THE BRAUER-MANIN OBSTRUCTION AND THE FIBRATION METHOD - LECTURE BY JEAN-LOUIS COLLIOT-THÉLÈNE

These are informal notes on the lecture I gave at IU Bremen on July 14th, 2005. Steve Donnelly prepared a preliminary set of notes, which I completed. [1]

1. The Brauer-Manin obstruction

For a scheme X, the Brauer group $\operatorname{Br} X$ is $\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$ ([9]). When X is a regular, noetherian, separated scheme, this coincides with the Azumaya Brauer group.

If F ia a field, then $\operatorname{Br}(\operatorname{Spec} F) = \operatorname{Br} F = \mathrm{H}^2(\mathrm{Gal}(\overline{F}/F), F^\times)$.

Let k be a field and X a k-variety. For any field F containing k, each element $A \in \operatorname{Br} X$ gives rise to an "evaluation map" $ev_A : X(F) \to \operatorname{Br} F$.

For a number field k, class field theory gives a fundamental exact sequence

$$0 \longrightarrow \operatorname{Br} k \longrightarrow \bigoplus_{\text{all } v} \operatorname{Br} k_v \xrightarrow{inv_v} \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

That the composite map from $\operatorname{Br} k$ to $\mathbb{Q}/\mathbb{Z}$ is zero is a generalization of the quadratic reciprocity law.

Now suppose X is a projective variety defined over a number field k, and $A \in \operatorname{Br} X$. One has the commutative diagram

$$\begin{array}{ccccc} X(k) & \longrightarrow & \prod_v X(k_v) & & \\ \downarrow{\scriptstyle ev_A} & & \downarrow{\scriptstyle ev_A} & & \\ \operatorname{Br} k & \longrightarrow & \oplus_v \operatorname{Br} k_v & \longrightarrow & \mathbb{Q}/\mathbb{Z} \end{array}$$

The second vertical map makes sense because, if X is projective, then for each fixed A, $ev_A : X(k_v) \to \operatorname{Br} k_v$ is the zero map for all but finitely many v.

Let $\Theta_A : \prod_v X(k_v) \to \mathbb{Q}/\mathbb{Z}$ denote the composed map. Thus $X(k) \subseteq \ker \Theta_A$ for all A. Given a subgroup $B \subseteq \operatorname{Br} X$, we write

$$X(\mathbb{A}_k)^B := \bigcap_{A \in B} \ker \Theta_A$$

(where, for X projective, $X(\mathbb{A}_k) = \prod_v X(k_v)$). With this notation one has the inclusions

$$X(k) \subseteq X(\mathbb{A}_k)^{\operatorname{Br} X} \subseteq X(\mathbb{A}_k)^B \subseteq X(\mathbb{A}_k)\,.$$

In this way, each $A \in \operatorname{Br} X$ potentially obstructs the existence of k-rational points on X.

Example: Let $V/\mathbb{Q}$ be the affine surface $0 \neq x^2 + y^2 = (3 - t^2)(t^2 - 2)$ and let $X \supset V$ be a smooth compactification. We take the element $A \in \operatorname{Br} X$ which is given on V by the quaternion algebra $(-1, 3 - t^2)$. One finds that for all places $v \neq 2$, $ev_A : X(k_v) \to \operatorname{Br} k_v$ is identically zero; however at $v = 2$, ev_A is identically $\frac{1}{2}$. Therefore $X(\mathbb{Q})$ must be empty.

The surface V is a special case of a surface defined by an affine equation $x^2 - ay^2 = P(t)$, with $a \in k^\times$ and $P(t) \in k[t]$ a polynomial of degree 4. A smooth projective model of such a surface is called a Châtelet surface. In 1984, Sansuc, Swinnerton-Dyer and I proved that

[1] References updated, May 4th, 2008; some hints at later literature, July 2012

if X is a Châtelet surface, then the condition $X(\mathbb{A}_k)^{\mathrm{Br}\,X} \neq \emptyset$ is a necessary and sufficient condition for the existence of a k-point. We actually showed that $X(k)$ is dense in $X(\mathbb{A}_k)^{\mathrm{Br}\,X}$ (under the diagonal embedding).

