic group, in such a manner that the map $H \to G$ becomes a group morphism with central kernel $\mathbb{G}_{m,k}$. The group $\mathbb{G}_{m,k}$ lies in the centre of H. Let $H \subset GL(V)$ be a faithful representation. The vector space V breaks up as a sum of vector spaces $W_\alpha \subset V$ upon which $\mathbb{G}_m$ acts by different characters and which are respected by the action of H. At least one of them has a nontrivial action of $\mathbb{G}_m$. Let us choose one such W. The composite map $\mathbb{G}_{m,k} \to H \to GL(W) \to \mathbb{G}_m$, where the last map is the determinant map, is nontrivial. Thus the inverse image of $SL(W)$ in H is a group H_1 which does not contain $\mathbb{G}_{m,k} \subset H$ hence has a finite intersection with $\mathbb{G}_{m,k}$. The image of the composite map $H_1 \to H \to G$, by dimension reason, is G. One then has a commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & A & \to & H_1 & \to & G & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \mathbb{G}_{m,k} & \to & H & \to & G & \to & 1 \end{array} \tag{4.2}$$

where A is a finite commutative k-group of multiplicative type. Thus the pull-back of the $\mathbb{G}_m$-torsor $H \to G$ to the finite cover $H_1 \to G$ has a section. Since the kernel of the map

$$\operatorname{Pic} G \to \operatorname{Pic} H_1$$

is killed by the degree of the map $H_1 \to G$, as it would be for any finite flat map, this shows that $\operatorname{Pic} G$ is a torsion group. (One could also invoke Prop. 3.6.)

Note that if we replace H_1 by its neutral component H_2 we find a central isogeny of connected groups $H_2 \to G$ such that the inverse image of L in $\operatorname{Pic} H_2$ vanishes.

As we have shown above, the group $\mathrm{Pic}\, G$ is a group of finite type. Thus $\mathrm{Pic}\, G$ is a finite group. □

Proposition 4.15. *Let k be an algebraically closed field of characteristic zero. Let G be a connected linear algebraic group over k.*

(i) Given an exact sequence

$$1 \to M \to G_1 \to G \to 1$$

with $M \subset G_1$ a (central) subgroup of multiplicative type, there is a natural exact sequence

$$0 \to \hat{G} \to \hat{G}_1 \to \hat{M} \to \mathrm{Pic}\, G \to \mathrm{Pic}\, G_1 \to 0.$$

In this sequence the map $\hat{M} \to \mathrm{Pic}\, G$ associates to a character $\chi : M \to \mathbb{G}_m$ the class of the $\mathbb{G}_m$-bundle $G_1 \times^M \mathbb{G}_m$, quotient of the product $G \times_k \mathbb{G}_m$ by the diagonal action of M.

(ii) There exists an exact sequence

$$1 \to \mu \to G_1 \to G \to 1$$

with $\mu \subset G_1$ a finite central subgroup (of multiplicative type) and with G_1 connected satisfying $\mathrm{Pic}\, G_1 = 0$.

Proof. Let $f : G_1 \to G$. If $B_1 \subset G_1$ is a Borel subgroup then $B = f(B_1)$ is a Borel subgroup of G (see [1] 11.14). Conversely, if $B \subset G$ is a Borel subgroup of B then $B_1 = f^{-1}(B)$ is a Borel subgroup of G_1. This uses the fact that the kernel M is in the centre of G_1. Let us fix B and B_1 as above. Let $T \subset B$ and $T_1 \subset B_1$ be the maximal tori. There is an induced exact sequence of groups of multiplicative type

$$1 \to M \to T_1 \to T \to 1.$$

(See again [1] 11.14). The fibration $G \to G/B$ is locally trivial (Gille 3, Thm. 4.1), so is the fibration $G_1 \to G_1/B_1$. Projection induces a natural isomorphism $G_1/B_1 \simeq G/B$. We have $\mathrm{Pic}\, B = 0$. The character group of B coincides with the character group of T. Using results from section 3 (Prop. 3.3) together with Rosenlicht's characterisation of units of connected linear algebraic groups we deduce the commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
0 & \to & \hat{G} & \to & \hat{T} & \to & \mathrm{Pic}\, G/B & \to & \mathrm{Pic}\, G & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \simeq & & \downarrow & & \\
0 & \to & \hat{G}_1 & \to & \hat{T}_1 & \to & \mathrm{Pic}\, G_1/B_1 & \to & \mathrm{Pic}\, G_1 & \to & 0 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & \hat{M} & & 0 & & & & \\
 & & & & \downarrow & & & & & & \\
 & & & & 0 & & & & & &
\end{array} \qquad (4.3)$$

The snake lemma then yields the exact sequence in statement (i).

Let $\chi : M \to \mathbb{G}_m$ be a character. There exists a character $\chi_1 : B_1 \to \mathbb{G}_m$ which induces χ on M. The composite map

$$G_1 \times^M \mathbb{G}_m \to G_1 \times^{B_1} \mathbb{G}_m \to G_1/B_1 \to G/B$$

coincides with the composite map

$$G_1 \times^M \mathbb{G}_m \to G \to G/B.$$

This implies that the class of $\mathbb{G}_m$-torsor $G_1 \times^M \mathbb{G}_m \to G$ over G is isomorphic to the class of the $\mathbb{G}_m$-torsor obtained by chasing through the diagram. This completes the proof of (i).

We know that the Picard group of G is finite. Suppose there exists an element z of prime order p in $\operatorname{Pic} G$. There exists a μ_p-torsor over G whose image in $\operatorname{Pic} G$ is the given nonzero class z (Kummer exact sequence in étale cohomology). The total space of this torsor is connected. Indeed if it were not, there would exist two multisections of coprime order and z would be zero. By the theorem discussed at the beginning of the subsection one may equip the k-variety X with the structure of an algebraic group in such a way that the map $X \to G$ becomes a homomorphism of algebraic groups with (central) kernel μ_p. The sequence in (i) yields an exact sequence

$$\hat{\mu}_p \to \operatorname{Pic} G \to \operatorname{Pic} X \to 0.$$

The last sentence of statement (ii) shows that the image of the map $\hat{\mu}_p \to \operatorname{Pic} G$ contains z. Thus the size of $\operatorname{Pic} X$ is smaller than the size of $\operatorname{Pic} G$.

If we iterate this process, we get a homomorphism of connected algebraic groups $G_1 \to G$ with finite kernel, which we know must be central in G_1, and with $\operatorname{Pic} G_1$=0. This proves statement (ii).

Here is an alternative argument which copes with positive characteristic. The group $\operatorname{Pic} G$ is finitely generated. Fix a system of generators. Let z be one of them. Represent it by a $\mathbb{G}_m$-torsor X over G. The same argument as above shows that X, which is clearly connected, may be equipped with the structure of an algebraic group with a homomorphism $X \to G$ with central kernel $\mathbb{G}_m$, and the map $\operatorname{Pic} G \to \operatorname{Pic} X$ kills the generator z. If one uses the fact that an extension of tori is a torus, one produces a homomorphism $G_1 \to G$ with kernel a central torus T. One then chooses a homomorphism $G_1 \to GL(V_1) \oplus \cdots \oplus GL(V_r)$ such that the torus T acts on each V_i by a nontrivial character and the product map of these characters has a finite kernel $\mu \subset T$. One then replaces G_1 by the connected component of 1 in the inverse image $G_2 \subset G_1$ of the product of the $SL(V_i)$. This gives an exact sequence

$$1 \to \mu \to G_2 \to G \to 1$$

(the map $G_2 \to G$ is onto for dimension reasons). From $\operatorname{Pic} G_1 = 0$ we conclude that the map $\operatorname{Pic} G \to \operatorname{Pic} G_2$ is zero. But this map is onto, as seen above. From a previous proposition we conclude $\operatorname{Pic} G_2 = 0$.

□

Remark 4.16. Let

$$1 \to \mu \to G_1 \to G \to 1$$

be an isogeny of connected linear algebraic groups. One then has the exact sequence

$$0 \to \hat{G} \to \hat{G}_1 \to \hat{\mu} \to \operatorname{Pic} G \to \operatorname{Pic} G_1 \to \operatorname{Pic} G \to 0.$$

This should be compared with isogenies of abelian varieties. If one has such an isogeny

$$1 \to \mu \to A_1 \to A \to 1$$

then there is a dual isogeny

$$1 \to \hat{\mu} \to \operatorname{Pic}^0 A \to \operatorname{Pic}^0 A_1 \to 1$$

where $\operatorname{Pic}^0 A$ denotes the dual abelian variety of an abelian variety A. Its k-points coincide with the subgroup of $\operatorname{Pic} A$ consisting of classes algebraically equivalent to zero. Thus one has an exact sequence

$$1 \to \hat{\mu} \to \operatorname{Pic} A \to \operatorname{Pic} A_1$$

but the last map is not onto.

Proposition 4.17. *Let k be an algebraically closed field of characteristic zero. Let G be a reductive group over k. The following properties are equivalent:*

(i) The group G is semisimple, i.e. there is no nontrivial connected abelian normal subgroup of G.

(ii) There is no normal subtorus $\mathbb{G}_m \subset G$.

(iii) The centre of G is a finite group scheme.

(iv) The group G is equal to its derived group.

(v) The group G is characterfree.

(vi) Any invertible function on G is constant.

Proof. The radical of a reductive group is known to be the maximal central torus. Thus (i), (ii) and (iii) are equivalent. Assume (ii). The structure theory of semisimple groups shows that such groups are spanned by images of homomorphisms $SL_2 \to G$ corresponding to the unipotent groups associated to roots. Since SL_2 is its own commutator subgroup, this shows (iv). That (iv) implies (v) is clear. If we have an exact sequence

$$1 \to \mathbb{G}_m \to G \to G_1 \to 1$$

then as seen above there is an exact sequence

$$\hat{G} \to \mathbb{Z} \to \operatorname{Pic} G_1.$$

Since $\operatorname{Pic} G_1$ is torsion (above) this implies $\hat{G} \neq 0$. Thus (v) implies (ii). Equivalence of (v) and (vi) is a consequence of Rosenlicht's result (section 1.2). □

Remark 4.18. The derived group of a reductive group G is reductive. Indeed it is normal in G thus its unipotent radical (= maximal normal connected solvable subgroup) is contained in the unipotent radical of G, hence is trivial. In particular, we see that the derived group of a reductive group is a semisimple group.

Proposition 4.19. *Let k be an algebraically closed field of characteristic zero. For a semisimple group G, the following properties are equivalent:*

(i) There is no nontrivial isogeny $G_1 \to G$.

(ii) $\operatorname{Pic} G = 0$.

(iii) There is no connected finite étale Galois cover $X \to G$.

Such a semisimple group G is said to be *simply connected*.

Proof. If there is a nontrivial element of primer order p in the finite group $\operatorname{Pic} G$ then as we have seen in the previous proof one may construct a nontrivial isogeny with kernel μ_p. Thus (i) implies (ii). Suppose we have a nontrivial isogeny

$$1 \to \mu \to G_1 \to G \to 1.$$

Since G is characterfree the exact sequence considered above gives an injection $\hat{\mu} \hookrightarrow \operatorname{Pic} G$. Thus (ii) implies (i). Miyanishi's result reduces the problem to the case of a connected abelian covering. One then applies Theorem 4.9 to equip the torsor with the structure of an isogeny. □

The following result is often stated without proof.

Proposition 4.20. *Let k be an algebraically closed field of characteristic zero and let G/k be a semisimple group.*

(i) There exists a unique semisimple simply connected group $\tilde{G}$ equipped with a central isogeny

$$1 \to \mu \to \tilde{G} \to G \to 1$$

and the group together with the map $\tilde{G} \to G$ is uniquely defined up to unique automorphism.

(ii) This sequence induces an isomorphism $\hat{\mu} \simeq \operatorname{Pic} G$.

(iii) Given a central isogeny $\tilde{G} \to G$ with $\tilde{G}$ simply connected and any isogeny $H \to G$ there is a unique homomorphism $\tilde{G} \to H$ such that the composite map $\tilde{G} \to H \to G$ is the given isogeny $\tilde{G} \to G$.

Proof. The iterative process described above shows that there exists an isogeny $\tilde{G} \to G$ with $\operatorname{Pic} \tilde{G} = 0$. Statement (ii) has already been established.

If we prove the statement in (iii) this will prove the unicity (up to unique isomorphism) of the cover $\tilde{G} \to G$. As for (ii) we have already established it. Let $H \to G$ be an isogeny with kernel ν. The fibre product $\tilde{G} \times_G H$ is a finite étale cover of $\tilde{G}$ with group ν. By the previous results any such cover completely breaks up as a disjoint union of varieties isomorphic to $\tilde{G}$. Let us choose the component which contains the elements $(1_{\tilde{G}}, 1_H)$. The second projection $\tilde{G} \times_G H \to H$ gives a morphism $\tilde{G} \to H$ which sends $1_{\tilde{G}}$ to 1_H. That this map is a homomorphism follows from the connectedness of $\tilde{G}$ and the fact that the composition $\tilde{G} \to H \to G$ is a homomorphism. □

Proposition 4.21. *Let k be an algebraically closed field of characteristic zero. Let G/k be a reductive group. The quotient of G by its centre is a semisimple group which is centreless. It is called the adjoint group of G.*

Proof. Let Z denote the centre of G. It is known to be a group of multiplicative type. Let G^{ad} denote the quotient of G by Z. We have the exact sequence

$$1 \to Z \to G \to G^{ad} \to 1.$$

If there was a normal $\mathbb{G}_a \subset G^{ad}$ then its inverse image $H \subset G$ would be a normal subgroup which would be a central extension

$$1 \to Z \to H \to \mathbb{G}_a \to 1.$$

Any such extension is split (exercise) as a sequence of groups. Thus there would exist a normal $\mathbb{G}_a$ in G, contradiction. Thus G^{ad} is reductive. Suppose it had a nontrivial centre M. That centre would be a group of multiplicative type. It would contain a μ_p for some prime p. The inverse image H of μ_p in G would be a normal subroup of G which would be a central extension of a μ_p (cyclic) by a group of multiplicative type, hence it would be a commutative group scheme M_1, clearly of multiplicative type, and stricty bigger than M. As a normal subgroup of multiplicative type in the connected group G, it would be central, contradiction. □

Let us show that the group SL_n $(n \geq 2)$ is a semisimple simply connected group.

First consider GL_n. This is an open set of affine space $\mathbb{A}^{n^2}$ given by the equation $det \neq 0$. This immediately implies $\operatorname{Pic} GL_n = 0$. The centre of GL_n is the diagonal $\mathbb{G}_m$. Thus GL_n is a reductive group.

It is a known fact that the determinant det is an irreducible polynomial. This implies that any character of GL_n is of the shape det^n for some $n \in \mathbb{Z}$.

We now look at the exact sequence:

$$1 \to SL_n \to GL_n \to \mathbb{G}_m \to 1,$$

given by the determinant. The homomorphism $GL_n \to \mathbb{G}_m$ admits a retraction: send an element $\lambda \in k^*$ to the diagonal element $(\lambda, 1, \ldots, 1) \in GL_n$. There thus exists an isomorphism of k-varieties $GL_n \simeq SL_n \times \mathbb{G}_m$. It is even an isomorphism of SL_n-torsors over $\mathbb{G}_m$. Note (exercise) that there can be no group isomorphism $GL_n \simeq SL_n \times \mathbb{G}_m$.

From section 3 we deduce $\operatorname{Pic} SL_n \oplus \operatorname{Pic} \mathbb{G}_m \simeq \operatorname{Pic} GL_n = 0$ hence $\operatorname{Pic} SL_n = 0$. We also deduce $\hat{SL_n} \oplus \mathbb{Z} \simeq \hat{GL_n}$, and s the map $\mathbb{Z} \to \hat{GL_n}$ is identity, thus $\hat{SL_n} = 0$.

The group SL_n, which is reductive (as a normal subgroup of a reductive group) is thus semisimple and simply connected.

The following basic commutative diagram of exact sequences of algebraic groups should always be kept in mind:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \mu_n & \to & SL_n & \to & PGL_n & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow = & & \\
1 & \to & \mathbb{G}_m & \to & GL_n & \to & PGL_n & \to & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbb{G}_m & = & \mathbb{G}_m & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array} \tag{4.4}$$

The left exact sequence is the Kummer sequence, the map $\mathbb{G}_m \to \mathbb{G}_m$ is $x \mapsto x^n$. The map $GL_n \to \mathbb{G}_m$ is given by the determinant. Hilbert's theorem 90 ensures that the two middle exact sequences define fibrations which are locally trivial for the Zariski topology.

