On the transcendental Brauer group

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Abstract

For a smooth and projective variety X over a field k of characteristic zero we prove the finiteness of the cokernel of the natural map from the Brauer group of X to the Galois-invariant subgroup of the Brauer group of the same variety over an algebraic closure of k. Under further conditions on k, e.g. over number fields, we give estimates for the order of this cokernel. We emphasise the rôle played by the exponent of the discriminant groups of the intersection pairing between the groups of divisors and curves modulo numerical equivalence.

Résumé

Soit X une variété projective et lisse sur un corps k de caractéristique zéro. Le groupe de Brauer de X s'envoie dans les invariants, sous le groupe de Galois absolu de k, du groupe de Brauer de la même variété considérée sur une clôture algébrique de k. Nous montrons que le quotient est fini. Sous des hypothèses supplémentaires, par exemple sur un corps de nombres, nous donnons des estimations sur l'ordre de ce quotient. L'accouplement d'intersection entre les groupes de diviseurs et de 1-cycles modulo équivalence numérique joue ici un rôle important.

Introduction

Let X be a smooth, projective and geometrically integral variety over a field k of characteristic zero. Let $\overline{k}$ be an algebraic closure of k. Let $\Gamma = \mathrm{Gal}(\overline{k}/k)$ and $\overline{X} = X \times_k \overline{k}$. The kernel of the natural map of Brauer groups $\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})$ is denoted by $\mathrm{Br}_1(X)$ and is called the *algebraic* Brauer group of X. The image of this map is called the *transcendental* Brauer group of X; it is a subgroup of the group of invariants $\mathrm{Br}(\overline{X})^\Gamma$. We thus have the inclusion of groups $\mathrm{Br}(X)/\mathrm{Br}_1(X) \subset \mathrm{Br}(\overline{X})^\Gamma$. One would like to compute these groups, for example, in connection with applications to the Brauer–Manin obstruction. The following question was recently discussed in [20] and [2].

If k is finitely generated over $\mathbb{Q}$, are these two groups finite?

In this paper we show that this double question reduces to a single one. For an arbitrary ground field k of characteristic 0 we prove that the cokernel of the natural map

$$\alpha : \mathrm{Br}(X) \to \mathrm{Br}(\overline{X})^\Gamma$$

is finite. Under further conditions on the ground field, for instance over number fields, we give estimates for the exponent and the order of this finite group.

The main tool of the paper is a natural complex (see Subsection 1.3)

$$\mathrm{Br}(X) \xrightarrow{\alpha} \mathrm{Br}(\overline{X})^\Gamma \xrightarrow{\beta} \mathrm{H}^2(k, \mathrm{Pic}(\overline{X}))$$

which for X with a k-point or for k a number field is an exact sequence. We have two approaches to the calculation of the (finite) image of β which give closely related though not identical estimates.

The first method is presented in Section 2. The main idea is to use the functoriality of the above complex with respect to morphisms of k-varieties and the triviality of the Brauer group of curves over an algebraically closed field (Tsen's theorem). It follows that the image of β has trivial restriction over any closed curve in X. This eventually leads to Theorems 2.1 and 2.2.

Our second approach exploits a general remark about differentials in the spectral sequence of composed functors. Let $\mathrm{Br}^0(\overline{X})$ be the maximal divisible subgroup of $\mathrm{Br}(\overline{X})$, and let $\mathrm{NS}(\overline{X})$ be the Néron–Severi group. We interpret the composed map

$$\mathrm{Br}^0(\overline{X})^\Gamma \hookrightarrow \mathrm{Br}(\overline{X})^\Gamma \xrightarrow{\beta} \mathrm{H}^2(k, \mathrm{Pic}(\overline{X})) \to \mathrm{H}^2(k, \mathrm{NS}(\overline{X})/_{\mathrm{tors}})$$

as the connecting homomorphism attached to a certain natural 2-extension of Γ-modules provided by the Kummer sequence, see Corollary 3.4. A theorem of Lieberman, recalled in Subsection 1.1, states that numerical and homological equivalences coincide on 1-dimensional algebraic cycles. (For surfaces Lieberman's result reduces to a classical theorem of Matsusaka.) Using this we show in Proposition 4.1 that the image of the above composite map is annihilated by the exponent of either of the two discriminant groups defined by the intersection pairing between the groups of divisors and curves on $\overline{X}$ modulo numerical equivalence. The estimates for the cokernel of α obtained by this method are proved in Theorems 4.2 and 4.3.

In Section 5 we discuss K3-surfaces and products of two curves. For such a surface with a k-point our result is particularly easy to state: the cokernel of α is annihilated by the exponent of the discriminant group defined by the intersection pairing on $\mathrm{NS}(\overline{X})/_{\mathrm{tors}}$, see Propositions 5.1 and 5.2.

T. Szamuely asked whether the finiteness result in Theorem 2.1 also holds for smooth, quasiprojective varieties. In Section 6 we give a positive answer in the case when the ground field k is finitely generated over $\mathbb{Q}$.

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1 Preliminaries

Let k be a field of characteristic 0 with an algebraic closure $\overline{k}$ and the absolute Galois group $\Gamma = \mathrm{Gal}(\overline{k}/k)$. Let X be a smooth, projective and geometrically integral variety over k. Let $\overline{X} = X \times_k \overline{k}$. Let $d = \dim(X)$.

If A is an abelian group we denote by $A[n]$ the set of elements $a \in A$ such that $na = 0$. For a prime number ℓ we denote by $A\{\ell\}$ the set of elements $a \in A$ such that $\ell^m a = 0$ for some $m \geq 1$.

1.1 Algebraic cycles

Let $\mathrm{CH}^i(\overline{X})$, $0 \leq i \leq d$, be the Chow group of codimension i cycles on $\overline{X}$, i.e., the group of linear combinations of irreducible subvarieties of codimension i with coefficients in $\mathbb{Z}$ modulo rational equivalence. We have $\mathrm{CH}^1(\overline{X}) = \mathrm{Pic}(\overline{X})$. Let $\mathrm{NS}(\overline{X})$ be the Néron–Severi group of $\overline{X}$, defined as the quotient of $\mathrm{Pic}(\overline{X})$ by its divisible subgroup $\mathrm{Pic}^0(\overline{X})$.

Since X is projective, the intersection index defines the Γ-equivariant bilinear form

$$\mathrm{CH}^i(\overline{X}) \times \mathrm{CH}^{d-i}(\overline{X}) \rightarrow \mathbb{Z}. \tag{1}$$

Let $N^i = \mathrm{Num}^i(\overline{X})$ be the group of codimension i cycles on $\overline{X}$ modulo numerical equivalence, defined as the quotient of $\mathrm{CH}^i(\overline{X})$ by the (left) kernel of the pairing (1). Write $N_i = N^{d-i}$. We obtain a Γ-equivariant bilinear form

$$N^i \times N_i \to \mathbb{Z} \tag{2}$$

with trivial left and right kernels. For every $i \geq 0$ the abelian group N^i is free and finitely generated. This follows from the existence of a Weil cohomology theory with coefficients in a field of characteristic zero, equipped with cycle maps transforming intersections of cycles into cup-product in cohomology [9, Thm. 3.5, p. 379].

The pairing (2) for $i = 1$ gives rise to the exact sequence of Γ-modules

$$0 \to N^1 \to \mathrm{Hom}(N_1, \mathbb{Z}) \to D \to 0, \tag{3}$$

which is the definition of the finite Γ-module D. This group is one of the two discriminant groups associated to the pairing $N^1 \times N_1 \to \mathbb{Z}$.

For all $i \geq 0$ we have the cycle class maps

$$\mathrm{CH}^i(\overline{X}) \longrightarrow \mathrm{H}^{2i}_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(i)),$$

see [12, Section VI.9] and [19, Cycle]. These maps transform cup-product in ℓ-adic cohomology into intersection of algebraic cycles, see [12, Prop. VI.9.5]. Let us define the Γ-modules

$$N^i_\ell = N^i \otimes \mathbb{Z}_\ell, \quad N_{i,\ell} = N_i \otimes \mathbb{Z}_\ell, \quad H^{2i}_\ell = \mathrm{H}^{2i}_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(i))/_{\text{tors}}.$$

For any cohomological theory with cycle maps for which intersection on cycles is compatible with cup-product on cohomology, homological equivalence implies numerical equivalence. It is a part of the "standard conjectures" that for any good cohomology theory, homological and numerical equivalences coincide. In the ℓ-adic étale set-up the ℓ-adic cycle map should factorise as

$$\mathrm{CH}^i(\overline{X}) \otimes \mathbb{Z}_\ell \to N^i_\ell \hookrightarrow H^{2i}_\ell. \tag{4}$$

This is true for $i = 1$ by a classical theorem of T. Matsusaka [11], who proved that $N^1 = \mathrm{NS}(\overline{X})/_{\text{tors}}$.

In the case of Betti cohomology, with cycle maps

$$\mathrm{CH}^i(X) \to \mathrm{H}^{2i}_{\text{Betti}}(X(\mathbb{C}), \mathbb{Q}(i)),$$

where $\mathbb{Z}(i) = \mathbb{Z}(2\pi\sqrt{-1})^{\otimes i}$, for $i = d-1$, this was proved by D. Lieberman [10, Cor. 1]. A more algebraic version is given by Kleiman [9, Remark 3.10]. These references prove more results, some of which rely on the Hodge index theorem. The case $i = d-1$ is simpler, as we now explain.

Proposition 1.1 *Let X be a smooth, connected, projective variety over $\mathbb{C}$ of dimension d. Define the homological equivalence on cycles via Betti cohomology with rational coefficients. Then the natural map*

$$CH_1(X)/\mathrm{hom} \longrightarrow CH_1(X)/\mathrm{num}$$

is an isomorphism.