One may wonder to which extent this result holds for other classes of varieties. That it could not extend to all varieties was foreseeable but it is only in 1999 that the first unconditional example of a smooth, projective, geometrically connected variety with $X(\mathbb{A}_k)^{\mathrm{Br}\,X} \neq \emptyset$ but $X(k) = \emptyset$ was produced (Skorobogatov, further work since then by Harari and Skorobogatov, see [16]).

For X a curve of genus one, if the Tate-Shafarevich group of the Jacobian of X is finite, then $X(\mathbb{A}_k)^{\mathrm{Br}\,X} \neq \emptyset$ implies $X(k) \neq \emptyset$. This observation is due to Manin (1970).

For curves of genus bigger than 1, quite surprisingly, it does not seem absurd to ask whether the same statement holds (see [18]).

2. Calculating the Brauer group

One would like to be able to compute $X(\mathbb{A}_k)^{\mathrm{Br}\,X}$. For this, a prerequisite is to compute the Brauer group $\mathrm{Br}\,X$, or a least a system of representants of $\mathrm{Br}\,X/\,\mathrm{Br}\,k$.

Suppose $\mathrm{char}\,k$ is zero, and X/k is smooth, projective and geometrically connected. We write $\overline{X} := X \times_k \overline{k}$, where $\overline{k}$ is an algebraic closure of k.

2.1. The "geometric" Brauer group.

For computing $\mathrm{Br}\,\overline{X}$, we have an exact sequence

$$0 \to (\mathbb{Q}/\mathbb{Z})^{b_2-\rho} \to \mathrm{Br}\,\overline{X} \to \mathrm{H}^3(\overline{X}, \mathbb{Z})_{tors} \to 0.$$

Here b_2 is the second Betti number, which one computes by using either l-adic cohomology $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_l)$ for an arbitrary prime l or by using classical cohomology $\mathrm{H}^2(X \times_k \mathbb{C}, \mathbb{Q})$ if an embedding $k \subset \mathbb{C}$ is given. The integer $\rho = \mathrm{rk}\,\mathrm{NS}\,\overline{X}$ is the rank of the geometric Néron-Severi group. The vanishing of $b_2 - \rho$ is equivalent to the vanishing of the coherent cohomology group $\mathrm{H}^2(X, O_X)$. The group $\mathrm{H}^3(\overline{X}, \mathbb{Z})_{tors}$ is a finite group, which one computes either as the direct sum over all primes l of the torsion in integral l-adic cohomology $\mathrm{H}^3_{\text{ét}}(\overline{X}, \mathbb{Z}_l)$ or as the torsion in classical cohomology $\mathrm{H}^3(X \times_k \mathbb{C}, \mathbb{Z})$ if an embedding $k \subset \mathbb{C}$ is given. If X is a curve, or if $\overline{X}$ is birational to a projective space, then $\mathrm{Br}\,\overline{X} = 0$.

Remarks

1. It is in general quite difficult to exhibit the Azumaya algebras on $\overline{X}$ corresponding to the divisible subgroup $(\mathbb{Q}/\mathbb{Z})^{b_2-\rho}$.
2. When k is a number field, it is an open question whether the group of fixed points $(\mathrm{Br}\,\overline{X})^{\mathrm{Gal}(\overline{k}/k)}$ is finite.

Complement (July 2012)
Further works on this topic :
Skorobogatov and Zarhin
Ieronymou, Skorobogatov and Zarhin
Colliot-Thélène et Skorobogatov
Hassett and Várilly-Alvarado

2.2. The "algebraic" Brauer group.

Define $\mathrm{Br}_1(X) := \ker[\mathrm{Br}\,X \to \mathrm{Br}\,\overline{X}]$. For computing this group, we have the exact sequence

$$0 \to \mathrm{Pic}\,X \to (\mathrm{Pic}\,\overline{X})^{\mathrm{Gal}(\overline{k}/k)} \to^* \mathrm{Br}\,k \to \mathrm{Br}_1\,X \to \mathrm{H}^1(k, \mathrm{Pic}\,\overline{X}) \to^* \mathrm{H}^3(k, \overline{k}^{\times})$$

where the maps marked with a $*$ are zero when $X(k)$ is nonempty. The group $\mathrm{H}^3(k, \overline{k}^\times)$ is trivial when k is a number field (this is a nontrivial result from class field theory).