The centre of SL_n is the diagonal μ_n, and the quotient of SL_n by its centre is the centreless adjoint group PGL_n.

Exercise Show that $SL_{n,\mathbb{C}}$ is not isomorphic to an affine space $\mathbb{A}^r_{\mathbb{C}}$. In the lecture I gave two methods.

(i) If there were such an isomorphism it would exist over a ring of finite type over $\mathbb{Z}$ and then by specializing at a maximal ideal one would get an isomorphism $SL_{n,\mathbb{F}} \simeq A^r_{\mathbb{F}}$ over some finite field $\mathbb{F} = \mathbb{F}_q$. One would then have a bijection between $SL_n(\mathbb{F})$ and $\mathbb{A}^n(\mathbb{F})$. But the order of $SL_n(\mathbb{F})$ is not a power of q. This argument is clearly useful in other contexts.

(ii) I discussed the example of $X = SL_{2,\mathbb{C}}$, that is the equation $ad - bc = 1$. The Bruhat decomposition here is extremely simple. One had an open set $U \subset X$ isomorphic to $\mathbb{A}^2 \times \mathbb{G}_m$ and the complement is a smooth closed set (given by $c \neq 0$) isomorphic to $\mathbb{A}^1 \times \mathbb{G}_m$. Among various possible cohomology theory one can take étale cohomology with coefficients $\mathbb{Z}/n$ and investigate $H^i(X, \mathbb{Z}/n)$ by means of the localization sequence (and the purity theorem). Ignoring twists, part of it is

$$H^2(U, \mathbb{Z}/n) \to H^1(F, \mathbb{Z}/n) \to H^3(X, \mathbb{Z}/n).$$

Now $H^2(U, \mathbb{Z}/n) = H^2(\mathbb{G}_m, \mathbb{Z}/n)$ using homotopy invariance and $H^2(\mathbb{G}_m, \mathbb{Z}/n) = 0$ for any number of reasons (we are over an algebraically closed field). But $H^1(F, \mathbb{Z}/n) = H^1(\mathbb{G}_m, \mathbb{Z}/n)$ by homotopy invariance and $H^1(\mathbb{G}_m, \mathbb{Z}/n) = \mathbb{Z}/n$ as one immediately checks using the Kummer sequence. Thus $H^3(X, \mathbb{Z}/n) \neq 0$. Closer examination shows that the localisation sequence yields $H^1(X, \mathbb{Z}/n) = 0$ and $H^2(X, \mathbb{Z}/n) = 0$. Had we started with PGL_2 and the Bruhat decomposition corresponding to projection $PGL_2 \to \mathbb{P}^1$ we would have had the same argument for $H^3(X, \mathbb{Z}/n)$ but here the maps $H^1(U, \mathbb{Z}/n) \to H^0(F, \mathbb{Z}/n)$ would not have been an isomorphism for n even.

Exercise. Develop the method to handle arbitrary semisimple simply connected groups and get Elie Cartan's result that $\pi_3(G(\mathbb{C})) = \mathbb{Z}$.

(iii) A variant is a result in algebraic K-theory, proved I think by various people (Deligne, Suslin): For G a semisimple simply connected absolutely almost simple split group G over a field, we have $H^1_{Zar}(G, \mathcal{K}_2) = \mathbb{Z}$. That result is a building block in the definition of the Rost invariant $H^1(k, G) \to H^3(k, \mathbb{Q}/\mathbb{Z}(2))$ for G an almost simple semisimple simply connected group over a field k.

(iv) The methods in (i) and (ii) are not totally unrelated ! Indeed one counts the number of rational points on a variety over a finite field $\mathbb{F}$ by computing alternate sums of traces of Frobenius acting on étale cohomology (with compact supports) of the variety over an algebraic closure of $\mathbb{F}$, with coefficients $\mathbb{Q}_l$ for l different from the characteristic.

4.6. **Arbitrary fields of characteristic zero.** Let k be a field of characteristic zero. This hypothesis will remain in force throughout the subsection. Let $\overline{k}$ denote an algebraic closure of k.

Let G/k be a connected linear algebraic group over k.

Over a field k of characteristic zero any algebraic k-group is smooth (Cartier).

One has a series of k-groups associated to the k-group G.

The unipotent radical $R_u(G) \subset G$ is the maximal normal connected solvable k-subgroup of G. One shows that $R_u(G) \times_k \overline{k} = R_u(\overline{G})$.

There is a natural exact sequence

$$1 \to R_u(G) \to G \to G^{red} \to 1.$$

This is the definition of the k-group G^{red} attached to G.

Any $\mathbb{G}_{a,k}$-torsor over an affine k-variety is trivial. This implies the same statement for a U-torsor over an affine k-variety.

Any unipotent group U/k is a successive extension by groups $\mathbb{G}_{a,k}$.

From this we deduce that the underlying k-variety of $R_u(G)$ is k-isomorphic to affine space $\mathbb{A}^m_k$ and that the k-variety G is k-isomorphic to the product of G^{red} and $R_u(G)$, hence to the product of G^{red} and an affine space $\mathbb{A}^m_k$.

By definition an algebraic k-group G is called reductive if $\overline{G}$ is reductive.

By the character group $\hat{G}$ of a k-group G we shall means the Galois module provided by the character group of $\overline{G}$.

By definition an algebraic k-group G is called semisimple if $\overline{G}$ is semisimple.

The group G^{red} is connected. It is reductive (consider the inverse image of a would-be normal $G_a \subset \overline{G}^{red}$ in $\overline{G}$).

Let now G be a reductive k-group. To this group we may associate:

The centre Z. This is the maximal central k-subgroup of multiplicative type in G.

The connected centre Z^0. This is the maximal normal k-torus $T \subset G$.

The derived group $[G, G] \subset G$, it is a semisimple group, denoted G^{ss}.

The quotient of G by $[G, G]$, denoted G^{tor}: this is the maximal quotient of G which is a group of multiplicative type.

The adjoint group G^{ad}, which is the quotient of G by its centre.

The natural map $G^{ss} \times T \to G$ is a k-isogeny.

All these data are stable under ground field extension, in particular by going over from k to $\overline{k}$. We let $g = Gal(\overline{k}/k)$ and $\overline{G} = G \times_k k$.

Proposition 4.22. *Given a semisimple k-group G there is a uniquely defined isogeny $G^{sc} \to G$ from a semisimple simply connected group k-group $G^{sc} \to G$. The kernel of* *this isogeny is the finite k-group μ of multiplicative type whose character group $\hat{\mu}$ is the Galois module given by* $\operatorname{Pic}\overline{G}$.

Proof. (Sketch)

Let $g = Gal(\overline{k}/k)$, and let μ as in the Proposition. we have the spectral sequence

$$E_2^{pq} = H^p(g, H^q_{et}(\mathbb{G}, \mu)) \Longrightarrow H^n_{et}(G, \mu).$$

Taking into account $H^0(\overline{G}, \mu) = \mu(\overline{k})$ and $\hat{\mu} = \operatorname{Pic}\overline{G}$ we get the long exact sequence

$$0 \to H^1(k, \mu) \to H^1(G, \mu) \to (H^1(\overline{G}, \mu))^g \to H^2(k, \mu) \to H^2(G, \mu).$$

The last arrow is an injection since $G(k) \neq \emptyset$. We have $H^1(\overline{G}, \mu) = Hom_{\mathbb{Z}}(\hat{\mu}, \operatorname{Pic}\overline{G})$ as g-modules. In

$$(H^1(\overline{G}, \mu))^g = Hom_g(\hat{\mu}, \operatorname{Pic}\overline{G}) = Hom_g(\operatorname{Pic}\overline{G}, \operatorname{Pic}\overline{G}),$$

we have the identity map, which corresponds to the simply connected cover of $\overline{G}$ (which is uniquely defined). Thus this class comes from an element in $H^1(G, \mu)$, which we may modify by adding an element in $H^1(k, \mu)$ so that its fibre at the point e_G is trivial. Let $H \to G$ be a corresponding torsor under μ. By the work in section 4, there is a group structure on H for which the map $H \to G$ is an isogeny with kernel μ. That the cover $\overline{H} \to \overline{G}$ is the simply connected cover of $\overline{G}$ is clear by going over to $\overline{k}$.

□

Proposition 4.23. *Let G/k be a connected linear algebraic group.*

(i) The group $\operatorname{Pic} G$ *is finite.*

(ii) If G is semisimple, the natural map $\operatorname{Pic} G \to \operatorname{Pic}\overline{G}$ *induces an isomorphism* $\operatorname{Pic} G \simeq (\operatorname{Pic}\overline{G})^g$.

(iii) If G is a k-torus T, the natural map $H^1(g, \overline{k}[G]^*) \to \operatorname{Pic} G$ induces an isomorphism $H^1(g, \hat{T}) \simeq \operatorname{Pic} T$.

Proof. From section 2 we get an exact sequence

$$0 \to H^1(g, \overline{k}[G]^*) \to \operatorname{Pic} G \to (\operatorname{Pic}\overline{G})^g \to H^2(g, \overline{k}[G]^*) \to H^2(g, \overline{k}(G)^*).$$

From the split exact sequence

$$1 \to \overline{k}^* \to \overline{k}[G]^* \to \hat{G} \to 1$$

we deduce $H^1(g, \overline{k}[G]^*) = H^1(g, \hat{G})$. Since $\hat{G}$ is a finitely generated torsion free g-module, the group $H^1(g, \hat{G})$ is finite. We already know that $\operatorname{Pic}\overline{G}$ is finite. Thus $\operatorname{Pic} G$ is finite. If G is semisimple, then $\overline{k}^* = \overline{k}[G]^*$ and since G has a k-point, from section 2 the above sequence yields (ii). As for (iii) it simply follows from $\operatorname{Pic}\overline{T} = 0$ for T a torus. □

5. Flasque resolutions of tori

Let g be a profinite group. Given a g-lattice M we have the dual g-lattice $M^0 = Hom_{\mathbb{Z}}(M, \mathbb{Z})$. The action is given by $(\sigma.f)(m) = f(\sigma^{-1}.m)$. The dual lattice of a permutation lattice is a permutation lattice.

A g-lattice M is called coflasque if $H^1(h, M) = 0$ for any closed subgroup h of g. A g-lattice M is called flasque if M^0 is coflasque.

A permutation lattice is both flasque and coflasque.

Let T be a k-torus and X a smooth k-compactification of T, that is a smooth, projective k-variety X in which T lies a dense open set.

It is known that such (equivariant) compactifications exist ([14]) in arbitrary characteristic. (It is much to be regretted that the same statement cannot be written for an arbitrary reductive k-group, although the experts are convinced it could be put together from the literature.)

Given the open set $T \subset X$ as above we have the basic exact sequence

$$\overline{k}[X]^* \to \overline{k}[T]^* \to Div_\infty \overline{X} \to \operatorname{Pic} \overline{X} \to \operatorname{Pic} \overline{T}$$

reduces to the short exact sequence of g-lattices

$$0 \to \hat{T} \to Div_\infty \overline{X} \to \operatorname{Pic} \overline{X} \to 0.$$

The middle term is a permutation module, it is the finitely generated free abelian group on the codimension 1 points of $\overline{X} \setminus \overline{T}$. Let us call it $\hat{P}$, and let us write $\operatorname{Pic} \overline{X} = \hat{F}$. We thus have the exact sequence of g-lattices

$$0 \to \hat{T} \to \hat{P} \to \hat{S} \to 0$$

and the dual sequence

$$0 \to \hat{S}^0 \to \hat{P}^0 \to \hat{T}^0 \to 0,$$

where $\hat{P}^0$ is a permutation lattice, since $\hat{P}$ is.

This sequence has been much investigated by Voskresenskiĭ.

Theorem 5.1. *(Voskresenskiĭ) Let k be a field, $\overline{k}$ a separable closure, $g = Gal(\overline{k}/k)$. T a k-torus and X a smooth k-compactification of T. Then the g-lattice* $\operatorname{Pic} \overline{X}$ *is flasque.*

Proof. For a k-torus T, the following conditions are equivalent: $\hat{T}^g = 0$ or $(\hat{T}^0)^g = 0$. Such a k-torus is called anisotropic (over k).

(a) Let us first assume that the k-torus T is anisotropic. If we apply Galois cohomology to the second sequence we find an exact sequence

$$(\hat{T}^0)^g \to H^1(g, \hat{S}^0) \to H^1(g, \hat{P}^0)$$

hence $H^1(g, \hat{S}^0) = 0$.

(b) Let T be an arbitrary k-torus. It is split by a finite Galois extension K/k with Galois group G. Let $N = \sum_{s \in G} s$. The image of multiplication by N on $\hat{T}$ is a sublattice of $\hat{T}$ with trivial G-action. Its kernel is a lattice L such that $L^G = 0$. Thus any k-torus T can be put in an exact sequence of k-tori

$$1 \to T_0 \to T \to T_1 \to 1$$

where T_0 is split, i.e. isomorphic as a k-torus to a product of copies of $\mathbb{G}_{m,k}$ and T_1 is anisotropic. Now Hilbert's theorem 90 implies that there is a k-birational equivalence between T and $T_0 \times_k T_1$. Let X_0, resp X_1 be smooth compactifications of T_0 resp. T_1. For X_0 we may take a projective space. The g-lattice $\operatorname{Pic} \overline{X}_0$ is the trivial g-lattice $\mathbb{Z}$. By the results in section 2 and 3 there exists an isomorphism of g-lattices

$$\operatorname{Pic} \overline{X} \oplus L_1 \simeq \operatorname{Pic} \overline{X}_1 \oplus \mathbb{Z} \oplus L_2,$$

where L_1 and L_2 are permutation g-lattices. From the result for the anisotropic torus T_1 we conclude $H^1(g, (\operatorname{Pic} \overline{X})^0) = 0$.

(c) Let now $K \subset \overline{k}$ be a field extension and let $h = Gal(\overline{k}/K)$. We may apply the previous argument to the K-torus T_K. This yields $H^1(h, (\operatorname{Pic} \overline{X})^0) = 0$. □

Recall ([10], [12], Gille's talks):

Proposition 5.2. *Let $0 \to L \to L_1 \to M_1 \to 0$ and $0 \to L \to L_2 \to M_2 \to 0$ be exact sequences of g-lattices. If L_1 and L_2 are permutation lattices and M_1 and M_2 are flasque then there is an isomorphism of g-lattices*

$$L_1 \oplus M_2 \simeq L_2 \oplus M_1.$$

Proposition 5.3. *(Endo-Miyata) Let G be a finite group. Let L be a G-module of finite type. For each subgroup $H \subset G$ choose a free $\mathbb{Z}$-lattice N_H with a surjective map $N_H \to M^H$. The kernel of the surjective map of G-lattices*

$$\oplus_{H \subset G} \mathbb{Z}[G/H] \otimes_{\mathbb{Z}} N_H \to M,$$

where H runs through the subgroups of G and M^H is considered as a G-lattice with trivial action, is a coflasque lattice.

By the duality $L \mapsto L^0$ for G-lattices we get a similar systematic way of constructing a flasque resolution of a given G-lattice M, that is an exact sequence of G-lattices

$$0 \to S \to P \to M \to 0$$

with S flasque and P permutation.

For later use let us also recall another result of Endo and Miyata. Given a G-module of finite type M there exists an exact sequence of G-modules

$$0 \to P \to S \to M \to 0$$

with S a flasque G-lattice and P a permutation lattice.

For all this see [12].

Note also that if G is quotient of g then a flasque resolution of a G-lattice L is a flasque resolution of L as a g-lattice.

Let g be a profinite group. Given a continuous discrete g-module M and $i = 1, 2$, one defines the group $Sha^i_\omega(M)$ as the subgroup of elements of $H^i(g, M)$ which vanish under restriction to arbitrary procyclic subgroups of g. Suppose g acts on M through a finite quotient G. Then $Sha^1_\omega(g, M) = Sha^1_\omega(G, M)$. If moreover M is a finitely generated g-lattice, then $Sha^2_\omega(g, M) = Sha^2_\omega(G, M)$.

Theorem 5.4. *Let k be a field, $\overline{k}/k$ a separable closure, $g = Gal(\overline{k}/k)$.*

(i) For any k-torus T there exists an exact sequence of g-lattices

$$0 \to \hat{T} \to \hat{P} \to \hat{F} \to 0$$

where $\hat{P}$ is a permutation module and $\hat{F}$ a flasque module.