Proof. We may assume $d = \dim(X) \geq 3$. Let $L \in CH^1(X)$ be the class of a hyperplane section. Multiplication by $L^{d-2} \in CH^{d-2}(X)$ induces a commutative diagram of $\mathbb{Q}$-vector spaces (where $A_{\mathbb{Q}} := A \otimes_{\mathbb{Z}} \mathbb{Q}$):

$$\begin{tikzcd}
CH^{d-1}(X)_{\mathbb{Q}}/\mathrm{num} & CH^{d-1}(X)_{\mathbb{Q}}/\mathrm{hom} \arrow[l] \arrow[r, hook] & Hdg^{d-2}(X,\mathbb{Q}) \arrow[r, hook] & H^{2d-2}(X,\mathbb{Q}(d-1)) \\
CH^1(X)_{\mathbb{Q}}/\mathrm{num} \arrow[u] & CH^1(X)_{\mathbb{Q}}/\mathrm{hom} \arrow[l] \arrow[r, hook] \arrow[u] & Hdg^2(X,\mathbb{Q}) \arrow[r, hook] \arrow[u] & H^2(X,\mathbb{Q}(1)) \arrow[u]
\end{tikzcd}$$

The leftward horizontal arrows are onto. All the rightward horizontal arrows are by definition injective. The fourth vertical map is an isomorphism by the hard Lefschetz theorem. The third vertical map, on Hodge classes, is then an isomorphism, because of the Hodge decomposition of the groups $H^i(X,\mathbb{C})$, and the fact that (p,q)-type classes go to $(p+d-2, q+d-2)$-type classes for $p = 0,1,2$. The map $CH^1(X)_{\mathbb{Q}}/\mathrm{hom} \to Hdg^2(X,\mathbb{Q})$ is an isomorphism by the Lefschetz (1,1)-theorem. All this implies that the second vertical arrow is also an isomorphism. The left vertical arrow is thus surjective. By definition, the two finite dimensional vector spaces $CH^1(X)_{\mathbb{Q}}/\mathrm{num}$ and $CH^{d-1}(X)_{\mathbb{Q}}/\mathrm{num}$ have the same dimension. The left vertical arrow is thus an isomorphism. The left bottom arrow is an isomorphism by Matsusaka's theorem. We conclude that the map $CH^{d-1}(X)_{\mathbb{Q}}/\mathrm{hom}\to CH^{d-1}(X)_{\mathbb{Q}}/\mathrm{num}$ is an isomorphism. This implies that the map $CH^{d-1}(X)/\mathrm{hom}\to CH^{d-1}(X)/\mathrm{num}$ is an isomorphism of finitely generated free abelian groups. QED

Let us now recall how several comparison theorems imply (4) for $i = d-1$, where X is a smooth, projective and geometrically integral variety over a field k of characteristic zero. Recall that for an algebraically closed field L containing k the Néron–Severi group of $X_L = X \times_k L$ does not depend on the field L, because it is the group of connected components of the Picard scheme $\mathrm{Pic}_{X_L/L}$. So we can use the notation N^1 without the risk of confusion.

Let C be a 1-cycle on $\overline{X}$ which is numerically equivalent to zero. There exist a subfield $K \subset \overline{k}$ finitely generated over $\mathbb{Q}$, a variety $\tilde{X}$ over K, and a

1-cycle $\tilde{C}$ on $\tilde{X}$ such that $X = \tilde{X} \times_K \overline{k}$ and $C = \tilde{C} \times_K \overline{k}$. Without loss of generality we can assume that N^1 is generated by the classes of reduced, absolutely irreducible, effective divisors $D_1, \ldots, D_r$ defined over K. We choose an embedding $K \subset \mathbb{C}$. Let $\overline{K}$ be the algebraic closure of K in $\mathbb{C}$, and let $\tilde{X}_{\overline{K}} = \tilde{X} \times_K \overline{K}$, $\tilde{X}_{\mathbb{C}} = \tilde{X} \times_K \mathbb{C}$. The cycle C goes to zero in $N_1 = \mathrm{Num}_1(\overline{X})$ if and only if C intersects trivially with $D_1, \ldots, D_r$. But then $\tilde{C}$ goes to zero in $\mathrm{Num}_1(\tilde{X}_{\mathbb{C}})$. By Prop. 1.1 $\tilde{C}$ goes to zero in the Betti cohomology group $\mathrm{H}^{2d-2}(\tilde{X}_{\mathbb{C}}, \mathbb{Q}(d-1))$.

The comparison theorem for étale and Betti cohomology ([18] XI, XVI, see also [12, Thm. III.3.12]) gives natural isomorphisms

$$\mathrm{H}^{2i}_{\text{ét}}(\tilde{X}_{\mathbb{C}}, \mathbb{Q}_\ell(i)) \cong \mathrm{H}^{2i}_{\text{Betti}}(\tilde{X}_{\mathbb{C}}(\mathbb{C}), \mathbb{Q}(i)) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell.$$

The cycle class maps transform cup-product in cohomology into intersection pairing of Chow groups.

Starting from this, one can prove that the Betti cycle map and the ℓ-adic cycle map with rational coefficients are compatible with these isomorphisms. A proof of this is sketched in [3, p. 21]. J. Riou showed us how a formal proof can be deduced from the uniqueness statement for cycle maps in [17, Prop. 1.2].

Since the natural map

$$\mathrm{H}^{2d-2}_{\text{ét}}(\tilde{X}_{\overline{K}}, \mathbb{Q}_\ell(d-1)) \longrightarrow \mathrm{H}^{2d-2}_{\text{ét}}(\tilde{X}_{\mathbb{C}}, \mathbb{Q}_\ell(d-1))$$

is an isomorphism of vector spaces over $\mathbb{Q}_\ell$ (cf. [12, Cor. VI.4.3]), the cycle class map sends $\tilde{C}$ to zero in $\mathrm{H}^{2d-2}_{\text{ét}}(\tilde{X}_{\overline{K}}, \mathbb{Q}_\ell(d-1))$. Extending the ground field from $\overline{K}$ to $\overline{k}$ we get (4) for $i = d-1$.

As recalled above, the pairing (1) is compatible with the cup-product pairing

$$\mathrm{H}^{2i}_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(i)) \times \mathrm{H}^{2d-2i}_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(d-i)) \to \mathbb{Z}_\ell$$

via the cycle class map. Hence we obtain the commutative diagram of pairings of Γ-modules

$$\begin{tikzcd} N^1 \arrow[d] & \times & N_1 \arrow[d] & \to & \mathbb{Z} \arrow[d] \\ H^2_\ell & \times & H^{2d-2}_\ell & \to & \mathbb{Z}_\ell \end{tikzcd} \tag{5}$$

with injective vertical maps. We shall use the following statement: the bottom pairing in (5) is perfect, i.e., it induces isomorphisms

$$H^2_\ell = \mathrm{Hom}_{\mathbb{Z}_\ell}(H^{2d-2}_\ell, \mathbb{Z}_\ell), \quad H^{2d-2}_\ell = \mathrm{Hom}_{\mathbb{Z}_\ell}(H^2_\ell, \mathbb{Z}_\ell).$$

L. Illusie tells us that this $\mathbb{Z}_\ell$-version can be proved using Deligne's $\mathbb{Z}_\ell$-adic formalism [4, §1.1]. Poincaré duality for the complex $R\Gamma(X, \mathbb{Z}/\ell^n)$ (see [18, XVIII]) gives rise to a perfect duality for the perfect complex $R\Gamma(X, \mathbb{Z}_\ell)$. Then one applies a universal coefficient theorem argument.

The exact sequence (3) gives rise to the exact sequence

$$0 \to N_\ell^1 \to \mathrm{Hom}_{\mathbb{Z}_\ell}(N_{1,\ell}, \mathbb{Z}_\ell) \to D\{\ell\} \to 0, \tag{6}$$

where the second arrow factors as

$$N_\ell^1 \to H_\ell^2 \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_\ell}(H_\ell^{2d-2}, \mathbb{Z}_\ell) \to \mathrm{Hom}_{\mathbb{Z}_\ell}(N_{1,\ell}, \mathbb{Z}_\ell).$$

1.2 The Brauer group

Let us recall Grothendieck's description of the Brauer group $\mathrm{Br}(\overline{X})$ from [5, III.8, p. 144-147]. Let $\rho = \dim_{\mathbb{Q}}(\mathrm{NS}(\overline{X}) \otimes \mathbb{Q})$ be the Picard number of $\overline{X}$, and let b_2 be the second Betti number of $\overline{X}$. Let us denote the maximal divisible subgroup of $\mathrm{Br}(\overline{X})$ by $\mathrm{Br}^0(\overline{X})$. There is an isomorphism of abelian groups

$$\mathrm{Br}^0(\overline{X}) \cong (\mathbb{Q}/\mathbb{Z})^{b_2 - \rho}.$$

The quotient $\mathrm{Br}(\overline{X})/\mathrm{Br}^0(\overline{X})$ is finite, more precisely there is an exact sequence of Γ-modules

$$0 \to \mathrm{Br}^0(\overline{X}) \to \mathrm{Br}(\overline{X}) \to \oplus_\ell \, \mathrm{H}^3_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))_{\mathrm{tors}} \to 0, \tag{7}$$

where ℓ runs through all prime numbers.

Let B_ℓ be the ℓ-adic Tate module of $\mathrm{Br}(\overline{X})$, defined as the inverse limit of $\mathrm{Br}(\overline{X})[\ell^m]$ over m. Note that B_ℓ is free as a $\mathbb{Z}_\ell$-module. The Galois module B_ℓ only controls the maximal divisible subgroup $\mathrm{Br}^0(\overline{X}) \subset \mathrm{Br}(\overline{X})$, in the sense that B_ℓ is also isomorphic to the Tate module of $\mathrm{Br}^0(\overline{X})$, and there is a canonical isomorphism of Γ-modules, cf. [5, II.8.1, p. 144]:

$$\mathrm{Br}^0(\overline{X}) \cong \oplus_\ell (B_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

The Kummer sequence

$$1 \to \mu_n \to \mathbb{G}_m \xrightarrow{x \mapsto x^n} \mathbb{G}_m \to 1 \tag{8}$$

gives rise to exact sequences of Γ-modules

$$0 \to \mathrm{Pic}(\overline{X})/\ell^m \to \mathrm{H}^2_{\text{ét}}(\overline{X}, \mu_{\ell^m}) \to \mathrm{Br}(\overline{X})[\ell^m] \to 0.$$

Since the divisible group $\mathrm{Pic}^0(\overline{X})$ goes to zero in $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mu_{\ell^m})$ we obtain the exact sequences

$$0 \to \mathrm{NS}(\overline{X})/\ell^m \to \mathrm{H}^2_{\text{ét}}(\overline{X}, \mu_{\ell^m}) \to \mathrm{Br}(\overline{X})[\ell^m] \to 0.$$

Passing to the inverse limit over m gives the exact sequence (8.7) of [5, III.8.2]:

$$0 \to \mathrm{NS}(\overline{X}) \otimes \mathbb{Z}_\ell \to \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1)) \to B_\ell \to 0. \tag{9}$$

The second arrow in (9) induces an isomorphism on torsion subgroups

$$(\mathrm{NS}(\overline{X}) \otimes \mathbb{Z}_\ell)_{\mathrm{tors}} = \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))_{\mathrm{tors}}.$$

Thus we deduce the exact sequence of finitely generated $\mathbb{Z}_\ell$-free Γ-modules

$$0 \to N^1_\ell \to H^2_\ell \to B_\ell \to 0. \tag{10}$$

As a sequence of $\mathbb{Z}_\ell$-modules, it is split. In particular, for any prime number ℓ the $\mathbb{Z}_\ell$-submodule $N^1_\ell \subset H^2_\ell$ is primitive, in the sense that the quotient is torsion-free. On tensoring (10) with $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ and taking the direct sum over all primes ℓ we obtain an exact sequence of Γ-modules

$$0 \to N^1 \otimes \mathbb{Q}/\mathbb{Z} \to \oplus_\ell \, (H^2_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \mathrm{Br}^0(\overline{X}) \to 0,$$

which gives rise to a 2-extension of Γ-modules

$$0 \to N^1 \to N^1 \otimes \mathbb{Q} \to \oplus_\ell \, (H^2_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \mathrm{Br}^0(\overline{X}) \to 0. \tag{11}$$

The following easy lemma will be used later on.