There are cases where it is easy to explicitly compute the group $\mathrm{H}^1(k, \mathrm{Pic}\, \overline{X})$ but where it is difficult to lift a given element of that group to an explicit element of $\mathrm{Br}_1(X)$: Even if one knows a 3-cocycle is a 3-coboundary, it is not easy to write it down as an explicit 3-coboundary. This may create difficulties for deciding whether a given $X(\mathbb{A}_k)^{\mathrm{Br}\, X}$ is empty or not. Such a delicate situation arises in the study of diagonal cubic surfaces ([6], [13]).

To get a hold on $\mathrm{H}^1(k, \mathrm{Pic}\, \overline{X})$ one uses the exact sequence of Galois-modules

$$0 \to \mathrm{Pic}^0_{X/k}(\overline{k}) \to \mathrm{Pic}\, \overline{X} \to \mathrm{NS}\, \overline{X} \to 0\,.$$

Here $\mathrm{NS}\, \overline{X}$ is of finite type. If $\mathrm{NS}\, \overline{X}_{tors} = 0$, then $\mathrm{H}^1(k, \mathrm{NS}\, \overline{X})$ is a finite group.

2.3. **Curves.** If $X = C$ is a curve, then $\mathrm{Br}_1 C = \mathrm{Br}\, C$ (as noted above), and the above exact sequence reads

$$0 \to \mathrm{Jac}\, C(\overline{k}) \to \mathrm{Pic}\, \overline{C} \to \mathbb{Z} \to 0.$$

Since $\mathrm{H}^1(k, \mathbb{Z}) = 0$, we thus have the exact sequence

$$(\mathrm{Pic}\, \overline{C})^{\mathrm{Gal}(\overline{k}/k)} \to \mathbb{Z} \to \mathrm{H}^1(k, \mathrm{Jac}\, C(\overline{k})) \to \mathrm{H}^1(k, \mathrm{Pic}\, \overline{C}) \to 0$$

which one may combine with the above long exact sequence. The group $\mathrm{H}^1(k, \mathrm{Jac}\, C(\overline{k}))$ classifies principal homogeneous spaces under $\mathrm{Jac}\, C = \mathrm{Pic}^0_{C/k}$. The map $\mathbb{Z} \to \mathrm{H}^1(k, \mathrm{Jac}\, C(\overline{k}))$ sends 1 to the class of the principal homogeneous space $\mathrm{Pic}^1_{C/k}$. If k is a number field, we thus have a surjective map from $\mathrm{Br}\, C$ to a quotient of $\mathrm{H}^1(k, \mathrm{Jac}\, C(\overline{k}))$. In practice, it is quite hard to lift an element of this quotient to an explicit element of $\mathrm{Br}\, C$.

Examples

1. If $C = \mathbb{P}^1_k$, then the natural map $\mathrm{Br}\, k \to \mathrm{Br}\, \mathbb{P}^1_k$ is an isomorphism.
2. If C is a smooth projective conic with no rational point, we have an exact sequence

$$0 \to \mathbb{Z}/2 \to \mathrm{Br}\, k \to \mathrm{Br}\, C \to 0$$
$$1 \mapsto [A_C]$$

where $[A_C] \in {}_2\mathrm{Br}\, k$ is the class corresponding to C.

2.4. **Residues.** Let A be a discrete valuation ring with field of fractions F and with residue field κ of characteristic zero. There is a natural "residue map" $\mathrm{Br}\, F \to \mathrm{H}^1(\kappa, \mathbb{Q}/\mathbb{Z})$ and an exact sequence

$$0 \to \mathrm{Br}\, A \to \mathrm{Br}\, F \to \mathrm{H}^1(\kappa, \mathbb{Q}/\mathbb{Z})\,.$$

Let k be a field of characteristic zero. Let X be a smooth, integral, k-variety with function field $k(X)$. Given a closed integral subvariety $Y \subset X$ of codimension 1, with function field $k(Y)$, we may consider the residue map $\mathrm{Br}\, k(X) \to \mathrm{H}^1(k(Y), \mathbb{Q}/\mathbb{Z})$. One then has (Grothendieck) the exact sequence

$$0 \to \mathrm{Br}\, X \to \mathrm{Br}\, k(X) \to \bigoplus_Y \mathrm{H}^1(k(Y), \mathbb{Q}/\mathbb{Z}),$$

where Y runs through all codimension 1 subvarieties of X as above.