(ii) The class of the g-lattice $\hat{F}$ is well defined up to addition of a permutation lattice. We shall denote it by $p(T)$ or $p(\hat{T})$.

(iii) The class $p(T)$ can be computed by a purely algebraic process.

(iv) If X is a smooth compactification of T, the natural sequence

$$0 \to \hat{T} \to Div_\infty \overline{X} \to \operatorname{Pic} \overline{X} \to 0$$

is a flasque resolution of $\hat{T}$. Thus $\operatorname{Pic} \overline{X}$ is in the class $p(T)$.

(v) Let T_1, T_2 be k-tori. Then $p(T_1) = p(T_2)$ if and only if the k-tori T_1 and T_2 are stably k-rationally equivalent, that is, for some integers r and r, there exists a k-birational equivalence between $T_1 \times_k \mathbb{G}^r_{m,k}$ and $T_2 \times_k \mathbb{G}^s_{m,k}$.

(vi) Let T_1, T_2, T_3 be k-tori. If T_3 is k-birational to $T_1 \times_k T_3$ then $p(T_3) = p(T_1) \oplus p(T_2)$. In other words, the class $p(T)$ is additive on k-birational equivalence classes of k-tori.

(vii) $p(T) = 0$ if and only if T is stably k-rational.

(viii) $p(T)$ is invertible if and only if there exists a k-torus T_1 such that $T \times_k T_1$ is k-birational to affine space.

(ix) Let K/k be the (finite) splitting field of ak-torus T. Let $G = Gal(K/k)$. let X be a smooth compactification of T. The k-birational invariant $H^1(g, \mathrm{Pic}\,\overline{X}) = H^1(g, p(T))$ is isomorphic to $Sha^2_\omega(G, \hat{T})$.

Proof. Points (i) to (iv) have been proved. Points (vi) and (vii) are direct consequences of (v). So is (viii), one one notices that the class of any invertible g-lattice L is of the shape $p(T)$ for some torus T (simply take the kernel of a surjection from a permutation lattice to L, for instance a permutation lattice Lwhich contains L as a direct factor).

So let us prove (v). If $T_1 \times \mathbb{G}_m^r$ is k-birational to $T_2 \times \mathbb{G}_m^r$, then one considers smooth compactifications X_1 of T_1 and X_2 of T_2. One then has a k-birational equivalence of smooth proper k-varieties between $X_1 \times_K \mathbb{P}^r_k$ and $X_2 \times_K \mathbb{P}^s_k$. One then applies Prop. 3.3 and get that the Galois modules $\mathrm{Pic}\,\overline{X}_1$ and $\mathrm{Pic}\,\overline{X}_1$ are equal up to addition of a permutation module. By (iv), this proves $p(T_1) = p(T_2)$.

Conversely, assume $p(T_1) = p(T_2)$. We can then produce a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \hat{T}_1 & \to & \hat{P}_1 & \to & \hat{S} & \to & 0 \\
 & & \uparrow= & & \uparrow & & \uparrow & & \\
0 & \to & \hat{T}_1 & \to & \hat{M} & \to & \hat{P}_2 & \to & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \hat{T}_2 & = & \hat{T}_2 & \to & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{5.1}$$

The g-lattice $\hat{M}$ is the fibre product of $\hat{P}_1$ and $\hat{P}_2$ above $\hat{S}$.

One then goes over to k-tori. One has the commutative diagram of exact sequences of k-tori

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & S & \to & P_1 & \to & T_1 & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow= & & \\
1 & \to & P_2 & \to & M & \to & T_1 & \to & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & T_2 & = & T_2 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array} \tag{5.2}$$

The k-tori P_1 and P_2 are dual of permutation modules. Thus there are quasitrivial tori, i.e. products of k-tori $R_{K/k}\mathbb{G}_{m,K}$ for various finite separable extensions K/k.

Lemma 5.5. *Let P be a quasitrivial k-torus.*

(i) The k-variety P is an open set of affine space over k, hence is k-rational.

(ii) If $Y \to X$ is a torsor over a k-variety under P, then it is Zariski locally trivial.

Proof. The first statement is obvious. Hilbert's theorem 90 together with Shapiro's lemma ensure that for for any field L containing k, and any such k-torus P, one has

$H^1(L, P) = 0$. The second statement follows from a variant of this. We use the fact that the Picard group of a semilocal ring is trivial. □

Thus for any exact sequence of k-tori

$$1 \to P \to M \to T \to 1$$

with P quasitrivial, the fibration $M \to T$ is split over the generic point of T (in fact everywhere locally) and the k-variety M is k-birational to the product $P \times_k T$.

If we apply this to the middle sequences in the above diagram we find that $T_1 \times_k P_2$ is k-birational to M which is k-birational to $T_2 \times_k P_1$. This proves (v).

Let us prove (ix). Consider a flasque resolution, for instance the one given by geometry,

$$0 \to \hat{T} \to \hat{P} \to \hat{S} \to 0.$$

Since $\hat{P}$ is a permutation module, we have $H^1(G, \hat{P}) = 0$. Given a G-lattice L there is a duality between the cohomolgy group $H^1(G, L)$ and the Tate cohomology group $\hat{H}^{-1}(G, L^0)$. For a cyclic group, Tate cohomology is periodic of period 2. From this we deduce that $H^1((g), \hat{S}) = 0$ for any element $g \in G$. One also checks that for a permutation module $\hat{P}$

$$Ker[H^2(G, \hat{P}) \to \prod_{g \in G} H^2(G, \hat{P})] = 0.$$

Indeed one uses Shapiro's lemma to reduce to the case $\hat{P} = \mathbb{Z}$ with trivial G-action and then the statement reduces to the fact that a homomorphism from G to $\mathbb{Q}/\mathbb{Z}$ which is trivial on elements of G is trivial. Applying these facts and cohomology to the flasque resolution yields the result. □

Let us discuss a basic example. Let K/k be a finite Galois extension of fields. Let $G = Gal(K/k)$. Let $T = R^1_{K/k}\mathbb{G}_m$ be the norm 1 torus. By definition we have the exact sequence of k-tori

$$1 \to T \to R_{K/k}\mathbb{G}_m \to \mathbb{G}_{m,k} \to 1.$$

The associated sequence of character groups is the sequence of G-lattices

$$0 \to \mathbb{Z} \to \mathbb{Z}[G] \to J_G \to 0,$$

where the map $\mathbb{Z} \to \mathbb{Z}[G]$ sends 1 to $N_G = \sum_{g \in G} g$. The dual sequence, in the sense of $L \mapsto L^0$, is the sequence

$$0 \to I_G \to \mathbb{Z}[G] \to \mathbb{Z} \to 0,$$

where the map $\mathbb{Z}[G] \to \mathbb{Z}$ is the augmentation map sending $\sum_{g \in G} n_g g \in \mathbb{Z}[G]$ to $\sum_{g \in G} n_g \in \mathbb{Z}$.

There is a natural map of G-lattices $\mathbb{Z}[G \times G] \to \mathbb{Z}[G]$ sending (g, h) to $g - h$. One easily checks that the image is $I_G \subset \mathbb{Z}[G]$. Let Q_G denote the kernel of this map. We thus have a long exact sequence

$$0 \to Q_G \to \mathbb{Z}[G \times G] \to \mathbb{Z}[G] \to \mathbb{Z} \to 0.$$

We may also consider the dual sequence, in the sense of lattices. It reads

$$0 \to \mathbb{Z} \to \mathbb{Z}[G] \to \mathbb{Z}[G \times G] \to S_G \to 0.$$

Proposition 5.6. *With the above notations, we have*
(i) The G-lattice Q_G is coflasque.
(ii)The G-lattice F_G is flasque.
(iii) $p(R^1_{K/k}\mathbb{G}_m) = p(J_G) = [F_G]$.
(iv) For any subgroup $H \subset G$, we have $H^1(H, p(R^1_{K/k}\mathbb{G}_m)) \simeq H^3(H, \mathbb{Z})$.

Proof. At this point these results easily follow from what we have done. See [10], Prop. 1, p. 183. Note that statement (iv) directly follows from the formula

$$H^1(H, p(R^1_{K/k}\mathbb{G}_m)) \simeq Sha^2_\omega(G, J_G)$$

and the exact sequence $0 \to \mathbb{Z} \to \mathbb{Z}[G] \to J_G \to 0$: we do not need to produce the flasque resolution to get this result. □

As a consequence, if K/k is a biquadratic extension, that is (assuming $char(k) \neq 2$) if $K = k(\sqrt{a}, \sqrt{b})$ with a, b, ab nonsquare in k, then $H^1(G, p(R^1_{K/k}\mathbb{G}_m)) = H^3(\mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{Z}) = \mathbb{Z}/2$. Thus this k-birational invariant is not zero and the k-torus $R^1_{K/k}\mathbb{G}_m)$ is not k-rational.

Thus we now have an example of a reductive k-group, in fact a torus, which is k-unirational (as all are) but which is not k-rational. This example has a long history (Chevalley, Voskresenskiĭ).

Remark 5.7. As Voskresenskiĭ remarked, the question of rationality of function fields of tori turned out to be related with a classical problem, often referred to as the (Emmy) Noether problem. Let k be a field and G be a finite group. One asks whether the field of invariants $(k(x_s)_{s\in G})^G$ is purely transcendental over k. For G an abelian group of order prime to the characteristic of k, that field is the function field of a certain k-torus T_G. Thus the technique above may help to decide whether $(k(x_s)_{s\in G})^G$ is purely transcendental or not.

If k is algebraically closed and G abelian then the field of invariants is purely transcendental.

For $k = \mathbb{Q}$ and $G = \mathbb{Z}/8$, Voskresenskiĭ showed $H^1(g, p(T_G)) \neq 0$, hence the field of invariants is not purely transcendental over $\mathbb{Q}$.

For $k = \mathbb{Q}$ and $G = \mathbb{Z}/47$, Swan showed $p(T_G) \neq 0$. The proof is harder, indeed in this case $p(T_G)$ is invertible.

Let $G = \mathbb{Z}/n$. Fix a primitive n-th root ζ of 1 in $\overline{k}$. The character group of the torus T_G is defined by the following exact sequence

$$0 \to \hat{T}_G \to \oplus_{i\in\mathbb{Z}/n}\mathbb{Z}.\zeta^i \to \mu_n \to 0,$$

where the map $\oplus_{i\in\mathbb{Z}/n}\zeta^i \to \mu_n$ sends $\sum_i n_i\zeta^i$ to $(\zeta)^{\sum n_i i}$. The middle term is clearly a permutation module for the action of the Galois group of $k(\zeta)/k$. The dual exact sequence of k-groups of mutiplicative type reads

$$1 \to \mathbb{Z}/n \to P \to T_G \to 1,$$

with P a quasitrivial torus. One checks that the field extension $k(P)/k(T_G)$ is the field extension $(k(x_s)_{s\in G})/(k(x_s)_{s\in G})^G$. From this sequence we see that the k-birational invariants $H^1(g, p(T_G))$, which we know is isomorphic to $Sha^2_\omega(g, \hat{T}_G)$, is isomorphic to $Sha^1_\omega(g, \mu_n)$.

In the lecture, I explained how for $k = \mathbb{Q}$ and $G = \mathbb{Z}/8$ Wang's counterexample to Grunwald's theorem yields a (seemingly) different proof of the non-$\mathbb{Q}$-rationality of $(\mathbb{Q}(x_g)_{g\in G})^G$.

For more on the whole topic, see Voskresenskiĭ's book [40].

A flasque resolution of a k-torus T is an exact sequence of k-tori

$$1 \to S \to P \to T \to 1,$$

with $\hat{P}$ a permutation module and $\hat{F}$ a flasque module. The existence and near unicity of such resolutions has been proven above: S is well defined up to multiplication by a quasitrivial torus.

At this point, let us explain the terminology "flasque". On smooth integral k-varieties, the functor $X \mapsto H^1_{et}(X, \mathbb{G}_m) = H^1_{Zar}(X, \mathbb{G}_m)$ satisifies the following property: if $U \subset X$ is a nonempty open set, then the restriction map $\mathrm{Pic}\, X \to \mathrm{Pic}\, U$ is surjective, i.e. $H^1_{et}(X, \mathbb{G}_{m,k}) \to H^1_{et}(U, \mathbb{G}_{m,k})$ is surjective. In this last statement, we cannot replace $\mathbb{G}_{m,k}$ by an arbitrary k-torus T, as easy examples show. There are "residues" which may prevent an element of $H^1_{et}(U, T)$ to come from an element of $H^1_{et}(X, T)$. This is analogue to the fact that $H^1_{et}(X, \mu_n)$ may not surject onto $H^1_{et}(U, \mu_n)$ because $k[X]^*/k[X]^{*n}$ need not surject onto $k[U]^*/k[U]^{*n}$ (think of $U = \mathbb{G}_m \subset \mathbb{A}^1$).

But we have

Proposition 5.8. *(CT-Sansuc) Let U be an open set of a smooth k-variety X. Let S/k be a flasque k-torus. Then the restriction map*

$$H^1_{et}(X, S) \to H^1_{et}(U, S)$$

is surjective.

Proof. See [10], Pop. 9 p. 194. □

Thus a torsor over U under a flasque torus S is isomorphic to the restriction of a a torsor over X under S.

In his lectures, Gille defined R-equivalence on the set of k-rational points of an arbitrary k-variety. This defines a set $X(k)/R$. Gille explained that for a connected k-group G this yields a group $G(k)/R$.

He also stated one of the main theorems of [10]:

Theorem 5.9. *Let*

$$1 \to S \to P \to T \to 1$$

be a flasque resolution of the k-torus T. Then the boundary map in Galois cohomology

$$T(k) \to H^1(k, S)$$

induces an isomorphism of groups

$$T(k)/R \xrightarrow{\simeq} H^1(k, S).$$

Proof. (Sketch, see [13], Lemme 1.6.2 p. 392) That the map is onto follows from $H^1(k, P) = 0$ (Hilbert's theorem 90), which thus gives an isomorphism $T(k)/im(P(k)) \simeq H^1(k, S)$. Because P is an open set of affine space, there is a surjective homomorphism $T(k)/im(P(k)) \to T(k)/R$. That the homomorphism $T(k)/im(P(k)) \simeq H^1(k, S)$ factors through $T(k)/R$ is a consequence of the fact that S is flasque. Indeed the whole thing reduces to the following statement: for an open set U of affine line $\mathbb{A}^1_k$, the composite map $H^1(k, S) \to H^1(\mathbb{A}^1_k, S) \to H^1(U, S)$ is onto. The first map would be onto for any k-torus F. That the second one is onto is a special case of the previous proposition. □

Once one has this result, one immediately the corollary.

Corollary 5.10. *Let T/k be a torus. If $T(K)/R = 1$ for any field K containing k then there exists a k-torus T' such that $T \times_k T'$ is k-rational.*

Proof. Consider a flasque resolution

$$1 \to S \to P \to T \to 1.$$

Let K be the function field of T. If we apply the hypothesis we find $H^1(K, F) = 0$ hence $P(K) \to T(K)$ onto. But that is saying the torsor $P \to T$ admits a section over an open set, hence the quasitrivial P, which is an open set of affine space, is k-briational to the product $T \times_k S$. This is enough to conclude. Note that in this case $\hat{S}$ is a direct factor of a permutation module. □

Whether a reductive k-group G which satisfies $G(K)/R = 1$ for any field K containing k is a k-birational direct factor of a k-rational variety is not known.

While we are on it, let us mention another remarkable property of flasque tori.

Theorem 5.11. *Let k be a field finitely generated over the prime field. Let S be a flasque k-torus. Then the group $H^1(k, S)$ is finite.*

Proof. See [10], Thm. 1, p. 192. The key ingredients are two classical finiteness theorems: the generalised unit theorem (combination of Dirichlet's units theorem and geometry) and the Mordell-Weil-Néron theorem (incorporating the finite generation of Néron-Severi groups). □

Remark 5.12. For a recent discussion of the finiteness theorems involved, see Bruno Kahn, Bulletin SMF, 2007.