Lemma 1.2 *Let F be a finite ℓ-primary torsion group, and let $n \geq 0$. Let A be a finite subquotient of $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^n \oplus F$. If the exponent of A is ℓ^m, then the order of A divides the product of ℓ^{mn} and the order of $F[\ell^m]$.*

Proof. The group A is a quotient of $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^r \oplus F' \subset (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^n \oplus F$, where F' is a finite group. Then A is a quotient of F'/ℓ^m. The order of F'/ℓ^m equals the order of $F'[\ell^m]$, which is a subgroup of $(\mathbb{Z}/\ell^m)^n \oplus F[\ell^m]$. QED

1.3 Basic exact sequence

Proposition 1.3 *Let X be a (not necessarily smooth or projective) scheme over a field k of characteristic zero.*

(i) *There is a complex that is functorial in X and k:*

$$\mathrm{Br}(X) \xrightarrow{\alpha} \mathrm{Br}(\overline{X})^{\Gamma} \xrightarrow{\beta} \mathrm{H}^2(k, \mathrm{Pic}(\overline{X})).$$

(ii) *Assume that $\mathrm{H}^0_{\text{ét}}(\overline{X}, \mathbb{G}_m) = \overline{k}^*$. Assume, moreover, that the natural map $\mathrm{H}^3_{\text{ét}}(k, \overline{k}^*) \to \mathrm{H}^3_{\text{ét}}(X, \mathbb{G}_m)$ is injective, which is the case when X has a k-point or when k is a number field. Then the above complex is an exact sequence, and we have* $\mathrm{Im}(\alpha) = \mathrm{Ker}(\beta)$ *and* $\mathrm{Coker}(\alpha) = \mathrm{Im}(\beta)$.

Proof. This follows from the Leray spectral sequence

$$E_2^{pq} = \mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(\overline{X}, \mathbb{G}_m)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathbb{G}_m). \tag{12}$$

Note that a k-point on X defines a section of the map $\mathrm{H}^3_{\text{ét}}(k, \overline{k}^*) \to \mathrm{H}^3_{\text{ét}}(X, \mathbb{G}_m)$; if k is a number field, then $\mathrm{H}^3_{\text{ét}}(k, \overline{k}^*) = 0$. QED

1.4 Restriction and corestriction

The following lemma is certainly well known, and is proved here for the sake of completeness.

Lemma 1.4 *Let X be a scheme over a field k of characteristic zero, and let $L \subset \overline{k}$ be a finite extension of k of degree n. There are restriction and corestriction homomorphisms*

$$\mathrm{res}_{L/k} : \mathrm{Br}(X) \to \mathrm{Br}(X_L), \qquad \mathrm{cores}_{L/k} : \mathrm{Br}(X_L) \to \mathrm{Br}(X)$$

such that $\mathrm{cores}_{L/k}(\mathrm{res}_{L/k}(x)) = nx$. *The following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Br}(X) \arrow[r, "\mathrm{res}_{L/k}"] \arrow[d, "\alpha"'] & \mathrm{Br}(X_L) \arrow[r, "\mathrm{cores}_{L/k}"] \arrow[d, "\alpha_L"'] & \mathrm{Br}(X) \arrow[d, "\alpha"'] \\
\mathrm{Br}(\overline{X})^{\Gamma} \arrow[r, hook] & \mathrm{Br}(\overline{X})^{\Gamma_L} \arrow[r, "\sigma"] & \mathrm{Br}(\overline{X})
\end{tikzcd}$$

Here $\Gamma_L = \mathrm{Gal}(\overline{k}/L)$, and $\sigma(x) = \sum \sigma_i(x)$, where $\sigma_i \in \Gamma$ are coset representatives of Γ/Γ_L.

Proof Let us recall the definition of $\mathrm{res}_{L/k}$ and $\mathrm{cores}_{L/k}$. Let $f : Y \to X$ be a finite flat morphism of connected smooth k-schemes, of degree n. Then we have morphisms of étale sheaves $\mathbb{G}_{m,X} \to f_*\mathbb{G}_{m,Y} \to \mathbb{G}_{m,X}$ defined on stalks by a natural injection and the norm map, respectively. The composition of the two maps coincides with raising to the power n. The functor f_* from the category of étale sheaves on Y to the category of étale sheaves on X is exact [12, Cor. II.3.6]. Thus the Leray spectral sequence gives an isomorphism $\mathrm{H}^p_{\text{ét}}(X, f_*\mathbb{G}_{m,Y}) \xrightarrow{\sim} \mathrm{H}^p_{\text{ét}}(Y, \mathbb{G}_{m,Y})$. Hence we obtain the desired maps

$$\mathrm{H}^p_{\text{ét}}(X, \mathbb{G}_{m,X}) \xrightarrow{\text{res}} \mathrm{H}^p_{\text{ét}}(Y, \mathbb{G}_{m,X}) \xrightarrow{\text{cores}} \mathrm{H}^p_{\text{ét}}(X, \mathbb{G}_{m,X})$$

whose composition is multiplication by n.

Now let X be a scheme over the field k. Let $L \subset \overline{k}$ be a field such that $[L : K] = n$, and let $Y = X_L = X \times_k L$. Clearly, we have the isomorphism $L \otimes_k \overline{k} \tilde{\to} \overline{k}^n$, where the components correspond to the n distinct embeddings of L into $\overline{k}$. By changing the base from X to $\overline{X}$ we get a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^p_{\text{ét}}(X, \mathbb{G}_m) \arrow[r, "\mathrm{res}_{L/k}"] \arrow[d] & \mathrm{H}^p_{\text{ét}}(X_L, \mathbb{G}_m) \arrow[r, "\mathrm{cores}_{L/k}"] \arrow[d] & \mathrm{H}^p_{\text{ét}}(X, \mathbb{G}_m) \arrow[d] \\
\mathrm{H}^p_{\text{ét}}(\overline{X}, \mathbb{G}_m) \arrow[r, hook] & \mathrm{H}^p_{\text{ét}}(\overline{X}, \mathbb{G}_m)^n \arrow[r] & \mathrm{H}^p_{\text{ét}}(\overline{X}, \mathbb{G}_m)
\end{tikzcd}$$

where the maps in the bottom row are the diagonal embedding and the product. The representation of the Galois group Γ in $\mathrm{H}^p_{\text{ét}}(\overline{X}, \mathbb{G}_m)^n$ is induced from the natural representation of Γ_L in $\mathrm{H}^p_{\text{ét}}(\overline{X}, \mathbb{G}_m)$. Thus passing to Γ-invariant subgroups, and taking $p = 2$, we obtain the statement of the lemma. QED

2 Approach via Brauer groups of curves

2.1 Finiteness

Theorem 2.1 *Let X be a smooth, projective and geometrically integral variety over a field k of characteristic zero. Then the cokernel of the natural map $\alpha : \mathrm{Br}(X) \to \mathrm{Br}(\overline{X})^\Gamma$ is finite.*

Proof. By Grothendieck's computation of $\mathrm{Br}(\overline{X})$ recalled in Subsection 1.2, for any prime power ℓ^n and any subquotient B of $\mathrm{Br}(\overline{X})$ the subgroup $B[\ell^n]$ is finite. Therefore, it is enough to show that $\mathrm{Coker}(\alpha)$ has finite exponent.

For this we can replace k by any finite extension. Indeed, if $k \subset L \subset \overline{k}$, $[L:k] = n$, then it follows from Lemma 1.4 that we have natural maps

$$\mathrm{Coker}(\alpha) \to \mathrm{Coker}(\alpha_L) \to \mathrm{Coker}(\alpha),$$

whose composition is the multiplication by n. It is therefore enough to prove that $\mathrm{Coker}(\alpha_L)$ has finite exponent.

In particular, we may assume that X has a k-point. By Proposition 1.3(ii) we have $\mathrm{Coker}(\alpha) = \mathrm{Im}(\beta)$. Let us show that $\mathrm{Im}(\beta)$ has finite exponent.

If $f : C \to X$ is a morphism, where C is a smooth, projective and geometrically integral curve over k, then the maps $f^* : \mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(\overline{C})$ and $f^* : \mathrm{Br}(\overline{X}) \to \mathrm{Br}(\overline{C})$ fit into the commutative diagram

$$\begin{tikzcd}
\mathrm{Br}(\overline{X})^{\Gamma} \arrow[r, "\beta_X"] \arrow[d] & \mathrm{H}^2(k, \mathrm{Pic}(\overline{X})) \arrow[d] \\
\mathrm{Br}(\overline{C})^{\Gamma} \arrow[r, "\beta_C"] & \mathrm{H}^2(k, \mathrm{Pic}(\overline{C}))
\end{tikzcd}$$

By a theorem of Tsen and Grothendieck ([5], Cor. 1.3, p. 90) we have $\mathrm{Br}(\overline{C}) = 0$. Hence

$(*)$ *For any* $f : C \to X$, *the group* $\mathrm{Im}(\beta_X)$ *is contained in the kernel of the right vertical map in the diagram.*

The degree map $\mathrm{Pic}(\overline{C}) \to \mathrm{NS}(\overline{C}) = \mathbb{Z}$ defines the exact sequence of Galois modules

$$0 \to \mathrm{Pic}^0(\overline{C}) \to \mathrm{Pic}(\overline{C}) \to \mathrm{NS}(\overline{C}) \to 0,$$

so that we have a commutative diagram with exact rows

$$\begin{tikzcd}
 & \mathrm{H}^2(k, \mathrm{Pic}^0(\overline{X})) \arrow[r] \arrow[d] & \mathrm{H}^2(k, \mathrm{Pic}(\overline{X})) \arrow[r] \arrow[d] & \mathrm{H}^2(k, \mathrm{NS}(\overline{X})) \arrow[d] \\
0 \arrow[r] & \mathrm{H}^2(k, \mathrm{Pic}^0(\overline{C})) \arrow[r] & \mathrm{H}^2(k, \mathrm{Pic}(\overline{C})) \arrow[r] & \mathrm{H}^2(k, \mathrm{NS}(\overline{C}))
\end{tikzcd} \tag{13}$$

The zero in the bottom row is due to the fact that $\mathrm{H}^1(k, \mathbb{Z}) = 0$.

Over the infinite field k, the Bertini theorem [7] for hyperplane sections of smooth projective varieties ensures that there exists a linear curve section $C \subset X$, defined over k, which is smooth and geometrically connected. A combination of the Bertini theorem and Zariski's connectedness theorem (see [6, Lemme 2.10, p. 210]) then implies that over an algebraic closure of k the inverse image under $C \to X$ of any connected finite étale cover of X is connected. In particular, the map of abelian varieties $\mathrm{Pic}^0_{X/k}\to\mathrm{Pic}^0_{C/k}$ has trivial kernel.