From the exactness of this sequence one deduces that $\mathrm{Br}\, X$ is a birational invariant for smooth, projective, integral k-varieties.

2.5. **The projective line.** Let us consider the special case $X = \mathbb{P}^1_k$. As noted above, $\operatorname{Br} \mathbb{P}^1_k = \operatorname{Br} k$. The short exact sequence above thus reads

$$0 \to \operatorname{Br} k \to \operatorname{Br} k(\mathbb{P}^1) \to \bigoplus_{P \in \mathbb{P}^1_k} \mathrm{H}^1(k_P, \mathbb{Q}/\mathbb{Z}),$$

where P runs through the closed points of $\mathbb{P}^1_k$ and k_P is the residue field at such a point P.

One may compute the cokernel of the last map : there is an exact sequence

$$0 \longrightarrow \operatorname{Br} k \longrightarrow \operatorname{Br} k(\mathbb{P}^1) \longrightarrow \bigoplus_{P \in \mathbb{P}^1_k} \mathrm{H}^1(k_P, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sum_P N_{k_P/k}} \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0,$$

where $N_{k_{P/k}}$ is the corestriction map.

2.6. **Conic bundles over the projective line.** Let X/k be a smooth, projective, geometrically connected surface equipped with a morphism $X \to \mathbb{P}^1_k$ whose generic fibre X_η is a smooth conic over $K = k(\mathbb{P}^1_k) = k(t)$. After performing k-birational transformations one may assume that for each closed point $P \in \mathbb{P}^1_k$, the fibre X_P is a conic over the residue field k_P, and that $X \to \mathbb{P}^1_k$ is relatively minimal. There are finitely many points $P \in \mathbb{P}^1_k$ for which X_P is not smooth. At such a point P, there is a quadratic extension F_P/k_P over which X_P splits into a pair of transversal lines. Write $F_P = k_P(\sqrt{a_P})$.

Let $A \in \operatorname{Br} K$ be the class of a quaternion algebra over K associated to the conic X_η/K, as in example 2 of section 2.3.

We shall assume that A does not come from $\operatorname{Br} k$. In the long exact sequence associated to $\mathbb{P}^1$ in section 2.5, for each closed point $P \in \mathbb{P}^1_k$, the residue $\delta_P(A) \in \mathrm{H}^1(k_P, \mathbb{Q}/\mathbb{Z})$ lies in $\mathrm{H}^1(F_P/k_P, \mathbb{Z}/2) = \mathbb{Z}/2$.

Using 2.3, 2.4 and 2.5, one shows that there is an exact sequence

$$0 \to \operatorname{Br} k \to \operatorname{Br} X \to (\oplus_P (\mathbb{Z}/2)_P)/(\{\delta_P(A)\}) \to k^\times/k^{\times 2}.$$

The last map sends the class of the element $1 \in (\mathbb{Z}/2)_P = \mathrm{H}^1(F_P/k_P, \mathbb{Z}/2) \subset \mathrm{H}^1(k_P, \mathbb{Q}/\mathbb{Z})$ to the class of $N_{k_P/k}(a_P) \in k^\times/k^{\times 2}$.

In this situation one may give explicit generators for $\operatorname{Br} X/\operatorname{Br} k$. They are given as the images under $\operatorname{Br} k(t) \to \operatorname{Br} k(X)$ of suitable linear combinations of elements of the shape $\mathrm{Cores}_{k_P/k}(t-\alpha_P, \beta_P) \in \operatorname{Br} k(t)$, where $k_P = k(\alpha_P)$, $\beta_P \in k_P^\times$, and $(t-\alpha_P, \beta_P)$ is a quaternion algebra over the field $k_P(t)$.