Remark 5.13. For a quasitrivial torus P, one has $H^1(k, P) = 0$ over any field so the result is trivial. Suppose that K/k is a finite extension of number fields. Let $R^1_{K/k}\mathbb{G}_m$ be the k-torus which fits into the exact sequence

$$1 \to R^1_{K/k}\mathbb{G}_m \to R_{K/k}\mathbb{G}_m \to \mathbb{G}_{m,k} \to 1,$$

where the map $R_{K/k}\mathbb{G}_m \to \mathbb{G}_{m,k}$ is given by the norm. Then $k^*/NK^* \simeq H^1(k, R^1_{K/k}\mathbb{G}_m)$. For K/k Galois classical tools of number theory (Chebotarev) show that this quotient is infinite. (Note: for general K/k it is also known that this quotient is infinite, but the proof uses the classification of finite simple groups.)

What is striking is that there are tori which lie in between these two extremes: $H^1(k, S)$ is finite, but need not be zero.

Corollary 5.14. *Let k be a field finitely generated over the prime field. Then for any k-torus T the quotient $T(k)/R$ is finite.*

We like to view this result as an analogue of the Mordell-Weil theorem.

Whether the same results holds for arbitrary reductive group is an open problem. The case where k is a number field was established by P. Gille. More on this below.

There are other fields k of interest for which one can prove that $H^1(k, S)$ is finite if S is a flasque torus. See [9], Theorem 3.4.

6. Flasque resolutions of connected linear algebraic groups

It has turned out relatively recently ([BoKu], [8]) that much of the theory developed for tori in the 70's extends to arbitrary connected linear algebraic groups.

For simplicity, we shall from the very beginning restrict attention to fields of characteristic zero. As usual, much of the theory can be developed in positive characteristic provided one restricts attention to reductive groups.

6.1. **The Picard group of a smooth compactification.**

Theorem 6.1. *(Borovoi–Kunyavskiĭ) Let k be a field, $char(k) = 0$. Let G/k be a connected algebraic group over k. Let X be a smooth, proper k-compactification of G. Then the Galois module $\mathrm{Pic}\,\overline{X}$ is flasque.*

Proof. Note that the char. zero assumption enables us to use Hironaka's theorem: any smooth k-variety admits a smooth compactification. One immediately reduces to the case where G is reductive.

If G is quasisplit, i.e. admits a Borel subgroup B defined over k, with maximal torus T, the Bruhat decomposition shows that G contains an open set k-isomorphic to the product of the k-torus T and an affine space. Let Y be a smooth k-compactification of T. By arguments already explained this implies that Galois module $\mathrm{Pic}\,\overline{X}$ up to addition of a permutation module is isomorphic to $\mathrm{Pic}\,\overline{Y}$. And we know that the latter one is flasque. Thus $\mathrm{Pic}\,\overline{X}$ is flasque (this easy argument is already in [10]).

Let now G/k be an arbitrary reductive group. Let Z denote the algebraic k-variety of Borel subgroups of G. This is k-variety which over $\overline{k}$ is isomorphic to $\overline{G}/B$ for $B \subset \overline{G}$ a Borel subgroup. Let $K = k(Z)$ denote the function field of this geometrically integral k-variety. Let $\overline{k}(Z)$ denote the function field of $\overline{Z}$. The Galois group of $\overline{k}(Z)/k(Z)$ coincides with the Galois group of $\overline{k}$ over k. Let $\overline{K}$ be an algebraic closure of $k(Z)$. Since $\overline{X}$ is a smooth, projective, rational $\overline{k}$-variety, we have isomorphisms

$$\mathrm{Pic}\,X \times_k \overline{k} \xrightarrow{\simeq} \mathrm{Pic}\,X \times_k \overline{k}(Z) \xrightarrow{\simeq} \mathrm{Pic}\,X \times_k \overline{K}$$

given by the natural maps. The $k(X)$-variety $Z \times_k k(Z)$ has an obvious rational point. Thus $G \times_k k(Z)$ contains a Borel subgroup over $k(Z)$. It follows that the The $Gal(\overline{K}/k(Z))$-lattice $\mathrm{Pic}\,X \times_k \overline{K}$ is flasque. But the action of $Gal(\overline{K}/k(Z))$ on $\mathrm{Pic}\,X \times_k \overline{K}$ is given by the the action of $Gal(\overline{k}(Z)/k(Z))$ on $\mathrm{Pic}\,X \times_k \overline{k}(Z)$, which is itself given by the action of $Gal(\overline{k}/k)$ on $\mathrm{Pic}\,X \times_k \overline{k}$. Thus the $Gal(\overline{k}/k)$-lattice $\mathrm{Pic}\,X \times_k \overline{k}$ is flasque. □

Remark 6.2. The trick of going to the generic point of some auxiliary geometrically integral k-variety is a very useful one.

We now proceed to the generalisation of flasque resolutions to arbitrary reductive groups.

6.2. **Quasitrivial groups.**

We need a substitute for the notion of a quasitrivial torus.

Definition 6.3. A smooth geometrically integral k-variety is called quasitrivial if it satisfies the two properties

(i) The Galois lattice $\overline{k}[X]^*/\overline{k}^*$ is a permutation lattice.

(ii) $\mathrm{Pic}\,\overline{X} = 0$.

Proposition 6.4. *Let X be a smooth quasitrivial k-variety.*

(i) Any open set of X is quasitrivial.

(ii) If T is a flasque k-torus or if $\overline{k}^* \xrightarrow{\simeq} \overline{k}[X]^*$ the restriction map $H^1(k,T) \to H^1(X,T)$ is an isomorphism.

(iii) Under any of the two previous hypotheses any torsor over X under T is isomorphic to the product of X and a principal homogeneous space E over k under T.

See [8], §1.

Definition 6.5. A reductive group H/k is called quasitrivial if its underlying k-variety is quasitrivial. This is equivalent to the combination of the two hypotheses:
(i) The character group $\hat{H}$ is a permutation module.
(ii) $\operatorname{Pic} \overline{H} = 0$.

Condition (i) is equivalent to the condition that H^{tor} is a quasitrivial torus.

Condition (ii) is equivalent to the condition that the semisimple group $H^{ss} = [H,H]$ is simply connected.

See [8], §2.

A typical example is the group $GL(D)$ for D an arbitrary central simple algebra over k.

Proposition 6.6. *Let*

$$1 \to G_1 \to G_2 \to G_3 \to 1$$

be an exact sequence of reductive groups. If G_1 and G_3 are quasitrivial, so is G_1.

See [8], §2. The proof is a direct application of Sansuc' exact sequence (§3).

6.3. **Flasque resolutions.**

Theorem 6.7. *Let G be a reductive k-group. There exists a flasque k-torus S, a quasitrivial linear algebraic group H and a central extension of k-groups*

$$1 \to S \to H \to G \to 1.$$

See [8], §3.

The proof uses the following fact: for an arbitrary k-group of multiplicative type M there exists an exact sequence of k-groups of multiplicative type

$$1 \to M \to S \to P \to 1$$

with S a flasque k-torus and P a quasitrivial torus. This is proved by a variant of the Endo–Miyata argument.

Such an exact sequence is called a flasque resolution of the group G.

Proposition 6.8. *Let G be a reductive k-group. Suppose given two flasque resolutions*

$$1 \to S_1 \to H_1 \to G \to 1$$

and

$$1 \to S_2 \to H_2 \to G \to 1.$$

Let P_i denote the quasitrivial torus H_i^{tor}. Then

(i) There exists an isomorphism of k-groups $S_1 \times_k H_2 \simeq S_2 \times_k H_1$.

(ii) The semisimple k-groups H_1^{ss} and H_2^{ss} are naturally isomorphic.

(iii) There exists an isomorphism of Galois modules

$$\hat{S}_1 \oplus \hat{P}_2 \simeq \hat{S}_2 \oplus \hat{P}_1.$$

(iv) Let T_* denotes the cocharacter group of a k-torus T. There exists a natural isomorphism between the Galois modules $Coker[S_{1,*} \to P_{1,*}]$ and $Coker[S_{2,*} \to P_{2,*}]$.

See [8], §3.

From a flasque resolution $1 \to S \to H \to G \to 1$ we can recover the group $\operatorname{Pic} G$.

Proposition 6.9. *Let G be a reductive k-group. Let $1 \to S \to H \to G \to 1$ be a flasque resolution of G. Let P be the quasitrivial torus H^{tor}. The kernel of the composite map $S \to H \to P$ is finite and there is a natural exact sequence*

$$\hat{P}^g \to \hat{S}^g \to \operatorname{Pic} G \to 0.$$

Proof. See [8], Prop. 3.3. □

Let $1 \to S \to H \to G \to 1$ be a flasque resolution of a reductive group G. From this sequence we deduce a commutative diagram of homomorphisms of k-groups

$$\begin{array}{ccccccccc} & & 1 & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \mu & \to & G^{sc} & \to & G^{ss} & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & S & \to & H & \to & G & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & M & \to & P & \to & G^{tor} & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 1 & & 1 & & 1 & & \end{array} \tag{6.1}$$

Here $P = H^{tor}$ is a quasitrivial torus, M is the kernel of the natural map $H^{tor} \to G^{tor}$. It is a k-group of multiplicative type. Inspection of the diagram shows that it is a quotient of S, hence is a k-torus.

In the case $G = PGL_n$, we have the flasque resolution

$$1 \to \mathbb{G}_m \to GL_n \to PGL_n \to 1$$

and the diagram above reduces to the one displayed earlier on: (here $PGL_n^{tor} = 1$).

$$\begin{array}{ccccccccc} & & 1 & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \mu_n & \to & SL_n & \to & PGL_n & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow = & & \\ 1 & \to & \mathbb{G}_m & \to & GL_n & \to & PGL_n & \to & 1 \\ & & \downarrow & & \downarrow & & & & \\ & & \mathbb{G}_m & = & \mathbb{G}_m & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 1 & & 1 & & & & \end{array} \tag{6.2}$$

and for D a central simple algebra over a field k one gets a similar diagram with $G = PGL(D)$.

Let q be a non degenerate quadratic form of rank at least 3 over a field k of characteristic different from 2. Another example is the following diagram discussed in Gille's lecture, and which involves the Clifford group $\Gamma^+(q)$.

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \mu_2 & \to & Spin(q) & \to & SO(q) & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow= & & \\
1 & \to & \mathbb{G}_m & \to & \Gamma^+(q) & \to & SO(q) & \to & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbb{G}_m & = & \mathbb{G}_m & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array} \tag{6.3}$$

In all these cases the k-group H happens to be a k-rational variety. But so do the groups G in each of these three cases.

When the rank of q is even, Gille also described the diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \mu_2 & \to & SO(q) & \to & PSO(q) & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow= & & \\
1 & \to & \mathbb{G}_m & \to & GO^+(q) & \to & PSO(q) & \to & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbb{G}_m & = & \mathbb{G}_m & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array} \tag{6.4}$$

This is not a diagram associated to a flasque resolution of $PSO(q)$.
Problem Write down a flasque resolution of $PSO(q)$.

6.4. **The algebraic fundamental group.**

Definition 6.10. Let $1 \to S \to H \to G \to 1$ be a flasque resolution of the reductive group G. Let $P = H^{tor}$. The algebraic fundamental group of G is the Galois module which is the cokernel of the map $S_* \to P_*$. It is denoted $\pi_1(G)$.

This definition makes sense since we have seen earlier that this cokernel does not depend on the flasque resolution of G. One checks that the map $S_* \to P_*$ is injective. We thus have:

Proposition 6.11. *There is an exact sequence*

$$0 \to S_* \to P_* \to \pi_1(G) \to 0.$$

It is a coflasque resolution of the Galois module $\pi_1(G)$.

By definition, $\pi_1(G)$ is a finitely generated Galois module. This is not a profinite group. There is a connexion with Grothendieck's profinite group (see below).

Such a group has been considered by Borovoi, with a different definition. One may show that the two definitions agree.

Proposition 6.12. *Let G/k be a reductive group. The finite abelian groups* $\operatorname{Pic} G$ *and* $(\pi_1(G)_g)_{tors}$ *are dual to each other.*

Proof. See [8], Prop. 6.3. □

Proposition 6.13. *Let G/k be a reductive group. Let μ denote the kernel of the isogeny $G^{sc} \to G^{ss}$. There is a natural exact sequence of Galois modules*

$$0 \to \mu(-1) \to \pi_1(G) \to G_*^{tor} \to 0.$$

Proof. See [8], Prop. 6.4. By definition $\mu(-1) = Hom_{\overline{k}}(\mu_N, \mu)$ for N a positive integer which is a multiple of the order of the k-group scheme μ. □

Remark 6.14. Thus for $G = T$ a torus, $\pi_1(T)$ is the group of cocharacters of T. For G a semisimple group, the group $\pi_1(G)$ is a twisted version of the group μ. For instance for $G = PGL_n$, we have $\pi_1(G) = \mathbb{Z}/n$ with the trivial action of the Galois group.

Corollary 6.15. *Let G/k be a reductive group.*
(i) G is semisimple if and only if $\pi_1(G)$ is finite.
(ii) G is (semisimple) simply connected if and only if $\pi_1(G) = 0$.
(iii) The group $\pi_1(G^{ss})$ coincides with the torsion of $\pi_1(G)$.
(iv) G is quasitrivial if and only if $\pi_1(G)$ is a permutation module.

Proof. See [8], Prop. 6.5. □

Proposition 6.16. *Assume $char(k) = 0$. The profinite completion of $\pi_1(G)$ is isomorphic, as a Galois module, with*

$$\pi_1^{Groth}(\overline{G})(-1) = Hom(\mathbb{Z}(1), \pi_1^{Groth}(\overline{G})).$$

Proof. See [8], Prop. 6.7. One proves $H^1(\overline{G}, \mu_n) \simeq Hom_{\mathbb{Z}}(\pi_1(G), \mathbb{Z}/n)$. □

Let $k = \mathbb{C}$. Xu Fei asks whether there is an isomorphism between $\pi_1(G)$ as defined above and the topologival $\pi_1^{top}(G(\mathbb{C}))$. This looks likely, the question is then to define a natural map between the two which will be an isomorphism.

6.5. **More geometry.** Assume $char(k) = 0$.

Let G/k be a reductive algebraic group and X a smooth k-compactification of G. Let $1 \to S \to H \to G \to 1$ be a flasque resolution of G. We have seen that the class of the flasque lattice $\hat{S}$ is well defined up to addition of a permutation module. On the other hand by the Borovoi–Kunyavskiĭ theorem the Galois lattice $\operatorname{Pic} \overline{X}$ is flasque. For G a k-torus we have seen that the class of $\operatorname{Pic} \overline{X}$ coincides with that of $\hat{S}$. It is harder to prove:

Theorem 6.17. *The Galois modules $\hat{S}$ and $\operatorname{Pic} \overline{X}$ are isomorphic up to addition of permutation lattices.*

Proof. [8], §5. The proof conjugates Theorem 4.11 of the present notes and the properties of "universal torsors" over rational varieties ([13]). I went through most of the argument. A key point is that the underlying variety of a universal torsor over X is a quasitrivial variety ([13]). □

This theorem implies ([8], Thm. 7.1 and Thm. 7.2, see also [Bo–Ku]).

Proposition 6.18. *Let G/k be a reductive algebraic group and $1 \to S \to H \to G \to 1$ a flasque resolution of G Let X be a smooth compactification of G. Then the k-birational invariant $H^1(g, \operatorname{Pic} \overline{X})$ is isomorphic to*
(i) the group $H^1(g, \hat{S})$;
(ii) the group $Sha^1_\omega(g, Hom_Z(\pi_1(G), \mathbb{Q}/\mathbb{Z}))$.
(iii) If $G = T$ is a k-torus, it is isomorphic to $Sha^2_\omega(k, \hat{T})$.
(iv) If G is semisimple with fundamental group μ, it is isomorphic to $Sha^1_\omega(k, \hat{\mu})$.

Remark 6.19. This proposition has several aspects.

(i) In principle, it enables to compute the k-birational invariant $H^1(k, \operatorname{Pic}\overline{X})$ without having to construct an explicit smooth compactification of G.

(ii) It shows that the groups $H^1(k, \hat{S})$ and $Sha^1_\omega(g, Hom_Z(\pi_1(G), \mathbb{Q}/\mathbb{Z}))$ are k-birational invariants of a k-group G which vanish if G is k-rational. It would be interesting to give a direct proof of that result.