By the Poincaré reducibility theorem [15, §19, Thm. 1] there exists an abelian subvariety $A\subset\mathrm{Pic}^0_{C/k}$ such that the natural map

$$\mathrm{Pic}^0_{X/k}\times A\to\mathrm{Pic}^0_{C/k}$$

is an isogeny of abelian varieties over k.

Since $\mathrm{H}^2(k,\mathrm{Pic}^0(\overline{C}))\to\mathrm{H}^2(k,\mathrm{Pic}(\overline{C}))$ is injective, this implies:

($**$) *The kernel of the composite map*

$$\mathrm{H}^2(k,\mathrm{Pic}^0(\overline{X}))\to\mathrm{H}^2(k,\mathrm{Pic}(\overline{X}))\to\mathrm{H}^2(k,\mathrm{Pic}(\overline{C}))$$

has finite exponent.

Since $N^1=\mathrm{NS}(\overline{X})/_{\mathrm{tors}}$ is a finitely generated free abelian group, we can choose finitely many, say m, curves in $\overline{X}$ such that the intersection pairing with the classes of these curves defines an injective group homomorphism $\iota: N^1\hookrightarrow\mathbb{Z}^m$. By taking normalisation we obtain m morphisms from smooth projective curves defined over $\overline{k}$ to $\overline{X}$. For the purpose of the proof we can replace k by a finite extension over which all of these curves are defined.

We now have morphisms $f_i: C_i\to X$, $i=1,\ldots,m$, from smooth, projective and geometrically irreducible curves to X defined over the ground field k. The maps induce a map of Γ-modules

$$\mathrm{NS}(\overline{X})\to\oplus_{i=1}^m\mathrm{NS}(\overline{C}_i)=\mathbb{Z}^m.$$

In view of (13), ($*$) and ($**$), to complete the proof it is enough to show that the kernel of the induced map $\mathrm{H}^2(k,\mathrm{NS}(\overline{X}))\to\mathrm{H}^2(k,\mathbb{Z}^m)$ has finite exponent. This map is the composite of two maps:

$$\mathrm{H}^2(k,\mathrm{NS}(\overline{X}))\to\mathrm{H}^2(k,N^1)\to\mathrm{H}^2(k,\mathbb{Z}^m).$$

It is enough to show that the kernel of each of these maps is of finite exponent.

From the cohomology sequence attached to the exact sequence of Γ-modules

$$0\to\mathrm{NS}(\overline{X})_{\mathrm{tors}}\to\mathrm{NS}(\overline{X})\to N^1\to 0$$

we deduce that the map

$$\mathrm{H}^2(k,\mathrm{NS}(\overline{X}))\to\mathrm{H}^2(k,N^1)$$

has its kernel annihilated by multiplication by the exponent of $\mathrm{NS}(\overline{X})_{\mathrm{tors}}$.

There exists a homomorphism $\mathbb{Z}^m\to N^1$ such that the composition of homomorphisms of abelian groups with trivial Galois action $N^1 \overset{\iota}{\hookrightarrow} \mathbb{Z}^m\to N^1$ is multiplication by a positive integer. The kernel of

$$\mathrm{H}^2(k,N^1)\to\mathrm{H}^2(k,\mathbb{Z}^m)$$

is annihilated by multiplication by this integer. QED

2.2 Upper bounds, I

Let δ_0 be the exponent of the finite group D defined in (3), and let ν_0 be the exponent of the finite group $\mathrm{NS}(\overline{X})_{\mathrm{tors}}$. Let α be the natural map $\mathrm{Br}(X)\to\mathrm{Br}(\overline{X})^\Gamma$.

Theorem 2.2 *Let X be a smooth, projective and geometrically integral variety over a field k of characteristic zero. Let L/k be a finite field extension such that the finitely generated abelian group $N_1=\mathrm{Num}_1(\overline{X})$ is generated by the classes of integral curves on $\overline{X}$ defined over L. Let $\lambda=[L:k]$.*

(i) *If* $\mathrm{H}^1(X,O_X)=0$ *and the map* $\mathrm{H}^3(k,\mathbb{G}_m)\to\mathrm{H}^3(X,\mathbb{G}_m)$ *is injective, then the exponent of* $\mathrm{Coker}(\alpha)$ *divides* $\lambda\delta_0\nu_0$.

(ii) *If k is a number field, the exponent of* $\mathrm{Coker}(\alpha)$ *divides* $2\lambda\delta_0\nu_0$, *and it divides* $\lambda\delta_0\nu_0$ *if k is totally imaginary.*

Proof. We follow the proof of Theorem 2.1 making necessary calculations along the way. We first apply Proposition 1.3. For any number field k we have $\mathrm{H}^3(k,\overline{k}^*)=0$, thus in this case we always have $\mathrm{Im}(\alpha)=\mathrm{Ker}(\beta)$, hence $\mathrm{Coker}(\alpha)=\mathrm{Im}(\beta)$. This also holds under the assumption that the map $\mathrm{H}^3(k,\mathbb{G}_m)\to\mathrm{H}^3(X,\mathbb{G}_m)$ is injective.

We choose finitely many, say m, integral curves $C_1,\ldots,C_m$ in $\overline{X}$ whose classes generate N_1, and then replace k by a finite extension L over which all of these curves are defined. The restriction-corestriction argument at the beginning of the proof of Theorem 2.1 shows that when we replace k by the finite extension L, of degree λ, the exponent of $\mathrm{Coker}(\alpha)$ divides the product of the exponent of $\mathrm{Coker}(\alpha_L)$ and λ. To prove the theorem, we may now assume $k=L$, that is $\lambda=1$.

Let $\mathrm{Pic}(\overline{X})\to\mathbb{Z}^m$ be the map given by restriction to the curves C_i, followed by the degree map on each curve. The proof of Theorem 2.1 shows that the image of β is contained in the kernel of the induced map

$$\mathrm{H}^2(k, \mathrm{Pic}(\overline{X})) \to \mathrm{H}^2(k, \mathbb{Z}^m).$$

The map $\mathrm{Pic}(\overline{X}) \to \mathbb{Z}^m$ factorises as follows

$$\mathrm{Pic}(\overline{X}) \to \mathrm{NS}(\overline{X}) \to N^1 \to \mathrm{Hom}(N_1, \mathbb{Z}) \to \mathbb{Z}^m.$$

We shall bound the exponent of each induced map on $\mathrm{H}^2(k, \cdot)$.

Sending each curve C_i to its class in N_1 yields an exact sequence of trivial Γ-modules

$$0 \to \mathbb{Z}^r \to \mathbb{Z}^m \to N_1 \to 0.$$

Dualising we obtain a split exact sequence of trivial Γ-modules

$$0 \to \mathrm{Hom}(N_1, \mathbb{Z}) \to \mathbb{Z}^m \to \mathbb{Z}^r \to 0.$$

The map

$$\mathrm{H}^2(k, \mathrm{Hom}(N_1, \mathbb{Z})) \to \mathrm{H}^2(k, \mathbb{Z}^m)$$

is therefore injective. From the exact sequence (3) we conclude that the kernel of

$$\mathrm{H}^2(k, N^1) \to \mathrm{H}^2(k, \mathrm{Hom}(N_1, \mathbb{Z}))$$

is annihilated by the exponent of $\mathrm{H}^1(k, D)$, hence by δ_0, the exponent of D.

As we have seen in the proof of the previous theorem, the kernel of the map

$$\mathrm{H}^2(k, \mathrm{NS}(\overline{X})) \to \mathrm{H}^2(k, N^1)$$

is killed by the exponent ν_0 of $\mathrm{NS}(\overline{X})_{\mathrm{tors}}$. Finally we have the exact sequence

$$0 \to \mathrm{Pic}^0_{X/k}(\overline{k}) \to \mathrm{Pic}(\overline{X}) \to \mathrm{NS}(\overline{X}) \to 0.$$

If $\mathrm{H}^1(X, O_X) = 0$, then $\mathrm{Pic}^0_{X/k} = 0$. When k is a number field and A is an abelian variety, we have $\mathrm{H}^2(k, A) = \oplus_v \mathrm{H}^2(k_v, A)$, where v runs through the real completions of k [13, Thm. 6.26 (c), p. 92]. Hence the exponent of $\mathrm{H}^2(k, A)$ is at most 2. This completes the proof of the theorem. QED

Remark. An explicit bound for the order of $\mathrm{Coker}(\alpha)$ immediately follows from Theorem 2.2 via Lemma 1.2.

3 Differentials

3.1 A general remark about differentials in spectral sequences

Let us recall the standard set-up of the spectral sequence of composed functors. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be abelian categories such that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. Let $G : \mathcal{A}{\rightarrow}\mathcal{B}$ and $F : \mathcal{B}{\rightarrow}\mathcal{C}$ be left exact additive functors such that G sends injective objects into F-acyclic. Then for every object $B \in \mathrm{Ob}(\mathcal{A})$ we have the spectral sequence

$$E_2^{pq} = (RF^p)(R^qG)B \Rightarrow R^{p+q}(FG)B. \tag{14}$$

Let

$$\partial_{p,q} : (R^pF)(R^qG)B \longrightarrow (R^{p+2}F)(R^{q-1}G)B$$

be the canonical maps in this spectral sequence.

Suppose we have an exact sequence in $\mathcal{A}$:

$$0{\rightarrow}A{\rightarrow}B{\rightarrow}C{\rightarrow}0. \tag{15}$$

Applying the right derived functors of G we get a long exact sequence in $\mathcal{B}$. Truncating it we obtain for any $q \geq 1$ the exact sequence

$$0{\rightarrow}B_1{\rightarrow}(R^{q-1}G)C{\rightarrow}(R^qG)A{\rightarrow}B_2{\rightarrow}0, \tag{16}$$

together with the surjective map $s : (R^{q-1}G)B{\rightarrow}B_1$ and the injective map $i : B_2{\rightarrow}(R^qG)B$. Let $\partial : (R^pF)B_2{\rightarrow}(R^{p+2}F)B_1$ be the connecting homomorphism defined by (16). Let

$$s_* = (R^{p+2}F)(s) : (R^{p+2}F)(R^{q-1}G)B{\rightarrow}(R^{p+2}F)B_1$$

be the map induced by s, and similarly let

$$i_* = (R^pF)(i) : (R^pF)B_2{\rightarrow}(R^pF)(R^qG)B$$

be the map induced by i.

Lemma 3.1 *We have* $\partial = s_*\partial_{p,q}i_*$.