Added July 2012. Since a conic bundle $X/\mathbb{P}^1_k$ contains a smooth conic $Y \subset X$, functoriality of the exact sequence

$$0 \to \operatorname{Pic} X \to (\operatorname{Pic} \overline{X})^{\mathrm{Gal}(\overline{k}/k)} \to^* \operatorname{Br} k \to \operatorname{Br}_1 X \to \mathrm{H}^1(k, \operatorname{Pic} \overline{X}) \to^* \mathrm{H}^3(k, \overline{k}^\times)$$

implies that the map $\mathrm{H}^1(k, \operatorname{Pic} \overline{X}) \to^* \mathrm{H}^3(k, \overline{k}^\times)$ *is zero.*

2.7. **Computing when no smooth projective model is available.** One is often confronted with the following problem : given a smooth, affine, geometrically connected variety U over a field k, compute the Brauer group of a smooth compactification X of U without knowing a single such smooth compactification. The point as far as local to global problems are concerned is that it is only the Brauer group of smooth compactifications which naturally produces obstructions to the existence of rational points. A preliminary question is to compute $\mathrm{H}^1(k, \operatorname{Pic} \overline{X})$ (also a birational invariant of smooth, projective varieties).

Assume $U = T$ is a k-torus, i.e. an algebraic group which over $\overline{k}$ becomes isomorphic to a product of multiplicative groups. To such a k-torus there is associated its character group

$\hat{T}$ (over $\overline{k}$). This is a g-lattice (g being the Galois group of $\overline{k}$ over k). For X any smooth k-compactification of T, one has

$$\mathrm{H}^1(k, \mathrm{Pic}\,\overline{X}) = \ker[\mathrm{H}^2(g, \hat{T}) \to \prod_h \mathrm{H}^2(h, \hat{T})],$$

where $h \subset g$ runs through all closed pro-cyclic subgroups of g – as a matter of fact the computation of this kernel may be done after going over only to a suitable finite Galois extension of k.

It seems hard to lift the elements of $\mathrm{H}^1(k, \mathrm{Pic}\,\overline{X})$ to explicit elements in $\mathrm{Br}\,X$. The situation gets worse if X is a smooth compactification of a principal homogeneous space U under T. We have the same formula for $\mathrm{H}^1(k, \mathrm{Pic}\,\overline{X})$ as above, but in this case for k arbitrary there is no reason why the map $\mathrm{Br}_1\,X \to \mathrm{H}^1(k, \mathrm{Pic}\,\overline{X})$ should be surjective. If k is a number field, the map is surjective but lifting seems nevertheless very hard. Hence it seems difficult to test the condition $X(\mathbb{A}_k)^{\mathrm{Br}\,X_c} \neq \emptyset$.

Probably the simplest nontrivial example is the norm 1 torus $T = R^1_{K/k}\mathbb{G}_m$ associated to a biquadratic extension $K = k(\sqrt{a}, \sqrt{b})/k$. In this case one finds $\mathrm{H}^1(k, \mathrm{Pic}\,\overline{X}) = \mathbb{Z}/2$. The same result holds if $X = X_c$ is a smooth compactification of a a principal homogeneous space of $R^1_{K/k}\mathbb{G}_m$, that is a variety $U = U_c$ given by an equation $\mathrm{Norm}_{K/k}(\mathbf{z}) = c$ for some $c \in k^{\times}$. If k is a number field, U_c has points in all completions of k, and X_c is a smooth compactification of U, then there exists some $A \in \mathrm{Br}\,X_c$ such that $X(\mathbb{A}_k)^{\mathrm{Br}\,X_c} = X(\mathbb{A}_k)^A$. But how to compute such an A in a systematic fashion ?

The question is important, since in this case it is known that $X_c(\mathbb{A}_k)^{\mathrm{Br}\,X_c} \neq \emptyset$ implies $X_c(k) \neq \emptyset$. The latter statement is a general fact for principal homogeneous spaces of connected linear algebraic groups (Sansuc [14]), and it holds more generally for smooth compactifications of homogeneous spaces under connected linear algebraic groups, at least when the geometric stabilizer group is connected (Borovoi).