(iii) Formula (ii) is a common generalisation of formula (iii), which we saw earlier, and a formula for a semisimple group G, which we now discuss. We start from a smooth compactification $G \subset X$. One the has the short exact sequence

$$0 \to Div_\infty \overline{X} \to \operatorname{Pic}\overline{X} \to \operatorname{Pic}\overline{G} \to 0$$

which reads

$$0 \to Div_\infty \overline{X} \to \operatorname{Pic}\overline{X} \to \hat{\mu} \to 0.$$

This leads to an isomorphism between $Sha^1_\omega(k, \operatorname{Pic}\overline{X})$ and $Sha^1_\omega(k, \hat{\mu}$. If we know $H^1(\mathfrak{H}, \operatorname{Pic}\overline{X}) = 0$ for procyclic closed subgroups $\mathfrak{H}$ of the Galois group, then we get

$$H^1(k, \operatorname{Pic}\overline{X}) \simeq Sha^1_\omega(k, \hat{\mu}).$$

The required vanishing is a consequence of the fact that $\operatorname{Pic}\overline{X}$ is flasque (Borovoi–Kunyavskiĭ, above).

(iv) We have already seen examples of k-tori which are not k-rational. The formula in (iv) enables us to give examples of semisimple k-groups which are not k-rational. Indeed let μ be a finite k-group of multiplicative type such that $Sha^1_\omega(k, \hat{\mu}) \neq 0$. Let K/k be the (finite) splitting field of $\hat{\mu}$. Then we may embed μ_K centrally into some $SL_{n,K}$, over K. One then central embeddings $\mu \subset R_{K/k}(\mu_K) \subset R_{K/k}SL_{n,K}$. Let G be the cokernel of the composed map. Then G is semisimple, its fundamental group is μ hence the k-birational invariant does not vanish on G, the k-group G is not k-rational. The nonvanishing of $Sha^1_\omega(k, \operatorname{Pic}\overline{X})$ is enough to conclude, one need not know that $\operatorname{Pic}\overline{X}$ is flasque.

The group which we have produced (using a well known method, see counterexamples to the Hasse principle in Serre's Cohomologie Galoisienne [35]) is neither simply connected not adjoint. Indeed in the simply connected case, for any smooth compactification X of G, the Galois module $\operatorname{Pic}\overline{X}$ is a permutation module. In the adjoint case, one may show that $\operatorname{Pic}\overline{X}$ is a direct factor of a permutation module. I do not know whether it may fail to be stably a permutation module (which would then prevent k-rationality).

For directions to totally explicit examples with $Sha^1_\omega(k, \hat{\mu}) \neq 0$, see [33], p. 35, [8] p. 188, also my paper in Inventiones math. 159 (2005) p. 601.

6.6. **Galois cohomology: arbitrary fields.** Let as before G/k be a reductive group over a field k of characteristic zero. Let

$$1 \to S \to H \to G \to 1$$

be a flasque resolution of G. The group H fits into the exact sequence

$$1 \to H^{ss} \to H \to H^{tor} \to 1,$$

the group H^{ss} is simply connected, the torus $P = H^{tor}$ is a quasitrivial torus.

Since S is central in H, the flasque resolution gives rise to an exact sequence of groups

$$1 \to S(k) \to H(k) \to G(k) \to H^1(k, S)$$

which is itself part of an exact sequence of pointed Galois cohomology sets

$$1 \to S(k) \to H(k) \to G(k) \to H^1(k,S) \to H^1(k,H) \to H^1(k,G) \to H^2(k,S) \quad (8.1)$$

The fibres of these various applications can be described more precisely (Serre, [35], chap. I, §5). In particular any fibre of the map $H^1(k,G) \to H^2(k,S)$ either is empty or is a quotient of a set $H^1(k, {}_cH)$ by an action of $H^1(k,S)$, the group ${}_cH$ being obtained from H by torsion by an element $c \in Z^1(k,S)$. One checks that ${}_cH$ is also a quasitrivial group.

We also have an exact sequence of groups

$$1 \to H^{ss}(k) \to H(k) \to H^{tor}(k)$$

which is part of an exact sequence of pointed sets

$$1 \to H^{ss}(k) \to H(k) \to H^{tor}(k) \to H^1(k,H^{ss}) \to H^1(k,H) \to H^1(k,H^{tor}).$$

Hilbert's theorem 90 enables us to rewrite this sequence of pointed sets as

$$1 \to H^{ss}(k) \to H(k) \to H^{tor}(k) \to H^1(k,H^{ss}) \to H^1(k,H) \to 1 \quad (8.2)$$

Theorem 6.20. *Let G/k be a reductive group over a field k.*

(i) A flasque resolution $1 \to S \to H \to G \to 1$ of a reductive k-group G induces an exact sequence of groups

$$H(k)/R \to G(k)/R \to Ker[H^1(k,S) \to H^1(k,H)] \to 1.$$

(ii) The quotient of $G(k)/R$ by the image of $H(k)$, that is the image of $G(k)$ in $H^1(k,S)$, is an abelian quotient of $G(k)/R$ which is independent of the chosen flasque resolution.

(iii) If k is finitely generated over the prime field, of is finitely generated over an algebraically closed field of char. zero, then this quotient is finite.

Proof. See [8], Thm. 8.1. The key point is that the map $G(k) \to H^1(k,S)$ mods out by R-equivalence, and this follows from the fact that $H \to G$ is a torsor over G under the *flasque* torus S. □

For G a reductive group over a field as in the previous theorem, it is an open problem whether the group $G(k)/R$ is finite. The previous proposition reduces the question to the case of a quasitrivial group. I do not know how to reduce the general problem to the case of a semisimple group. For a special class of fields, which includes the case of totally imaginary number fields, we can manage, thanks to work of P. Gille [15], [16], [17].

6.7. **Galois cohomology: good fields of cohomological dimension 2.** Let us start with a definition.

By a *good field of cohomological dimension at most 2* we shall mean a field k of characteristic zero which satisfies the following properties:

(a) The cohomological dimension of k is at most 2.

(b) Over any finite extension K of k, index and exponent of central simple algebras over K coincide.

(c) The maximal abelian extension of k is of cohomological dimension 1 (this hypothesis is used only when the groups under consideration contain some factor of type E_8).

Totally imaginary number fields satisfy these properties. Nonarchimedan local fields satisfy these properties. A theorem of de Jong says that function fields in (at most) 2 variables over an algebraically closed field satisfy these properties. For each of these three classes of fields, for any flasque k-torus S, the group $H^1(k, S)$ is finite.

The following result is due to Gille ([15], [16], [17]). A minor variation is the general hypothesis on the field k in (iv) and (v), which appeared in [9] and [8].

Let k be a field. Let F be a covariant functor from k-algebras to sets. Let $R_{0,1}$ be the semi-local ring of the affine line $\mathbb{A}^1_k$ at the points 0 and 1. Two elements $a, b \in F(k)$ are called elementarily equivalent if there exists an element $\alpha \in F(R_{0,1})$ such that $\alpha(0) = a$ and $\alpha(1) = b$. R-equivalence on $F(k)$ is the equivalence relation generated by this elementary relation.

Theorem 6.21. *(Gille) Let k be a field. Let*

$$1 \to G \to H \to T \to 1$$

be an exact sequence of connected reductive k-groups with T a torus. For any commutative k-algebra let $C(A) = Im[H(A) \to T(A)] = Ker[T(A) \to H^1_{\text{ét}}(A, G)]$.

(i) There is an exact sequence

$$G(k)/R \to H(k)/R \to C(k)/RC(k) \to 1.$$

(ii) For any finite field extension L/k we have inclusion of groups $N_{K/k}(RC(K)) \subset RC(k) \subset RT(k) \subset T(k)$.

(iii) If G is simply connected and K/k is a field extension such that G_K is quasisplit then $N_{K/k}(RT(K)) \subset RC(k) \subset C(k) \subset T(k)$.

(iv) Assume that G is simply connected and that k is a good field of cohomological dimension at most 2. Then Galois cohomology of the above sequence yields an isomorphism $H(k)/R \stackrel{\simeq}{\to} T(k)/R$.

(v) Let H be a quasitrivial group over a field k which is a good field of cohomological dimension at most 2. Then $H(k)/R = 1$.

Proof. (Sketch) I refer to Gille's last lecture (and the references therein) for the proofs of (i), (ii), (iii).

For k as in (iv) and (v), Serre's conjecture II holds, i.e. $H^1(k, G) = 0$ for any semisimple simply connected group G. For k a p-adic field, this is Kneser's theorem; for general fields one uses the Merkur'ev-Suslin theorem. There is work of Suslin, Bayer-Parimala, Gille, Chernousov. See [9], Thm. 1.2, for more details.

This implies $C(k) = T(k)$.

Given a k-variety X one associates the subgroup $N_X(k) \subset k^*$ spanned by the images of the subgroups $N_{K/k}K^* \subset k^*$ for all the finite field extensions K/k such that $X(K) \neq \emptyset$. Let X_G denote the variety of Borel subgroups of G.

For $G = SL(D)$, with D a central simple algebra, $N_{X_G}(k)$ is the image of the reduced norm $D^* \to k^*$. The quotient $k^*/N_{X_G}(k)$ thus coincides with $H^1(k, SL(D))$. As mentioned above, this set is here reduced to one element. Thus $N_{X_G}(k) = k^*$. We have a similar result if we replace $\mathbb{G}_{m,k}(k) = k^*$ by $P(k)$ for P a quasitrivial torus: the subgroup of $P(k)$ spanned by the norms of all $P(K)$ for K/k finite with $X_G(K) \neq \emptyset$ is the whole of $P(k)$.

For an arbitrary semisimple simply connected group G, Gille [16] shows that Serre's conjecture II, known with our hypotheses, implies the same result .

Let

$$1 \to S \to P \to T \to 1$$

be a flasque resolution of the k-torus T. For any extension K/k, the group $RT(K) \subset T(K)$ is the image of $P(K)$.

If we apply (iii), we see that the subgroup of $T(k)$ spanned by the $N_{K/k}(RT(K))$ for K/k runnning through the extensions such that G_K is quasisplit coincides with the subgroup of $T(k)$ spanned by the norms in $T(k)$ of the images of $P(K)$ in $T(K)$ for K/k running through such extensions. By what we have just seen, for k as in the Theorem, this is the image of $P(k)$ in $T(k)$, thus $P(k) = RT(k)$ is included in $RC(k)$. But the inclusion $RC(k) \subset RT(k)$ is obvious.

Thus $C(k)/RC(k) = T(k)/RT(k)$. Combining this with (i) we get the exact sequence

$$G(k)/R \to H(k)/R \to T(k)/R \to 1.$$

For a good field of cohomological dimension at most 2, and G a semisimple simply connected group, one has $G(k)/R = 1$.This result again has a long history, for which we refer to [9], Thm. 4.5.

Together with the previous exact sequence, this proves (iv).

As for (v), this is a direct consequence of (iv).

□

The following result is essentially due to Gille.

Corollary 6.22. *Let k be a good field of cohomological dimension at most 2. Let G/k be a reductive group and*

$$1 \to S \to H \to G \to 1$$

be a flasque resolution. Then

(i) The induced map $G(k)/R \to H^1(k, S)$ is an isomorphism.

(ii) If k a totally imaginary number field, or a p-adic field, or a function field in at most two variables over an algebraically closed field, the group $G(k)/R$ is a finite abelian group.

Proof. This immediately follows from the two previous results, and the finiteness of $H^1(k, S)$ for a flasque torus over a field as in (ii), already mentioned. □

Remark 6.23. The flasque resolutions of groups as presented here were defined in [8] after Gille's work, so the presentation by Gille was slightly different.

Proposition 6.24. *Let k be a field, G a connected reductive group and*

$$1 \to S \to H \to G \to 1$$

a flasque resolution of G. Let $P = H^{tor}$. Such a resolution induces a map

$$H^1(k, G) \to Ker[H^2(k, S) \to H^2(k, P)].$$

A fibre of this map either is empty or is a quotient of a set $H^1(k,_c H^{ss})$ for a suitable semisimple simply connected group ${}_cH^{ss}$. There is a natural map

$$Ker[H^2(k, S) \to H^2(k, P)] \to Hom(\operatorname{Pic} G, \operatorname{Br} k).$$

The composite map

$$H^1(k, G) \to Hom(\operatorname{Pic} G, \operatorname{Br} k)$$

does not depend on the choice of the flasque resolution of G. These maps induce a complex of pointed sets

$$H^1(k, H^{ss}) \to H^1(k, G) \to Hom(\operatorname{Pic} G, \operatorname{Br} k).$$

Proof. See [8], Prop. 8.2. and Prop. 8.3. Among other things, the proof uses the exact sequence

$$(\hat{P})^g \to (\hat{S})^g \to \operatorname{Pic} G \to 0$$

(Prop. 6.9 above). □

A variant of the next proposition first appeared in [2].

Proposition 6.25. *Let k be a field, G/k a connected reductive group and*

$$1 \to S \to H \to G \to 1$$

a flasque resolution of G. Let $P = H^{tor}$. Assume that k is a good field of cohomological dimension at most 2. Such a resolution induces a bijection between the set $H^1(k, G)$ and the group $Ker[H^2(k, S) \to H^2(k, P)]$.

Proof. See [8], Thm. 8.4. The surjectivity statement requires the work in [9] and heavy noncommutative Galois cohomology (bands). □

6.8. **Galois cohomology: Number fields.** Let G be a connected reductive group over a number field k.

There are three basic topics we are interested in here.

(i) Problems about R-equivalence. Structure of the group $G(k)/R$, finiteness, size, comparison between the global group $G(k)/R$ and the local groups $G(k_v)/R$.

(ii) Problems of *weak approximation.* Given a finite set S of places of k, is the image of the diagonal map

$$G(k) \to \prod_{v \in S} G(k_v)$$

dense (for the product of the local topologies) ?

(iii) The Hasse principle. If a principal homogeneous space (= torsor) E under G has rational points in all completions k_v of k, does it have a point in k ? The set $H^1(k, G)$ classified all such torsors up to nonunique isomorphism. The Hasse principle holds for all torsors if and only if the natural map

$$H^1(k, G) \to \prod_v H^1(k_v, G)$$

is trivial.

6.8.1. *Simply connected groups, quasitrivial groups.* I will first list the basic arithmetic tools which will be used as black boxes in the study of this problem for arbitrary linear algebraic groups. As the reader will see in the next subsubsection, Galois cohomology together with class field theory reduces the study of the above problem to the case of semisimple simply connected groups, a case which must be handled by direct methods.

Theorem 6.26. *(Kneser, Bruhat–Tits) Let G be a semisimple, simply connected group over a nonarchimedean local field k. Then $H^1(k, G) = 0$, any torsor under G is trivial, i.e. has a k-point.*

Kneser proved this theorem by a case by case argument. Bruhat and Tits gave a unified proof. This was discussed by Gille in his lectures.

As indicated in the previous section, the result holds in the more general context of good fields of cohomological dimension 2 (work of Merkur'ev-Suslin, Bayer–Parimala, Gille, Chernousov). In this general context only a case by case argument is known.

Theorem 6.27. *(Kneser, Platonov) Let G be a semisimple, simply connected group over a number field k. Then weak approximation for G holds with respect to an arbitrary finite set S of places of k.*

I will come back to this theorem in a later lecture.

Theorem 6.28. *(Eichler, Landherr, Kneser, Harder, Chernousov) Let G/k be a semisimple, simply connected group over a number field k. Then the Hasse principle holds for torsors under G. More precisely, the diagonal map*

$$H^1(k,G) \to \prod_{v \text{ real}} H^1(k_v,G),$$

where v runs through the real places of k, is a bijection.

I will not discuss the long proof of this deep theorem, which requires a case by case discussion. I refer to Kneser's Tata lecture notes for classical groups and to the book [32] (Platonov and Rapinchuk) for the general case.

Theorem 6.29. *Let H be a quasitrivial reductive group over a field k. Then*
(i) For k a p-adic field, $H^1(k,H) = 1$.
(ii) For k a number field, weak approximation holds for H.
(iii) For k a number field, the diagonal map

$$H^1(k,H) \to \prod_{v \text{ real}} H^1(k_v,H)$$

is a bijection of finite sets.

Proof. See [8], Prop. 9.2. Note that the proof of (ii) is an elementary case of the fibration method. It uses both the Hasse principle and weak approximation for the simply connected group H^{ss}, and it uses weak approximation for the quasitrivial torus P.

□

Theorem 6.30. *(Combination of work of many people) Let G be a semisimple, simply connected group over a number field k.*
(i) The group $G(k)/R$ is finite.
(iii) If the group G is almost k-simple and isotropic, $G(k)/R = 1$.
(iii) If k is totally imaginary, $G(k)/R = 1$.
(iv) If G has no factor of type E_6, then $G(k)/R = 1$.
(v) For H/k a quasitrivial reductive group over a number field the analogues of these four results hold.