Proof. There exists a short exact sequence

$$0 \to A^{\cdot} \to B^{\cdot} \to C^{\cdot} \to 0$$

of injective resolutions of A, B, C, respectively. Let $a_n : A^n \to A^{n+1}$ be the differentials in $A^{\cdot}$, and similarly for $B^{\cdot}$ and $C^{\cdot}$. We have the commutative diagram

$$\begin{tikzcd}
 & A^{q-1}/\mathrm{Im}(a_{q-2}) \arrow[r] \arrow[d] & B^{q-1}/\mathrm{Im}(b_{q-2}) \arrow[r] \arrow[d] & C^{q-1}/\mathrm{Im}(c_{q-2}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ker}(a_q) \arrow[r] & \mathrm{Ker}(b_q) \arrow[r] & \mathrm{Ker}(c_q) &
\end{tikzcd}$$

Applying the snake lemma we obtain the exact sequence

$$(R^{q-1}G)A \to (R^{q-1}G)B \to (R^{q-1}G)C \to (R^qG)A \to (R^qG)B \to (R^qG)C.$$

Truncating it we get (16). Chasing the diagram one checks that (16) is equivalent to the 2-extension

$$0 \to (R^{q-1}G)B \to B^{q-1}/b_{q-2}(B^{q-2}) \to \mathrm{Ker}(b_q) \to (R^qG)B \to 0 \tag{17}$$

pulled back via $i : B_2 \to (R^qG)B$ and pushed out via $s : (R^{q-1}G)B \to B_1$. By definition, the canonical map $\partial_{p,q}$ is the connecting homomorphism

$$(R^pF)(R^qG)B \longrightarrow (R^{p+2}F)(R^{q-1}G)B$$

defined by (17), hence $s_*\partial_{p,q}i_* = \partial$. QED

3.2 Applications to the Brauer group

The Kummer sequence (8) gives rise to the 2-extension of Γ-modules

$$0 \to \mathrm{Pic}(\overline{X})/\mathrm{Pic}(\overline{X})[n] \to \mathrm{Pic}(\overline{X}) \to \mathrm{H}^2_{\text{ét}}(\overline{X}, \mu_n) \to \mathrm{Br}(\overline{X})[n] \to 0, \tag{18}$$

where the second arrow is defined by multiplication by n on $\mathrm{Pic}(\overline{X})$.

Proposition 3.2 *The following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Br}(\overline{X})[n]^{\Gamma} \arrow[r, "\partial"] \arrow[d] & \mathrm{H}^2(k, \mathrm{Pic}(\overline{X})/\mathrm{Pic}(\overline{X})[n]) \\
\mathrm{Br}(\overline{X})^{\Gamma} \arrow[r, "\beta"] & \mathrm{H}^2(k, \mathrm{Pic}(\overline{X})) \arrow[u]
\end{tikzcd}$$

Here ∂ is the connecting homomorphism defined by (18), *and the vertical arrows are the obvious natural maps.*

Proof. In the set-up of (14) and Lemma 3.1, let now $\mathcal{A}$ be the category of étale sheaves on X, let $\mathcal{B}$ be the category of continuous discrete Γ-modules, and let $\mathcal{C}$ be the category of abelian groups. Let $G = \pi_*$, where $\pi : X \to \mathrm{Spec}(k)$ is the structure morphism. Let $F(M) = M^\Gamma$. Let $A = \mu_{n,X}$, $B = C = \mathbb{G}_{m,X}$, and let (15) be the Kummer sequence (8). Take $p = 0$ and $q = 2$. The associated sequence (16) is precisely the sequence (18). It remains to apply Lemma 3.1: the left vertical map in the diagram is i_*, the bottom horizontal map is $\beta = \partial_{0,2}$, and the right vertical map is s_*. QED

It is easy to check that the exact sequence

$$0 \to \mathrm{Pic}^0(\overline{X}) \to \mathrm{Pic}(\overline{X}) \to \mathrm{NS}(\overline{X}) \to 0$$

gives rise to the exact sequence

$$0 \to \mathrm{Pic}^0(\overline{X})/\mathrm{Pic}^0(\overline{X})[n] \to \mathrm{Pic}(\overline{X})/\mathrm{Pic}(\overline{X})[n] \to \mathrm{NS}(\overline{X})/\mathrm{NS}(\overline{X})[n] \to 0.$$

The divisible subgroup $\mathrm{Pic}^0(\overline{X}) \subset \mathrm{Pic}(\overline{X})$ is contained in the kernel of $\mathrm{Pic}(\overline{X}) \to \mathrm{H}^2_{\text{ét}}(\overline{X}, \mu_n)$, hence (18) gives rise to the 2-extension of Γ-modules

$$0 \to \mathrm{NS}(\overline{X})/\mathrm{NS}(\overline{X})[n] \to \mathrm{NS}(\overline{X}) \to \mathrm{H}^2_{\text{ét}}(\overline{X}, \mu_n) \to \mathrm{Br}(\overline{X})[n] \to 0, \tag{19}$$

where the second arrow is induced by multiplication by n on $\mathrm{NS}(\overline{X})$.

Corollary 3.3 *The following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Br}(\overline{X})[n]^\Gamma \arrow[r, "\partial"] \arrow[d] & \mathrm{H}^2(k, \mathrm{NS}(\overline{X})/\mathrm{NS}(\overline{X})[n]) \\
\mathrm{Br}(\overline{X})^\Gamma \arrow[r, "\beta"] & \mathrm{H}^2(k, \mathrm{Pic}(\overline{X})) \arrow[u]
\end{tikzcd}$$

Here ∂ is the connecting homomorphism defined by (19)*, and the vertical arrows are the obvious natural maps.*

Proof. This immediately follows from Proposition 3.2. QED

Corollary 3.4 *The following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Br}^0(\overline{X})^\Gamma \arrow[r, "\partial"] \arrow[d] & \mathrm{H}^2(k, N^1) \\
\mathrm{Br}(\overline{X})^\Gamma \arrow[r, "\beta"] & \mathrm{H}^2(k, \mathrm{Pic}(\overline{X})) \arrow[u]
\end{tikzcd}$$

Here ∂ is the connecting homomorphism defined by (11)*, and the vertical arrows are the obvious natural maps.*

Proof. Let n be a positive integer divisible by ν_0, the exponent of $\mathrm{NS}(\overline{X})_{\mathrm{tors}}$. For such an n the exact sequence (19) takes the form

$$0 \to N^1 \to \mathrm{NS}(\overline{X}) \to \mathrm{H}^2_{\text{ét}}(\overline{X}, \mu_n) \to \mathrm{Br}(\overline{X})[n] \to 0,$$

the map $N^1 \to \mathrm{NS}(\overline{X})$ being induced by multiplication by n on $\mathrm{NS}(\overline{X})$. Let us write $n = \prod_\ell n_\ell$, where n_ℓ is a power of the prime ℓ.

Let $P_\ell = \mathrm{NS}(\overline{X})\{\ell\}$, and let $\mathrm{Im}(P_\ell)$ be the image of P_ℓ under the composite map

$$\mathrm{NS}(\overline{X}) \to \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1)) \to \mathrm{H}^2_{\text{ét}}(\overline{X}, \mu_{n_\ell}).$$

We have the following commutative diagram of Γ-modules, with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & N^1 \arrow[r] \arrow[d, equal] & N^1 \otimes \mathbb{Q} \arrow[r] & \oplus_\ell (H^2_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell) \arrow[r] & \mathrm{Br}^0(\overline{X}) \arrow[r] & 0 \\
0 \arrow[r] & N^1 \arrow[r, "\times n"] \arrow[d, equal] & N^1 \arrow[r] \arrow[u] \arrow[d, equal] & \oplus_\ell H^2_\ell / n_\ell \arrow[r] \arrow[u] \arrow[d] & \mathrm{Br}^0(\overline{X})[n] \arrow[r] \arrow[u] \arrow[d] & 0 \\
0 \arrow[r] & N^1 \arrow[r, "\times n"] \arrow[d, equal] & N^1 \arrow[r] & \oplus_\ell \mathrm{H}^2_{\text{ét}}(\overline{X}, \mu_{n_\ell})/\mathrm{Im}(P_\ell) \arrow[r] & \mathrm{Br}(\overline{X})[n] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N^1 \arrow[r, "\times n"] & \mathrm{NS}(\overline{X}) \arrow[r] \arrow[u] & \mathrm{H}^2_{\text{ét}}(\overline{X}, \mu_n) \arrow[r] \arrow[u] & \mathrm{Br}(\overline{X})[n] \arrow[r] & 0.
\end{tikzcd}$$

The exact sequence in the first row is obtained by tensoring (10) with $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ and then taking the direct sum over all primes ℓ. To obtain the exact sequence in the second row, tensor (10) with $\mathbb{Z}/n_\ell$ and then take the direct sum over all primes ℓ. The *vertical* map $N^1 \to N^1 \otimes \mathbb{Q}$ sends x to $x \otimes \frac{1}{n}$. All other vertical arrows are natural maps.

Using this diagram, we deduce from Corollary 3.3 that the restriction of the composite map

$$\mathrm{Br}(\overline{X})^\Gamma \xrightarrow{\beta} \mathrm{H}^2(k, \mathrm{Pic}(\overline{X})) \longrightarrow \mathrm{H}^2(k, N^1)$$

to $\mathrm{Br}^0(\overline{X})^\Gamma \subset \mathrm{Br}(\overline{X})^\Gamma$, is the connecting homomorphism defined by (11), the top 2-extension in the diagram. QED

4 Approach via transcendental cycles

In this section X is a smooth, projective and geometrically integral variety of dimension d over a field k of characteristic zero.

4.1 Lattices of algebraic and transcendental cycles

Let ℓ be a prime. If M is a $\mathbb{Z}_\ell$-module, we write $M^* = \mathrm{Hom}_{\mathbb{Z}_\ell}(M, \mathbb{Z}_\ell)$. Recall from Subsection 1.1 the following commutative diagram of Γ-equivariant pairings

$$\begin{tikzcd}
N^1_\ell \arrow[r, phantom, "\times"] \arrow[d] & N_{1,\ell} \arrow[r] \arrow[d] & \mathbb{Z}_\ell \arrow[d, equal] \\
H^2_\ell \arrow[r, phantom, "\times"] & H^{2d-2}_\ell \arrow[r] & \mathbb{Z}_\ell
\end{tikzcd}$$

with *injective* vertical maps (by Matsusaka's and Lieberman's theorems). Moreover, the bottom pairing induces isomorphisms $H^2_\ell = (H^{2d-2}_\ell)^*$ and $H^{2d-2}_\ell = (H^2_\ell)^*$. By the exact sequence (6) and the remark after it we see that the composition

$$N^1_\ell \to H^2_\ell \xrightarrow{\sim} (H^{2d-2}_\ell)^* \to N^*_{1,\ell}$$

is an injective map with cokernel $D\{\ell\}$, where D is the finite abelian group defined in (3). In particular, this is an isomorphism if ℓ does not divide $\delta = |D|$, the order of D.