Coming back to the case of equations $\mathrm{Norm}_{K/k}(\mathbf{z}) = c$, in the case $K = k(\sqrt{a}, \sqrt{b})$, Sansuc [15] gives an algorithm to decide whether $X_c(\mathbb{A}_k)^{\mathrm{Br}\,X_c} \neq \emptyset$. It would be interesting to understand this algorithm better.

It is natural to study varieties which are given as the total space of a one-parameter family of principal homogeneous spaces. A special but already difficult case is that of varieties given by an affine equation

$$\mathrm{Norm}_{K/k}(\mathbf{z}) = P(t)$$

where K/k is a finite field extension and $P(t)$ a polynomial in one variable. In [5] the group $\mathrm{H}^1(k, \mathrm{Pic}\,\overline{X})$ for smooth projective models X of varieties defined by such an equation was computed for many cases, but it could not computed in all cases. See the questions raised at the end of section 2 of [5].

Added July 2012. Further work on this section. Borovoi–Kunyavskiĭ, CT–Kunyavskiĭ, Demarche, Borovoi–Demarche, D. Wei, CT (on Orsay webpage).

3. The fibration method

Let k be a number field and X/k a smooth, projective, integral variety equipped with a dominant k-morphism $X \to \mathbb{P}^1_k$ whose generic fibre X_η is absolutely irreducible. For $P \in \mathbb{P}^1$ let X_P denote the fibre above P (so X_P is defined over k_P).

Question : *Assume that the smooth fibres over k-points of $\mathbb{P}^1_k$ satisfy the Hasse principle, or at least that the Brauer-Manin obstruction to the Hasse principle for these fibres is the only one. Does it follow that the Brauer-Manin obstruction to the Hasse principle is the only obstruction for X ?*

It is not reasonable to expect a positive answer in the general case, for instance for pencils of curves of genus one with multiple fibres. Here is a context in which one may conjecture a positive answer :

The generic fibre X_η is a smooth compactification of a (connected) homgeneous space of a connected algebraic group $G/k(\mathbb{P}^1)$, and

(i) the geometric stabilizer for this action is a connected group;

(ii) all fibres of $X \to \mathbb{P}^1_k$ have at least one component of multiplicity one.

(Condition (ii) is automatic when G is a connected linear algebraic group.)

We call a closed point $P \in \mathbb{P}^1$ a *good* point if X_P contains a multiplicity one component which is absolutely irreducible over k_P. Set

$$\delta := \sum_{\text{bad } P} [k_P : k] .$$

Theorem 3.1. *Suppose that $\delta = 0$, or that $\delta = 1$ and f has a section over $\overline{k}$. If the smooth fibres of f satisfy the Hasse principle, then X satisfies the Hasse principle.*

We have the following theorem of D. Harari.

Theorem 3.2. [10] [11] *Suppose $\delta \leq 1$ and f has a section over $\overline{k}$. Also suppose* $\operatorname{Pic} \overline{X_\eta}$ *is torsion free, and* $\operatorname{Br} \overline{X_\eta}$ *is finite. Assume that for all $P \in \mathbb{P}^1(k)$ for which X_P is smooth,*

$$X_P(\mathbb{A}_k)^{\mathrm{Br X_P}} \neq \emptyset \implies X_P(k) \neq \emptyset .$$

Then $X(\mathbb{A}_k)^{\operatorname{Br} X} \neq \emptyset$ implies $X(k) \neq \emptyset$.

The following corollary had been known for some time.

Corollary 3.3. *Assume that whenever X is a smooth intersection of two quadrics in $\mathbb{P}^4$, one has*

$$X(\mathbb{A}_k)^{\mathrm{BrX}} \neq \emptyset \implies X(k) \neq \emptyset .$$

Then the Hasse principle holds for every smooth intersection of two quadrics in a projective space $\mathbb{P}^n$ with $n \geq 5$.

Recall Schinzel's hypothesis, which asserts the following: given any finite set of irreducible polynomials $\{p_1(x), \ldots, p_n(x)\} \subset \mathbb{Z}[x]$ satisfying the trivial necessary conditions, there are infinitely many positive integers m for which $p_1(m), \ldots, p_n(m)$ are all prime.