Proof. (Indications)
(i) This is an immediate consequence of a general (ergodic) theorem of Margulis: if G/k is an almost simple simply connected group G over a number field k then any normal subgoup of $G(k)$ either is contained in the centre of $G(k)$ or is of finite index in $G(k)$.

The next results rely on a case by case discussion.

(ii) This is essentially a consequence of the Kneser–Tits conjecture, according to which for an isotropic absolutely almost simple group G, the group $G(k)$ is generated by its unipotent subgroups. Over an arbitrary field, this conjecture is wrong (counterexamples were provided by Platonov). Over a number field it has been proved for all simple groups except for some of type E_6. There is a long history. That

$G(k)/R = 1$ also for type E_6 was proved by Chernousov and Timoshenko ([3], Thm. 2.12).

(iii) This is a special case of a result which holds for arbitrary good fields of cohomological dimension at most 2, and which involves the work of many people (for references see [9], Thm. 4.5.)

(iv) For this result and its relation with the Platonov-Margulis conjecture on the structure of normal noncentral subgroups of $G(k)$, see [32] and [3].

(v) Starting from the previous results the proof is achieved by a variant of the method of Thm. 6.21. See [17].

□

A curious question remains: in the E_6-case, over a formally real number field k, is the group $G(k)/R$ commutative ?

6.8.2. *General reductive groups.* We first discuss R-equivalence. The following theorem, except for the presentation by means of flasque resolutions of groups, is due to P. Gille ([15], [17]).

Theorem 6.31. *Let G be a reductive group over a number field k. The group $G(k)/R$ is finite. More precisely, a flasque resolution*

$$1 \to S \to H \to G \to 1$$

induces an exact sequence of finite groups

$$H^{ss}(k)/R \to G(k)/R \to H^1(k, S) \to 1,$$

where H^{ss}, the derived group of H, is a simply connected group.

Proof. See [15], [17] and [8], Thm. 9.3. The Hasse principle for torsors of quasitrivial groups is used to prove that the map $G(k) \to H^1(k, S)$ is surjective (only the real places are involved, and for such a place the flasque k_v-torus S_{k_v} is quasitrivial, hence $H^1(k_v, S) = 0$). □

We turn to weak approximation.

The following result elaborates on the one given in [8], Thm. 9.4 (i).

Theorem 6.32. *Let G be a reductive group over a number field k. Let*

$$1 \to S \to H \to G \to 1$$

be a flasque resolution of G. Let Σ be a finite set of places of k. Then

(i) The closure of the image of the map $G(k) \to \prod_{v\in\Sigma} G(k_v)$ coincides with the inverse image under the map $\prod_{v\in\Sigma} G(k_v) \to \prod_{v\in\Sigma} H^1(k_v, S)$ of the image of the map $H^1(k, S) \to \prod_{v\in\Sigma} H^1(k_v, S)$.

(ii) The closure of the image of the map $G(k) \to \prod_{v\in\Sigma} G(k_v)$ is a normal subgroup. The quotient $A_\Sigma(G)$ of $\prod_{v\in\Sigma} G(k_v)$ by the closure of the diagonal image of $G(k)$ is a finite abelian group isomorphic to the cokernel of $H^1(k, S) \to \prod_{v\in\Sigma} H^1(k_v, S)$.

(iii) The set Σ_0 of places v such that $H^1(k_v, S) \neq 0$ is finite. It is contained in the set of places which are ramified in the splitting field of the k-torus S.

(iv) For any finite set Σ containing Σ_0, the natural projection $A_\Sigma(G) \to A_{\Sigma_0}(G)$ is an isomorphism of finite abelian groups. One therefore writes $A(G) = A_{\Sigma_0}(G)$. The finite abelian group $A(G)$ is isomorphic to the cokernel of the diagonal map of finite groups

$$H^1(k, S) \to \prod_{v\in\Sigma_0} H^1(k_v, S).$$

(v) For any finite set Σ_1 of places of k such that $\Sigma_0 \cap \Sigma_1 = \emptyset$, weak approximation holds for Σ_1: the diagonal map

$$G(k) \to \prod_{v \in \Sigma_1} G(k_v)$$

has dense image.

Proof. One uses the commutative diagram of exact sequences

$$\begin{array}{ccccccc} H(k) & \to & G(k) & \to & H^1(k,S) & \to & H^1(k,H) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \prod_{v\in\Sigma} H(k_v) & \to & \prod_{v\in\Sigma} G(k_v) & \to & \prod_{v\in\Sigma} H^1(k_v,S) & \to & \prod_{v\in\Sigma} H^1(k_v,H) \end{array} \tag{6.5}$$

The maps $G(k) \to H^1(k,S)$ and $G(k_v) \to H^1(k_v,S)$ are onto. One thus has the commutative diagram of exact sequences

$$\begin{array}{ccccccc} H(k) & \to & G(k) & \to & H^1(k,S) & \to 1 \\ \downarrow & & \downarrow & & \downarrow & \\ \prod_{v\in\Sigma} H(k_v) & \to & \prod_{v\in\Sigma} G(k_v) & \to & \prod_{v\in\Sigma} H^1(k_v,S) & \to & 1 \end{array} \tag{6.6}$$

Weak approximation at any finite set of places for the quasitrivial group H enables one to prove all statements. For more details, see [8], Thm. 9.4. (i) □

Remark 6.33. (a) The nonobvious fact that the closure of $G(k)$ in $\prod_{v\in\Sigma} G(k_v)$ is normal follows directly from our formalism.

(b) Statement (v) says that G satisfies weak-weak approximation, i.e. weak approximation anywhere outside of a fixed finite set of places of k. This property still holds for homogeneous spaces of a connected linear algebraic group when the geometric stabilizers are connected. It is an open question whether it holds when the geometric stabilizers are finite. If the answer were positive, any finite group would be a Galois group over $\mathbb{Q}$. This is a famous open question.

We now discuss kernel and cokernel of the map

$$H^1(k,G) \to \prod_{\text{all } v} H^1(k_v,G)$$

for an arbitrary reductive group G. The question about the kernel is that of the Hasse principle for torsors of G. The question regarding the cokernel was considered by Kottwitz around 1986.

Theorem 6.34. *Let G be a reductive group over a number field k. Let*

$$1 \to S \to H \to G \to 1$$

be a flasque resolution of G.

(i) This sequence induces a bijection between the set $Sha^1(k,G)$ and the finite abelian group $Ker[H^2(k,S) \to \prod_{\text{all } v} H^2(k_v,S)]$.

(ii) This sequence induces an exact sequence of pointed sets

$$H^1(k,G) \to \oplus_{\text{all } v} H^1(k_v,G) \to Hom(\operatorname{Pic} G, \mathbb{Q}/\mathbb{Z}).$$

Proof. The injection in (i) follows from what has been done so far. The proof of the other statements uses further tools, due to Kneser, Harder, Sansuc and Borovoi. See [8], Thm. 9.4. □

Remark 6.35. Satement (i) is a variant of a result of Sansuc ([33]).

Statement (ii) is a variant of a theorem of Kottwitz. Its proof builds upon the toric version of class field theory, due to Tate and Nakayama, as applied to the torus S. The sequence itself is a "reciprocity sequence" which in the case of G a torus boils down to a direct consequence of Tate-Nakayama.

Tate-Nakayama is also used in the proof of the following result ([8], Thm. 9.4.)

Corollary 6.36. *(Sansuc) Let G be a reductive group over a number field. Let X be a smooth compactification of G. There is an exact sequence of finite abelian groups*

$$0 \to A(G) \to Hom(H^1(k, \operatorname{Pic} \overline{X}), \mathbb{Q}/\mathbb{Z}) \to Sha^1(k, G) \to 0.$$

For G a torus, this sequence was found by Voskresenskiĭ. Sansuc extended the result to arbitrary (connected, linear) groups.

Remark 6.37. The occurrence of the group $H^1(k, \operatorname{Pic} \overline{X})$ is due to the fact that this group coincides with the group $H^1(k, \hat{S})$, as we have seen earlier.

This sequence shows that the purely algebraic group $H^1(k, \operatorname{Pic} \overline{X})$, which appeared as a k-birational invariant earlier in these notes, controls both weak approximation for G and the Hasse principle for torsors under G.

When this group vanishes, weak approximation and the Hasse principle holds. Such is the case for instance for G adjoint.

On the opposite side, if for a given "algebraic" type of G with $H^1(k, \operatorname{Pic} \overline{X}) \neq 0$, one may expect arithmetic versions of this given algebraic type with lack of weak approximation and other ones where the Hasse principle for torsors fails.

The simplest example is given by the k-torus $T = R^1_{K/k}\mathbb{G}_m$ for K/k a biquadratic extension. In this case $\operatorname{Br} X/\operatorname{Br} k = \mathbb{Z}/2$. In this case $A(T) = 0$ and $Sha^1(k, T) = \mathbb{Z}/2$ if and only if all decomposition groups in $Gal(K/k)$ are cyclic. Example: $k = \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt{13}, \sqrt{17})$. If one decomposition group is equal to the whole group $Gal(K/k)$, then $A(T) = \mathbb{Z}/2$ and $Sha^1(k, T) = 0$. Example: $k = \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt{-1}, \sqrt{2})$.

Exhibiting totally concrete examples, in this case finding an explicit element in $k = \mathbb{Q}$ which is a norm from K everywhere locally but not globally requires further work. It is in general easier to give explicit counterexamples to weak approximation, especially at some (not predetermined) finite set of places (rather than at just one place). See the exercises at the end of the book "Algebraic number theory" (editors Cassels and Fröhlich).

As we have already mentioned several times, the group $H^1(k, \operatorname{Pic} \overline{X})$ is related to the Brauer group of X. In the next section I will discuss the Brauer group of an arbitrary algebraic variety X and its relation to weak approximation and the Hasse principle for X.

7. The Brauer-Manin obstruction for arbitrary varieties

Since I have written several surveys on this topic ([4], [5], [6], [7]), I do not write up detailed notes for this section.

See also a short introduction to the topic by Harari [22].

For futher study, read Skorobogatov's book [37].

7.1. **The Brauer group of a field.** Definition. Central simple algebras and Galois cohomology. For K/k cyclic, isomorphism $k^*/NK^* \simeq Ker[\operatorname{Br} k \to \operatorname{Br} K]$.

7.2. **Number fields.** The three basic approximation theorems for a number field.

Weak approximation.

Strong approximation (generalization of Chinese remainder theorem for $\mathbb{Z}$).

Theorem 7.1. *Let k be a number field, S a finite set of places of k, for each $v \in S$ an element $\lambda_v \in k_v$, $\varepsilon > 0$. Let v_0 be a place of k. Then there exists a $\lambda \in k$ and*
(i) $|\lambda - \lambda_v|_v < \varepsilon$ for all $v \in S$,
(ii) $v(\lambda) \geq 0$ at any finite place $v \notin S \cup v_0$.

Note that we have the choice of the v_0.

Dirichlet's theorem on primes in an arithmetic progression.

Theorem 7.2. *Let k be a number field, S a finite set of finite places of k, for each $v \in S$ an element $\lambda_v \in k_v^*$, $\varepsilon > 0$. Then there exists a $\lambda \in k^*$ and a finite place $v_0 \in k$ such that*
(i) $|\lambda - \lambda_v|_v < \varepsilon$ for all $v \in S$,
(ii) λ is positive at all real completions of k,
(iii) λ is a unit at any finite place $v \notin S \cup v_0$ and $v_0(\lambda) = 1$.

Here we do not have the choice of the v_0.

There is a further approximation theorem of the last kind which is of interest, where one approximates at the real places at the expense of losing control at a certain number of predetermined finite places. The key tool to get this result from Dirichlet's result is a theorem of Waldschmidt.

There is another basic tool, it is Tchebotarev's theorem, a special case of which asserts the following.

Let K/k be a finite extension of number fields. There are infinitely many places v of k which are split in K, i.e. the k_v-algebra $K \otimes_k k_v$ is isomorphic to a product of copies of k_v.

7.3. The Brauer group of a local field and of a global field. The fundamental exact sequence of class field theory:

$$0 \to \mathrm{Br}\, k \to \oplus_v \mathrm{Br}\, k_v \to \mathbb{Q}/\mathbb{Z} \to 0.$$

How this sequence contains the law of quadratic reciprocity. Application to the Hasse principle for conics. Application to the Hasse principle for Severi-Brauer varieties. Weak approximation for these varieties, since for these $X(k) \neq \emptyset$ implies X isomorphic to projective space.

7.4. Birational invariance of various properties. Lang-Nishimura lemma: Over any field k, existence of a k-point is a k-birational invariant of smooth proper geometrically integral k-varieties.

Implicit function theorem over a local field: smooth maps induce maps on local points which locally for the analytic topology admit a section.

Weak approximation is a k-birational invariant of smooth geometrically integral k-varieties.

Hasse principle is a k-birational invariant of smooth proper geometrically integral k-varieties.

Insist on assuming existence of nonsingular solutions in all completions k_v.

7.5. Quadratic forms and one-parameter families of quadrics: Hasse's theorem and further positive results. The Hasse principle and weak approximation for quadrics of arbitrary dimension.

Hasse's proof for the passage from 3 to 4 variables: use of Dirichlet's theorem on primes in an arithmetic progression, in the form mentioned above.

Note that the same proof gives the result for a system

$$a_1x_1^2 + b_1y_1^2 = \cdots = a_ix_i^2 + b_iy_i^2 = \cdots = a_nx_n^2 + b_ny_n^2 \neq 0$$

with all $a_i, b_i \in k^*$. The variety above is a principal homogeneous space under the k-torus T given by the equation

$$x_1^2 + a_1b_1y_1^2 = \cdots = x_i^2 + a_ib_iy_i^2 = \cdots = x_n^2 + a_nb_ny_n^2 \neq 0.$$

From the Voskresenskiĭ exact sequence we must have the purely algebraic statement $Sha^2_\omega(\hat{T}) = 0$. Exercise: prove this directly.

Passage from 4 to more variables.

More generally, proof for affine equations of the type

$$\sum_{i=1}^{3} a_ix_i^2 = P(t_1, \ldots, t_n)$$

with $a_i \in k^*$ and $P(t_1, \ldots, t_n) \neq 0$.

Extending Hasse's result. Families of quadrics of dimension at least 3.

Equations of the type

$$\sum_{i=1}^{n} a_i(t_1, \ldots, t_n)x_i^2 = 0$$

with all $a_i \in k[t_1, \ldots, t_n]$ nonzero and $n \geq 5$. Proof uses weak approximation for $\mathbb{A}^n_k$.

Can replace $\mathbb{A}^n_k$ or any base which is smooth and satisfies Hasse principle and weak approximation.

For $n = 4$, equations of the type

$$\sum_{i=1}^{4} a_i(t)x_i^2 = 0$$

with all $a_i(t) \in k[t]$ and the product of the $a_i(t)$ *squarefree.* Then the Hasse principle and weak approximation hold. Proof uses strong approximation, Tsen's theorem and Tchebotarev's theorem.

7.6. Families of quadrics: counterexamples. For $n = 3$ and $n = 4$, there are counterexamples to the Hasse principle and to weak approximation for equations of the type

$$\sum_{i=1}^{n} a_i(t)x_i^2 = 0$$

with all $a_i(t) \neq 0$.

Let $K = \mathbb{Q}(\sqrt{-1})$.

Counterexample to the Hasse principle

$$y^2 + z^2 = (3 - t^2)(t^2 - 2) \neq 0.$$

Counterexample to weak approximation. Let us consider the surface X

$$y^2 + z^2 = t(t - 1)(t - 3) \neq 0.$$

There are rational points, for instance with $t = 5$. The set of $\mathbb{Q}$-rational points on X is actually Zariski dense on this variety. This may be seen in a number of ways. In fact it is known that a normal cubic surface in $\mathbb{P}^3_k$ (k field of char. zero, but the result is more general) with a nonsingular rational points is unirational over its ground field. Consider the function $t-1$. At all places p but the real and the dyadic place it takes only the value $1 \in Q_p^*/NK_p^*$. On the reals it takes the two values in $\mathbb{R}^*/NC^*$. On the 2-adics it takes the two values in Q_2^*/NK_2^*. Above $t = 7$ we find the conic

$$y^2 + z^2 = 2^3.3.7.$$

This has points in $\mathbb{Q}_2$ and $\mathbb{R}$ (but not in $\mathbb{Q}_3$ and $\mathbb{Q}_7$). Let M_2 be a $\mathbb{Q}_2$-point and M_∞ be an $\mathbb{R}$-point. Then the pair $(M_2, M_\infty) \in X(\mathbb{Q}_2) \times X(\mathbb{R})$ cannot be approximated by a $\mathbb{Q}$-rational point of X.