As we have seen in Subsection 1.2 the subgroup $N^1_\ell \subset H^2_\ell$ is primitive. However, Kollár (see [22], Thm. 14) showed that $N_{1,\ell}$ is not necessarily a primitive subgroup of H^{2d-2}_ℓ. This can be remedied as follows. First of all, if ℓ does not divide δ, the natural map $(H^{2d-2}_\ell)^* \to N^*_{1,\ell}$ is surjective, hence $N_{1,\ell}$ is primitive in H^{2d-2}_ℓ. For every ℓ we define the Γ-module M_ℓ as the saturation of $N_{1,\ell}$ in H^{2d-2}_ℓ, in other words,

$$M_\ell \;=\; H^{2d-2}_\ell \;\cap\; N_{1,\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \;\subset\; H^{2d-2}_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

If ℓ does not divide δ, then $M_\ell = N_{1,\ell}$. Now we define the Γ-module M as the subgroup of $N_1 \otimes \mathbb{Q}$ consisting of the elements that map to M_ℓ under the natural map

$$N_1 \otimes \mathbb{Q} \longrightarrow N_1 \otimes \mathbb{Q}_\ell \cong M_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell,$$

for every prime ℓ. Thus $N_1 \subset M \subset N_1 \otimes \mathbb{Q}$. We also obtain a Γ-equivariant bilinear form $N^1 \times M \to \mathbb{Q}$. By tensoring with $\mathbb{Z}_\ell$ for every prime ℓ we see that this is actually an integral bilinear form

$$N^1 \times M \to \mathbb{Z},$$

which extends the intersection pairing form on $N^1 \times N_1$. It gives the exact sequence of Γ-modules

$$0 \to N^1 \to \mathrm{Hom}(M, \mathbb{Z}) \to E \to 0, \tag{20}$$

which is the definition of the finite Γ-module E, and for each ℓ it gives the exact sequence

$$0 \to N^1_\ell \to M^*_\ell \to E\{\ell\} \to 0. \tag{21}$$

It is clear that E is a Γ-submodule of D, and $D/E = \mathrm{Hom}(M/N_1, \mathbb{Q}/\mathbb{Z})$, hence $|D/E| = |M/N_1|$. Note that if $d = 2$, i.e., X is a surface, then $N_{1,\ell} = N^1_\ell \subset H^2_\ell$ is primitive, hence $M = N_1$ and $D = E$.

Let $S_\ell \subset H^2_\ell$ be the orthogonal complement to $N_{1,\ell}$ (or to M_ℓ). We let $T_\ell \subset H^{2d-2}_\ell$ be the orthogonal complement to N^1_ℓ with respect to the cup-product pairing. Dualising the exact sequence of finitely generated, $\mathbb{Z}_\ell$–free, Γ-modules

$$0 \to T_\ell \to H^{2d-2}_\ell \to (N^1_\ell)^* \to 0$$

we obtain the exact sequence of finitely generated, $\mathbb{Z}_\ell$–free, Γ-modules

$$0 \to N^1_\ell \to H^2_\ell \to T^*_\ell \to 0. \tag{22}$$

It gives a canonical identification $T^*_\ell = B_\ell$, where B_ℓ is the Tate module of the Brauer group defined in Subsection 1.2.

Since $M_\ell \subset H^{2d-2}_\ell$ is a primitive subgroup, cup-product gives the following exact sequence:

$$0 \to S_\ell \to H^2_\ell \to M^*_\ell \to 0. \tag{23}$$

The composite map $N^1_\ell \subset H^2_\ell \xrightarrow{\sim} (H^{2d-2}_\ell)^* \to M^*_\ell \to (N_{1,\ell})^*$ is injective. Thus $S_\ell \cap N^1_\ell = 0$. Using (22) and (23) we see that for every prime ℓ we have canonical isomorphisms of Γ-modules

$$E\{\ell\} = M^*_\ell/N^1_\ell = H^2_\ell/(N^1_\ell \oplus S_\ell) = T^*_\ell/S_\ell.$$

We thus have a natural exact sequence

$$0 \to M^*_\ell/N^1_\ell \to (S_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell) \oplus (N^1_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H^2_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell \to 0. \tag{24}$$

The following commutative diagram of Γ-modules with exact rows and columns will be useful to us:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & M^*_\ell/N^1_\ell \arrow[r] \arrow[d] & S_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell \arrow[r] \arrow[d] & \mathrm{Br}^0(\overline{X})\{\ell\} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N^1_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell \arrow[r] \arrow[d] & H^2_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell \arrow[r] \arrow[d] & \mathrm{Br}^0(\overline{X})\{\ell\} \arrow[r] & 0 \\
 & M^*_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell \arrow[r, equal] \arrow[d] & M^*_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd} \tag{25}$$

The middle row, respectively, column, here is (22), respectively, (23), tensored with $\mathbb{Q}_\ell/\mathbb{Z}_\ell$. The rest of the diagram follows from (24).

Taking the direct sum over all primes ℓ we obtain from (25) the equivalence of 2-extensions

$$\begin{tikzcd}
0 \arrow[r] & N^1 \arrow[r] \arrow[d, equal] & \mathrm{Hom}(M,\mathbb{Z}) \arrow[r] \arrow[d] & \oplus(S_\ell\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell/\mathbb{Z}_\ell) \arrow[r] \arrow[d] & \mathrm{Br}^0(\overline{X}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N^1 \arrow[r] & N^1\otimes\mathbb{Q} \arrow[r] & \oplus(H^2_\ell\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell/\mathbb{Z}_\ell) \arrow[r] & \mathrm{Br}^0(\overline{X}) \arrow[r] & 0
\end{tikzcd}$$

The top extension is the Yoneda product of 1-extensions of Γ-modules (20) and

$$0\to E\to\oplus\,(S_\ell\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell/\mathbb{Z}_\ell)\to\mathrm{Br}^0(\overline{X})\to 0 \tag{26}$$

Let us denote by $\partial_1:\mathrm{Br}^0(\overline{X})^\Gamma\to\mathrm{H}^1(k,E)$ and $\partial_2:\mathrm{H}^1(k,E)\to\mathrm{H}^2(k,N^1)$ the differentials defined by these 1-extensions.

Proposition 4.1 *The composed map*

$$\mathrm{Br}^0(\overline{X})^\Gamma\hookrightarrow\mathrm{Br}(\overline{X})^\Gamma\xrightarrow{\beta}\mathrm{H}^2(k,\mathrm{Pic}(\overline{X}))\to\mathrm{H}^2(k,N^1)$$

coincides, up to sign, with the composed map

$$\mathrm{Br}^0(\overline{X})^\Gamma\xrightarrow{\partial_1}\mathrm{H}^1(k,E)\xrightarrow{\partial_2}\mathrm{H}^2(k,N^1).$$

In particular, the image of $\beta(\mathrm{Br}^0(\overline{X})^\Gamma)$ in $\mathrm{H}^2(k,N^1)$ is annihilated by the exponent of E.

Proof. We have seen that (11) is equivalent to the Yoneda product of (20) and (26), so the proposition follows from Corollary 3.4. QED

4.2 Upper bounds, II

Let us denote the order of the finite group $\oplus_\ell\mathrm{H}^3_{\text{ét}}(\overline{X},\mathbb{Z}_\ell(1))_{\text{tors}}$ by γ, and its exponent by γ_0. Let ε_0 be the exponent of the finite group E defined in (20). The integer ε_0 divides δ_0, which is the exponent of the finite group D defined in (3). Recall that ν is the order of the finite group $\mathrm{NS}(\overline{X})_{\text{tors}}$, and ν_0 is its exponent.

If $d=2$, i.e., X is a surface, then $\mathrm{H}^2_{\text{ét}}(\overline{X},\mathbb{Z}_\ell(1))_{\text{tors}}$ is dual to $\mathrm{H}^3_{\text{ét}}(\overline{X},\mathbb{Z}_\ell(1))_{\text{tors}}$, hence $\gamma=\nu$ and $\gamma_0=\nu_0$. Here $N^1=N_1$, and we have the symmetric bilinear pairing

$$N^1\times N^1\to\mathbb{Z},$$

with trivial kernel. The integer $\delta = |D|$ is the absolute value of the determinant of this pairing. The groups D and E coincide, hence $\delta = \varepsilon$ and $\delta_0 = \varepsilon_0$.

Here is our main result over a general field of characteristic zero.

Theorem 4.2 *Let X be a smooth, projective and geometrically integral variety over a field k of characteristic zero such that $\mathrm{H}^1(X, O_X) = 0$. Assume that the canonical map $\mathrm{H}^3(k, \overline{k}^*) \to \mathrm{H}^3_{\text{ét}}(X, \mathbb{G}_m)$ is injective (for example, X has a k-point). Then we have the following statements.*

(i) *The exponent of the cokernel of*

$$\alpha : \mathrm{Br}(X) \to \mathrm{Br}(\overline{X})^\Gamma$$

divides $\gamma_0 \varepsilon_0 \nu_0$. The order of $\mathrm{Coker}(\alpha)$ divides $\gamma(\varepsilon_0\nu_0)^{b_2-\rho}$.

(ii) *If X is a surface, the exponent of $\mathrm{Coker}(\alpha)$ divides $\delta_0\nu_0^2$; the order of $\mathrm{Coker}(\alpha)$ divides $\nu(\delta_0\nu_0)^{b_2-\rho}$.*

Proof. Under our hypotheses $\mathrm{Coker}(\alpha) = \mathrm{Im}(\beta)$ by Proposition 1.3, so we only need to estimate the size of $\beta(\mathrm{Br}(\overline{X})^\Gamma)$. From (7) we deduce the exact sequence

$$0 \to \mathrm{Br}^0(\overline{X})^\Gamma \to \mathrm{Br}(\overline{X})^\Gamma \to \oplus_\ell \, \mathrm{H}^3_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))^\Gamma_{\text{tors}}.$$

This implies that $|\beta(\mathrm{Br}(\overline{X})^\Gamma)|$ divides $\gamma|\beta(\mathrm{Br}^0(\overline{X})^\Gamma)|$, and the exponent of $\beta(\mathrm{Br}(\overline{X})^\Gamma)$ divides the product of γ_0 and the exponent of $\beta(\mathrm{Br}^0(\overline{X})^\Gamma)$.

By Proposition 4.1, the group $\varepsilon_0.\beta(\mathrm{Br}^0(\overline{X})^\Gamma)$ is a subgroup of

$$\mathrm{Ker}[\mathrm{H}^2(k, \mathrm{Pic}(\overline{X})) \to \mathrm{H}^2(k, N^1)].$$

We have the short exact sequence

$$\mathrm{H}^2(k, \mathrm{Pic}^0(\overline{X})) \to \mathrm{H}^2(k, \mathrm{Pic}(\overline{X})) \to \mathrm{H}^2(k, \mathrm{NS}(\overline{X}))$$

and the short exact sequence

$$\mathrm{H}^2(k, \mathrm{NS}(\overline{X})_{\text{tors}}) \to \mathrm{H}^2(k, \mathrm{NS}(\overline{X})) \to \mathrm{H}^2(k, N^1).$$

Our assumption $\mathrm{H}^1(X, O_X) = 0$ implies $\mathrm{Pic}^0(\overline{X}) = 0$, hence the exponent of $\varepsilon_0.\beta(\mathrm{Br}^0(\overline{X})^\Gamma)$ divides ν_0. Thus the exponent of $\beta(\mathrm{Br}^0(\overline{X})^\Gamma)$ divides $\varepsilon_0.\nu_0$. The group $\beta(\mathrm{Br}^0(\overline{X})^\Gamma)$ is a subquotient of $(\mathbb{Q}/\mathbb{Z})^{b_2-\rho}$. Lemma 1.2 then gives a bound for the order of $\beta(\mathrm{Br}^0(\overline{X})^\Gamma)$, from which then follows the claimed bound for the order of $\beta(\mathrm{Br}(\overline{X})^\Gamma)=\mathrm{Coker}(\alpha)$. The statement for a surface then follows from the general facts recalled at the beginning of this subsection. QED

When k is a number field, we have a result for arbitrary varieties.