The following result encompasses various earlier results (Sansuc and the speaker, Serre, Swinnerton-Dyer) :

Theorem 3.4. [8] *Assume Schinzel's hypothesis. With notation and hypotheses as in the beginning of the section, suppose*

1. *the Hasse principle holds for smooth fibres of $f : X \to \mathbb{P}^1$, and*
2. *for all $P \in \mathbb{P}^1_k$, there is a component $Z_P \subseteq X_P$ of multiplicity one such that the algebraic closure of k_P in $k(Z_P)$ is* **abelian** *over k_P.*

Then $X(\mathbb{A}_k)^{\operatorname{Br} X} \neq \emptyset \implies X(k) \neq \emptyset$.

One wonders whether the same theorem holds when the Hasse principle hypothesis on the smooth fibres is replaced by the hypothesis that the Brauer-Manin obstruction to the Hasse principle for these is the only one. This question is open already when the generic fibre X_η satisfies the conditions of Theorem 3.2, for instance when the generic fibre is birational to a principal homogeneous space of a connected linear algebraic group. The **abelianity** condition in hypothesis (2) has so far prevented us from establishing such a result.

Here is a special case of this question. Let K/k be a finite field extension, $P(t) \in k[t]$ a nonconstant polynomial of degree at least 2. Let X be a smooth projective model of the affine variety defined by the equation

$$\mathrm{Norm}_{K/k}(\mathbf{z}) = P(t).$$

Does $X(\mathbb{A}_k)^{\mathrm{Br}\, X} \neq \emptyset$ imply $X(k) \neq \emptyset$?

When K/k is cyclic, and the Schinzel hypothesis is granted, Theorem 3.4 yields a positive answer.

When $P(t)$ is separable and split of degree 2, and $k = \mathbb{Q}$, the answer is in the affirmative. The proof ([12] [5]) combines descent theory and the circle method.

In each of the following special cases, even at the expense of assuming the Schinzel hypothesis, the answer is not known :

1. The extension K/k is cubic but not Galois, and P is of degree 4 (or more).
2. $K = k(\sqrt{a}, \sqrt{b})/k$ is a biquadratic extension, and P is of degree 3 (or more).
3. $K = k(\sqrt{a}, \sqrt{b})/k$ is a biquadratic extension, and $P(t) = c(t^2 - a)$ for some $c \in k^\times$.

Starting with Swinnerton-Dyer's paper [19], work has been done on varieties with a pencil of homogeneous spaces of curves of genus one ([7], [20], [21], [17], [22]). In all these works, one assumes that Tate-Shafarevich groups of elliptic curves are finite. Indeed, one takes the case of curves for granted. In many of these papers, one also assumes the Schinzel hypothesis.

In the two papers [21] [17] however, one does without the Schinzel hypothesis; more precisely, in this case the Schinzel hypothesis is used in the only case when it has been established, that of one polynomial of degree one : in this case, this is Dirichlet's theorem on primes in an arithmetic progression.

The reader will find more detailed introductions to the fibration method and its applications in the surveys [1] [2] [3] and in the notes [4]. These reports do not cover the developments [17] [22].

Added July 2012. Further work on this section : Assuming Schinzel's hypothesis, Dasheng WEI gets case 1. above. Browning–Heath-Brown discuss case 3. Derenthal–Smeets–Wei show that in case 3. above, with $c \notin k^{\times 2}$, the Hasse principle and weak approximation hold.

References

[1] J.-L. Colliot-Thélène, L'arithmétique des variétés rationnelles, Annales de la Faculté des sciences de Toulouse, vol. **1**, numéro 3 (1992) 295–336.

[2] J.-L. Colliot-Thélène, The Hasse principle in a pencil of algebraic varieties, *Number theory (Tiruchirapalli, 1996)*, Contemp. Math. vol. **210**, Amer. Math. Soc. (Providence, RI, 1998), pp. 19–39.

[3] J.-L. Colliot-Thélène, Points rationnels sur les fibrations, in *Higher dimensional varieties and rational points* (Budapest, 2001), Bolyai Soc. Math. Stud. **12**, Springer, Berlin, 2003, pp. 171–221.