There are subtler examples of surfaces $X/\mathbb{Q}$ with affine model

$$y^2 + z^2 = xQ(x)$$

with $Q(x)$ irreducible polynomial of degree 2 where the topological space $X(\mathbb{R})$ has two connected components but there are rational points only in one of these components.

From a counterexample to the Hasse principle of the shape $y^2 - az^2 = P(t)$ one gets a counterexample to the Hasse principle of the shape $x_1^2 - ax_2^2 = P(t)(x_3^2 - ax_4^2)$.

Mention of existence of many other counterexamples in the literature. Curves of genus 1, curves of higher genus, cubic surfaces.

7.7. The Brauer group of a scheme: algebra and geometry. Definition. How to compute the Brauer group. Residues. Purity theorem for the Brauer group. Computation of the Brauer group of a conic bundle. See [7].

Application to a smooth projective model X for surfaces of the shape $y^2 - az^2 = R(t)$, with $a \in k^*$.

If $R(t) = P(t)Q(t)$ is separable and P and Q are both of even degree, then the quaternion algebra $(a, P(t)$ over the surface $y^2 - az^2 = R(t) \neq 0$ is unramified above any smooth model of this surface. Proof by showing that it has residue zero at any DVR in the function field of the surface and use of purity. If P and Q are irreducible over k, the class $(a, P(t)$ generates the quotient $\mathrm{Br}\, X/\mathrm{Br}\, k$.

If $R(t) = P(t)Q(t)$ is separable and P and Q are both irreducible of odd degree, then $\mathrm{Br}\, X/\mathrm{Br}\, k$=0.

From now on k denotes a number field.

7.8. The Brauer-Manin obstruction.

Proposition 7.3. *For X/k proper and $A \in \mathrm{Br}\, X$ for almost all v the image of $ev_A : X(k_v) \to \mathrm{Br}\, k_v$ is reduced to 0.*

Example: The algebra $(a, P(t))$ over

$$y^2 - az^2 = P(t)Q(t) \neq 0$$

with PQ separable, P and Q of even degree.

Definition of the Brauer-Manin set $X(\mathbb{A}_k)^{\mathrm{Br}\, X}$ for a smooth, projective, geometrically integral k-variety. The basic inclusion

$$X(k)^{cl} \subset X(A_k)^{\mathrm{Br}\, X},$$

where $X(k)^{cl}$ denotes the closure of the set $X(k)$ in the adèles $X(A_k)$ of X.

Comparison with the argument in the two counterexamples above.

If the equation has solutions in all k_v and there is no Brauer-Manin obstruction, then there exists $\alpha \in k^*$ such that the system

$$y_1^2 - az_1^2 = \alpha P(t) \neq 0,\ y_2^2 - az_2^2 = \alpha^{-1} Q(t) \neq 0$$

has solutions in all k_v.

The following proposition is in some sense a converse of the above proposition.

Theorem 7.4. *(Harari). Let $U \subset X$ be a nonempty Zariski open set of a smooth geometrically integral variety. If $A \in \operatorname{Br} U$ does not come from $\operatorname{Br} X$ then there are infinitely many places v such that the image of the map $ev_A : U(k_v) \to \operatorname{Br} k_v$ is not reduced to one element.*

Proof. See [20]. A simpler proof is given in [6], §1. □

Two examples:

(i) The quaternion algebra (a, t) over $U = Speck[t, t^{-1}]$, for $a \in k^*$ not a square.

(ii) The quaternion algebra $(a, P(t))$ over

$$y^2 - az^2 = P(t)Q(t) \neq 0$$

with a not a square, PQ separable , P and Q of odd degree.

Exercise. Let

$$\sum_{i=1}^{3} a_i(t) x_i^2 = 0$$

be a family of conics. If for almost all $r \in k$ the equation,

$$\sum_{i=1}^{3} a_i(r) x_i^2 = 0$$

has a nontrivial solution in k then he original equation has a nontrivial solution with all $x_i \in k[t]$.

The last proposition may look rather negative. It can be put to good use thanks to:

Theorem 7.5. *(Harari's formal lemma). Let U be a nonempty Zariski open set of a smooth geometrically integral variety X. Let $B \subset \operatorname{Br} U$ be a finite subgroup. Let $\{P_v\} \in U(A_k)$. Assume that for each $\alpha \in \operatorname{Br} U \cap \operatorname{Br} X$ we have*

$$\sum_{v \in \Omega_k} \alpha(P_v) = 0.$$

Then for any finite set S of places of k there exists $\{M_v\} \in U(A_k)$ such that $M_v = P_v$ for $v \in S$ and such that for each $\beta \in B$ we have

$$\sum_{v \in \Omega_k} \alpha(P_v) = 0.$$

Proof. The proof, indeed, is a formal consequence of the previous result. See [20]. See also [6], §1. □

Example: Equation

$$y^2 - az^2 = P(t)Q(t) \neq 0$$

with PQ separable and irreducible, P and Q of odd degree. In this case, if X is a smooth projective model, then $H^1(k, \operatorname{Pic} \overline{X}) = 0$.

If the equation has solutions in all k_v then there exists $\alpha \in k^*$ such that the system

$$y_1^2 - az_1^2 = \alpha P(t) \neq 0,\ y_2^2 - az_2^2 = \alpha^{-1} Q(t) \neq 0$$

has solutions in all k_v.

7.9. The fibration method. From now on k is a number field.

We have a dominant morphism $f : X \to Y$ of smooth proper k-varieties, with (smooth) geometrically integral generic fibre.

We have information on the behaviour of k-points of Y and on the behaviour of k-points on the fibres of f above k-points of Y. We want to extract information on the k-points of the total space X.

Example: conic bundles over $\mathbb{P}^1_k$.

We assume that the base Y satisfies the Hasse principle and weak approximation. For instance the case $Y = \mathbb{P}^1_k$ is already very interesting.

For the fibres of f, on the arithmetic side, we will assume anything between:

(a) The fibres above k-points of a nonempty open set of Y satisfy the Hasse principle and weak approximation.

(b) There is a Zariski-dense set of k-points $m \in Y(k)$ such that for the fibre X_m/k above m in this set we have $X_m(k)^{cl} \subset X_m(A_k)^{\mathrm{Br}\, X_m}$.

The question is then whether for the total space the Hasse principle and weak approximation hold, or at least whether for the total space the Brauer-Manin set coincides with the closure of the set of k-points.

As we shall see, the answer depends very much on the structure of the bad fibres of $f : X \to Y$.

We start with some easy cases, which were used in [11].

Proposition 7.6. *Assume that f has a section over k. If there exists a nonempty Zariski open set U of Y such that weak approximation holds for the fibres X_m for m in $U(k)$, then weak approximation holds for the total space X.*

Proposition 7.7. *Assume that f has a section over k and that Y contain a nonempty open set U set isomorphic to an open set of $\mathbb{A}^n_k$. If there exists a Hilbert set H of points m in $U(k)$ such that the fibres X_m for $m \in H$ satisfy weak approximation, then X satisfies weak approximation.*

Indeed, it is known that any Hilbert set in $\mathbb{A}^n(k)$ is dense in any finite product $\prod_{v \in S} \mathbb{A}^n(k_v)$. A stonger version (due to T. Ekedahl) asserts a strong approximation result for any Hilbert set $H \subset \mathbb{A}^n(k)$.

Proposition 7.8. *Assume that all the fibres of $f : X \to Y$ are geometrically integral, or more generally are split, i.e. contain a component of multiplicity one which is geometrically integral. Assume the Hasse principle and weak approximation hold for Y.*

(i) If there exists a nonempty open set $U \subset Y$ such that the Hasse principle (resp. weak approximation) hold for Y_m with $m \in U(k)$ then the Hasse principle (resp. weak approximation) hold for X.

(ii) Assume that Y is k-rational. If there exists a Hilbert set H of points of points in $Y(k)$ such that for $m \in H$ the Hasse principle (resp. weak approximation) holds for Y_m then the Hasse principle (resp. weak approximation) holds for X.

The proof is not difficult. The hypothesis on the fibres is fundamental. That statement played an important rôle in [11].

What what one would really like to have, say with $Y = \mathbb{P}^1_k$, is a theorem saying that if the Brauer-Manin obstruction is the only one for the fibres then it is also the only one for the total space.

In his thesis [20] and in further work [21], Harari proved the following theorems. The proof builds up on Harari's "formal lemma" mentioned earlier.

Theorem 7.9. *(Harari) Let X/k be a smooth, proper, geometrically integral k-variety and $f : X \to \mathbb{P}^1_k$ be a dominant k-morphism with geometrically integral generic fibre. Assume that all fibres over $\mathbb{A}^1_k \subset \mathbb{P}^1_k$ are split, which is the case if they are geometrically integral. Assume that f has a section over $\overline{k}$. Assume that the Picard group of the geometric generic fibre is finitely generated and torsionfree and that the Brauer group of the geometric generic fibre is a finite group. Assume that for all m in a Hilbert set H of $\mathbb{A}^1(k) = k$ we have*

$$X_m(\mathbb{A}_k)^{\mathrm{Br}\, X_m} \neq \emptyset \implies X_m(k) \neq \emptyset$$

resp. the stronger hypothesis: $X_m(k)$ is dense in $X_m(\mathbb{A}_k)^{\mathrm{Br}\, X_m}$.

Then

$$X(\mathbb{A}_k)^{\mathrm{Br}\, X_m} \neq \emptyset \implies X(k) \neq \emptyset,$$

resp. $X(k)$ is dense in $X(\mathbb{A}_k)^{\mathrm{Br}\, X_m}$.

The assumption on the Picard group and the Brauer group of generic fibre are fullfilled by smmoth complete intersections of dimension at least 3 in projective space. The hypothesis rules out the case of fibrations into curves of genus at least one.

Getting a similar result over $\mathbb{P}^n_k$ for $n \geq 2$ turned out to be delicate. See [36], [21]. However in the presence of a section, the situation is much simpler.

Theorem 7.10. *(Harari)*[20][21] *Let $f : X \to Y$ be a dominant k-morphism of smooth, projective, geometrically integral varieties, with geometrically integral generic fibre. Assume that f has a section. Assume that Y is k-rational. Assume that the Picard group of the geometric generic fibre is finitely generated and torsionfree and that the Brauer group of the geometric generic fibre is a finite group. Assume there exists a Hilbert set H of points in $Y(k)$ such that for any $m \in H$ the fibre X_m/k is smooth and satisfies: $X_m(k)$ is dense in $X_m(\mathbb{A}_k)^{\mathrm{Br}\, X_m}$. Then $X(k)$ is dense in $X(\mathbb{A}_k)^{\mathrm{Br}\, X}$.*

7.10. The descent method. This is to some extent the opposite of the previous method. Typically one starts with a smooth, proper, geometrically integral k-variety X over a number field k, one assumes the Brauer-Manin set is not empty. One shows this implies the existence of at least one k-variety Y with a dominant map $Y \to X$ (in practice, $Y \to X$ is a torsor under a commutative algebraic group) such that Y has points in all completions and, although in general of a higher dimension, is arithmetically simpler than X: in very favourable cases it should satisfy the Hasse principle.

This has been successful for Châtelet surfaces, i.e. surfaces with affine model

$$y^2 - az^2 = P(t)$$

for P a polynomial of degree 4 [11] and more generally for conic bundles over the projective line with at most 4 geometric singular fibres. The case of conic bundles with 5 geometric singular fibres has also been handled (Salberger and Skorobogatov), the proof uses a further technique, related to the study of zero-cycles of degree 1, due to Salberger. Some cases of surfaces $y^2 - az^2 = P(t)$ with $P(t)$ of degree 6 have been handled by Swinnerton-Dyer.

7.11. The conditional method based on Schinzel's hypothesis. Schinzel's hypothesis, an extension of Dirichlet's theorem on primes in an arithmetic progression.

In 1978, Sansuc and I saw that if one accepted Schinzel's hypothesis, then one could prove the Hasse principle for an equation

$$y^2 - az^2 = P(t)$$

with $P(t)$ irreducible, by mimicking Hasse's proof of the Hasse principle for quadratic forms in 4 variables starting from the 3 variables case. Extensions were later given by Serre. The following theorem (CT–Swinnerton-Dyer, Crelle, 1994) is the best result achieved in this direction.

Theorem 7.11. *Assume Schinzel's hypothesis. Let $f : X \to \mathbb{P}^1_k$ be a flat proper map, with X/k smooth and generic fibre geometrically integral. Assume:*

(i) The Hasse principle and weak approximation hold for smooth fibres of f.

(ii) For each closed point $P \in \mathbb{P}^1_k$, with residue field k_P, there is a component $Z_P \subset X_P$ of multiplicity one such that the algebraic closure of k_P in the function field $k(Z_P)$ is **abelian**.

Then $X(k)$ is dense in $X(A_k)^{\mathrm{Br}\,X}$.

The simplest example. If $P(t) \in k[t]$ is an irreducible polynomial, if Schinzel's hypothesis holds, then the Hasse principle and weak approximation hold for the surface $y^2 - az^2 = P(t)$. In the case where P is reducible and has factors of odd degree, the proof that the Brauer-Manin condition implies the existence of a rational point uses Harari's formal lemma.

The theorem applies more generally to conic bundles over $\mathbb{P}^1_k$, to families of Severi-Brauer varieties over $\mathbb{P}^1_k$ (this case had been considered by Serre).

Here are two serious problems

(1) We have not been able to eliminate the hypothesis of abelianity in the above theorem.

(2) We do not know how to prove a similar theorem under the weaker assumption that $X_m(k)$ is dense in $X_m(A_k)^{\mathrm{Br}\,X_m}$ for smooth fibres $X_m, m \in \mathbb{P}^1(k)$ (compare Harari's result in the case where all fibres but at most one are split).

The two problems are related.

This is a nuisance. For instance this prevents us from handling such simple equations as

$$N_{K/k}(\sum_{i=1}^{4} x_i\omega_i) = P(t)$$

for K/k a biquadratic extension with basis $\omega_i, i = 1, \ldots, 4$ over k and $P(t)$ a polynomial of degree 3.

8. Fibrations on linear algebraic groups

This section will be improved

Let G be a reductive algebraic group over a field k. There are two natural ways to fibre the underlying variety.

The first one is via the variety of tori.

The second one is via the adjoint representation of G on itself.

Both have been used in the study of the arithmetic of linear algebraic groups.

8.1. The variety of tori. Let G be a reductive group over k. Let T be a fixed maximal k-torus of G. Let $N \subset G$ denote the normalizer of T. let $X = N/T$. This is finite étale k-group scheme. Consider the map

$$\varphi : G/T \times T \to G/N \times G$$

defined by

$$(gT, t) \mapsto (gN, gtg^{-1}).$$

Given an element $n \in N$, the map

$$(gT, t) \mapsto (gn^{-1}T, ntn^{-1})$$

induces a left action of W on $G/T \times T$. We have

$$\varphi(w.(gT, t)) = \varphi((gT, t)).$$

Let $H \subset G/N \times G$ denote the image of φ.

The k-variety G/N is the variety of maximal k-tori of G. For any field F containing k, the set $(G/N)(F)$ is in bijection with the set of maximal F-tori in G.

The points of $H(F)$ are given by a pair consisting of a maximal F-torus and an F-point in that torus. The fibration $H \to G/N$ is the family of tori in G.

Inside H we have the open set H^{reg} whose points are given by a pair consisting of a maximal torus T_1 and a regular (semisimple) element $t_1 \in T_1$. The centralizer of such an element x_1 is the maximal torus T_1. The projection map $H \to G$ induces an open embedding $H^{reg} \subset G$. Thus H is k-birational to G.

The cover $G/T \to G/N$ is a torsor under the finite étale k-group scheme W. So is the cover $G/T \times T \to H$.

One has the cartesian square

$$\begin{array}{ccc} G/T \times T & \to & H \\ \downarrow & & \downarrow \\ G/T & \to & G/N \end{array} \tag{8.1}$$

A theorem of Chevalley asserts that the variety G/N is a k-rational variety (see Gille's first lecture).

Suppose that k is algebraically closed. The above diagram shows that the generic torus T_η splits over a field extension with group $W = W(k)$. It can be showed that the extension $k(G/T)/k(G/N)$ indeed is the (smallest) splitting field of the generic torus T_η.