Theorem 4.3 *Let X be a smooth, projective and geometrically integral variety over a number field k. Then we have the following statements.*

(i) *The exponent of the cokernel of*

$$\alpha:\mathrm{Br}(X)\to\mathrm{Br}(\overline{X})^\Gamma$$

divides $2\gamma_0\varepsilon_0\nu_0$, and it divides $\gamma_0\varepsilon_0\nu_0$ if k is totally imaginary. The order of $\mathrm{Coker}(\alpha)$ *divides $\gamma(2\varepsilon_0\nu_0)^{b_2-\rho}$, and it divides $\gamma(\varepsilon_0\nu_0)^{b_2-\rho}$ if k is totally imaginary.*

(ii) *If X is a surface, the exponent of* $\mathrm{Coker}(\alpha)$ *divides $2\delta_0\nu_0^2$ and it divides $\delta_0\nu_0^2$ if k is totally imaginary; the order of* $\mathrm{Coker}(\alpha)$ *divides $\nu(2\delta_0\nu_0)^{b_2-\rho}$, and it divides $\nu(\delta_0\nu_0)^{b_2-\rho}$ if k is totally imaginary.*

Proof. In this case $\mathrm{H}^3(k,\overline{k}^*)=0$. We follow the proof of Theorem 4.2. It is enough to note that $\mathrm{H}^2(k,\mathrm{Pic}^0(\overline{X}))$ is a finite group of exponent 2, and it is zero when k is totally imaginary [13, Thm. 6.26 (c), p. 92]. The statement for a surface then follows as before. QED

Remark We have the isomorphism

$$\mathrm{NS}(\overline{X})_{\mathrm{tors}}=\oplus_\ell \mathrm{H}^2_{\text{ét}}(\overline{X},\mathbb{Z}_\ell(1))_{\mathrm{tors}}.$$

The Poincaré duality implies that the finite abelian groups $\mathrm{H}^2_{\text{ét}}(\overline{X},\mathbb{Z}_\ell(1))_{\mathrm{tors}}$ and $\mathrm{H}^{2d-1}_{\text{ét}}(\overline{X},\mathbb{Z}_\ell(d-1))_{\mathrm{tors}}$ are dual to each other.

The following proposition is a useful complement to Theorem 4.3. For a number field k we denote by k_v the completion of k at a non-archimedean place v, and by k_v^{nr} the maximal unramified extension of k_v. If S is a finite set of primes of k and E a finite Γ-module, we let $\mathrm{H}^1_S(k,E)$ be the subgroup of $\mathrm{H}^1(k,E)$ consisting of the elements unramified outside S, that is, the intersection of kernels of the natural restriction maps $\mathrm{H}^1(k,E)\to\mathrm{H}^1(k_v^{\mathrm{nr}},E)$ for all $v\notin S$.

Proposition 4.4 *Let X be a smooth, projective and geometrically integral variety over a number field k with good reduction outside a finite set S of* *primes of k. Let E be the finite Γ-module defined in* (20)*. Let T_E be the set of primes of k dividing the order of E. Then*

$$\partial_1(\mathrm{Br}^0(\overline{X})^\Gamma) \subset \mathrm{H}^1_{S\cup T_E}(k,E).$$

Proof. Let $I_v = \mathrm{Gal}(\overline{k}_v/k_v)$ be the inertia subgroup. It is well known that if v is a prime of good reduction, and the residual characteristic of k_v is different from ℓ, then the natural action of I_v on $\mathrm{H}^2_{\text{ét}}(\overline{X},\mu_{\ell^m})$, $m\geq 1$, and hence on $\mathrm{H}^2_{\text{ét}}(\overline{X},\mathbb{Z}_\ell(1))$, is trivial (by the smooth base change theorem, see [12, Cor. VI.4.2]). This implies, by the Kummer sequence, that I_v acts trivially on $\mathrm{Br}(\overline{X})\{\ell\}$. But I_v also acts trivially on $S_\ell \subset H^2_\ell$, hence the differential

$$\partial : \mathrm{Br}^0(\overline{X})\{\ell\}^{I_v} \longrightarrow \mathrm{H}^1(k_v^{\mathrm{nr}}, E\{\ell\})$$

defined by the exact sequence of I_v-modules (the top row of (25))

$$0\to E\{\ell\}\to S_\ell\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell/\mathbb{Z}_\ell\to \mathrm{Br}^0(\overline{X})\{\ell\}\to 0,$$

is zero. Hence, for $v\notin S\cup T_E$ the image of $\mathrm{Br}^0(\overline{X})^\Gamma$ in $\mathrm{H}^1(k,E)$ is in the kernel of the restriction map to $\mathrm{H}^1(k_v^{\mathrm{nr}},E)$. QED

5 Applications to surfaces

Proposition 5.1 *Let X be a $K3$ surface over a field k of characteristic zero, such that the map $\mathrm{H}^3(k,\overline{k}^*)\to\mathrm{H}^3_{\text{ét}}(X,\mathbb{G}_m)$ is injective (for example, k is a number field, or X has a k-point). Let α be the natural map $\mathrm{Br}(X)\to\mathrm{Br}(\overline{X})^\Gamma$. Then the exponent of $\mathrm{Coker}(\alpha)$ divides δ_0, and the order divides $\delta_0^{b_2-\rho}$.*

Proof. For a K3 surface, $\mathrm{H}^1(X,O_X)=0$ and the Néron–Severi group $\mathrm{NS}(\overline{X})$ is torsion-free, so $\nu=1$. The statement is a special case of Theorem 4.2 (ii). QED

Examples

1. Let $X\subset\mathbb{P}^3_k$ be a diagonal quartic surface over a field of characteristic zero. It is well known that $\delta=64$ and $\delta_0=8$, see [16]. This already implies that any element of odd order in $\mathrm{Br}(\overline{X})^\Gamma$ comes from $\mathrm{Br}(X)$. Furthermore, $b_2=22$ and $\rho=20$. If we follow the proof of Theorem 4.2, we see that $\mathrm{Coker}(\alpha)$ is a subquotient of $(\mathbb{Q}_2/\mathbb{Z}_2)^2$. Since its exponent divides 8, $\mathrm{Coker}(\alpha)$ is isomorphic to a subgroup of $(\mathbb{Z}/8)^2$.

2. Let k be a field of characteristic zero. Suppose $X \subset \mathbb{P}^g_k$ is a 'very general' K3 surface, that is, $\mathrm{NS}(\overline{X}) \cong \mathbb{Z}$ is generated by the hyperplane section class H. We have $\delta = (H.H) = 2g-2$. The exact sequence (3) now becomes

$$0 \to \mathbb{Z} \to \mathbb{Z} \to D \to 0,$$

where $E = D = \mathbb{Z}/(2g-2)$ with trivial Γ-action. The proof of Theorem 4.2 gives an injection

$$\mathrm{Coker}(\alpha) \hookrightarrow (\mathbb{Z}/(2g-2))^{21}.$$

Since $\mathrm{H}^1(k,\mathbb{Z}) = 0$, the map $\partial_1 : \mathrm{H}^1(k,D) \to \mathrm{H}^2(k,\mathbb{Z}) = \mathrm{H}^2(k,\mathrm{NS}(\overline{X})) = \mathrm{H}^2(k,\mathrm{Pic}(\overline{X}))$ is injective. If k is a number field, S is the set of bad reduction primes for X, and T is the set of primes dividing $2g-2$, then Proposition 4.4 gives an injective homomorphism

$$\mathrm{Coker}(\alpha) \hookrightarrow \mathrm{H}^1_{S\cup T}(k,\mathbb{Z}/(2g-2)).$$

When X is a product of two curves we have a result similar to Theorem 4.2 (ii) but in a case when $\mathrm{H}^1(X,O_X) \neq 0$.

Proposition 5.2 *Let $X = C_1 \times C_2$ be a product of two smooth, projective and geometrically integral curves over a field k of characteristic zero. Let J_1 and J_2 be the Jacobians of C_1 and C_2, respectively. Assume X has a k-point. Then we have the following statements.*

(i) *The exponent of the cokernel of* $\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})^\Gamma$ *divides* δ_0.

(ii) *If* $\mathrm{Hom}_{\overline{k}}(\overline{J}_1, \overline{J}_2) = 0$, *then* $\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})^\Gamma$ *is surjective.*

Proof. The following facts are well known, see [14, §1; Cor. 6.2]. The natural map of Γ-modules

$$(p_1^*, p_2^*) : \mathrm{Pic}(\overline{C}_1) \oplus \mathrm{Pic}(\overline{C}_2) \longrightarrow \mathrm{Pic}(\overline{X})$$

is a split injection, a retraction of which is given by the choice of a k-point $M = (M_1, M_2) \in X(k) = C_1(k) \times C_2(k)$. This induces an isomorphism of Γ-modules

$$\mathrm{Pic}^0(\overline{C}_1) \oplus \mathrm{Pic}^0(\overline{C}_2) \xrightarrow{\sim} \mathrm{Pic}^0(\overline{X}).$$

The cokernel of (p_1^*, p_2^*) is the finitely generated $\mathbb{Z}$-free Γ-module $\mathrm{Hom}_{\overline{k}-\mathrm{grp}}(\overline{J}_1, \overline{J}_2)$. There is an induced exact sequence of finitely generated $\mathbb{Z}$-free Γ-modules

$$0 \to \mathrm{NS}(\overline{C}_1) \oplus \mathrm{NS}(\overline{C}_2) \to \mathrm{NS}(\overline{X}) \to \mathrm{Hom}_{\overline{k}-\mathrm{grp}}(\overline{J}_1, \overline{J}_2) \to 0,$$

that is

$$0\to\mathbb{Z}\oplus\mathbb{Z}\to\mathrm{NS}(\overline{X})\to\mathrm{Hom}_{\overline{k}-\mathrm{grp}}(\overline{J}_1,\overline{J}_2)\to 0,$$

with a splitting associated to the point M. The k-point $M\in X(k)$ thus gives rise to a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^2(k,\mathrm{Pic}^0(\overline{X})) \arrow[r] \arrow[d, "\simeq"] & \mathrm{H}^2(k,\mathrm{Pic}(\overline{X})) \arrow[d] \\
\mathrm{H}^2(k,\mathrm{Pic}^0(\overline{C}_1)\oplus\mathrm{Pic}^0(\overline{C}_2)) \arrow[r, hook] & \mathrm{H}^2(k,\mathrm{Pic}(\overline{C}_1)\oplus\mathrm{Pic}(\overline{C}_2))
\end{tikzcd} \tag{27}$$

The injection in the bottom row is due to the fact that $\mathrm{H}^1(k,\mathbb{Z})=0$.