[4] J.-L. Colliot-Thélène, Sackler lectures, Tel-Aviv University, November 2003. Available at http://www.math.u-psud.fr/ colliot/liste-prepub.html

[5] J.-L. Colliot-Thélène, D. Harari et A. N. Skorobogatov, Valeurs d'un polynôme à une variable représentées par une norme, in *Number theory and algebraic geometry*, 69–89, London Math. Soc. Lecture Note Ser. **303**, Cambridge Univ. Press, Cambridge, 2003.

[6] J.-L. Colliot-Thélène, D. Kanevsky et J.-J. Sansuc, Arithmétique des surfaces cubiques diagonales, in *Diophantine approximation and transcendence theory*, Bonn, 1985 (ed. G. Wüstholz), Lecture Notes in Mathematics **1290** (Springer, Berlin, 1987), pp. 1–108.

[7] J.-L. Colliot-Thélène, A. N. Skorobogatov et Sir Peter Swinnerton-Dyer, Rational points and zero-cycles on fibred varieties: Schinzel's hypothesis and Salberger's device, J. reine angew. Math. (Crelle) **495** (1998), 1–28.

[8] J.-L. Colliot-Thélène, A. N. Skorobogatov and Sir Peter Swinnerton-Dyer, Hasse principle for pencils of curves of genus one whose Jacobians have rational 2-division points, Invent. math. **134** (1998), no. 3, 579–650.

[9] A. Grothendieck, Le groupe de Brauer I, II, III. Exemples et compléments, *Dix exposés sur la cohomologie des schémas,* North-Holland (Amsterdam); Masson (Paris, 1968), pp. 67–188.

[10] D. Harari, Méthode des fibrations et obstruction de Manin, Duke Math. J., **75** (1994), 221–260.

[11] D. Harari, Flèches de spécialisations en cohomologie étale et applications arithmétiques, Bull. Soc. Math. France, **125** (1997), 143–166.

[12] R. Heath-Brown and A. N. Skorobogatov, Rational solutions of certain equations involving norms, Acta Math. **189** (2002), no. 2, 161–177.

[13] A. Kresch and Yu. Tschinkel, On the arithmetic of del Pezzo surfaces of degree 2, Proc. London Math. Soc. (3) **89** (2004), no. 3, 545–569.

[14] J.-J. Sansuc, Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres, J. reine angew. Math. (Crelle) **327** (1981), 12–80.

[15] J.-J. Sansuc, À propos d'une conjecture arithmétique sur le groupe de Chow d'une surface rationnelle, Séminaire de théorie des nombres de Bordeaux, 1981/1982, Exp. No. **33**, 38 pp., Univ. Bordeaux I, Talence, 1982.

[16] A. N. Skorobogatov, *Torsors and rational points*, Cambridge Tracts in Mathematics **144**, Cambridge University Press, Cambridge, 2001.

[17] A. N. Skorobogatov and Sir Peter Swinnerton-Dyer, 2-descent on elliptic curves and rational points on certain Kummer surfaces. Adv. Math. **198** (2005), no. 2, 448–483.

[18] M. Stoll, Finite descent obstructions and rational points on curves. Algebra Number Theory 1 (2007), no. 4, 349–391.

[19] Sir Peter Swinnerton-Dyer, Rational points on certain intersections of two quadrics. Abelian varieties (Egloffstein, 1993), 273–292, de Gruyter, Berlin, 1995.

[20] Sir Peter Swinnerton-Dyer, Arithmetic of diagonal quartic surfaces. II. Proc. London Math. Soc. (3) **80** (2000), no. 3, 513–544. Corrigenda: Proc. London Math. Soc. (3) **85** (2002), no. 3, 564.

[21] Sir Peter Swinnerton-Dyer, The solubility of diagonal cubic surfaces. Ann. Sci. École Norm. Sup. (4) **34** (2001), no. 6, 891–912.

[22] O. Wittenberg, Intersections de deux quadriques et pinceaux de courbes de genre 1. Lecture Notes in Mathematics **1901**. Springer, Berlin, 2007.

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