References for this section: Voskresenskiĭ's book.

8.2. The Steinberg map. Let G/k be a semisimple group.

Let $k[C] = k[G]^G$ be the ring of invariants under the adjoint action of G on itself: an element g of G acts by $x \mapsto gxg^{-1}$. We thus have an affine k-scheme C and a morphism $G \to C = G//G$. This is called the Steinberg map (see [39]).

Simplest example: the map $SL_2 \to \mathbb{A}^1$ sending a matrix to its trace. Next example: the map $SL_n \to \mathbb{A}^{n-1}$ sending a matrix to the coefficients of its characteristic polynomial.

Let $T \subset G$ be a maximal torus. There is a natural map $k[G]^G \to k[T]^N = k[T]^W$. This map is an isomorphism (Chevalley). One thus has $T/W \xrightarrow{\simeq} C$.

One then considers the map

$$G/T \times T \to G$$

given by

$$(gT, t) \mapsto gtg^{-1}.$$

One has the commutative diagram

$$\begin{array}{ccccc} G/T \times T & \rightarrow & & & G \\ \downarrow & & & & \downarrow \\ T & \rightarrow & T/W & \stackrel{\simeq}{\rightarrow} & C \end{array} \tag{8.2}$$

Both horizontal maps are generically W-torsors. Over the open set $C^{reg} \subset C$ corresponding to semisimple regular elements of G, all the restrictions of the maps are smooth. The fibre of $G \to C$ over a k-point in this open set is a G-homogeneous space with geometric stabilizer a torus of maximal rank.

If G is simply connected, the ring $k[T]^N$ is a polynomial ring, that is C is k-isomorphic to affine space $\mathbb{A}^r_k$, where r is the (geometric) rank of G.

See [39] p. 62 in the split case. See [27] for the nonsplit case.

Question : Over a field k of characteristic zero, is any twisted form of affine space $\mathbb{A}^n_k$ k-isomorphic to affine space $\mathbb{A}^n_k$?

Over a nonperfect field k, there are twisted forms of the affine line which are not isomorphic to affine line $\mathbb{A}^1_k$. Indeed such is the case for the complement in $\mathbb{P}^1_k$ of a closed point whose residue field isa nontrivial purely inseparable extension of k.

If the group G/k is split, then the Steinberg map has a section over k. This result of Steinberg (loc.cit.) is a generalization of the existence of the companion matrix. The existence of a section over k implies that the group G/k is quasisplit. For most types of groups this is a sufficient condition. See [39]. (Is this always a suffiicent condition ?)

Remark There is a Lie algebra version of the Steinberg map, which goes under the name of adjoint representation, and has been much discussed (Chevalley, Kostant). One fixes a maximal k-torus $T \subset G$. One lets $\mathfrak{h} \subset \mathfrak{g}$ be the corresponding Lie algebras. The group G acts on $\mathfrak{g}$ by the adjoint representation.

$$\begin{array}{ccccc} G/T \times \mathfrak{h} & \rightarrow & & & \mathfrak{g} \\ \downarrow & & & & \downarrow \\ \mathfrak{h} & \rightarrow & \mathfrak{h}/W & \stackrel{\simeq}{\rightarrow} & \mathfrak{g}//G. \end{array} \tag{8.3}$$

The k-variety $\mathfrak{h}/W \stackrel{\simeq}{\rightarrow} \mathfrak{g}//G$ is an affine space $\mathbb{A}^r_k$.

8.3. Summary. Let G/k be a semisimple group. We have the basic diagram, where the upper maps are induced by projection of $G/T \times T$ to the first factor and the bottom maps are induced by the projection to the second factor.

$$\begin{array}{ccccc} G/T & \rightarrow & G/N & & \\ \uparrow & & \uparrow & & \\ G/T \times T & \rightarrow & H & \rightarrow & G \\ \downarrow & & \downarrow & & \downarrow \\ T & \rightarrow & T/W & \stackrel{\simeq}{\rightarrow} & C. \end{array} \tag{8.4}$$

The morphism $H \to G$ is birational.

For G simply connected, the K-variety C is k-isomorphic to affine space.

8.4. The variety of tori and weak approximation.

Theorem 8.1. *(Kunyavskiĭ–Skorobogatov, appendix to* [36]*)*

(i) Let G be a reductive group over a number field k. Let K be the function field of the variety of tori. Let T_η denote the generic torus. If $Sha^2_\omega(g, \hat{T}_\eta) = 0$ then weak approximation holds for G.

(ii) If G is semisimple and simply connected, then weak approximation holds for G.

Part (i) follows from the above discussion together with the fact that if T is a torus over a number field such that $Sha^2_\omega(k, \hat{T}) = 0$ then weak approximation holds for T (we have seen this earlier in this lecture course).

Part (ii) follows from the statement that in the simply connected case we have $Sha^2_\omega(g, \hat{T}_\eta) = 0$. This is a result of Kunyavskiĭ and Voskresenskiĭ in the case A_n and of Klyachko [24] in the general case.

This proof looks quite different from the classical proofs of weak approximation for simply connected groups. The original paper by Kneser [25] gave a case by case proof for the classical groups. The papers by Kneser [26] and [27] are devoted to the proof of the strong approximation theorem for simply connected groups, but proves en route that weak approximation holds ([27] p. 152). Reduction theory is used in the process. The proof in the book [32] also uses reduction theory, moreover it uses some measure theory. More on the papers by Kneser below.

Harari's general theorem on fibrations with a section, applied to the family $H \to G/N$ yields:

Theorem 8.2. *(Harari* [20]*) Let G be a connected linear algebraic group and $G \subset X$ a smooth compactification. Then*

(i) $X(k)$ is dense in $X(\mathbb{A}_k)^{\mathrm{Br}\, X}$.

(ii) If $H^1(k, \mathrm{Pic}\, \overline{X}) = 0$, then G satisfies weak approximation.

(iii) If G is simply connected, or is adjoint, or is absolutely almost simple, weak approximation holds for G.

Proof. (Sketch) The proof of (i) simply uses the fact that for a smooth compactification T_c of a torus T we know that $T_c(k)$ is dense in $T_c(\mathbb{A}_k)^{\mathrm{Br}\, T_c}$. This is essentially a consequence of Tate-Nakayama theory, together with an easy identification of certain maps in Galois cohomology.

Since X is a rational variety over $\overline{k}$ we have $\mathrm{Br}\, X/\mathrm{Br}\, k \simeq H^1(k, \mathrm{Pic}\, \overline{X})$. This proves (ii).

As for (iii) we know that $H^1(k, \mathrm{Pic}\, \overline{X}) = 0$ holds for each of the given types of groups. (For G simply connected, $\mathrm{Pic}\, \overline{X}$ is a permutation module.) □

Note that this proof does not require knowledge of the splitting structure of the generic torus, as opposed to the proof by Kunyavskiĭ and Skorobogatov. Moreover it gives the best result for arbitrary groups: one can thus do without reductions to the case of semisimple or quasitrivial groups. No need of flasque resolutions, except for tori.

8.5. The Steinberg map and strong approximation. In [26] and [27], Kneser proves the strong approximation theorem for semisimple groups.

The key case of the strong approximation theorem is the following statement.

Theorem 8.3. *Let G be an absolutely almost simple and simply connected group over a number field k. Let v_0 be a place such that $G(k_{v_0})$ is not compact (i.e. $G_{k_{v_0}}$ is isotropic). Then the set $G(k).G(k_{v_0})$ is dense in the group $G(A_k)$ of adèles of G.*

There are several proofs available, which are not totally independent: Kneser generalizing earlier results of Eichler; Platonov (see [32]; Margulis (Margulis in his book takes weak approximation for granted).

I shall describe the main steps of Kneser's proof in the special case of $G = SL(D)$, the special linear group of a central simple algebra D over a number field k of degree n. The result is then a theorem of Eichler. I shall follow Kneser's proof, as given in [26]. As Kneser points out, this gives a good idea of the general case, dealt with in [27]. I will try to point out which arguments extend to general groups.

Kneser's proof is a fibration method. The fibration is given by the Steinberg map, which in this case is very classical, it is the map

$$\varphi : SL(D) \to \mathbb{A}_k^{n-1}$$

sending an element to the coefficient of the reduced characteristic polynomial. Let us write $G = SL(D)$ and $Y = \mathbb{A}_k^{n-1}$. We thus have the map

$$\varphi : G \to Y.$$

An element in $G(k)$ is called regular if its reduced characteristic polynomial has no multiple factor. The set of such elements is a nonempty Zariski open set $G^{reg} \subset G$. The restriction of φ to G^{reg} is smooth. We denote by Y^{reg} the image of G^{reg} in Y. The fibre of φ above a k-point of Y^{reg} is a homogeneous space under G. The geometric stabilizers are maximal tori in $G \times_k \overline{k}$. Furthermore, if $T \subset G$ is a maximal k-torus, the restriction of φ to $T^{reg} \subset G$ defines a finite étale map $T^{reg} \to Y^{reg}$. Something special to the situation is the description of the maximal tori in $G = SL(D)$. They are of the shape $T = R^1_{F/k}G_m$, where $F \subset D$ is a separable commutative subalgebra of (maximal) rank n. Such algebras are classified by $H^1(k, \mathfrak{S}_n)$. The character group of a torus $R_{F/k}G_m$ is a (direct sum) of permutation modules of $\mathfrak{S}_n$, the basis corresponding to the various k-homomorphisms $K \to \overline{k}$. The character group of $T = R^1_{F/k}G_m$ is the cokernel of the diagonal inclusion $\mathbb{Z} \to \hat{T}$ where the map is $1 \to N_{K/k}$.

We fix an integral structure on G, over the ring O of integers of k. As usual, O_v denotes the completion of O at a finite place v.

Let v_0 be as in the theorem. Let S be finite set of places of k, $v_0 \notin S$. Let $U \subset G(A_k)$ be an open set which is a product of open sets $U_v \subset G(k_v)$, almost all of them equal to $G(O_v)$. Set $U_{v_0} = G(k_{v_0})$. To prove the density property we may take Let U_v for $v \in S$ small enough so that all elements in $U_v \subset G(k_v)$ are regular.

To prove the theorem it is enough to prove the following claim :

Claim *There exists a point in $G(k)$ which lies in each $U_v \subset G(k_v)$.*

The proof consists of the following steps.

(1) Find $m \in Y^{reg}(k)$ such that the fibre $Y_m \subset G$ contains points in U_v for each place v except possibly at the place v_0.

(2) Find $m \in Y^{reg}(k)$ such that the fibre $Y_m \subset G$ contains points in U_v for each place v (that is, it also possesses a point in $G(k_{v_0})$.)

(3) Find $m \in Y^{reg}(k)$ such that the fibre $Y_m \subset G$ contains points $g_v \in U_v$ for each place v and contains a k-point $g \in G(k)$.

(4) Show that g and the adèle $\{g_v\}_{v\in\Omega}$ are conjugate under the action of the adelic group D^*_A.

(5) There exists a *compact* open set $K = \prod_{v\in\Omega} K_v \subset D^*_A$ with $1 \in K_v$ for each v and K_v equal to $D^*_{O_v}$ for almost all v, such that $D^*_A = D^*.D^*_{k_{v_0}}.K^{-1}$.

(6) There exists a point $g \in G(k)$ which for each place v of k belongs to the open set $U_v^{K_v}$ of $G(k_v)$ which is the image of the map

$$U_v \times K_v \to G(k_v)$$

$$(g,k) \mapsto kgk^{-1}.$$

Now for each $v \neq v_0$ the $U_v^{K_v}$ as U_v varies are a base of neighbourhoods for $a_v \in G(k_v)$.

Renaming the U_v' s, this is enough to conclude.

Let us now discuss the various steps.

For almost all places v of k, the algebra D_{k_v} is split. Over such places v, the Steinberg map has a section. Thus for such places the map $G(k_v) \to Y(k_v)$ is surjective. Moreover for almost all places v the map $G(O_v) \to Y(O_v)$ is onto.

This argument is general, at least for inner forms. If G is a semisimple simply connected group, for almost all places v, the variety of Borel subgroups of G has a k_v-point, hence, at least for inner forms, the Steinberg map has a section over k_v ([39]). To get the statement over O_v, one looks at the scheme of Borel subgroups over a suitable open set of $SpecO$ and one applies Hensel's lemma.

For any place v, the map $G^{reg}(k_v) \to Y(k_v)$ is open. Thus each $\varphi(U_v)$ is open in $Y(k_v)$.

Using strong approximation for affine space we conclude that there exists a k-point m as in (1).

We could conclude that there is an m as in (2) if we had $\varphi(G(k_{v_0})) = Y(k_{v_0})$. That is the case for instance if D is split at v_0, which indeed would be the case if D was of prime degree. In the general case, an extra effort is needed to establish (2). The starting point is that the hypothesis $G(k_{v_0})$ not compact implies the existence of a k_{v_0}-homomorphism of groups

$$\mathbb{G}_{m,k_{v_0}} \subset G_{k_{v_0}}.$$

In the next instalment of these notes, I will try to explain how this enables one to establish (2). For the time being I refer to [26] p. 195, Hilfssatz 3.3 for the case $G = SL(D)$ and to [27] p. 191 for arbitrary simply connected groups.

This gives (2).

Getting from (2) to (3) is a Hasse principle problem. In the present case, the statement is :

For any k-point $m \in Y^{reg}(k)$ the fibre Y_m satisfies the Hasse principle.

This is a consequence of Kneser's Hilfssatz 3.4 ([26]) :

A regular polynom of degree n in $k[t]$ (with leading coefficient s 1 and constant coefficient $(-1)^n$) is the reduced characteristic polynom of an element of $SL(D)$ if and only if it is so locally. Kneser proves this by using the purely algebraic fact ([26], Hilfssatz 4.1):

Over a field k, a regular polynomial $P \in k[t]$ is the reduced characteristic polynom of an element in D if and only if $D \otimes_k k[t]/P(t)$ is totally split.

One then uses the Hasse principle for the Brauer group of finite extension of a number field.

This gives (2) in the case $G = SL(D)$.

For G/k an arbitrary semisimple simply connected group G one cannot hope for the Hasse principle for all fibres Y_m.

What Kneser shows in [27] is that there are at least many k-points $m \in Y^{reg}(k)$ with the property (2) such that Y_m satisfies the Hasse principle.

The proof of Kneser is rather involved (Proof of intermediate step, p. 193-196).

Let us explain how this can be seen from the point of view of the Brauer-Manin obstruction.

Let $m \in Y^{reg}(k)$. The fibre Y_m, as explained earlier, is a homogeneous space of G. The geometric stabilizers are tori.

Given a simply connected group G/k a G-homogeneous space Z whose geometric stablizers are tori, there is a natural k-torus associated to the situation. Borovoi has shown (Harari might explain this in his lectures) that if $Sha^1_{\omega}(k, \hat{T}) = 0$ then the Hasse principle (and weak approximation) hold for Z. For this he uses "highly twisted tori". These are precisely the tori such that the image of the Galois group on the character group of T contains the Weyl group of G relative to T. This builds upon the Hasse principle for $H^1(k, G)$.

In the case $G = SL(D)$, for any $m \in Y^{reg}(k)$ we have $T_m = R^1_{F_m/k}\mathbb{G}_m$ for some separable commutative algebra F_m/k of degree n and one easily proves $Sha^1_{\omega}(k, \hat{T_m}) = 0$.

It is very likely that Kneser's result in the general case (whose details are not given in [27]) (see p. 194) can be reached by proving:

Let Y_η be the generic fibre of $G \to Y$. Let T_η be the associated torus. Then $Sha^1_{\omega}(k(\mathbb{A}^n), \hat{T}_\eta) = 0$ for G simply connected.

Then an argument combining Hilbert's irreducibility theorem (the variant with strong approximation on $\mathbb{A}^n$, as established by Ekedahl) then produces an $m \in Y^{reg}(k)$ such that (3) holds.

This argument is the analogue for the Steinberg fibration of the argument which Kunyavskiĭ and Skorobogatov apply to the torus fibration to prove weak approximation.

Once we have (3), (4) follows essentially formally, using the fact that two regular elements in D^*_F with the same reduced characteristic polynomial are conjugate over F. In [27] there is no indication how to prove an analogous result for arbitrary semisimple simply connected groups G.

Statement (5) is a consequence of reduction theory.

Statement (6) is an immediate consequence of (5).

This completes the proof.

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