We proceed as in the proof of Theorem 2.1. By Proposition 1.3, for $i=1,2$, we have commutative diagrams of exact sequences

$$\begin{tikzcd}
\mathrm{Br}(X) \arrow[r] \arrow[d] & \mathrm{Br}(\overline{X})^{\Gamma} \arrow[r, "\beta_X"] \arrow[d] & \mathrm{H}^2(k,\mathrm{Pic}(\overline{X})) \arrow[d] \\
\mathrm{Br}(C_i) \arrow[r] & \mathrm{Br}(\overline{C}_i)^{\Gamma} \arrow[r, "\beta_{C_i}"] & \mathrm{H}^2(k,\mathrm{Pic}(\overline{C}_i))
\end{tikzcd}$$

By Tsen's theorem we have $\mathrm{Br}(\overline{C}_i)=0$. Thus

$$\beta(\mathrm{Br}(\overline{X})^{\Gamma})\subset\mathrm{Ker}[\mathrm{H}^2(k,\mathrm{Pic}(\overline{X}))\to\mathrm{H}^2(k,\mathrm{Pic}(\overline{C}_1)\oplus\mathrm{Pic}(\overline{C}_2))].$$

Since $\mathrm{NS}(\overline{X})$ is torsion-free, we see (see the Remark after Theorem 4.3) that $\mathrm{H}^3_{\text{ét}}(\overline{X},\mathbb{Z}_\ell(1))$ is torsion-free for any ℓ, which implies $\mathrm{Br}^0(\overline{X})=\mathrm{Br}(\overline{X})$. By Proposition 4.1 the image of $\mathrm{Br}(\overline{X})^{\Gamma}=\mathrm{Br}^0(\overline{X})^{\Gamma}$ in $\mathrm{H}^2(k,N^1)=\mathrm{H}^2(k,\mathrm{NS}(\overline{X}))$ is annihilated by δ_0. Thus any element in $\delta_0.\beta(\mathrm{Br}(\overline{X})^{\Gamma})\subset\mathrm{H}^2(k,\mathrm{Pic}(\overline{X}))$ comes from $\mathrm{H}^2(k,\mathrm{Pic}^0(\overline{X}))$ and goes to zero in $\mathrm{H}^2(k,\mathrm{Pic}(\overline{C}_1)\oplus\mathrm{Pic}(\overline{C}_2))$. From (27) we conclude $\delta_0.\beta(\mathrm{Br}(\overline{X})^{\Gamma})=0$. This proves (i).

If $\mathrm{Hom}_{\overline{k}-\mathrm{grp}}(\overline{J}_1,\overline{J}_2)=0$, then $\mathrm{NS}(\overline{C}_1)\oplus\mathrm{NS}(\overline{C}_2)\tilde{\longrightarrow}\mathrm{NS}(\overline{X})$ and $\delta=1$. QED

Examples

1. In the case where $C_1=E$ and $C_2=E'$ are elliptic curves not isogenous over $\overline{k}$, compare with [21, Prop. 3.3].

2. If $C_1=C_2$ is an elliptic curve E without complex multiplication over $\overline{k}$, then $\mathrm{NS}(\overline{X})$ is a free abelian group of rank 3, generated by the classes of $E\times\{0\}$ and $\{0\}\times E$ and the diagonal Δ. We have $\delta=2$, $b_2=6$, $\rho=3$, hence the cokernel of $\alpha:\mathrm{Br}(X)\to\mathrm{Br}(\overline{X})^{\Gamma}$ is isomorphic to a subgroup of $(\mathbb{Z}/2)^3$. See [21, Prop. 4.3] for an example with $\mathrm{Coker}(\alpha)\neq 0$.

6 Open varieties

This section partially answers a question raised by T. Szamuely.

Proposition 6.1 *Let k be a field of finite type over $\mathbb{Q}$. Let U be a smooth, quasiprojective k-variety. Then $\mathrm{H}^1_{\text{ét}}(\overline{U},\mathbb{Q}/\mathbb{Z})^\Gamma$ is a finite group.*

Proof. This is a reformulation of a special case of a result of Katz and Lang [8, Thm. 1, p. 295]. One may also give a proof along the following more general lines. One may assume that U/k is geometrically connected. By Hironaka's theorem, there exists a smooth, projective, geometrically integral k-variety X which contains U as a dense open set. Let $Z = X \setminus U$, and let $F \subset Z$ be the singular locus of Z. Let $X^0 = X \setminus F$ and $Z^0 = Z \setminus F$. Since X^0 and F^0 are smooth, the localisation sequences for étale cohomology with finite coefficients and purity give rise to the exact sequences of Γ-modules

$$0\to \mathrm{H}^1_{\text{ét}}(\overline{X}^0,\mathbb{Q}_\ell/\mathbb{Z}_\ell)\to \mathrm{H}^1_{\text{ét}}(\overline{U},\mathbb{Q}_\ell/\mathbb{Z}_\ell)\to \mathrm{H}^0(\overline{Z}^0,\mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)).$$

Since F is of codimension at least 2 in X, the inclusion $X^0\to X$ induces an isomorphism

$$\mathrm{H}^1_{\text{ét}}(\overline{X},\mathbb{Q}_\ell/\mathbb{Z}_\ell) = \mathrm{H}^1_{\text{ét}}(\overline{X}^0,\mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

We thus get an exact sequence

$$0\to \mathrm{H}^1_{\text{ét}}(\overline{X},\mathbb{Q}_\ell/\mathbb{Z}_\ell)^\Gamma\to \mathrm{H}^1_{\text{ét}}(\overline{U},\mathbb{Q}_\ell/\mathbb{Z}_\ell)^\Gamma\to \mathrm{H}^0(\overline{Z}^0,\mathbb{Q}_\ell/\mathbb{Z}_\ell(-1))^\Gamma.$$

The smooth k-variety Z^0 decomposes as a disjoint union $Z^0 = \cup_i Z_i$ of connected smooth k-varieties. Let k_i denote the integral closure of k in the function field $k(Z_i)$. We choose a k-embedding $k_i \subset \overline{k}$, and let $\Gamma_i = \mathrm{Gal}(\overline{k}/k_i)$. The group $\mathrm{H}^0(\overline{Z}^0,\mathbb{Q}_\ell/\mathbb{Z}_\ell(-1))^\Gamma$ is the direct sum $\oplus_i(\mathbb{Q}_\ell/\mathbb{Z}_\ell(-1))^{\Gamma_i}$. Each group $(\mathbb{Q}_\ell/\mathbb{Z}_\ell(-1))^{\Gamma_i}$ is finite; moreover, it is zero for almost all ℓ. Indeed, this statement is immediately reduced to the following statement: if k is a number field and $\Gamma = \mathrm{Gal}(\overline{k}/k)$, then $(\mathbb{Q}_\ell/\mathbb{Z}_\ell(-1))^\Gamma$ is finite, and is zero for almost all ℓ. One is then reduced to checking that $\mathrm{H}^1_{\text{ét}}(\overline{X},\mathbb{Q}_\ell/\mathbb{Z}_\ell)^\Gamma$ is finite, and is zero for almost all ℓ. This follows from the proper base change theorem and the Weil conjectures (cf. [1, Thm. 1.5]). QED

Theorem 6.2 *Let k be a field of finite type over $\mathbb{Q}$, and let U be a smooth, quasiprojective, geometrically integral k-variety. Then we have the following statements.*

(i) *The quotient* $\mathrm{Br}(\overline{U})^{\Gamma}/\mathrm{Im}(\mathrm{Br}(U))$ *is a finite group.*

(ii) *If* U *is a surface, and the* ℓ*-adic Tate conjecture for divisors holds for a smooth compactification of* U*, then* $\mathrm{Br}(\overline{U})\{\ell\}^{\Gamma}$ *is finite.*

(iii) *If the* ℓ*-adic Tate conjecture for divisors holds for a smooth compactification* X *of* U*, and if, moreover, the Galois module* $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1))$ *is semisimple, then* $\mathrm{Br}(\overline{U})\{\ell\}^{\Gamma}$ *is finite.*

Proof. We follow the beginning of proof of Proposition 6.1 and use the same notation. The localisation sequences for étale cohomology with finite coefficients and purity give rise to the exact sequence of Γ-modules

$$0\to\mathrm{Br}(\overline{X}^0)\to\mathrm{Br}(\overline{U})\to\mathrm{H}^1_{\text{ét}}(\overline{F}^0, \mathbb{Q}/\mathbb{Z}).$$

Since the codimension of F in X is at least 2, the purity theorem for the Brauer group shows that the restriction map $\mathrm{Br}(\overline{X})\to\mathrm{Br}(\overline{X}^0)$ is an isomorphism. We thus have an exact sequence of Γ-modules

$$0\to\mathrm{Br}(\overline{X})\to\mathrm{Br}(\overline{U})\to\mathrm{H}^1_{\text{ét}}(\overline{F}^0, \mathbb{Q}/\mathbb{Z}).$$

Taking invariants under Γ, we get an exact sequence

$$0\to\mathrm{Br}(\overline{X})^{\Gamma}\to\mathrm{Br}(\overline{U})^{\Gamma}\to\mathrm{H}^1_{\text{ét}}(\overline{F}^0, \mathbb{Q}/\mathbb{Z})^{\Gamma}.$$

By Proposition 6.1, the group $\mathrm{H}^1_{\text{ét}}(\overline{F}^0, \mathbb{Q}/\mathbb{Z})^{\Gamma}$ is finite. Statements (ii) and (iii) are then a consequence of the finiteness of $\mathrm{Br}(\overline{X})\{\ell\}^{\Gamma}$, which holds under the respective hypotheses of (ii) and (iii), see [2, Prop. 4.1].

Functoriality gives a commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Br}(\overline{X})^{\Gamma} \arrow[r] & \mathrm{Br}(\overline{U})^{\Gamma} \arrow[r] & \mathrm{H}^1_{\text{ét}}(\overline{F}^0, \mathbb{Q}/\mathbb{Z})^{\Gamma} \\
& \mathrm{Br}(X) \arrow[r] \arrow[u] & \mathrm{Br}(U) \arrow[u] &
\end{tikzcd}$$

By Theorem 2.1, the quotient $\mathrm{Br}(\overline{X})^{\Gamma}/\mathrm{Im}(\mathrm{Br}(X))$ is finite. By Proposition 6.1, the group $\mathrm{H}^1_{\text{ét}}(\overline{F}^0, \mathbb{Q}/\mathbb{Z})^{\Gamma}$ is finite. This implies that the quotient $\mathrm{Br}(\overline{U})^{\Gamma}/\mathrm{Im}(\mathrm{Br}(U))$ is finite as well, which is statement (i). QED

Remark We do not know whether (i) still holds over an arbitrary field k of characteristic zero.

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